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Models of expansions of $\mathbb{N}$ with no end extensions

Saharon Shelah*

Einstein Institute of Mathematics, Hebrew University of Jerusalem, Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel
Department of Mathematics, Hill Center-Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854, United States of America

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We deal with models of Peano arithmetic (specifically with a question of Ali Enayat). The methods are from creature forcing. We find an expansion of $\mathbb{N}$ such that its theory has models with no (elementary) end extensions. In fact there is a Borel uncountable set of subsets of $\mathbb{N}$ such that expanding $\mathbb{N}$ by any uncountably many of them suffice. Also we find arithmetically closed $\mathcal{A}$ with no ultrafilter on it with suitable definability demand (related to being Ramsey).

1 Introduction

Recently, Ali Enayat solved a long standing problem on models of Peano arithmetic that appeared as [4, Problem 7] (among other results):

Theorem 1.1 (*See* [1]) *For some arithmetically closed family $\mathcal{A}$ of subsets of ω, the model $\mathbb{N}_{\mathcal{A}} = (\mathbb{N}, A)_{A\in\mathcal{A}}$ has no conservative extension* (*i.e., one in which the intersection of any definable subset with $\mathbb{N}$ belongs to $\mathcal{A}$*).

Motivated by this result he asked:

Question 1.2 *Is there $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that some model of $\mathrm{Th}(\mathbb{N}_{\mathcal{A}})$ has no elementary end extension?*

In other words, this question asks whether the countability demand in the MacDowell-Specker theorem is necessary: this classical theorem says that if T is a theory in a countable vocabulary $\tau = \tau_T$ extending $\tau(\mathbb{N}) = \{0, 1, +, \times\}$ and T contains $\mathrm{PA}(\tau)$, then any model of T has an (elementary) end extension; Gaifman extended this theorem in several ways, e.g., having minimal extensions (see [4] on it). The present author extended it in [10] in another way: we do not need addition and multiplication, i.e., any model of T has an elementary end extension when τ is a countable vocabulary, $\{0, <\} \subseteq \tau$, T is a (first order) theory in $\mathbb{L}(\tau)$, T says that $<$ is a linear order with 0 first, every element x has a successor $S(x)$, and all cases of the induction scheme belong to T. Mills [5] proved that there is a countable non-standard model of PA with uncountable vocabulary such that it has no elementary end extension.

We answer Question 1.2 positively in §5, we give a sufficient condition in §3 and deal with a relevant forcing in §4. In fact we get an uncountable Borel set $\mathbf{B} \subseteq \mathcal{P}(\mathbb{N})$ such that if $B_\alpha \in \mathbf{B}$ for $\alpha < \alpha_*$ are pairwise distinct and α_* is uncountable, then $\mathrm{Th}(\mathbb{N}, B_\alpha)_{\alpha<\alpha_*}$ satisfies the conclusion.

Furthermore, Enayat asked in [1]:

Question 1.3 *Can we prove in ZFC that there is an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathcal{A}$ carries no minimal ultrafilter?*

He proved it for the stronger notion of 2-*Ramsey ultrafilter*. We shall deal with the problem later (see [9]); here we prove that there is an arithmetically closed Borel set $\mathbf{B} \subseteq \mathcal{P}(\mathbb{N})$ such that any expansion

* e-mail: shelah@math.huji.ac.il

$\mathbb{N}$ by any uncountably many members of $\mathbf{B}$ has such a property, i.e., the family of definable subsets of $\mathbb{N}^+$ carry no 2.5-*Ramsey ultrafilter*.

Note that

(∗) if $N \neq \mathbb{N}$ is a model of PA which has no cofinal minimal extension, then on $\mathrm{StSy}(N)$ there is no minimal ultrafilter, see Definitions 2.2, 2.3(1).

Finally, Enayat asked:

Question 1.4 *For a Borel set $\mathcal{A} \subseteq \mathcal{P}(\omega)$,*

(a) *does the model $\mathbb{N}_{\mathcal{A}}$ have a conservative end extension? This is what is answered here (in the light of the previous paragraph).*

(b) *Suppose further that $\mathcal{A}$ is arithmetically closed. Is $(\mathcal{A} \cap [\omega]^{\aleph_0}, \supseteq)$ a proper forcing notion?*

The results in this paper solve Question 1.4(a); Question 1.4(b) is solved jointly with Enayat in [2]. Enayat suggests that if we succeed to combine an example for "$\mathrm{StSy}(N)$ has no minimal ultrafilter" and Kaufman and Schmerl [3], then we should be able to solve the "there is N with no cofinal minimal extension" [4, Problem 2]. Note that our claim on the creature forcing gives suitable kinds of Ramsey theorems.

2 Notation & definitions

Notation 2.1

(1) As usual in set theory, ω is the set of natural numbers. Let $\mathrm{pr} : \omega \times \omega \longrightarrow \omega$ be the standard pairing function (i.e., $\mathrm{pr}(n, m) = \binom{n+m}{2} + n$, so one–to–one onto two–place function).

(2) Let $\mathcal{A}$ denote a subset of $\mathcal{P}(\omega)$.

(3) The Boolean algebra generated by $\mathcal{A} \cup [\omega]^{<\aleph_0}$ will be denoted by $\mathrm{BA}(\mathcal{A})$.

