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REGRESSIVE PARTITION RELATIONS FOR INFINITE CARDINALS

ANDRAS HAJNAL, AKIHIRO KANAMORI AND SAHARON SHELAH

ABSTRACT. The regressive partition relation, which turns out to be important in incompleteness phenomena, is completely characterized in the transfinite case. This work is related to Schmerl [**S**], whose characterizations we complete.

Regressive functions arise naturally in the study of infinite cardinals, from Fodor's well-known lemma to contexts involving large cardinals (for example, the n-subtle cardinals of Baumgartner [**B2**]). In Kanamori and McAloon [**KM**], a regressive function version of the theorem of Erdös and Rado [**ER**] on canonical partitions was miniaturized and shown to be independent of Peano arithmetic. This result in turn reverberated to the infinite context to raise new questions; here we completely characterize the corresponding partition symbol for infinite cardinals. In contrast to the Erdös-Rado Theorem for the ordinary partition symbol, we show that these partition relations actually provide a characterization of cardinals in the finite Mahlo hierarchy. Thus, just as with the finite miniaturization, an elementary combinatorial property leads to a necessary transcendence. Our work confirms some speculations in McAloon [**M**], where an infinitary analogue of the Paris-Harrington partition relation is considered.

After we had already established some characterizations, we became aware of the close relationship of this work to the results of Schmerl [**S**]. The third author then saw how to sharpen the characterization of Schmerl's property as well as ours, and this paper is written so as to approach these optimal results most directly. The sharpening uses ideas of Todorcevic [**T**] who noted that our 3.4 for $n = 0$ can be derived directly from his work.

In §1, we begin the study of our partition symbol and establish the straightforward positive results about Mahlo cardinals. In §2, we develop some technical formulations and lemmata. Finally, we apply this work in §3 to establish the optimal results on the necessity of Mahlo cardinals. We discuss the connections with Schmerl [**S**] at the end.

We would like to thank Jim Schmerl for several expository suggestions, particularly for providing the statement of 3.3.

1. Preliminaries. We first formulate the regressive partition symbol: Let X be a set of ordinals and n a natural number. If f is a function with domain $[X]^n$, we write $f(\alpha_0, \ldots, \alpha_{n-1})$ for $f(\{\alpha_0, \ldots, \alpha_{n-1}\})$, with the understanding that $\alpha_0 < \cdots < \alpha_{n-1}$. Such a function is called *regressive* iff $f(\alpha_0, \ldots, \alpha_{n-1}) < \alpha_0$

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whenever $\alpha_0 < \cdots < \alpha_{n-1}$ are all from X and $\alpha_0 > 0$. There is a natural notion of homogeneity for such a function f: $H \subseteq X$ is *min-homogeneous for* f iff whenever $\alpha_0 < \cdots < \alpha_{n-1}$ and $\beta_0 < \cdots < \beta_{n-1}$ are all from H, $\alpha_0 = \beta_0$ implies $f(\alpha_0, \ldots, \alpha_{n-1}) = f(\beta_0, \ldots, \beta_{n-1})$. In other words, f on an n-tuple from H only depends on the first element. We write $X \to (\gamma)^n_{\text{reg}}$ iff whenever f on $[X]^n$ is regressive, there is an $H \in [X]^\gamma$ min-homogeneous for f.

The following is a simple case of the motivating result of Erdös and Rado [**ER**] on canonical partitions.

THEOREM 1.1. *For any* $n \in \omega$, $\omega \to (\omega)^n_{\text{reg}}$.

In Kanamori and McAloon [**KM**], the following miniaturization of this result is shown to be independent of Peano arithmetic:

(*) For any $k, n \in \omega$ there is an $m \in \omega$ such that $m \to (k)^n_{\text{reg}}$.

Spurred by these results, we turned to the study of the transfinite case. First, we state a simple proposition that relates our symbol to well-known concepts; its (b) subsumes 1.1.

PROPOSITION 1.2. (α) *If* $\eta \to (\gamma)^n_{\text{reg}}$ *and* $\text{cf}(\gamma) > \delta$, *then* $\eta \to (\gamma)^n_\delta$.
(b) $\eta \to (\eta)^n_{\text{reg}}$ *for every* $n \in \omega$ *iff* $\eta \to (\eta)^2_{\text{reg}}$ *is* ω *or weakly compact.*

PROOF. For (b), that $\eta \to (\eta)^2_{\text{reg}}$ implies $\eta \to (\eta)^2_2$ follows from (a). Conversely, there is a direct argument that $\eta \to (\eta)^{n+1}_3$ implies $\eta \to (\eta)^n_{\text{reg}}$. Alternatively, the standard proof by induction on n using the Tree Property shows that $\eta \to (\eta)^n_{\text{reg}}$ for every $n \in \omega$.

