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Nonproper products

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Abstract

We show that there exist two proper creature forcings having a simple (Borel) definition, whose product is not proper. We also give a new condition ensuring properness of some forcings with norms.

1. *Introduction*

In Rosłanowski and Shelah [4], a theory of forcings built with the use of norms was developed and a number of conditions to ensure the properness of the resulting forcings was given. However, it is not clear how sharp those results really are and this problem was posed in Shelah [7, Question 4.1]. In particular, he asked about the properness of the forcing notion

$$\mathbb{Q} = \left\{ \langle w_n : n < \omega \rangle : w_n \subseteq 2^n,\ w_n \neq \emptyset \text{ and } \lim_{n\to\omega} |w_n| = \infty \right\}$$

ordered by $\bar{w} \leqslant \bar{w}' \quad \Leftrightarrow \quad (\forall n \in \omega)(w'_n \subseteq w_n)$. In Section 2, we give a general criterion for collapsing the continuum to $\aleph_0$ and then in Corollary 2.8 we apply it to the forcing $\mathbb{Q}$, just showing that it is not proper.

That the property of properness is not productive, that is, is not preserved under taking products, has been observed by Shelah long ago (see [6, XVII, 2.12]). However, his examples are somewhat artificial and certainly it would be desirable to know of some rich enough subclass of proper forcings that is productively closed. It was a natural conjecture put forth by Zapletal, that the class of definable, say analytic or Borel, proper forcings would have this property. Actually, it was only proved recently by Spinas [8] that finite powers of the Miller rational perfect set forcing and finite powers of the Laver forcing notion are proper. These are two of the most frequently used forcings in the set theory of the reals. However, in this paper we will show that this phenomenon does not extend to all forcing notions defined in the setting of norms on possibilities. In Section 4 of the paper, we give an example of a forcing notion with norms which, by the theory developed in Section 2, is not proper and yet it can be decomposed as a product of two proper forcing notions of a very similar type, and both of which have a Borel definition. The properness of the factors is a consequence of a quite general theorem presented in Section 3 (Theorem 3.3). It occurs that a strong version of halving from [4, Section 2.2] implies the properness of forcing notions of the type $\mathbb{Q}^*_\infty(K, \Sigma)$. More on applications of halving can be found in Kellner and Shelah [2, 3] and Rosłanowski and Shelah [5].

Notation. Most of our notation is standard and compatible with that of classical textbooks on Set Theory (like Bartoszyński and Judah [1]). However, in forcing we keep the convention that a *stronger condition is the larger one*.

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In this paper, $\mathbf{H}$ will stand for a function with domain ω and such that $(\forall m\in\omega)(2\leqslant|\mathbf{H}(m)|<\omega)$. We also assume that $0\in\mathbf{H}(m)$ (for all $m\in\omega$); if it is not the case, then we fix an element of $\mathbf{H}(m)$ and we use it whenever appropriate notions refer to 0.

Creature background: Since our results are stated for creating pairs with several special properties, below we present a somewhat restricted context of the creature forcing, introducing *good creating pairs*.

DEFINITION 1.1. (1) *A creature for* $\mathbf{H}$ is a triple

$$t=(\mathbf{nor},\mathbf{val},\mathbf{dis})=(\mathbf{nor}[t],\mathbf{val}[t],\mathbf{dis}[t])$$

such that $\mathbf{nor}\in\mathbb{R}^{\geqslant 0}$, $\mathbf{dis}\in\mathcal{H}(\omega_1)$, and, for some integers $m^t_{\rm dn}<m^t_{\rm up}<\omega$,

$$\emptyset\neq\mathbf{val}\subseteq\left\{\langle u,v\rangle\in\prod_{i<m^t_{\rm dn}}\mathbf{H}(i)\times\prod_{i<m^t_{\rm up}}\mathbf{H}(i):u\vartriangleleft v\right\}.$$

The family of all creatures for $\mathbf{H}$ is denoted by $\mathrm{CR}[\mathbf{H}]$.

(2) Let $K\subseteq\mathrm{CR}[\mathbf{H}]$ and $\Sigma:K\to\mathcal{P}(K)$. We say that (K,Σ) is a *good creating pair* for $\mathbf{H}$ whenever the following conditions are satisfied for each $t\in K$:

(a) [*Fullness*] $\mathrm{dom}(\mathbf{val}[t])=\prod_{i<m^t_{\rm dn}}\mathbf{H}(i)$;

(b) $t\in\Sigma(t)$ and if $s\in\Sigma(t)$, then $\mathbf{val}[s]\subseteq\mathbf{val}[t]$ and so also $m^s_{\rm dn}=m^t_{\rm dn}$ and $m^s_{\rm up}=m^t_{\rm up}$;

(c) [*Transitivity*] If $s\in\Sigma(t)$, then $\Sigma(s)\subseteq\Sigma(t)$.

(3) A good creating pair (K,Σ) is

(a) *local* if $m^t_{\rm up}=m^t_{\rm dn}+1$ for all $t\in K$;

(b) *forgetful* if, for every $t\in K$, $v\in\prod_{i<m^t_{\rm up}}\mathbf{H}(i)$ and $u\in\prod_{i<m^t_{\rm dn}}\mathbf{H}(i)$, we have

$$\langle v{\restriction}m^t_{\rm dn},v\rangle\in\mathbf{val}[t]\quad\Rightarrow\quad\langle u,u^\frown v{\restriction}[m^t_{\rm dn},m^t_{\rm up})\rangle\in\mathbf{val}[t],$$

(c) *strongly finitary* if, for each $i<\omega$, we have

$$|\mathbf{H}(i)|<\omega\quad\text{and}\quad|\{t\in K:m^t_{\rm dn}=i\}|<\omega.$$

(4) If $t_0,\dots,t_n\in K$ are such that $m^{t_i}_{\rm up}=m^{t_{i+1}}_{\rm dn}$ (for $i<n$) and $w\in\prod_{i<m^{t_0}_{\rm dn}}\mathbf{H}(i)$, then we let

$$\mathrm{pos}(w,t_0,\dots,t_n)\stackrel{\rm def}{=}\left\{v\in\prod_{j<m^{t_n}_{\rm up}}\mathbf{H}(j):w\vartriangleleft v\ \&\ (\forall i\leqslant n)(\langle v{\restriction}m^{t_i}_{\rm dn},v{\restriction}m^{t_i}_{\rm up}\rangle\in\mathbf{val}[t_i])\right\}.$$

If K is forgetful and $t\in K$, then we also define

$$\mathrm{pos}(t)=\{v{\restriction}[m^t_{\rm dn},m^t_{\rm up}):\langle v{\restriction}m^t_{\rm dn},v\rangle\in\mathbf{val}[t]\}.$$

Note that if K is forgetful, then to describe a creature in K it is enough to give $\mathrm{pos}(t),\mathbf{nor}[t]$ and $\mathbf{dis}[t]$. This is how our examples will be presented (as they all will be forgetful). Also, if K is additionally local, then we may write $\mathrm{pos}(t)=A$ for some $A\subseteq\mathbf{H}(m^t_{\rm dn})$ with a natural interpretation of this abuse of notation.

If $w,t_0,\dots,t_n$ are as in Definition 1.1(4) and $s_i\in\Sigma(t_i)$ for $i\leqslant n$, and $u\in\mathrm{pos}(w,s_0,\dots,s_k)$, $k<n$, then $\mathrm{pos}(u,s_k,\dots,s_n)\subseteq\mathrm{pos}(w,t_0,\dots,t_n)$ (remember Definition 1.1(2b)).