(4) Let D denote a non-principal ultrafilter on $\mathcal{A}$. When $\mathcal{A}$ is not a sub-Boolean-Algebra of $\mathcal{P}(\omega)$, this means that $D \subseteq \mathcal{A}$ and there is a unique non-principal ultrafilter D' on the Boolean algebra $\mathrm{BA}(\mathcal{A})$ such that $D = D' \cap \mathcal{A}$. (In 2.3 this extension makes a difference.)

(5) Let τ denote a vocabulary extending $\tau_{\mathrm{PA}} = \tau_{\mathbb{N}} = \{0, 1, +, \times, <\}$, usually countable.

(6) $\mathrm{PA}_\tau = \mathrm{PA}(\tau)$ is Peano arithmetic for the vocabulary τ.

(7) A model N of $\mathrm{PA}(\tau)$ is *ordinary* if $N{\restriction}\tau_{\mathrm{PA}}$ extends $\mathbb{N}$; usually our models will be ordinary.

(8) $\varphi(N, \bar{a})$ is $\{b : N \models \varphi[b, \bar{a}]\}$, where $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_N)$ and $\bar{a} \in {}^{\ell g(\bar{y})}N$.

(9) $\mathrm{Per}(A)$ is the set (or group) of permutations of A.

(10) For sets u, v of ordinals let $\mathrm{OP}_{v,u}$, "the order preserved function from u to v", be defined by:

$$\mathrm{OP}_{v,u}(\alpha) = \beta \text{ if and only if}$$
$$\beta \in v, \alpha \in u \quad \text{and} \quad \mathrm{otp}(v \cap \beta) = \mathrm{otp}(u \cap \alpha).$$

(11) We say that $u, v \subseteq \mathrm{Ord}$ form a Δ-system pair when $\mathrm{otp}(u) = \mathrm{otp}(v)$ and $\mathrm{OP}_{v,u}$ is the identity on $u \cap v$.

Definition 2.2

(1) For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we let
ar-cl$(\mathcal{A}) = \{B \subseteq \omega : B$ is first order definable in $(\mathbb{N}, A_1, \dots, A_n)$ for some $n < \omega$ and $A_1, \dots, A_n \in \mathcal{A}\}$.
The set ar-cl$(\mathcal{A})$ is called *the arithmetic closure of $\mathcal{A}$*.

(2) For a model N of $\mathrm{PA}(\tau)$ let *the standard system of N* be

$$\mathrm{StSy}(N) = \{\varphi(N', \bar{a}) \cap \mathbb{N} : \varphi(x, \bar{y}) \in \mathbb{L}(\tau) \quad \text{and} \quad \bar{a} \in {}^{\ell g(\bar{y})}N\}$$

for any ordinary model N' isomorphic to N.

Definition 2.3 Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

(1) For $h \in {}^{\omega}\omega$ let $\mathrm{cd}(h) = \{\mathrm{pr}(n, h(n)) : n < \omega\}$, where pr is the standard pairing function of ω, see 2.1(1).

(2) An ultrafilter D on $\mathcal{A}$, is called *minimal* when:
if $h \in {}^{\omega}\omega$ and $\mathrm{cd}(h) \in \mathcal{A}$, then for some $X \in D$ we have that $h{\restriction}X$ is either constant or one-to-one.

(3) An ultrafilter D on $\mathcal{A}$ is called *Ramsey* when:
if $k < \omega$ and $h : [\omega]^k \longrightarrow \{0, 1\}$ and $\mathrm{cd}(h) \in \mathcal{A}$, then for some $X \in D$ we have $h{\restriction}[X]^k$ is constant.
Similarly we define k-Ramsey ultrafilters.

(4) D is called 2.5-Ramsey or *self-definably closed* when:

if $\bar{h} = \langle h_i : i < \omega\rangle$ and $h_i \in {}^{\omega}(i + 1)$ and $\mathrm{cd}(\bar{h}) = \{\mathrm{pr}(i, \mathrm{pr}(n, h_i(n)) : i < \omega, n < \omega\}$ belongs to $\mathcal{A}$, then for some $g \in {}^{\omega}\omega$ we have:

$$\mathrm{cd}(g) \in \mathcal{A} \quad \text{and} \quad (\forall i)[g(i) \leq i \wedge \{n < \omega : h_i(n) = g(i)\} \in D];$$

this follows from 3-Ramsey and implies 2-Ramsey.

(5) D is *weakly definably closed* when:

if $\langle A_i : i < \omega\rangle$ is a sequence of subsets of ω and $\{\mathrm{pr}(n, i) : n \in A_i \text{ and } i < \omega\} \in \mathcal{A}$, then $\{i : A_i \in D\} \in \mathcal{A}$, (follows from 2-Ramsey); Kirby called it "definable"; Enayat uses "iterable".

Definition 2.4 For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\mathbb{N}_{\mathcal{A}}$ be $\mathbb{N}$ expanded by a unary relation A for every $A \in \mathcal{A}$, so formally it is a $\tau_{\mathcal{A}}$–model, $\tau_{\mathcal{A}} = \tau_{\mathbb{N}} \cup \{P_A : A \in \mathcal{A}\}$, but below if we use $\mathcal{A} = \{A_t : t \in X\}$, then we actually use $\{P_t : t \in X\}$.