For exponent $n = 2$, the next result is a simple variation of the Erdös-Rado Theorem for the ordinary partition symbol and is the best possible by that theorem and 1.2(a).

THEOREM 1.3. $(2^\kappa)^+ \to (\kappa^+ + 1)^2_{\text{reg}}$.

PROOF. Suppose that f on $[(2^\kappa)^+]^2$ is regressive, and let $<_T$ be the usual corresponding Erdös-Rado tree on $(2^\kappa)^+$. That is, $\alpha <_T \beta$ iff $\alpha < \beta$ and $f(\xi, \alpha) = f(\xi, \beta)$ whenever $\xi <_T \alpha$. Notice that if B is a chain through the tree and β and $\bar{\beta}$ are two immediate successors of B, then, since β and $\bar{\beta}$ are incomparable, there must be a $\xi \in B$ such that $f(\xi, \beta) \neq f(\xi, \bar{\beta})$. But then this ξ must be the maximum point of B; otherwise if $\xi <_T \alpha \in B$, then $f(\xi, \beta) = f(\xi, \alpha) = f(\xi, \bar{\beta})$.

We can now show by induction that every level $\delta < \kappa^+$ of the tree has cardinality $\leq 2^\kappa$. The above argument shows that every chain without a maximum point has at most one immediate successor, so that any limit level $\delta < \kappa^+$ must inductively have cardinality $\leq (2^\kappa)^\delta = 2^\kappa$. For successor $\delta < \kappa^+$, the above argument shows that each α in level $\delta - 1$ has at most $|\alpha| \leq 2^\kappa$ immediate successors since f is regressive, so level δ must inductively have cardinality $\leq 2^\kappa \cdot 2^\kappa = 2^\kappa$. The argument is now complete, since level κ^+ must be nonempty, and any element there induces a chain corresponding to a $\kappa^+ + 1$ length min-homogeneous set.

Baumgartner has pointed out that 1.2 and 1.3 are special cases of his results on canonical partition relations in [**B1**].

To achieve positive relations for exponents $n \geq 3$, we shall need cardinals in the Mahlo hierarchy. Recall that κ is 0-*Mahlo* iff κ is strongly inaccessible and

$(n+1)$-*Mahlo* iff every closed unbounded subset of κ contains an n-Mahlo cardinal. For the inductive argument and later correlations, it will be convenient to verify a stronger relation:

THEOREM 1.4. *If κ is n-Mahlo, $\gamma < \kappa$, and $X \subseteq \kappa$ is unbounded, then $X \to (\gamma)^{n+2}_{\mathrm{reg}}$.*

PROOF. For $n = 0$, we can argue as in 1.3 that the corresponding Erdös-Rado tree is a κ-tree; that is, every level $\delta < \kappa$ of the tree has cardinality $< \kappa$. Thus, for every $\gamma < \kappa$ there is a chain of length γ, and we are done.

Proceeding by induction, suppose now that κ is $(n+1)$-Mahlo, $\gamma < \kappa$, $X \subseteq \kappa$ is unbounded, and f on $[X]^{n+3}$ is regressive. Again, let $<_T$ be the corresponding Erdös-Rado tree on X. That is, $\alpha <_T \beta$ iff $\alpha < \beta$ and $f(\xi_0, \dots, \xi_{n+1}, \alpha) = f(\xi_0, \dots, \xi_{n+1}, \beta)$ whenever $\xi_0 <_T \cdots <_T \xi_{n+1} <_T \alpha$. Now let $h\colon \kappa \leftrightarrow X$ be the increasing enumeration of X. Since $<_T$ can again be seen to be a κ-tree, if rank_T denotes the corresponding rank function, the set $C = \{\varsigma < \kappa \mid \gamma < \varsigma,\ h''\varsigma \subseteq \varsigma$, and if $\mathrm{rank}_T(h(\xi)) < \varsigma$, then $\xi < \varsigma$ must be closed unbounded. Let $\lambda \in C$ be n-Mahlo. Then we can apply the inductive hypothesis to $\overline{X} = \{\alpha \in X \mid \alpha <_T h(\lambda)\} \in [\lambda]^\lambda$ and the function $\bar{f}$ on $[X]^{n+2}$ defined by $\bar{f}(\xi_0, \dots, \xi_{n+1}) = f(\xi_0, \dots, \xi_{n+1}, h(\lambda))$ to complete this argument.

It turned out that this result is also observed in a different notation in Schmerl [**S**]. He also noticed that one can go a bit further: If κ is n-Mahlo, $m \in \omega$, and $X \subseteq \kappa$ is unbounded, then $X \to (m)^{n+3}_{\mathrm{reg}}$.

2. Technicalities. This section is devoted to some technical considerations and several lemmata. They serve to isolate the salient features that push through the main inductive arguments which establish our characterizations.