DEFINITION 1.2. Let (K,Σ) be a good creating pair for $\mathbf{H}$. We define a forcing notion $\mathbb{Q}^*_\infty(K,\Sigma)$ as follows.

A condition in $\mathbb{Q}^*_\infty(K,\Sigma)$ is a sequence $p=(w^p,t^p_0,t^p_1,t^p_2,\dots)$ such that

(a) $t^p_i\in K$ and $m^{t^p_i}_{\rm up}=m^{t^p_{i+1}}_{\rm dn}$ (for $i<\omega$) and

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(b) $w \in \prod_{i<m^{t^p_0}_{\rm dn}} \mathbf{H}(i)$ and $\lim_{n\to\infty} \mathbf{nor}[t^p_n] = \infty$.

The relation $\leqslant$ on $\mathbb{Q}^*_\infty(K,\Sigma)$ is given by: $p \leqslant q$ if and only if, for some $i<\omega$, we have $w^q \in \mathrm{pos}(w^p, t^p_0, \ldots, t^p_{i-1})$ (if $i=0$ this means $w^q = w^p$) and $t^q_n \in \Sigma(t^p_{n+i})$ for all $n<\omega$.

For a condition $p \in \mathbb{Q}^*_\infty(K,\Sigma)$ we let $i(p) = \mathrm{lh}(w^p)$.

2. *Collapsing creatures*

We will show here that very natural forcing notions of type $\mathbb{Q}^*_\infty(K,\Sigma)$ (for a big local and finitary creating pair (K,Σ)) collapse $\mathfrak{c}$ to $\aleph_0$, in particular answering [**7**, Question 4.1]. The main ingredient of the proof is similar to the 'negative theory' presented in [**4**, Section 1.4], and Definition 2.1 should be compared with [**4**, Definition 1.4.4] (but the two properties are somewhat incomparable).

DEFINITION 2.1. Let $h : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ be a nondecreasing unbounded function and let (K,Σ) be a good creating pair for $\mathbf{H}$. We say that (K,Σ) is *sufficiently h-bad* if there are sequences $\bar{m} = \langle m_i : i<\omega\rangle$, $\bar{A} = \langle A_i : i<\omega\rangle$ and $\bar{F} = \langle F_i : i<\omega\rangle$ such that

(α) $\bar{m}$ is a strictly increasing sequence of integers, $m_0 = 0$, and

$$(\forall t \in K)(\exists i<\omega)(m^t_{\rm dn} = m_i \ \&\ m^t_{\rm up} = m_{i+1});$$

(β) A_i are finite nonempty sets;

(γ) $F_i = (F^0_i, F^1_i) : A_i \times \prod_{m<m_{i+1}} \mathbf{H}(m) \longrightarrow A_{i+1} \times 2$;

(δ) if $i<\omega$, $t\in K$, $m^t_{\rm dn} = m_i$ and $\mathbf{nor}[t] > 4$, then there is $a \in A_i$ such that
for every $x \in A_{i+1}\times 2$, for some $s_x \in \Sigma(t)$ we have
$\mathbf{nor}[s_x] \geqslant \min\{h(\mathbf{nor}[t]), h(i)\}$ and
$(\forall u \in \prod_{m<m_i} \mathbf{H}(m))(\forall v \in \mathrm{pos}(u, s_x))(F_i(a,v) = x)$.

PROPOSITION 2.2. *Suppose that $h : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ is a nondecreasing unbounded function, and (K,Σ) is a strongly finitary good creating pair for $\mathbf{H}$. Assume also that (K,Σ) is sufficiently h-bad. Then the forcing notion $\mathbb{Q}^*_\infty(K,\Sigma)$ collapses $\mathfrak{c}$ onto $\aleph_0$.*

Proof. The proof is similar to that of Rosłanowski and Shelah [**4**, Proposition 1.4.5], but for the reader's convenience we present it fully.

Let $\bar{m}, \bar{A}$ and $\bar{F}$ witness that (K,Σ) is sufficiently h-bad. For $i<\omega$ and $a \in A_i$ we define $\mathbb{Q}^*_\infty(K,\Sigma)$-names $\dot\rho_{i,a}$ (for a real in 2^ω) and $\dot\eta_{i,a}$ (for an element of $\prod_{j\geqslant i} A_j$) as follows:

$$\Vdash_{\mathbb{Q}^*_\infty(K,\Sigma)} \text{`}\dot\eta_{i,a}(i) = a \text{ and } \dot\eta_{i,a}(j) = F^0_{j-1}(\dot\eta_{i,a}(j-1), \dot W{\restriction}m_j) \text{ for } j>i\text{'}$$

and

$$\Vdash_{\mathbb{Q}^*_\infty(K,\Sigma)} \text{`}\dot\rho_{i,a}{\restriction}i \equiv 0 \text{ and } \dot\rho_{i,a}(j) = F^1_j(\dot\eta_{i,a}(j), \dot W{\restriction}m_{j+1}) \text{ for } j \geqslant i\text{'}.$$

Above, $\dot W$ is the canonical name for the generic function in $\prod_{i<\omega}\mathbf{H}(i)$, that is, $p \Vdash_{\mathbb{Q}^*_\infty(K,\Sigma)}$ '$w^p \lhd \dot W \in \prod_{i<\omega}\mathbf{H}(i)$'. We are going to show that

$$\Vdash_{\mathbb{Q}^*_\infty(K,\Sigma)} \text{`}(\forall r \in 2^\omega \cap \mathbf{V})(\exists i<\omega)(\exists a\in A_i)(\forall j \geqslant i)(\dot\rho_{i,a}(j) = r(j))\text{'}.$$

To this end, suppose that $p \in \mathbb{Q}^*_\infty(K,\Sigma)$ and $r \in 2^\omega$. Passing to a stronger condition if needed, we may assume that $(\forall j<\omega)(\mathbf{nor}[t^p_j] > 4)$. Let $i<\omega$ be such that $\mathrm{lh}(w^p) = m_i$; then also $m^{t^p_j}_{\rm dn} = m_{i+j}$ for $j<\omega$ (remember Definition 2.1(α)).

Fix $k<\omega$ for a moment. By downward induction on $j \leqslant k$ choose $s^k_j \in \Sigma(t^p_j)$ and $a^k_j \in A_{i+j}$ such that

(a) $\mathbf{nor}[s^k_j] \geqslant \min\{h(\mathbf{nor}[t^p_j]), h(i+j)\}$ for all $j \leqslant k$;
(b) $(\forall u \in \prod_{m<m_{i+k}} \mathbf{H}(m))(\forall v \in \mathrm{pos}(u, s^k_k))(F^1_{i+k}(a^k_k, v) = r(i+k))$;
(c) for $j < k$:

$$\left(\forall u \in \prod_{m<m_{i+j}} \mathbf{H}(m)\right)(\forall v \in \mathrm{pos}(u, s^k_j))(F^1_{i+j}(a^k_j, v) = r(i+j) \ \&\ F^0_{i+j}(a^k_j, v) = a^k_{j+1}).$$

(Plainly it is possible by Definition 2.1(δ).)