Definition 2.5 Let N be a model of $T \supseteq \mathrm{PA}(\tau)$, $\tau = \tau_T$.

(1) We say that N^+ is an end extension of N when:
   (a) $N \prec N^+$,
   (b) if $a \in N$ and $b \in N^+ \setminus N$, then $N^+ \models a < b$.

(2) We say N^+ is a conservative [end] extension of N whenever (a),(b) hold and
   (c) if $\varphi(x, \bar{y}) \in \mathbb{L}(\tau)$, $\bar{b} \in {}^{\ell g(\bar{y})}(N^+)$, then $\varphi(N^+, \bar{b}) \cap N$ is a definable subset of N.

We may ask: How is creature forcing relevant? Do we need Rosłanowski–Shelah [6]?

The creatures (and creatures forcing) we deal with fit [6], but instead of CS iteration it suffices for us to use a watered down version of creature iteration. That is here it is enough to define $\mathbb{Q}_u$ for finite $u \subseteq \mathrm{Ord}$ such that:

(a)$_1$ $\mathbb{Q}_u$ is a creature forcing with generic $\langle \underset{\sim}{t}_\alpha : \alpha \in u\rangle$; this restriction implies that cases irrelevant in full forcing where we have to use countable u, are of interest here; hence we can use creature forcing rather than iterated creature forcing.

(a)$_2$ In §4, $\mathbb{Q}_u$ is a good enough ${}^{\omega}\omega$–bounding creature forcing, so we have continuous reading of names.

(a)$_3$ We are used to do it above a countable models N of ZFC^-, and this seems more transparent. But actually asking on the Δ_n–type of the generic over $\mathbb{N}$ suffices. That is, we can, e.g., by Δ_{n+7} formula over $\mathbb{N}$ find, e.g., a condition $p \in \mathbb{Q}_u$ such that any $\bar{t} \in \mathbf{B}_p$, e.g., a branch in the tree its Δ_n-type over $\mathbb{N}$, i.e., the Δ_n-theory of $(\mathbb{N}, \bar{t})$, so t_ℓ acts as a predicate (we can think of $\mathbf{B}_u$ as $\subseteq {}^{u}({}^{\omega}2)$).

Here the construction is by forcing over a countable $N_* \prec (\mathcal{H}(\chi), \in)$. Note that there is no problem to add $\mathcal{A}^* := N_* \cap \mathcal{P}(\omega)$. So we can prove the results for $\mathcal{A} =$ (countable) $\cup$ (perfect). To improve it to perfect we need to force for PA by induction on n for Σ_n formulas.

(a)$_4$ Note: for this it is fine if in every $p \in \mathbb{Q}_u$ the total number of commitments of the form "ρ is a member of $\varrho_x(i)$" is finite.

(b)$_1$ We can use $u_n = {}^{n}2$, just a notational change, we would like to choose p_n by induction on $n < \omega$ such that:
   (α) $p_n \in \mathbb{Q}_{u_2}$,

(β) p_n is such that for $\bar{t} \in \mathbf{B}_{p_n}$ the Σ_n–theory of $(\mathbb{N}, \bar{t})$ can be read continuously on p,

(γ) if $h : {}^n 2 \longrightarrow {}^{n+1}2$ is such that $(\forall \rho \in {}^n 2)(h(\rho) \restriction n = \rho)$, then $h(p_n) = p_n \restriction \mathrm{Rang}(h)$ both defined naturally (can make one duplicating at a time).

(b)$_2$ In (b)$_1$, the set $\bigcup\{\varrho_x(i) : x \in p\}$ grows from p_n to p_{n+1}, i.e., here we need the major point in the choice of $\mathrm{nor}^0_x(C)$; however we do not need to diagonalize over it as in the proof about $\mathbb{Q}_u$.

(c)$_1$ However, in §4 we can define full creature iterated forcing, i.e. using countable support; it is of interest but irrelevant here;

(c)$_2$ but some cases of such creature forcing may look like: look at

$$\mathbf{T}' = \bigcup \left\{ \prod_{k<n} (i+1) : n < \omega \right\},$$

and the ideal

$$\left\{ A \subseteq \prod_{i<\omega} (i+1) : A = \bigcup_{n<\omega} A_n \text{ and } (\forall n < \omega)(\forall \eta \in \mathbf{T}')(\exists \nu \in \mathrm{suc}_{\mathbf{T}'}(\eta))(\forall \eta \in A_n)[\neg(\nu \lhd \eta)] \right\}.$$

(c)$_3$ In the cases in which (c)$_2$ is relevant, we get a Borel set $\mathbf{B}$ such that $(\mathbb{N}, t)_{t \in \mathbf{B}} \ldots$, but not "for every $\aleph_1$ members of $\mathbf{B}$ we have …".

(d) Actually, what we use are iterated creature forcing, but as we deal only with $\mathbb{Q}_u$, u finite, so here we need not rely on the theory of creature iteration.