First of all, we formulate a technical hypothesis, due to the third author, which we shall preserve throughout the induction in order to obtain the optimal results. $W(n, X)$ is the following proposition, where we continue to take X a set of ordinals and $n \in \omega$.

> $W(n, X)$: *There is an $f\colon [X]^{n+2} \to \omega$ and a g regressive on $[X]^{n+3}$ such that whenever $H \subseteq X$ is both homogeneous for f (in the usual sense) and min-homogeneous for g, then setting $f''[H]^{n+2} = \{k\}$, $|H| \leq k$.*

The particular bound $|H| \leq k$ is immaterial. In fact, for any unbounded $h\colon \omega \to \omega$, $W(n, X)$ is equivalent to the formulation with $|H| \leq h(k)$ instead: just renumber the range values of f appropriately. Also, note that whenever A is a set such that $\omega - A$ is infinite, then we can require $\mathrm{range}(f) \cap A = \emptyset$: simply let $e\colon \omega \to (\omega - A)$ be the increasing enumeration and compose f with e. Finally, it is easy to see that $W(n, X)$ is a strong negation of our partition symbol:

PROPOSITION 2.1. *$W(n, X)$ implies $(X - \omega) \not\to (\omega)^{n+3}_{\mathrm{reg}}$.*

PROOF. Suppose that the pair $\langle f, g \rangle$ exemplifies $W(n, X)$. For any $\alpha \geq \omega$, fix a bijection $b_\alpha\colon \omega \times \alpha \to \alpha$. Now define h on $[X - \omega]^{n+3}$ by

$$h(\alpha_0, \dots, \alpha_{n+2}) = b_{\alpha_0}(f(\alpha_1, \dots, \alpha_{n+2}), g(\alpha_0, \dots, \alpha_{n+2})),$$

so that h is regressive. If, to the contrary, $(X-\omega) \to (\omega)^{n+3}_{\text{reg}}$, there would be an $H \in [X-\omega]^\omega$ min-homogeneous for h. But then $H - \{\min(H)\}$ is infinite, homogeneous for f, and min-homogeneous for g, contradicting the choice of $\langle f, g\rangle$.

With respect to the previous proposition and for a future correlation, we note that in most cases only final segments matter for our partition symbol.

PROPOSITION 2.2. *If $\gamma \geq \omega$ and $X \cap \eta \not\to (\gamma)^n_{\text{reg}}$, then $X \to (\gamma)^n_{\text{reg}}$ iff $(X-\eta) \to (\gamma)^n_{\text{reg}}$.*

PROOF. Let h exemplify $X \cap \eta \not\to (\gamma)^n_{\text{reg}}$. In the nontrivial direction, if f is regressive on $[X-\eta]^n$, we must find a min-homogeneous set of ordertype γ. First define an auxiliary g on $[X]^n$ as follows:

$$g(\alpha_0, \dots, \alpha_{n-1}) = \begin{cases} f(\alpha_0, \dots, \alpha_{n-1}) & \text{if } \eta \leq \alpha_0, \\ h(\alpha_0, \dots, \alpha_{n-1}) & \text{if } \alpha_{n-1} < \eta, \text{ else}, \\ 1 \quad \text{if } 1 < \alpha_0 < \dots < \alpha_i < \eta \leq \alpha_{i+1} \text{ and } i \text{ is even}, \\ 0 \quad \text{otherwise}. \end{cases}$$

g is regressive, so by hypothesis there is an $H \in [X]^\gamma$ min-homogeneous for g. $H - \eta \neq \emptyset$ by the second clause of g, so H can have at most one element of $\eta - 2$, by the third and fourth clauses of g. Thus, $H - \eta$ still has ordertype, γ since $\gamma \geq \omega$, and is min-homogeneous for f.

The next several lemmata are preservation results for $W(n, X)$. Similar results hold for the negation of our partition symbol and lead to a characterization, but the structure of the main inductive arguments needs the preservation of something like $W(n, X)$ for obtaining optimal results.

LEMMA 2.3. *Suppose $W(n, X \cap \xi)$ holds for every $\xi \in X$. Then $W(n+1, X)$ holds.*

PROOF. For each $\xi \in X$, let the pair $\langle f_\xi, g_\xi\rangle$ exemplify $W(n, X \cap \xi)$. Define f on $[X]^{n+3}$ by $f(\alpha_0, \dots, \alpha_{n+2}) = f_{\alpha_{n+2}}(\alpha_0, \dots, \alpha_{n+1}) + 1$. If g on $[X]^{n+4}$ is analogously defined from the g_ξ's then it is straightforward to verify that $\langle f, g\rangle$ exemplifies $W(n+1, X)$.