Since, for each $j < \omega$, both $\Sigma(t^p_j)$ and A_{i+j} are finite, we may use König's Lemma to pick an increasing sequence $\bar{k} = \langle k(\ell) : \ell < \omega\rangle$ such that

$$a^{k(\ell+1)}_j = a^{k(\ell')}_j \quad \text{and} \quad s^{k(\ell+1)}_j = s^{k(\ell')}_j,$$

for $\ell < \ell' < \omega$ and $j \leqslant k(\ell)$. Put $w^q = w^p$ and $t^q_j = s^{k(j+1)}_j$, $b_j = a^{k(j+1)}_j$ for $j < \omega$. Easily, $q = (w^q, t^q_0, t^q_1, t^q_2, \ldots)$ is a condition in $\mathbb{Q}^*_\infty(K, \Sigma)$ stronger than p. Also, by clause (c) of the choice of s^k_j, we clearly have

$$(\forall j < \omega)(\forall v \in \mathrm{pos}(w^q, t^q_0, \ldots, t^q_j))(F^0_{i+j}(b_j, v) = b_{j+1} \ \&\ F^1_{i+j}(b_j, v) = r(i+j)).$$

Hence,

$$q \Vdash_{\mathbb{Q}^*_\infty(K,\Sigma)} \text{`}(\forall j < \omega)(\dot\eta_{i,b_0}(i+j) = b_j \ \&\ \dot\rho_{i,b_0}(i+j) = r(i+j))\text{'},$$

completing the proof. □

Lemma 2.3. *Suppose that positive integers N, M, d satisfy $(N-2)\cdot 2^M < d$. Let A, B be finite sets such that $|A| \geqslant 2^M$ and $|B| \leqslant N$. Then there is a mapping $\hat{F} : A \times {}^dM \to B$ with the property that:*

($\circledast$) *if $2 \leqslant \ell \leqslant M$, $\langle c_i : i < d\rangle \in \prod_{i<d}[M]^\ell$, then there is $a \in A$ such that, for every $b \in B$, for some $c^b_i \in [c_i]^{\lfloor \ell/2\rfloor}$ (for $i < d$), we have*

$$\left(\forall u \in \prod_{i<d} c^b_i\right)(\hat{F}(a, u) = b).$$

Proof. Plainly we may assume that $|A| = 2^M$ and $|B| = N \geqslant 2$, and then we may pretend that $A = {}^M2$ and $B = N$.

For $h \in A = {}^M2$ and $u \in {}^dM$ we let $\hat{F}(h, u) < N$ be such that

$$\hat{F}(h, u) \equiv \sum_{i<d} h(u(i)) \mod N.$$

This defines the function $\hat{F} : A \times {}^dM \to B = N$, and we are going to show that it has the property stated in ($\circledast$). To this end, suppose that $2 \leqslant \ell \leqslant M$ and $\langle c_i : i < d\rangle \in \prod_{i<d}[M]^\ell$. For each $i < d$ we may choose $h_i \in A$ so that

$$|(h_i)^{-1}[\{0\}] \cap c_i| \geqslant \lfloor \ell/2\rfloor \quad \text{and} \quad |(h_i)^{-1}[\{1\}] \cap c_i| \geqslant \lfloor \ell/2\rfloor.$$

Then, for some $h \in A$ and $I \subseteq d$, we have $|I| \geqslant d/2^M$ and $h_i = h$ for $i \in I$. For $i \in d \setminus I$ we may pick $c^*_i \in [c_i]^{\lfloor \ell/2\rfloor}$ and $j_i < 2$ such that $h{\restriction}c^*_i \equiv j_i$.

Now suppose $b \in B$. Take a set $J \subseteq I$ such that

$$|J| + \sum_{i\in d\setminus I} j_i \equiv b \mod N$$

(possible as $|I| \geqslant d/2^M > N-2$, so $|I| \geqslant N-1$). By our choices, we may pick $c^b_i \in [c_i]^{\lfloor \ell/2\rfloor}$ (for $i \in I$) such that

if $i \in J$, then $h\restriction c_i^b \equiv 1$, and
if $i \in I \setminus J$, then $h\restriction c_i^b \equiv 0$.
For $i \in d \setminus I$ we let $c_i^b = c_i^*$ (selected earlier). It should be clear that then

$$\left(\forall u \in \prod_{i<d} c_i^b\right) (\hat{F}(h,u) = b),$$

as needed. □

EXAMPLE 2.4. Let $\bar{m} = \langle m_i : i < \omega\rangle$ be an increasing sequence of integers such that $m_0 = 0$ and $m_{i+1} - m_i > 4^{i+3}$. Let $h(\ell) = \lfloor \ell/2 \rfloor$ for $\ell < \omega$.

For $j < \omega$ we let $\mathbf{H}^0_{\bar{m}}(j) = i + 2$, where i is such that $m_i \leqslant j < m_{i+1}$. Let $K^0_{\bar{m}}$ consist of all (forgetful) creatures $t \in \mathrm{CR}[\mathbf{H}^0_{\bar{m}}]$ such that

(1) $\mathbf{dis}[t] = \langle i^t, \langle Z^t_j : m_{i^t} \leqslant j < m_{i^t+1}\rangle\rangle$ for some $i^t < \omega$ and $\emptyset \neq Z^t_j \subseteq \mathbf{H}^0_{\bar{m}}(j)$ (for $m_{i^t} \leqslant j < m_{i^t+1}$);
(2) $\mathbf{nor}[t] = \min\{|Z^t_j| : m_{i^t} \leqslant j < m_{i^t+1}\}$;
(3) $\mathrm{pos}(t) = \prod_{j\in[m_{i^t}, m_{i^t+1})} Z^t_j$.

Finally, for $t \in K^0_{\bar{m}}$ we let

$$\Sigma^0_{\bar{m}}(t) = \{s \in K^0_{\bar{m}} : i^t = i^s \ \&\ (\forall j \in [m_{i^t}, m_{i^t+1}))(Z^s_j \subseteq Z^t_j)\}.$$

Then $(K^0_{\bar{m}}, \Sigma^0_{\bar{m}})$ is a strongly finitary and sufficiently h-bad good creating pair for $\mathbf{H}^0_{\bar{m}}$. Consequently, the forcing notion $\mathbb{Q}^*_\infty(K^0_{\bar{m}}, \Sigma^0_{\bar{m}})$ collapses $\mathfrak{c}$ onto $\aleph_0$.

Proof. It should be clear that $(K^0_{\bar{m}}, \Sigma^0_{\bar{m}})$ is a strongly finitary good creating pair for $\mathbf{H}^0_{\bar{m}}$. To show that it is sufficiently h-bad, let $A_i = {}^{i+2}2$, $B_i = A_{i+1} \times 2 = {}^{i+3}2 \times 2$ and $M_i = i + 2$. Since $|B_i| \cdot 2^{M_i} = 2^{i+4+i+2} < m_{i+1} - m_i \stackrel{\mathrm{def}}{=} d_i$, we may apply Lemma 2.3 for $A = A_i$, $B = B_i$, $M = M_i$ and $d = d_i$ to get functions $\hat{F}_i : A_i \times {}^{d_i}M_i \to B_i$ with the property (⊛) (for those parameters). For $a \in A_i$ and $v \in \prod_{j<m_{i+1}} \mathbf{H}^0_{\bar{m}}(j)$, we interpret $F_i(a, v)$ as $\hat{F}_i(a, u)$ where $u \in {}^{d_i}(i + 1)$ is given by $u(j) = v(m_i + j)$ for $j < d_i$. It is straightforward to show that $\bar{m}$, $\bar{A} = \langle A_i : i < \omega\rangle$ and $\bar{F} = \langle F_i : i < \omega\rangle$ witness that $(K^0_{\bar{m}}, \Sigma^0_{\bar{m}})$ is h-bad. □

The above example (together with Proposition 2.2) easily gives the answer to [7, Question 4.1]. To show how our problem reduces to this example, let us recall the following.