3 Models of theories of expansions of $\mathbb{N}$ with no end extensions

Theorem 3.1

(1) *For some $\mathcal{A} \subseteq \mathcal{P}(\omega)$ some model of $\mathrm{Th}(\mathbb{N}_{\mathcal{A}})$ has no end extension.*

(2) *There is an uncountable Borel set $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that for any uncountable $\mathcal{A}' \subseteq \mathcal{A}$ the theory $T := \mathrm{Th}(\mathbb{N}_{\mathcal{A}'})$ has a model with no end extension.*

(3) *In fact, any model N of T such that the naturally associated tree (set of levels N, the set of nodes of level $n \in N$ is $({}^n 2)^N$) has no undefinable branch is O.K.; such models exist by* [11].

(4) *Moreover, without loss of generality, the set of subsets of $\mathbb{N}$ definable in $\mathbb{N}_{\mathcal{A}}$ is Borel.*

The proof is split into a series of definitions and claims finding a sufficient condition proved in Sections 3 and 4. More specifically, Theorem 3.5(b) gives a sufficient condition which is proved in Proposition 5.6.

Definition 3.2

(1) Let sequences $\bar{n}^* = \langle n^*_i : i < \omega \rangle$ and $\bar{k}^* = \langle k^*_i : i < \omega \rangle$ be such that $n^*_0 = 0$, $n^*_i \ll k^*_{i+1} \ll n^*_{i+1}$ for $i < \omega$. We can demand that the ranges of $\bar{n}^*, \bar{k}^*$ are definable in $\mathbb{N}$ even by a bounded formula. In fact, in our computations later we put $n^*_i = \beth(30i + 30)$ (for $i > 0$) and $k^*_i = \beth(30i + 20)$, where $\beth(0) = 1$, $\beth(i+1) = 2^{\beth(i)}$. We also let $n_*(i) = n^*_i$.

(2) Let $\mathcal{Y}_\ell = \{\pi : \pi \text{ is a permutation of } {}^{n_*(\ell)}2\}$ and $\mathbf{T}_n = \{\langle \pi_\ell : \ell < n \rangle : \pi_\ell \in \mathcal{Y}_\ell \text{ for } \ell < n\}$ and $\mathbf{T} = \bigcup\{\mathbf{T}_n : n < \omega\}$. For $\varkappa \in \mathbf{T}_n$ we keep the convention that $\varkappa = \langle \pi^\varkappa_\ell : \ell < n \rangle$ (unless otherwise stated).

(3) For $\varkappa \in \mathbf{T}$ let $<_\varkappa$ be the following partial order:

(a) $\mathrm{Dom}(<_\varkappa) = \bigcup \{ {}^{n_*(i)}2 : i < \ell g(\varkappa) \}$;

(b) $\eta <_\varkappa \nu$ if and only if they are from $\mathrm{Dom}(<_\varkappa)$ and for some $i < j$ we have $\eta \in {}^{n_*(i)}2$, $\nu \in {}^{n_*(j)}2$ and $\pi^\varkappa_i(\eta) \lhd \pi^\varkappa_j(\nu)$.

Let $t_\varkappa = (\mathrm{Dom}(<_\varkappa), <_\varkappa)$ for $\varkappa \in \mathbf{T}$.

(4) Let $\mathbf{T}_\omega$ be $\lim_\omega(\mathbf{T})$, i.e.,

$$\mathbf{T}_\omega = \{\langle \pi_i : i < \omega\rangle : \pi_i \text{ is a permutation of } {}^{n_*(i)}2 \text{ for } i < \omega\}$$

and for $\varkappa \in \mathbf{T}_\omega$ let $\varkappa{\restriction}n = \langle \pi_i^\varkappa : i < n\rangle$.
We interpret $\varkappa \in \mathbf{T}_\omega$ as the tree $t_\varkappa := \left(\bigcup_{i<\omega} {}^{n_*(i)}2, <_\varkappa\right)$, where $<_\varkappa = \bigcup\{<_{\varkappa\restriction n} : n < \omega\}$, so $t = t_\varkappa$ is $(\mathrm{Dom}(t), <_t)$.

(5) Let F be a one–to–one function from $\bigcup\{{}^{n_*(i)}2 : i < \omega\}$ onto ω, defined in $\mathbb{N}$ (i.e., the functions $n \mapsto \ell g(F^{-1}(n))$ and $(n, i) \mapsto (F^{-1}(n))(i)$ are definable in $\mathbb{N}$ even by a bounded formula) such that F maps each ${}^{n_*(i)}2$ onto an interval. Then clearly F^{-1} is a one–to–one function from $\mathbb{N}$ onto $\bigcup\{{}^{n_*(i)}2 : i < \omega\}$. If $\bar{n}^*, \bar{k}^*$ are not definable in $\mathbb{N}$ then we mean definable in $(\mathbb{N}, \bar{n}^*, \bar{k}^*)$, considering $\bar{n}^*, \bar{k}^*$ as unary functions.

(6) For $\varkappa \in \mathbf{T}_\omega$ let $<_\varkappa^*$ be $\{(F(\eta), F(\nu)) : \eta <_\varkappa \nu\}$ and $A_\varkappa = \{\mathrm{pr}(n_1, n_2) : n_1 <_\varkappa^* n_2\}$ and let $t_\varkappa^* = (\omega, <_\varkappa^*)$; similarly $t_\varkappa^*$ for $\varkappa \in \mathbf{T}$.