LEMMA 2.4. *Suppose $X \subseteq \eta$, where $\mathrm{cf}(\eta) = \omega$ and $W(n, X \cap \xi)$ holds for every $\xi < \eta$. Then $W(n, X)$ holds.*

PROOF. Fix a sequence $\langle \xi_k \mid k \in \omega\rangle$ cofinal in η with $\xi_0 = 0$, and let $\langle f_k, g_k\rangle$ exemplify $W(n, X \cap \xi_k)$ for every $k \in \omega$. By a remark just before 2.1, we can suppose that no natural number of form $3^s 5^t$ is in the range of any f_k. Now define $f\colon [X]^{n+2} \to \omega$ by

$$f(\alpha_0, \dots, \alpha_{n+1}) = \begin{cases} f_{k+1}(\alpha_0, \dots, \alpha_{n+1}) & \text{if } \xi_k \leq \alpha_0 < \dots < \alpha_{n+1} < \xi_{k+1}, \\ & \text{for some } k, \text{ else}, \\ 3^i 5^k \quad \text{where } \xi_k \leq \alpha_0 < \dots < \alpha_i < \xi_{k+1} \leq \alpha_{i+1}. \end{cases}$$

Also, define g regressive on $[X]^{n+3}$ by

$$g(\alpha_0, \dots, \alpha_{n+2}) = \begin{cases} g_{k+1}(\alpha_0, \dots, \alpha_{n+2}) & \text{if } \xi_k \leq \alpha_0 < \dots < \alpha_{n+2} < \xi_{k+1}, \\ & \text{for some } k, \text{ else}, \\ 0 \quad \text{otherwise}. \end{cases}$$

It can now be seen that $\langle f, g\rangle$ exemplifies $W(n, X)$: If $H \subseteq X$ is homogeneous for f and has at least $n+3$ elements, then the second clause of f insures that there must be some k such that $H \subseteq [\xi_k, \xi_{k+1})$. Hence, by definition of g, we can invoke $W(n, X \cap \xi_{k+1})$.

LEMMA 2.5. *If there is an* η *such that* $W(n, X \cap \eta)$ *and* $|X| \leq 2^{|\eta|}$, *then* $W(n, X)$.

PROOF. Let $\langle f, g\rangle$ exemplify $W(n, X \cap \eta)$. As before, we can assume that the range of f is disjoint from $\{0, 1, n+4\}$. As $|X| \leq 2^{|\eta|}$, let $\{A_\alpha \mid \alpha \in X - \eta\}$ be distinct subsets of η, and for $\alpha < \beta$ both in $X - \eta$ let $\delta(\alpha, \beta)$ be the least element in the symmetric difference $(A_\alpha - A_\beta) \cup (A_\beta - A_\alpha)$. Finally, define $F\colon [X]^{n+2} \to \omega$ by

$$F(\alpha_0, \ldots, \alpha_{n+1}) = \begin{cases} f(\alpha_0, \ldots, \alpha_{n+1}) & \text{if } \alpha_{n+1} < \eta, \\ n+4 & \text{if } \eta \leq \alpha_0, \text{ else}, \\ 0 & \text{if } \alpha_i < \eta \leq \alpha_{i+1} \text{ and } i \text{ is even}, \\ 1 & \text{otherwise.} \end{cases}$$

Also, define G on $[X]^{n+3}$ by

$$G(\alpha_0, \ldots, \alpha_{n+2}) = \begin{cases} g(\alpha_0, \ldots, \alpha_{n+2}) & \text{if } \alpha_{n+2} < \eta, \\ \delta(\alpha_1, \alpha_2) & \text{if } \eta \leq \alpha_0, \\ 0 & \text{otherwise.} \end{cases}$$

G is regressive, and we can deduce that $\langle F, G\rangle$ exemplifies $W(n, X)$: Suppose $H \subseteq X$ is homogeneous for F and min-homogeneous for G with at least $n+3$ elements. If $H \subseteq \eta$, then we are done. If $H \subseteq X - \eta$, then by the second clause of G it is not difficult to deduce that $|H| \leq n+4$. Finally, we can easily derive a contradiction in the remaining cases $F''[H]^{n+2} = \{0\}$ or $\{1\}$.

LEMMA 2.6. *Suppose that* $C^* \subseteq C$ *are both closed unbounded subsets of some limit ordinal* η. *If* $W(n, C^*)$ *and* $W(n, C \cap \xi)$ *for every* $\xi \in C^*$, *then* $W(n, C)$.

PROOF. For each $\alpha \in C$, set

$$\psi(\alpha) = \begin{cases} \sup(C^* \cap \alpha) & \text{if } \alpha > \text{ least element of } C^*, \\ 0 & \text{otherwise.} \end{cases}$$

We next define the *type* of a member of $[C]^{n+2}$ according to C^*: If $\alpha_0 < \cdots < \alpha_{n+1}$ are all in C, let $\{\xi_0, \ldots, \xi_k\}$ enumerate the set $\{\psi(\alpha_i) \mid i \leq n+1\}$ in increasing order and set $r_j = |\{i | \psi(\alpha_i) = \xi_j\}|$ for $j \leq k$. Then the *type* of $\{\alpha_0, \ldots, \alpha_{n+1}\}$ is $\langle r_0, \ldots, r_k\rangle$, which we can assume through coding is one natural number.