DEFINITION 2.5 (see [4, Definition 4.2.1]). Suppose $0 < m < \omega$ and, for $i < m$, we have $t_i \in \mathrm{CR}[\mathbf{H}]$ such that $m^{t_i}_{\mathrm{up}} \leqslant m^{t_{i+1}}_{\mathrm{dn}}$. Then we define *the sum of the creatures* t_i as a creature $t = \Sigma^{\mathrm{sum}}(t_i : i < m)$ such that (if well defined, then):

(a) $m^t_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$, $m^t_{\mathrm{up}} = m^{t_{m-1}}_{\mathrm{up}}$;
(b) $\mathbf{val}[t]$ is the set of all pairs $\langle h_1, h_2\rangle$ such that:
 $\mathrm{lh}(h_1) = m^t_{\mathrm{dn}}$, $\mathrm{lh}(h_2) = m^t_{\mathrm{up}}$, $h_1 \vartriangleleft h_2$,
 and $\langle h_2\restriction m^{t_i}_{\mathrm{dn}}, h_2\restriction m^{t_i}_{\mathrm{up}}\rangle \in \mathbf{val}[t_i]$ for $i < m$,
 and $h_2\restriction[m^{t_i}_{\mathrm{up}}, m^{t_{i+1}}_{\mathrm{dn}})$ is identically zero for $i < m - 1$;
(c) $\mathbf{nor}[t] = \min\{\mathbf{nor}[t_i] : i < m\}$;
(d) $\mathbf{dis}[t] = \langle t_i : i < m\rangle$.

If, for all $i < m - 1$, we have $m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}}$, then we call the sum *tight*.

DEFINITION 2.6. Let (K, Σ) be a local good creating pair for $\mathbf{H}$, let $\bar{m} = \langle m_i : i < \omega\rangle$ be a strictly increasing sequence with $m_0 = 0$. We define the *$\bar{m}$-summarization* $(K^{\bar{m}}, \Sigma^{\bar{m}}, \mathbf{H}^{\bar{m}})$ *of* $(K, \Sigma, \mathbf{H})$ as follows:

(1) $\mathbf{H}^{\bar{m}}(i)=\prod_{m=m_i}^{m_{i+1}-1}\mathbf{H}(m)$;
(2) $K^{\bar{m}}$ consists of all tight sums $\Sigma^{\rm sum}(t_\ell:m_i\leqslant\ell<m_{i+1})$ such that $i<\omega$, $t_\ell\in K$, $m^{t_\ell}_{\rm dn}=\ell$;
(3) if $t=\Sigma^{\rm sum}(t_\ell:m_i\leqslant\ell<m_{i+1})\in K^{\bar{m}}$, then $\Sigma^{\bar{m}}(t)$ consists of all creatures $s\in K^{\bar{m}}$ such that $s=\Sigma^{\rm sum}(s_\ell:m_i\leqslant\ell<m_{i+1})$ for some $s_\ell\in\Sigma(t_\ell)$ (for $\ell=m_i,\ldots,m_{i+1}-1$).

PROPOSITION 2.7. *Assume that (K,Σ) is a local good creating pair for $\mathbf{H}$, and $\bar{m}=\langle m_i:i<\omega\rangle$ is a strictly increasing sequence with $m_0=0$. Then:*
(1) *$(K^{\bar{m}},\Sigma^{\bar{m}})$ is a good creating pair for $\mathbf{H}^{\bar{m}}$;*
(2) *the forcing notion $\mathbb{Q}^*_\infty(K^{\bar{m}},\Sigma^{\bar{m}})$ can be embedded as a dense subset of the forcing notion $\mathbb{Q}^*_\infty(K,\Sigma)$ (so the two forcing notions are equivalent).*

COROLLARY 2.8. *Let $\mathbf{H}:\omega\to\omega$ be increasing, $\mathbf{H}(0)\geqslant 2$, and let $g:\mathbb{R}^{\geqslant 0}\to\mathbb{R}^{\geqslant 0}$ be an unbounded nondecreasing function. We define $(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$ as follows: $K^{\mathbf{H}}_g$ consists of all creatures $t\in{\rm CR}[\mathbf{H}]$ such that*
(1) $\mathbf{dis}[t]=\langle i^t,A^t\rangle$ *for some* $i^t<\omega$ *and* $\emptyset\neq A^t\subseteq\mathbf{H}(i^t)$;
(2) $\mathbf{nor}[t]=g(|A^t|)$, $m^t_{\rm dn}=i^t$, $m^t_{\rm up}=i^t+1$ *and* ${\rm pos}(t)=A^t$.

For $t\in K^{\mathbf{H}}_g$ we let

$$\Sigma^{\mathbf{H}}_g(t)=\{s\in K^{\mathbf{H}}_g:i^t=i^s\ \&\ A^s\subseteq A^t\}.$$

Then $(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$ is a local strongly finitary good creating pair for $\mathbf{H}$. The forcing notion $\mathbb{Q}^*_\infty(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$ collapses $\mathfrak{c}$ onto $\aleph_0$. In particular, the forcing notion $\mathbb{Q}$ defined in Section 1 is not proper.

Proof. Let $p\in\mathbb{Q}^*_\infty(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$. Plainly, $\lim_{i\to\infty}|A^{t^p_i}|=\infty$, so we may find a condition $q\geqslant p$ and an increasing sequence $\bar{m}=\langle m_i:i<\omega\rangle$ such that
(1) $m_0=0$, $m_1={\rm lh}(w^q)$, $m_{i+1}-m_i>4^{i+3}$;
(2) if $m_i\leqslant m^{t^q_k}_{\rm dn}<m_{i+1}$, then $|A^{t^q_k}|=i+2$.

Now we define a condition q^* in $\mathbb{Q}^*_\infty((K^{\mathbf{H}}_g)^{\bar{m}},(\Sigma^{\mathbf{H}}_g)^{\bar{m}})$ by

$$w^{q^*}=w^q,\qquad t^{q^*}_i=\Sigma^{\rm sum}(t^q_k:m_{i+1}\leqslant k<m_{i+2})\quad(\text{for } i<\omega).$$

The forcing notion $\mathbb{Q}^*_\infty(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$ above the condition q is equivalent to the forcing notion $\mathbb{Q}^*_\infty((K^{\mathbf{H}}_g)^{\bar{m}},(\Sigma^{\mathbf{H}}_g)^{\bar{m}})$ above q^*. Plainly, $\mathbb{Q}^*_\infty((K^{\mathbf{H}}_g)^{\bar{m}},(\Sigma^{\mathbf{H}}_g)^{\bar{m}})$ above q^* is isomorphic to $\mathbb{Q}^*_\infty(K^0_{\bar{m}},\Sigma^0_{\bar{m}})$ of Example 2.4 above the minimal condition r with $w^r=w^{q^*}$. The assertion follows now by the last sentence of Example 2.4. □

REMARK 2.9. (1) If, for example, $g(x)=\log_2(x)$, then the creating pair $(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$ is big (see [**4**, Definition 2.2.1]), and we may even get 'a lot of bigness'. Thus, the bigness itself is not enough to guarantee properness of the resulting forcing notion.