(7) For $\mathbf{S} \subseteq \mathbf{T}_\omega$ let $\mathcal{A}_\mathbf{S} = \{A_\varkappa : \varkappa \in \mathbf{S}\}$ and let $\mathbf{A}_\mathbf{S}$ be the arithmetic closure of $\mathcal{A}_\mathbf{S}$ recalling 2.2(1).

Proposition 3.3 *For $\varkappa \in \mathbf{T}_\omega$, in $(\mathbb{N}, A_\varkappa)$ we can define $<_\varkappa^*$ and*

$$(\mathbb{N}, A_\varkappa) \models \text{“}<_\varkappa^* \textit{ is a tree with set of levels } \mathbb{N}, \textit{ set of elements } \mathbb{N} \textit{ and each level finite} (= \textit{bounded in } \mathbb{N}, \textit{ even an interval}).\text{”}$$

Of course, $t_\varkappa$ and $t_\varkappa^* = (\omega, <_\varkappa^*)$ are isomorphic trees. Note that in $\mathbb{N}$ we can interpret the finite set theory $\mathcal{H}(\aleph_0)$.

Our aim is to construct objects with the following properties.

Definition 3.4

(1) We say $\mathbf{T}_\omega^*$ is *strongly pcd* (perfect cone disjoint) whenever:
$\mathbf{T}_\omega^*$ is a perfect subset of $\mathbf{T}_\omega$ such that:
$\boxtimes^{\mathrm{st}}_{\mathbf{T}_\omega^*}$ if $n < \omega$ and $\varkappa_0, \varkappa_1, \ldots, \varkappa_n \in \mathbf{T}_\omega^*$ with no repetitions and for $\ell = 0, 1$, η_ℓ is an ω–branch of $t_{\varkappa_\ell}^*$ which is definable in $(\mathbb{N}, A_{\varkappa_\ell}, A_{\varkappa_2}, \ldots, A_{\varkappa_n})$, then η_0, η_1 belong to disjoint cones (in their respective trees) which means that:
$(\boxdot)$for some level n the sets

$$\{a : a \text{ is } <_{t_\ell}^*\text{-above the member of } \eta_\ell \text{ of level } n\} \subseteq \mathbb{N}$$

for $\ell = 0, 1$ are disjoint.

(2) We say $\mathbf{T}_\omega^*$ is *weakly pcd* (perfect cone disjoint) whenever:
$\mathbf{T}_\omega^*$ is a perfect subset of $\mathbf{T}_\omega$ such that:
$\boxtimes^{\mathrm{wk}}_{\mathbf{T}_\omega^*}$ for every n and $\varphi(x, \bar{y}_\ell) \in \mathbb{L}(\tau_{\mathrm{PA}} + \{P_0, \ldots, P_n\})$ there is $i(*)$ such that *if*
– $i \in [i(*), \omega)$ and $\varkappa_{m,\ell} \in \mathbf{T}_\omega^*$ for $m \le n$, $\ell = 0, 1$,
– $\varkappa_{0,0} \ne \varkappa_{0,1}$ and
– $\varkappa_{m_1,\ell_1}{\restriction}i = \varkappa_{m_2,\ell_2}{\restriction}i$ if and only if $m_1 = m_2$, and
– $P_0, \ldots, P_n$ are unary predicates, $\varphi = \varphi(x, \bar{y}, P_0, \ldots, P_n) \in \mathbb{L}(\tau_{\mathrm{PA}} + \{P_0, \ldots, P_n\})$, and $\bar{b}_\ell \in {}^{\ell g(\bar{y})}\mathbb{N}$, $\varphi(x, \bar{b}_\ell, A_{\varkappa_{0,\ell}}, \ldots, A_{\varkappa_{n,\ell}})$ define in $(\mathbb{N}, A_{\varkappa_{0,\ell}}, \ldots, A_{\varkappa_{n,\ell}})$ a branch B_ℓ of $t_{\varkappa_{0,\ell}}^*$ for $\ell = 0, 1$

then the branches B_0, B_1 have disjoint cones (in their respective trees).

(3) Conditions $\otimes^{\mathrm{wk}}_{\mathbf{T}_\omega^*}$ and $\otimes^{\mathrm{st}}_{\mathbf{T}_\omega^*}$ are defined like $\boxtimes^{\mathrm{wk}}_{\mathbf{T}_\omega^*}$, $\boxtimes^{\mathrm{st}}_{\mathbf{T}_\omega^*}$ above replacing “have disjoint cones” (i.e., $(\boxdot)$) by “have bounded intersection”, which means that
$(\odot)$ for some a the sets $\{b \in \eta_0 : b \text{ is of level} > a\}$ and $\{b \in \eta_1 : b \text{ is of level} > a\}$ are disjoint.
Then we define *weakly pbd* and *strongly pbd* (where *pbd* stands for *perfect branch disjoint*) in the same manner as pcd above, replacing $\boxtimes^{\mathrm{wk}}_{\mathbf{T}_\omega^*}$, $\boxtimes^{\mathrm{st}}_{\mathbf{T}_\omega^*}$ by $\otimes^{\mathrm{wk}}_{\mathbf{T}_\omega^*}$ and $\otimes^{\mathrm{st}}_{\mathbf{T}_\omega^*}$, respectively.