Now let $\langle f, g\rangle$ exemplify $W(n, C^*)$ and $\langle f_\xi, g_\xi\rangle$ exemplify $W(n, C \cap \xi)$ for every $\xi \in C^*$. We can assume that the ranges of f and the f_ξ's do not contain any natural number coding a type. Define F on $[C]^{n+2}$ by

$$F(\alpha_0, \ldots, \alpha_{n+1}) = \begin{cases} f(\psi(\alpha_0), \ldots, \psi(\alpha_{n+1})) & \text{if } 0 < \psi(\alpha_0) < \cdots < \psi(\alpha_{n+1}), \\ f_\xi(\alpha_0, \ldots, \alpha_{n+1}) & \text{if } \psi(\alpha_0) = \cdots = \psi(\alpha_{n+1}), \\ & \text{where } \xi \text{ is the next element of } C^* \\ & \text{after } \psi(\alpha_0), \\ \text{type of } \{\alpha_0 \ldots, \alpha_{n+1}\} & \text{otherwise.} \end{cases}$$

Also define G on $[C]^{n+3}$ by

$$G(\alpha_0, \ldots, \alpha_{n+2}) = \begin{cases} g(\psi(\alpha_0), \ldots, \psi(\alpha_{n+2})) & \text{if } 0 < \psi(\alpha_0) < \cdots < \psi(\alpha_{n+2}), \\ g_\xi(\alpha_0, \ldots, \alpha_{n+2}) & \text{if } \psi(\alpha_0) = \cdots = \psi(\alpha_{n+2}), \\ & \text{where } \xi \text{ is the next element of } C^* \\ & \text{after } \psi(\alpha_0), \\ 0 & \text{otherwise.} \end{cases}$$

G is regressive, and we can deduce that $\langle F, G\rangle$ exemplifies $W(n, C)$: Suppose H is homogeneous for F with at least $n+3$ elements. By using the third clause of F, we can easily deduce that ψ must be either one-to-one or constant on $[H]^{n+2}$. Hence, the argument is complete by definition of G.

We will also need a version of 2.6 that deals directly with or partition symbol.

LEMMA 2.7. *Suppose that $n \geq 3$ and $C^* \subseteq C$ are both closed unbounded subsets of some limit ordinal η such that $C \cap \omega = \emptyset$. If $C^* \not\to (\gamma)^n_{\text{reg}}$ and $C \cap \xi \not\to (\gamma)^n_{\text{reg}}$ for every $\xi \in C^*$, then $C \not\to (\gamma)^n_{\text{reg}}$.*

PROOF. Let ψ and *type* be as in 2.6. Let g exemplify $C^* \not\to (\gamma)^n_{\text{reg}}$ and g_ξ exemplify $C \cap \xi \not\to (\gamma)^n_{\text{reg}}$ for every $\xi \in C^*$. Since $C \cap \omega = \emptyset$, we can assume that the ranges of g and the g_ξ's do not contain any number coding a type.

We can now define G on $[C]^n$ as follows.

$$G(\alpha_0, \ldots, \alpha_{n-1}) = \begin{cases} g(\psi(\alpha_0), \ldots, \psi(\alpha_{n-1})) & \text{if } \psi(\alpha_0) < \cdots < \psi(\alpha_{n-1}), \\ g_\xi(\alpha_0, \ldots, \alpha_{n-1}) & \text{if } \psi(\alpha_1) = \cdots = \psi(\alpha_{n-1}), \\ & \text{where } \xi \text{ is the next element of } C^* \\ & \text{after } \psi(\alpha_1), \\ \text{type of } \{\alpha_0, \ldots, \alpha_{n-1}\} & \text{otherwise.} \end{cases}$$

(In the second clause, that we start with $\psi(\alpha_1)$ is not a misprint; that $n \geq 3$ is called upon here.) G is regressive, so suppose that $H \subseteq C$ is min-homogeneous for G. We can assume that H has at least $n+1$ elements, and we can let $\beta_0 < \beta_1$ be its least two elements.

Assume first that $\psi(\beta_0) = \psi(\beta_1)$. If there were a further $\beta \in H$ such that $\psi(\beta_1) < \psi(\beta)$, then there would be two sequences of length n from H, both starting with β_0 and with different types—one with β_1 and one without. This is contradictory, so ψ must be constant on H. Thus, by the second clause of G, H cannot have ordertype γ.

Assume next that $\psi(\beta_0) < \psi(\beta_1)$. Suppose first that there were a further $\beta \in H$ such that $\psi(\beta_1) < \psi(\beta)$. Then if ψ were not one-to-one on H, one can again generate two appropriate sequences of length n from H, both starting with β_0 and with different types, to derive a contradiction. Thus, ψ must be one-to-one on H, and by the first clause of G, H cannot have ordertype γ.