(2) Forcing notions of the form $\mathbb{Q}^*_\infty(K,\Sigma)$ are special cases of $\mathbb{Q}^*_f(K,\Sigma)$ (see [**4**, Definition 1.1.10 and Section 2.2]). However, if the function f is growing very fast (much faster than $\mathbf{H}$), then our method does not apply. Let us recall that if (K,Σ) is simple, finitary and big and has the halving property, and $f:\omega\times\omega\to\omega$ is $\mathbf{H}$-fast (see [**4**, Definition 1.1.12]), then $\mathbb{Q}^*_f(K,\Sigma)$ is proper. Thus, one may wonder if we may omit halving; can the forcing notion $\mathbb{Q}^*_f(K^{\mathbf{H}}_g,\Sigma^{\mathbf{H}}_g)$ be proper for $\mathbf{H}$ and f suitably 'fast'?

3. *Properness from halving*

It was shown in [4, Theorem 2.2.11] that halving and bigness (see [4, Definitions 2.2.1, 2.2.7]) imply properness of the forcings $\mathbb{Q}^*_f(K,\Sigma)$ (for fast f). It occurs that if we have a stronger version of halving, then we may get the properness of $\mathbb{Q}^*_\infty(K,\Sigma)$ even without any bigness assumptions.

Definition 3.1. Let (K,Σ) be a forgetful good creating pair.

(1) Let $t \in K$ and $\varepsilon > 0$. We say that a creature $t^* \in \Sigma(t)$ is an ε-**half** of t if the following hold:

(i) $\mathbf{nor}[t^*] \geqslant \mathbf{nor}[t] - \varepsilon$;

(ii) if $s \in \Sigma(t^*)$ and $\mathbf{nor}[s] > 1$, then we can find $t_0 \in \Sigma(t)$ such that

$$\mathbf{nor}[t_0] \geqslant \mathbf{nor}[t] - \varepsilon \quad \text{and} \quad \mathrm{pos}(t_0) \subseteq \mathrm{pos}(s).$$

(2) Let $\bar{\varepsilon} = \langle \varepsilon_i : i < \omega \rangle$ be a sequence of positive real numbers and $\bar{m} = \langle m_i : i < \omega \rangle$ be a strictly increasing sequence of integers with $m_0 = 0$. We say that the pair (K,Σ) has the $(\bar{\varepsilon},\bar{m})$-*halving property* if, for every $t \in K$ with $m_i \leqslant m^t_{\mathrm{dn}}$ and $\mathbf{nor}[t] \geqslant 2$, there exists an ε_i-**half** of t in $\Sigma(t)$.

Definition 3.2. Let (K,Σ) be a good creating pair. Suppose that $p \in \mathbb{Q}^*_\infty(K,\Sigma)$ and $I \subseteq \mathbb{Q}^*_\infty(K,\Sigma)$ is open dense. We say that p *essentially belongs to* I, written $p \in^* I$, if there exists $i_* < \omega$ such that, for every $v \in \mathrm{pos}(w^p, t^p_0, \dots, t^p_{i_*-1})$, we have $(v, t^p_{i_*}, t^p_{i_*+1}, t^p_{i_*+2}, \dots) \in I$.

Note that if $I \subseteq \mathbb{Q}^*_\infty(K,\Sigma)$ is open dense, $p \in^* I$ and $p \leqslant q$, then also $q \in^* I$.

Theorem 3.3. *Let $\bar{\varepsilon} = \langle \varepsilon_i : i < \omega \rangle$ be a decreasing sequence of positive numbers and $\bar{m} = \langle m_i : i < \omega \rangle$ be a strictly increasing sequence of integers with $m_0 = 0$. Assume that, for each $i < \omega$,*

$$\left| \prod_{n<m_i} \mathbf{H}(n) \right| \leqslant 1/\varepsilon_i.$$

Let (K,Σ) be a good creating pair for $\mathbf{H}$ and suppose that (K,Σ) is local and forgetful and has the $(\bar{\varepsilon},\bar{m})$-halving property. Then the forcing notion $\mathbb{Q}^*_\infty(K,\Sigma)$ is proper.

Proof. We start with two technical claims.

Claim 3.4. *Let $a \geqslant 2$ and $I \subseteq \mathbb{Q}^*_\infty(K,\Sigma)$ be open dense. Furthermore, suppose that $p \in \mathbb{Q}^*_\infty(K,\Sigma)$ and $i < \omega$ is such that $i(p) \leqslant m_i$ and $\mathbf{nor}[t^p_n] > a$ for every $n \geqslant m_i - i(p)$. Finally, let $v \in \prod_{n<m_i} \mathbf{H}(n)$. Then there exists $q \in \mathbb{Q}^*_\infty(K,\Sigma)$ such that*

(a) *$p \leqslant q$, $w^p = w^q$ and $t^p_n = t^q_n$ for every $n < m_i - i(p)$;*

(b) *$\mathbf{nor}[t^q_n] \geqslant a - \varepsilon_i$ for every $n \geqslant m_i - i(p)$;*

(c) *either, letting $q^{[v]} = (v, t^q_{m_i-i(p)}, t^q_{m_i-i(p)+1}, t^q_{m_i-i(p)+2}, \dots)$, $q^{[v]} \in^* I$ or else there is no $r \geqslant q^{[v]}$ such that $r \in I$, $w^r = v$ and $\mathbf{nor}[t^r_n] > 1$ for every n.*

Proof of the claim. We know that (K,Σ) has the $(\bar{\varepsilon},\bar{m})$-halving property and therefore, for each $n \geqslant m_i - i(p)$, we may choose an ε_i-half $t^{q_0}_n \in \Sigma(t^p_n)$ of t^p_n. For $n < m_i - i(p)$ put $t^{q_0}_n = t^p_n$ and let $w^{q_0} = w^p$. This defines a condition $q_0 = (w^{q_0}, t^{q_0}_0, t^{q_0}_1, t^{q_0}_2, \dots) \in \mathbb{Q}^*_\infty(K,\Sigma)$. Plainly, (a) and (b) hold for q_0 instead of q. Now if there is no $r \geqslant q_0^{[v]}$ with $r \in I$, $w^r = v$ and $\mathbf{nor}[t^r_n] > 1$ for every $n < \omega$, we can let $q = q_0$. Hence, we may assume that such $r = (w^r, t^r_0, t^r_1, t^r_2, \ldots)$ does exist.

Pick $j < \omega$ large enough such that $\mathbf{nor}[t^r_n] \geqslant a - \varepsilon_i$ for every $n \geqslant j$. Now we define $q \in \mathbb{Q}^*_\infty(K,\Sigma)$:

(1) $w^q = w^p$, $t^q_n = t^p_n$ for $n < m_i - i(p)$;
(2) $t^q_n = t^r_{n-m_i+i(p)}$ for $n \geqslant m_i - i(p) + j$;
(3) for $m_i - i(p) \leqslant n < m_i - i(p) + j$ let $t^q_n \in \Sigma(t^p_n)$ be such that

$$\mathbf{nor}[t^q_n] \geqslant \mathbf{nor}[t^p_n] - \varepsilon_i \geqslant a - \varepsilon_i \quad \text{and} \quad \mathrm{pos}(t^q_n) \subseteq \mathrm{pos}(t^r_{n-m_i+i(p)})$$

(exists by the halving property).