(4) Omitting strongly/weakly means weakly.

One may now ask if the existence of pcd/pbd (Definition 3.4) can be proved and if this concept helps us. We shall prove the existence of pbd in Sections 3 and 4, specifically in Proposition 5.6. The existence of pcd remains an open question. Below we argue that objects of this kind are usefull to prove Theorem 3.1.

Theorem 3.5

(a) *If* $\mathbf{T}^*_\omega$ *is a pcd, i.e., it is a perfect subset of* $\mathbf{T}_\omega$ *satisfying* $\boxtimes^{\rm wk}_{\mathbf{T}^*_\omega}$ *from Definition* 3.4*, then* $\mathcal{A} = \mathcal{A}_{\mathbf{T}^*_\omega}$ *(see Definition* 3.2(7)*) as is required in Theorem* 3.1*.*

(b) *Even if* $\mathbf{T}^*_\omega$ *is a pbd then* $\mathcal{A} = \mathcal{A}_{\mathbf{T}^*_\omega}$ *as is required in Theorem* 3.1*.*

Proof. (a) We will deal with each part of Theorem 3.1. First we give details for Theorem 3.1(3).

For $\varkappa \in \mathbf{T}^*_\omega$ recall

$$A_\varkappa = \{\mathrm{pr}(F(\eta), F(\nu)) : \eta <^*_\varkappa \nu\} \subseteq \mathbb{N}$$

and $\mathcal{A} = \{A_\varkappa : \varkappa \in \mathbf{T}^*_\omega\} \subseteq \mathcal{P}(\omega)$. Assume $\mathcal{A}' \subseteq \mathcal{A}$ is uncountable and let $T = T_{\mathcal{A}'} = \mathrm{Th}(\mathbb{N}_{\mathcal{A}'})$ and $\tau_{\mathcal{A}'}$ be its vocabulary. Then by [11] the theory T has a model M in which definable trees (we are interested just in the case the set of levels being M with the order $<^M$) have no undefinable branches, so, in particular (and this is enough)

if $\varkappa \in \mathcal{A}$, then $(<^*_\varkappa)^M$ has no undefinable branch

(i.e., as in [11], branches mean full branches, "visiting" every level). Note that "the a-th level of $(M, (<^*_\varkappa)^M)$" does not depend on $\varkappa$.

Assume towards contradiction M^+ is an (elementary) end-extension of M and let $b^* \in M^+ \setminus M$. Now consider any $A_\varkappa \in \mathcal{A}$ so $(<^*_\varkappa)^M$ is naturally definable in M and

$M \models$ " for every element a serving as level,
$\langle\{c : b <_\varkappa c\} : b$ is of level a in the tree $t_\varkappa$, i.e. $(M, (<^*_\varkappa)^M)\rangle$
is a partition of $\{x : x$ is of $<^*_\varkappa$-level $> a\}$ to finitely many sets, "

the finite is in the sense of M of course.

As M^+ is an end-extension of M—recalling Definition 3.2(5)–it follows that the level of b^* in M^+ is above M and b^* defines a branch of $(M, (<^*_\varkappa)^M)$ which we call $\eta_\varkappa = \langle b^\varkappa_a : a \in M\rangle$. That is $b^\varkappa_a$ is the unique member of M of level a such that $M^+ \models$" $b^\varkappa_a \leq^*_\varkappa b^*$."

By the choice of M the branch $\eta_\varkappa$, i.e., $\{b^\varkappa_a : a \in M\}$ is a definable subset of M, say by $\varphi_\varkappa(x, \bar{d}_\varkappa)$ where $\varphi_\varkappa(x, \bar{y}_\varkappa) \in \mathbb{L}(\tau_{\mathcal{A}'})$ and $\bar{d}_\varkappa \in {}^{\ell g(\bar{y}_\varkappa)}M$. Now by the assumptions on $\mathcal{A}, \mathcal{A}', T$ there are $s_{\varkappa,1}, \ldots, s_{\varkappa,n_\varkappa} \in \mathbf{T}^*_\omega \setminus \{\varkappa\}$ with no repetitions, hence $A_{s_{\varkappa,n}} \in \mathcal{A}' \setminus \{A_\varkappa\}$ for $n = 1, \ldots, n_\varkappa$, and in $\varphi_\varkappa(x, \bar{y}_\varkappa)$ only $A_{s_{\varkappa,1}}, \ldots, A_{s_{\varkappa,n_\varkappa}}$ and $A_\varkappa$ appear (i.e., the predicates $P_{s_{\varkappa,1}}, \ldots, P_{s_{\varkappa,n_\varkappa}}, P_\varkappa$ corresponding to them and $\tau_{\rm PA}$, of course). Let $s_{\varkappa,0} = \varkappa$ and we write $\varphi'_\varkappa = \varphi'_\varkappa(x, \bar{y}_\varkappa, \bar{P}_\varkappa)$, where $\bar{P}_\varkappa = \langle P_{s_{\varkappa,\ell}} : \ell \leq n_\varkappa\rangle$ and $\varphi'_\varkappa$ has non-logical symbols only from $\tau_{\rm PA}$ and so $\varphi'_\varkappa = \varphi''_\varkappa(x, \bar{y}_\varkappa) \in \mathbb{L}(\tau_{\rm PA} \cup \{P_\ell : \ell \leq n_\varkappa\})$, that is $\varphi'_\kappa(x, \bar{y}_\varkappa)$ when we substitute P_ℓ for $P_{s_{\varkappa,\ell}}$ for $\ell \leq n_\varkappa$.