In the remaining case of $\psi(\beta_0) < \psi(\beta_1)$ with $\psi(\beta) = \psi(\beta_1)$ for every further $\beta \in H$, we can invoke the second clause of G to again show that H cannot have ordertype γ. This completes the proof.

3. Characterizations. With the work of §2 in hand, we can now establish our characterizatons. The next theorem takes the first step beyond 1.3 and is itself the basis step of the general inductive argument. The main line of argument is related to the third author's recent work on the consistency of $2^\omega \to [\omega_1]^2_3$, which in turn was influenced by Todorcevic's recent result $\omega_1 \not\to [\omega_1]^2_{\omega_1}$.

THEOREM 3.1. *If for some limit ordinal η, $C \subseteq \eta$ is closed unbounded and contains no inaccessible cardinals, then $W(0, C)$.*

PROOF. We proceed by induction, considering the different possibilities for η. For any $\eta \leq \omega$, the result is trivially true. Next, suppose that $\text{cf}(\eta) = \omega$. If first of all C has ordertype $\delta + \omega$ for some limit ordinal δ then the result follows by induction and 2.5. Otherwise, for arbitrarily large $\xi < \eta$, $C \cap \xi$ is closed unbounded in ξ; then the result follows by induction and 2.4.

Suppose now that $\mathrm{cf}(\eta) > \omega$, and for some $\xi < \eta$ we have $|C| \leq 2^{\xi}$. Since there is a $\bar{\xi}$ such that $\xi \leq \bar{\xi} < \eta$ and $C \cap \bar{\xi}$ is closed unbounded in $\bar{\xi}$, the result follows by induction and 2.5.

Finally, it remains to consider the case of η a strong limit cardinal such that $\mathrm{cf}(\eta) > \omega$. Here, $C^* = \{\alpha \in C \mid \alpha \text{ is a singular cardinal}\}$ is also closed unbounded since C contains no inaccessibles. It now suffices to establish $W(0, C^*)$, for then the result follows by induction and 2.6.

To do this, let us first define sets C_α for $\alpha \in C^*$ as follows: If α is a limit point of C^*, let C_α be a closed unbounded subset of α of ordertype $\mathrm{cf}(\alpha)$ such that $\mathrm{cf}(\alpha) < \min(C_\alpha)$. If α is not a limit point of C^*, set $C_\alpha = \{\sup(C^* \cap \alpha)\}$.

Next, set $\sigma(\alpha, \beta) = \min(C_\beta - \alpha) \geq \alpha$ for $\alpha < \beta$ both in C^*. Then, inductively define $\tau_m(\alpha, \beta)$ as follows: Set $\tau_0(\alpha, \beta) = \beta$. If $\tau_m(\alpha, \beta)$ is defined and $> \alpha$, set $\tau_{m+1}(\alpha, \beta) = \sigma(\alpha, \tau_m(\alpha, \beta))$. Since the $\tau_m(\alpha, \beta)$'s form a descending sequence of ordinals, let $k \in \omega$ be a maximal such that $\tau_k(\alpha, \beta)$ is defined. For further reference, notice that if $\alpha < \beta < \gamma$ are all in C^* then

$$\text{(1)} \qquad \text{if } m \leq k \text{ and } \beta \leq \tau_m(\alpha, \gamma), \text{ then } \tau_m(\alpha, \gamma) = \tau_m(\beta, \gamma).$$

Set $F(\alpha, \beta) = 4k + 2$.

Finally, define three functions F_0, F_1 and F_2 on $[C^*]^3$ as follows.

$$F_0(\alpha, \beta, \gamma) = \max\{m \mid \tau_m(\alpha, \gamma) \geq \beta\},$$

$$F_1(\alpha, \beta, \gamma) = \begin{cases} 1 & \text{if } \mathrm{cf}(\tau_m(\alpha, \gamma)) < \alpha, \text{ where } m = F_0(\alpha, \beta, \gamma), \\ 0 & \text{otherwise,} \end{cases}$$

$$F_2(\alpha, \beta, \gamma) = \begin{cases} 1 + \text{ordertype } (C_{\tau_m(\alpha,\gamma)} \cap \beta) & \text{if this is } < \alpha, \\ & \text{where } m = F_0(\alpha, \beta, \gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Since C^* consists of cardinals, we can faithfully code F_0, F_1, and F_2 into one function G regressive on $[C^*]^3$. We will now establish that $\langle F, G \rangle$ exemplifies $W(0, C^*)$: Suppose that $H \subseteq C^*$ is homogeneous for F and min-homogeneous for G. In particular, $F''[H]^2 = \{4k + 2\}$ for some $k \in \omega$. By min-homogeneity for G, whenever $\alpha \in H$, there are $m(\alpha) < k$ and $i(\alpha) < 2$ such that if $\beta, \gamma \in H$ and $\alpha < \beta < \gamma$, then $F_0(\alpha, \beta, \gamma) = m(\alpha)$ and $F_1(\alpha, \beta, \gamma) = i(\alpha)$.