Clearly $p \leqslant q$ and (a), (b) hold. Also, for every $u \in \mathrm{pos}(v, t^q_{m_i-i(p)}, \ldots, t^q_{m_i-i(p)+j})$ we have $q^{[u]} \geqslant r$, and hence $q^{[u]} \in I$, as I is open. Consequently, $q^{[v]} \in^* I$. □

CLAIM 3.5. *Let $a \geqslant 3$ and $I \subseteq \mathbb{Q}^*_\infty(K,\Sigma)$ be open dense. Suppose that $p \in \mathbb{Q}^*_\infty(K,\Sigma)$ and $i < \omega$ is such that $i(p) \leqslant m_i$ and $\mathbf{nor}[t^p_n] > a$ for every $n \geqslant m_i - i(p)$. Then there exists $q \in \mathbb{Q}^*_\infty(K,\Sigma)$ such that*

(a) $p \leqslant q$, $w^p = w^q$ and $t^p_n = t^q_n$ for $n < m_i - i(p)$;
(b) $\mathbf{nor}[t^q_n] \geqslant a - 1$ for every $n \geqslant m_i - i(p)$;
(c) for every $v \in \prod_{n<m_i} \mathbf{H}(n)$, either $q^{[v]} \in^* I$, or else there is no $r \in I$ such that $r \geqslant q^{[v]}$, $w^r = v$ and $\mathbf{nor}[t^r_n] > 1$ for all n.

Proof of the claim. Let $\langle v_l : l < k\rangle$ enumerate $\prod_{n<m_i} \mathbf{H}(n)$; thus $k \leqslant 1/\varepsilon_i$. Applying Claim 3.4 k times, it is straightforward to construct a sequence $\langle q_l : l \leqslant k\rangle \subseteq \mathbb{Q}^*_\infty(K,\Sigma)$ such that

(1) $q_0 = p$, $q_l \leqslant q_{l+1}$, $w^{q_l} = w^p$ and $t^{q_l}_n = t^p_n$ for every $n < m_i - i(p)$;
(2) $\mathbf{nor}[t^{q_l}_n] \geqslant a - l \cdot \varepsilon_i$ for every $n \geqslant m_i - i(p)$;
(3) $\langle q_l, q_{l+1}, v_l, a - l\cdot\varepsilon_i\rangle$ are like $\langle p, q, v, a\rangle$ in Claim 3.4.

Then clearly $q = q_k$ is as desired. □

We argue now that the forcing notion $\mathbb{Q}^*_\infty(K,\Sigma)$ is proper. So suppose that N is a countable elementary submodel of $(\mathcal{H}(\chi), \in)$ (for some sufficiently large regular cardinal χ), $K, \Sigma, \ldots \in N$. Let $p \in N \cap \mathbb{Q}^*_\infty(K,\Sigma)$ and let $\langle I_\ell : \ell < \omega\rangle$ list with ω-repetitions all open dense subsets of $\mathbb{Q}^*_\infty(K,\Sigma)$ from N.

By induction on $\ell < \omega$, we choose integers i_ℓ and conditions $p_\ell \in N \cap \mathbb{Q}^*_\infty(K,\Sigma)$ as follows. We set $p_0 = p$ and $i_0 > i(p)$ is such that $\mathbf{nor}[t^{p_0}_n] > 3$ for all $n \geqslant m_{i_0} - i(p)$.

Now assume that we have defined $p_\ell \in N \cap \mathbb{Q}^*_\infty(K,\Sigma)$ and $i_\ell < \omega$ so that $w^p = w^{p_\ell}$ and $\mathbf{nor}[t^{p_\ell}_n] > 3 + \ell$ for every $n \geqslant m_{i_\ell} - i(p)$. Applying Claim 3.5 (inside N) to $3+\ell, I_\ell, p_\ell, i_\ell$ here standing for a, I, p, i there, we may find a condition $p_{\ell+1} \in N \cap \mathbb{Q}^*_\infty(K,\Sigma)$ such that

(a)$_\ell$ $p_\ell \leqslant p_{\ell+1}$, $w^p = w^{p_\ell} = w^{p_{\ell+1}}$ and $t^{p_\ell}_n = t^{p_{\ell+1}}_n$ for all $n < m_{i_\ell} - i(p)$;
(b)$_\ell$ $\mathbf{nor}[t^{p_{\ell+1}}_n] \geqslant 2 + \ell$ for every $n \geqslant m_{i_\ell} - i(p)$;
(c)$_\ell$ for every sequence $v \in \prod_{n<m_{i_\ell}} \mathbf{H}(n)$, if there exists $r \in I_\ell$ such that $r \geqslant p^{[v]}_{\ell+1}$, $w^r = v$ and $\mathbf{nor}[t^r_n] > 1$ for every n, then $p^{[v]}_{\ell+1} \in^* I_\ell$.

Then we choose $i_{\ell+1} > i_\ell$ so that $\mathbf{nor}[t^{p_{\ell+1}}_n] > 3 + \ell + 1$ for all $n \geqslant m_{i_{\ell+1}} - i(p)$.

After the inductive construction is carried out, we let q be the natural fusion determined by the p_ℓ (so $w^q = w^p$ and $t^q_n = t^{p_\ell}_n$ whenever $n < m_{i_\ell} - i(q)$). Plainly, $q \in \mathbb{Q}^*_\infty(K,\Sigma)$ (remember

$(a)_{\ell+1}+(b)_\ell$) and it is stronger than all p_ℓ (for $\ell<\omega$). Let us show that q is $(N,\mathbb{Q}^*_\infty(K,\Sigma))$-generic. To this end, suppose $I\in N$ is a dense open subset of $\mathbb{Q}^*_\infty(K,\Sigma)$ and $r\in\mathbb{Q}^*_\infty(K,\Sigma)$ is stronger than q. Pick a condition $r_0=(v,t^{r_0}_0,t^{r_0}_1,t^{r_0}_2,\ldots)\geqslant r$ and $\ell<\omega$ such that

$(*)$ $r_0\in I$, $I=I_\ell$ and $\mathrm{lh}(v)=m_{i_\ell}$;

$(**)$ $\mathbf{nor}[t^{r_0}_n]>1$ for every $n<\omega$.

Then $r_0\geqslant q^{[v]}\geqslant p^{[v]}_{\ell+1}$. Therefore, by $(c)_\ell$, we see that $p^{[v]}_{\ell+1}\in^* I_\ell$ and hence we may find $u\in\mathrm{pos}(v,t^{r_0}_0,\ldots,t^{r_0}_k)$ (for some $k<\omega$) such that $p^{[u]}_{\ell+1}\in I_\ell$. Then $p^{[u]}_{\ell+1}\in N\cap I$ is compatible with r.

Note that the above argument shows also that, for every open dense subset $I\in N$ of $\mathbb{Q}^*_\infty(K,\Sigma)$, the set $\{q^{[v]}: v\in\prod_{n<m_i}\mathbf{H}(n)\ \&\ i<\omega\}\cap I$ is predense above q. □

4. *A nonproper product*

Here, we give an example of two proper forcing notions $\mathbb{Q}^*_\infty(K^1,\Sigma^1)$ and $\mathbb{Q}^*_\infty(K^2,\Sigma^2)$ such that their product $\mathbb{Q}^*_\infty(K,\Sigma)$ collapses $\mathfrak{c}$ onto $\aleph_0$.