For $A_\varkappa \in \mathcal{A}$ let

$$m_\varkappa = \min\{m : s_{\varkappa,\ell}{\restriction}m \text{ for } \ell = 0, \ldots, n_\varkappa \text{ are pairwise distinct}\}.$$

Hence for some $\varphi_*(x, \bar{y}_*), n_*, m_*, \bar{s}_*$ the set

$$\mathcal{A}_2 = \{A_\varkappa \in \mathcal{A} : \varphi'_\varkappa = \varphi_*,\ \bar{y}_\varkappa = \bar{y}_*,\ \text{ so } n_\varkappa = n_*,\ m_\varkappa = m_* \text{ and } \langle s_{\varkappa,\ell}{\restriction}m_* : \ell = 0, \ldots, n_*\rangle = \bar{s}_*\}$$

is uncountable. Let $i(*) \geq m_*$ be as guaranteed by $\boxtimes^{\rm wk}_{\mathbf{T}^*_\omega}$, so for some uncountable $\mathcal{A}_3 \subseteq \mathcal{A}_2$ for some $\bar{s}_{**}$ we have that $\langle s_{\varkappa,\ell}{\restriction}i(*) : \ell = 1, \ldots, n_*\rangle = \bar{s}_{**}$ whenever $A_\varkappa \in \mathcal{A}_3$. As $\mathcal{A}$ is uncountable, clearly for some $A_{\varkappa_1} \neq A_{\varkappa_2} \in \mathcal{A}$ we have $\{\varkappa_1, \varkappa_2\}$ is disjoint to $\{s_{\varkappa_\ell,m} : m = 1, \ldots, n_{\varkappa_\ell}$ and $\ell = 1, 2\}$.

So by $\boxtimes^{\rm wk}_{\mathbf{T}^*_\omega}$ from Definition 3.4 for some $a \in M$ we have

($\boxdot$) $M \models$ "$\{c : b^{\varkappa_1}_a <^*_{\varkappa_1} c\} \cap \{c : b^{\varkappa_2}_a <^*_{\varkappa_2} c\} = \emptyset$."

[Why? Because $\mathbb{N}_{\mathcal{A}'} \models$" $(\forall \bar{y}_{\varkappa_1})(\forall \bar{y}_{\varkappa_2})$ [if $\varphi_{\varkappa_\ell}(-, \bar{y}_{\varkappa_\ell})$ define a branch of $t^*_{\varkappa_\ell}$ for $\ell = 1, 2$, then there are x_1, x_2 such that $\varphi_{\varkappa_1}(x_1, \bar{y}_{\varkappa_1}) \wedge \varphi_{\varkappa_2}(x_2, \bar{y}_{\varkappa_2}) \wedge \neg(\exists z)[x_1 \leq_{t^*_{\varkappa_1}} z \wedge x_2 \leq_{t^*_{\varkappa_2}} z]]$ ".] But in M^+ the elements b^* belong to both, contradiction to $M \prec M^+$. Now, parts (2), (3) of Theorem 3.1 follow and so does part (1).

(4) For this, cf. [2]. For an alternative proof: When is $\mathcal{B} = \{A \subseteq \mathbb{N} : A$ is definable in $\mathbb{N}_{\mathcal{A}}\}$ Borel? As we can shrink $\mathbf{T}^*_\omega$, without loss of generality there is a function $g \in {}^\omega\omega$ such that for every $f \in {}^\omega\omega$ definable in $\mathbb{N}_{\mathcal{A}}$, we have $f <_{J^{\rm bd}_\omega} g$, i.e., $(\forall^\infty i))(f(i) < g(i))$. This suffices (in fact if we prove 3.4 using forcing notion $\mathbb{Q}_u$, where each $\mathbb{Q}_u$ is ${}^\omega\omega$-bounding this will be true for $\mathbf{T}^*_\omega$ itself and we do this in §3; moreover we have continuous reading for every such f (as a function of $(A_{\varkappa_0}, \ldots, A_{\varkappa_{n-1}})$ for some $\varkappa_0, \ldots, \varkappa_{n-1} \in \mathbf{T}^*_\omega$).

In order to prove (b), we repeat the proof of (a) above until the choice of $\{\varkappa_1, \varkappa_2\}$ (right before ($\boxdot$)), but we replace the rest of the arguments for Theorem 3.1(3) by the following.

So by $\otimes^{\rm wk}_{\mathbf{T}^*_\omega}$ of Definition 3.4(3), for some $a_* \in M$ we have

($\odot$) $M \models$ "the sets $\{b^{\varkappa_1}_a : a_* < a\}, \{b^{\varkappa_2}_a : a_* < a\}$ are disjoint."