To verify $W(0, C^*)$, let us assume to the contrary that $|H| \geq 4k + 3$. Since there are $2k$ possible pairs $\langle m(\alpha), i(\alpha) \rangle$, by the Pigeonhole Principle there must be three elements $\alpha_0 < \alpha_1 < \alpha_2$ among the first $4k + 1$ elements of H such that $m(\alpha_0) = m(\alpha_1) = m(\alpha_2) = $ fixed m and $i(\alpha_0) = i(\alpha_1) = i(\alpha_2) = $ fixed i. Let α_3 and α_4 be two elements of H above α_2.

Set $\rho = \tau_m(\alpha_1, \alpha_4)$. Since $F_0(\alpha_1, \alpha_2, \alpha_4) = m$,

$$\text{(2)} \qquad \alpha_1 \leq \tau_{m+1}(\alpha_1, \alpha_4) < \alpha_2 \leq \rho.$$

Thus

$$\text{(3)} \qquad C_\rho \cap [\alpha_1, \alpha_2) \neq \emptyset,$$

since $\tau_{m+1}(\alpha_1, \alpha_4)$ is a member of this set by (2). By (1) and (2), we also have $\rho = \tau_m(\alpha_2, \alpha_4)$. Since $C_\rho \cap \mathrm{cf}(\rho) = \emptyset$ by the definition of the C_α's it follows from

(3) that $i = F_1(\alpha_2, \alpha_3, \alpha_4) = 1$. Since $i(\alpha_0) = i = 1$, we can now conclude from (3) that $F_3(\alpha_0, \alpha_1, \alpha_4) = 1 + \text{ordertype } (C_\rho \cap \alpha_1) \neq 1 + \text{ordertype } (C_\rho \cap \alpha_2) = F_3(\alpha_0, \alpha_2, \alpha_4)$, contradicting min-homogeneity.

All of the work has now been done for the overall inductive result.

THEOREM 3.2. *If for some limit ordinal η, $C \subseteq \eta$ is closed unbounded and contains no n-Mahlo cardinals, then $W(n, C)$.*

PROOF. We proceed by induction on n. The case $n = 0$ is 3.1. Assume that the result has already been established for n, and suppose that $C \subseteq \eta$ is closed unbounded and contains no $(n+1)$-Mahlo cardinals. We shall now establish by induction on ξ that

$$(*) \qquad W(n, C \cap \xi) \quad \text{for every } \xi \in C.$$

This together with 2.3 implies $W(n+1, C)$, so the proof would be complete.

$(*)$ is trivially true for ξ the minimum element of C. Also, if ς and ξ are consecutive elements of C and $W(n, C \cap \varsigma)$, then $W(n, C \cap \xi)$ by 2.5. The only remaining case to consider is when ξ is a limit point of C. Then $\xi \in C$, so ξ is not $(n+1)$-Mahlo. Thus, there is a closed unbounded $D \subseteq C \cap \xi$ containing no n-Mahlo cardinals. By the inductive hypothesis on n, $W(n, D)$. Thus, by the inductive hypothesis on ξ and 2.6, $W(n, C \cap \xi)$.

We can now state several summarizing characterizations. The first formulation was suggested to us by Schmerl and subsumes the others.

THEOREM 3.3. *For any $\gamma \geq \omega$, the following are equivalent*:

(a) $X \to (\gamma)^{n+3}_{\text{reg}}$.

(b) *Either* (i) *there is an $(n+1)$-Mahlo $\kappa > \gamma$ such that $X \cap \kappa$ is unbounded in κ, or* (ii) *γ is ω or weakly compact and $X \cap \gamma$ is unbounded in γ.*

PROOF. (b)$\to$(a) is immediate by 1.4 and 1.2(b). Assume now that (a)$\to$(b) is false, and fix a pair $\langle X, \alpha \rangle$ with $\eta = \sup(X)$ the least possible such that $X \to (\gamma)^{n+3}_{\text{reg}}$, yet (b) fails. Note that η must be a limit ordinal by 2.2. Let $C \subseteq \eta$ be the $<$-closure of X, i.e. the closed unbounded subset of η consisting of the members of X together with the limit points of X. Since (b) fails, $\eta > \gamma$, else γ would be ω or weakly compact by 1.2(b), so η cannot be $(n+1)$-Mahlo. Thus, there is a closed unbounded $C^* \subseteq C - \omega$ which does not contain any n-Mahlo cardinals. By 2.1 and 3.2, $C^* \not\to (\gamma)^{n+3}_{\text{reg}}$. Also, for any $\xi < \eta$, $(C \cap \xi) \not\to (\gamma)^{n+3}_{\text{reg}}$ by the minimality of η, since C is just the closure of X. Hence, $(C - \omega) \not\to (\gamma)^{n+3}_{\text{reg}}$ by 2.7. We can also conclude that $C \not\to (\gamma)^{n+3}_{\text{reg}}$ by 2.2, since either $\gamma > \omega$, or else $C \cap \omega = X \cap \omega$ is bounded below ω by the failure of (b). But this is a contradiction, since $X \subseteq C$.