Throughout this section, we write log instead of $\log_2$.

Definition 4.1. Let $x,i\in\mathbb{R}$, $x>0$, $i\geqslant 0$ and $k\in\omega\setminus\{0\}$. We let

$$f_k(x,i)=\frac{\log(\log(\log(x))-i)}{k}$$

in the case that all three logarithms are well defined and attain a value at least 1. In all other cases we define $f_k(x,i)=1$.

Lemma 4.2. (1) $f_k(x/2,i)\geqslant f_k(x,i)-1/k$;

(2) *letting* $j=(\log(\log(x))+i)/2$, *if* $f_k(x,i)\geqslant 2$, *then* $f_k(x,j)=f_k(x,i)-1/k$;

(3) *letting* j *as in* (2), *if* $\min\{f_k(x,i),f_k(y,j)\}>1$, *then* $f_k(y,i)\geqslant f_k(x,i)-1/k$;

(4) *if* $x\geqslant 2^{2^{4+i}}$ *and* z *such that* $\log(\log(z))=(\log(\log(x))+i)/2$, *then* $f_k(z,i)=f_k(x,i)-1/k$.

Proof. (1) Note that, for $x\geqslant 2$, we have

$$(*)\qquad \log(x-1)\geqslant\log(x)-1.$$

Indeed, $x\geqslant 2$ implies $x-1\geqslant x/2$. Applying log to both sides, we get $\log(x-1)\geqslant\log(x/2)=\log(x)-1$.

If $x<2^{2^{2+i}}$, then $\log(\log(\log(x))-i)<1$ (if at all defined), and $f_k(x,i)=1=f_k(x/2,i)$. So assume $x\geqslant 2^{2^{2+i}}$. Then $\log(x)\geqslant 2^{2+i}\geqslant 2$ and $\log(\log(x))-i\geqslant 2$, and hence we may apply $(*)$ with $\log(x)$ and $\log(\log(x))-i$ and obtain

$$\log\left(\log\left(\log\left(\frac{x}{2}\right)\right)-i\right)$$
$$=\log(\log(\log(x)-1)-i)\geqslant\log(\log(\log(x))-1-i)\geqslant\log(\log(\log(x))-i)-1.$$

By dividing both sides by k, we arrive at (1).

(2) Note that $f_k(x,i)\geqslant 2$ implies $\log(\log(x))-i\geqslant 4$ and hence $\log(\log(x))-j=(\log(\log(x))-i)/2\geqslant 2$. Consequently,

$$f_k(x,j)=\frac{\log(\log(\log(x))-j)}{k}=\frac{\log((\log(\log(x))-i)/2)}{k}$$
$$=\frac{\log(\log(\log(x))-i)-1}{k}=f_k(x,i)-\frac{1}{k}.$$

(3) By assumption we have $\log(\log(y)) - j \geqslant 0$. By plugging in the definition of j and adding $j - i$ to both sides, we obtain $\log(\log(y)) - i \geqslant \frac{1}{2}(\log(\log(x)) - i)$ and hence $\log(\log(\log(y)) - i) \geqslant \log(\log(\log(x)) - i) - 1$. After dividing by k, we reach (3).

(4) Note that

$$f_k(z,i) = \frac{1}{k}\log(\log(\log(z)) - i) = \frac{1}{k}\log((\log(\log(x)) - i)/2)$$
$$= \frac{1}{k}[\log(\log(\log(x)) - i) - 1] = f_k(x,i) - \frac{1}{k}. \qquad \square$$

We are going to modify the example in Corollary 2.8 and Example 2.4.

Let $\bar{m} = \langle m_i : i < \omega\rangle$ be an increasing sequence of integers such that $m_0 = 0$ and $m_{i+1} - m_i > 4^{i+3}$. For $j < \omega$ let $\mathbf{H}(j) = i + 2$, where i is such that $m_i \leqslant j < m_{i+1}$, and let $g(x) = x$. The local good creating pair $(K_g^{\mathbf{H}}, \Sigma_g^{\mathbf{H}})$ introduced in Corollary 2.8 is denoted by (K^1, Σ^1). By Example 2.4 we know that $((K^1)^{\bar{m}}, (\Sigma^1)^{\bar{m}})$ (see Definition 2.6) is sufficiently bad and hence (by Proposition 2.7) the forcing $\mathbb{Q}^*_\infty(K^1, \Sigma^1)$ collapses $\mathfrak{c}$ into $\aleph_0$.

Recall that, for a creature $t \in K^1$, we have

(1) $\mathbf{dis}[t] = \langle i^t, A^t\rangle$ for some $i^t < \omega$ and $\emptyset \neq A^t \subseteq \mathbf{H}(i^t)$;

(2) $\mathbf{nor}[t] = |A^t|$ and $\mathrm{pos}(t) = A^t$.

Let $l_n = |\mathbf{H}(n)|$ and

$$k_n = \lfloor \sqrt{\max\{k \in \omega \setminus \{0\} : f_k(l_n, 0) > 1\}} \rfloor \quad \text{if } l_n > 2^{2^{16}},$$

and $k_n = 2$ if $l_n \leqslant 2^{2^{16}}$. Certainly we have $\lim_{n\to\infty} l_n = \infty$ and therefore $\lim_{n\to\infty} k_n = \infty$ as well (and the sequence $\langle k_n : n < \omega\rangle$ is nondecreasing). Note also that $\lim_{n\to\infty} f_{k_n}(l_n, 0) = \infty$.

Definition 4.3. Let K consist of all creatures $t \in \mathrm{CR}[\mathbf{H}]$ such that

(1) $\mathbf{dis}[t] = \langle m^t, A^t, i^t\rangle$ for some $m^t < \omega$ and $\emptyset \neq A^t \subseteq \mathbf{H}(m^t)$, and $i^t \in \omega$, $0 \leqslant i^t \leqslant \log(\log(l_{m^t}))$;

(2) $\mathbf{nor}[t] = f_{k_{m^t}}(|A^t|, i^t)$, $m^t_{\mathrm{dn}} = m^t$, $m^t_{\mathrm{up}} = m^t + 1$ and $\mathrm{pos}(t) = A^t$.

For $t \in K$ we let

$$\Sigma(t) = \{s \in K : m^s = m^t \ \&\ A^s \subseteq A^t \ \&\ i^s \geqslant i^t\}.$$

Lemma 4.4. *The pair (K, Σ) is a local forgetful strongly finitary good creating pair for $\mathbf{H}$. The forcing notion $\mathbb{Q}^*_\infty(K, \Sigma)$ collapses $\mathfrak{c}$ to $\aleph_0$.*

Proof. It is straightforward to check that $(K^{\bar{m}}, \Sigma^{\bar{m}})$ inherits the sufficient badness of $((K^1)^{\bar{m}}, (\Sigma^1)^{\bar{m}})$ (remember Lemma 4.2(1)). Then use Proposition 2.7. $\square$

We are now going to define the desired factoring $\mathbb{Q}^*_\infty(K, \Sigma) \simeq \mathbb{P}^0 \times \mathbb{P}^1$ into proper factors $\mathbb{P}^0, \mathbb{P}^1$. For this we recursively define an increasing sequence $\bar{n} = \langle n_i : i < \omega\rangle$ so that $n_0 = 0$ and n_{i+1} is large enough such that

$$k_{n_{i+1}} \geqslant 2 \cdot \prod_{j < n_i} \mathbf{H}(j).$$

We put $U^0 = \bigcup_{i<\omega}[n_{2i}, n_{2i+1})$ and $U^1 = \bigcup_{i<\omega}[n_{2i+1}, n_{2i})$ and we let $\pi^0 : \omega \to U^0$ and $\pi^1 : \omega \to U^1$ be the increasing enumerations.