The following characterization of $(n+1)$-Mahlo cardinals is now a consequence of 3.3, or of 1.4, 2.1, and 3.2 directly. Todorcevic noted that the case $n = 0$ can be derived directly from his work in [**T**].

THEOREM 3.4. *The following are equivalent*:

(a) *κ is $(n+1)$-Mahlo.*

(b) *For any $\gamma < \kappa$ and unbounded $X \subseteq \kappa$, $X \to (\gamma)^{n+3}_{\text{reg}}$.*

(c) *For any closed unbounded $C \subseteq \kappa$, $C \to (\omega)^{n+3}_{\text{reg}}$.*

3.3(c) is optimal, in the sense that ω cannot be replaced by any $m \in \omega$, by the remark at the end of §1.

We can also take a more dynamic approach: the following is another consequence of 3.3.

THEOREM 3.5. *If $\gamma > \omega$, then the least η such that $\eta \to (\gamma)^{n+3}_{\text{reg}}$ is the least $(n+1)$-Mahlo cardinal $\geq \gamma$.*

As an immediate corollary, we have another characterization:

THEOREM 3.6. *The following are equivalent for $\kappa > \omega$.*
(a) *κ is $(n+1)$-Mahlo or a limit of $(n+1)$-Mahlo cardinals.*
(b) *For every $\gamma < \kappa$, $\kappa \to (\gamma)^{n+3}_{\text{reg}}$.*

We conclude by making some remarks on the connection of our results to the work of Schmerl [**S**]. Schmerl and Shelah [**SS**] deals with a model-theoretic transfer theorem which involves combinatorial properties of cardinals high in the Mahlo hierarchy. Schmerl [**S**] established that for n-Mahlo cardinals these properties provide a characterization. If F is an ordinal-valued function with domain a set of ordinals X, then f on $[X]^n$ is *F-regressive* iff $f(\alpha_0, \ldots, \alpha_{n-1}) < F(\alpha_0)$ whenever $F(\alpha_0) > 0$. Schmerl's property $P(n, \alpha)$ of κ, stated in our terminology, is: For every cardinal-valued function $F\colon \kappa \to \kappa$ and every F-regressive function on $[\kappa]^{n-1}$, there is a min-homogeneous set for f of ordertype α. Note that for regular κ, $\kappa \in P(n, \alpha)$ iff for any unbounded $X \subseteq \kappa$, $X \to (\alpha)^{n+1}_{\text{reg}}$. Thus, our study turns out to be a variant, motivated by regressive functions. Considering only $F =$ the identity map on κ does simplify the development and leads to clear inductive arguments involving closed unbounded sets. Schmerl essentially provided the following characterizations, stated in our terminology, for finite min-homogeneous sets.

THEOREM. *The following are equivalent*:
(a) *κ is n-Mahlo.*
(b) *For any $m \in \omega$ and unbounded $X \subseteq \kappa$, $X \to (m)^{n+3}_{\text{reg}}$.*
(c) *For any closed unbounded $C \subseteq \kappa$, $C \to (n+5)^{n+3}_{\text{reg}}$.*

However, Schmerl did not complete his characterization of $P(n, \alpha)$ for every n and α; we switched to the properties $W(n, X)$ in order to obtain the optimal results. In particular, our results confirm a conjecture from [**S**] by establishing $P(n+2, \omega)$ implies κ is $(n+1)$-Mahlo in a sharp sense and fill in the question marks in the chart on p. 290 of [**S**].

In developing some Π^1_2 "Borel diagonalization" propositions about reals equiconsistent with the existence of n-Mahlo cardinals, H. Friedman [**F**] relied on the combinatorial work of Schmerl [**S**]. Thus, regressive partition relations provide a unifying approach to two incompleteness phenomena: the finite version is equivalent to the Paris-Harrington proposition (see Kanamori and McAloon [**KM**]), and the transfinite version leads to Friedman's result.

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MATHEMATICAL INSTITUTE, HUNGARIAN ACADEMY OF SCIENCES, BUDAPEST, HUNGARY

DEPARTMENT OF MATHEMATICS, BOSTON UNIVERSITY, BOSTON, MASSACHUSETTS 02215

DEPARTMENT OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM, ISRAEL

DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, VANCOUVER, CANADA

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903