Definition 4.5. Let $\ell \in \{0, 1\}$. We define $\mathbf{H}^\ell = \mathbf{H} \circ \pi^\ell$ and we introduce K^ℓ, Σ^ℓ as follows:

(1) K^ℓ consists of all creatures $t \in \mathrm{CR}[\mathbf{H}^\ell]$ such that

(i) $\mathbf{dis}[t] = \langle m^t, A^t, i^t\rangle$ for some $m^t < \omega$ and $\emptyset \neq A^t \subseteq \mathbf{H}^\ell(m^t)$, and $i^t \in \omega$, $0 \leqslant i^t \leqslant \log(\log(l_n))$, where $n = \pi^\ell(m^t)$;
(ii) $m^t_{\rm dn} = m^t$, $m^t_{\rm up} = m^t + 1$, $\mathrm{pos}(t) = A^t$ and $\mathbf{nor}[t] = f_{k_n}(|A^t|, i^t)$ (where again $n = \pi^\ell(m^t)$);

(2) for $t \in K^\ell$ we let

$$\Sigma^\ell(t) = \{s \in K^\ell : m^s = m^t \ \&\ A^s \subseteq A^t \ \&\ i^s \geqslant i^t\}.$$

Lemma 4.6. (1) *For $\ell \in \{0,1\}$, (K^ℓ, Σ^ℓ) is a local forgetful good creating pair for $\mathbf{H}^\ell$.*

(2) *Let $\bar m^0 = \langle m^0_i : i < \omega\rangle$ and $\bar\varepsilon^0 = \langle \varepsilon^0_i : i < \omega\rangle$ be such that $\pi^0(m^0_i) = n_{2i}$ and $\varepsilon^0_i = 2/k_{n_{2i}}$. Then (K^0, Σ^0) has the $(\bar\varepsilon^0, \bar m^0)$-halving property.*

(3) *Let $\bar m^1 = \langle m^1_i : i < \omega\rangle$ and $\bar\varepsilon^1 = \langle \varepsilon^1_i : i < \omega\rangle$ be such that $\pi^1(m^1_i) = n_{2i+1}$ and $\varepsilon^1_i = 2/k_{n_{2i+1}}$. Then (K^1, Σ^1) has the $(\bar\varepsilon^1, \bar m^1)$-halving property.*

Proof. (1) The proof should be clear.

(2) Let $t \in K^0$, $\mathbf{nor}[t] \geqslant 2$, $\mathbf{dis}[t] = \langle m, A, i^*\rangle$. Let $n = \pi^0(m) \geqslant n_{2i}$ (so $m^0_i \leqslant m = m^t_{\rm dn}$). Define $j = (\log(\log(|A|)) + i^*)/2$ and let z be such that $\log(\log(z)) = j$. Certainly, $k_n \geqslant 2$ and $f_{k_n}(|A|, i^*) \geqslant 2$, so $\log(\log(|A|)) - i^* \geqslant 16$ and hence $i^* < j \leqslant \lceil j\rceil < \log(\log(|A|)) \leqslant \log(\log(l_n))$. Let $t^* \in K^0$ be such that $\mathbf{dis}[t^*] = \langle m, A, \lceil j\rceil\rangle$. Clearly $t^* \in \Sigma^0(t)$. We are going to argue that t^* is an ε^0_i-half of t (in (K^0, Σ^0)).

By $(*)$ of the proof of Lemma 4.2(1) and then by Lemma 4.2(2), we have

$$\begin{aligned}\mathbf{nor}[t^*] &= f_{k_n}(|A|, \lceil j\rceil) = \frac{1}{k_n}\log(\log(\log(|A|)) - \lceil j\rceil)\\ &\geqslant \frac{1}{k_n}\log((\log(\log(|A|)) - j) - 1) \geqslant \frac{1}{k_n}(\log((\log(\log(|A|)) - j)) - 1)\\ &= f_{k_n}(|A|, j) - \frac{1}{k_n} = f_{k_n}(|A|, i^*) - \frac{2}{k_n} \geqslant \mathbf{nor}[t] - \varepsilon^0_i.\end{aligned}$$

Now let $s \in \Sigma^0(t^*)$ be such that $\mathbf{nor}[s] > 1$. Let $\mathbf{dis}[s] = \langle m, A', i'\rangle$, thus $A' \subseteq A$ and $i' \geqslant \lceil j\rceil \geqslant j$. Let $t_0 \in K^0$ be such that $\mathbf{dis}[t_0] = \langle m, A', i^*\rangle$. Then $t_0 \in \Sigma^0(t)$ and $\mathrm{pos}(t_0) = A' = \mathrm{pos}(s)$. Also, $\mathbf{nor}[s] > 1$ implies $\log(\log(|A'|)) > i' \geqslant j$. By the definition of z we conclude $|A'| > z$. Noting that $|A| > 2^{2^{4+i^*}}$, we apply Lemma 4.2(4) to obtain

$$\mathbf{nor}[t_0] = f_{k_n}(|A'|, i^*) \geqslant f_{k_n}(z, i^*) = f_{k_n}(|A|, i^*) - \frac{1}{k_n} \geqslant \mathbf{nor}[t] - \varepsilon^0_i.$$

(3) The proof is similar to that of (2) above. □

Corollary 4.7. (1) *The forcing notions $\mathbb{Q}^*_\infty(K^\ell, \Sigma^\ell)$ (for $\ell = 0, 1$) are proper.*

(2) *Let $\mathbb{Q} = \{p \in \mathbb{Q}^*_\infty(K, \Sigma) : i(p) = n_i,\ i < \omega\}$. Then $\mathbb{Q}$ is a dense suborder of $\mathbb{Q}^*_\infty(K, \Sigma)$ and it is isomorphic with a dense suborder of the product $\mathbb{Q}^*_\infty(K^0, \Sigma^0) \times \mathbb{Q}^*_\infty(K^1, \Sigma^1)$. Consequently, the latter forcing collapses $\mathfrak{c}$ to $\aleph_0$.*

Proof. (1) Let $\bar m^0, \bar\varepsilon^0$ be as in Lemma 4.6(2). By the choice of $\bar n$ we have

$$\left|\prod_{n < m^0_i} \mathbf{H}^0(n)\right| = \left|\prod\left\{\mathbf{H}(j) : j \in \bigcup_{\ell < i}[n_{2\ell}, n_{2\ell+1})\right\}\right| \leqslant \prod_{j < n_{2i-1}} \mathbf{H}(i) \leqslant \frac{1}{2}k_{n_{2i}} = 1/\varepsilon^0_i.$$

Consequently, Theorem 3.3 and Lemma 4.6(1) and (2) imply that $\mathbb{Q}^*_\infty(K^0, \Sigma^0)$ is proper.

Similarly for $\mathbb{Q}^*_\infty(K^1, \Sigma^1)$.

(2) The proof should be clear. □

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