(Remember that all the trees we consider have the same levels.) But in M^+ the element b^* belongs to both definable branches contrary to $M \prec M^+$. □

Theorem 3.6

(1) *If $\mathbf{T}^*_\omega$ is a strong pcd, i.e., it is a perfect subset of $\mathbf{T}_\omega$ satisfying $\boxtimes^{st}_{\mathbf{T}^*_\omega}$ from* 3.4, *and $\mathcal{A} \subseteq \{A_\varkappa : \varkappa \in \mathbf{T}^*_\omega\}$ is uncountable, then there is no weakly definably closed ultrafilter on* ar-cl$(\mathcal{A})$*, see Definition* 2.3(5)*.*

(2) *Above, we may replace "pcd" with "pbd".*

(3) *Without loss of generality,* ar-cl$(\mathbf{T}^*_\omega)$ *is a Borel set.*

P r o o f. (1) Assume towards contradiction that a pair $(\mathcal{A}, D)$ forms a counterexample. Let $M = \mathbb{N}_{\mathcal{A}}$ and let M^+ be an $\aleph_2$–saturated elementary extension of M and let $b^* \in M^+$ realizes the type

$$p^* = \{\varphi(x, \bar{a}) : \ \varphi(x, \bar{y}) \in \mathbb{L}(\tau_M), \bar{a} \in {}^{\ell g(\bar{y})}M \quad \text{and} \quad \{b \in M : M \models \varphi[b, \bar{a}]\} \text{ includes some member of } D\}.$$

Clearly p^* is a set of formulas over M, finitely satisfiable in M and even a complete type over M.

Now, for every $\varkappa$ such that $A_\varkappa \in \mathcal{A}$ and $i < \omega$ we consider a function $g_{\varkappa,i}$ definable in M as follows:

$(*)_1$ $g_{\varkappa,i}(c)$ is:
- (α) b if c is of $<^*_\varkappa$–level $\geq i$ in $(\mathbb{N}, <_\varkappa)$ and b is of $<^*_\varkappa$–level i and $b \leq^*_\varkappa c$;
- (β) c if c is of $<^*_\varkappa$-level $< i$ in $(\mathbb{N}, <_\varkappa)$.

Clearly $g_{\varkappa,i}$ is definable in $(\mathbb{N}, A_\varkappa)$, the range of $g_{\varkappa,i}$ is finite, so $g_{\varkappa,i} \restriction B_{\varkappa,i}$ is constant for some $B_{\varkappa,i} \in \{g^{-1}_{\varkappa,i}\{x\} : x \in \mathrm{Rang}(g_{\varkappa,i})\} \cap D$. As all co-finite subsets of $\mathbb{N}$ belong to D, also $B_{\varkappa,i}$ cannot be a singleton member of level $\neq i$. Hence for some $b_{\varkappa,i}$ of level i for $<^*_\varkappa$ we have $B_{\varkappa,i} \subseteq \{c : b_{\varkappa,i} \leq^*_\varkappa c\}$. Now moreover for some formula $\varphi_\varkappa(x_0, x_1, x_2) \in \mathbb{L}(\tau_{\rm PA} + P_\varkappa)$, for each $i \in \mathbb{N}$ the formula $\varphi_\varkappa(x_0, x_1, i)$ defines $g_{\varkappa,i}(x_1) = x_1$. By the "weakly definable closed" (see Definition 2.3(5)), $\{b_{\varkappa,i} : i < \omega\}$ is definable in $\mathbb{N}_{\mathcal{A}}$.

Now we continue as in the proof of Theorem 3.5.

(2) Similarly.

(3) As in Theorem 3.5 (for Theorem 3.1(4)). □

4 The (iterated) creature forcing

We continue the previous section, so we use notation as there, see Definitions 3.2 and 3.4. In particular, $n^*_0 = 0$, $n_*(i) = n^*_i = \beth(30i + 30)$ (for $i > 0$) and $k^*_i = \beth(30i + 20)$. We also set $\ell^*_i = \beth(30i + 10)$.

Definition 4.1 For $i < \omega$ and a finite set u of ordinals we define:

(A) OB_i^u is the set of all triples (f, g, e) such that ($\mathrm{Per}(A)$ stands for the set of permutations of A):
  (a) $f, g \in {}^u(\mathrm{Per}({}^{n_*(i)}2))$;
  (b) if $i - 1 = j \geq 0$ and $\alpha \in u$, then $(f(\alpha)(\rho))\restriction n_j^* = (g(\alpha)(\rho))\restriction n_j^*$ for all $\rho \in {}^{n_*(i)}2$,
  (c) e is a function with domain u such that for each $\alpha \in u$

$$e(\alpha) : \mathrm{Per}\big({}^{n_*(i-1)}2\big) \longrightarrow \mathrm{Per}\big({}^{n_*(i)}2\big) \times \mathrm{Per}\big({}^{n_*(i)}2\big).$$

  Above, we stipulate $n_*(i-1) = 0$ if $i = 0$. Also, let us note that some triples will never be used, only $\bigcup\{\mathrm{suc}(x) : x \in \mathrm{OB}_i^u\}$ and we should iterate.

(B) For $x \in \mathrm{OB}_i^u$ we let $x = (f_x, g_x, e_x)$ and $i = \mathbf{i}(x)$ and $u = \mathrm{supp}(x)$.

(C) For $x \in \mathrm{OB}_i^u$ we set

$$\begin{aligned}\mathrm{suc}(x) &= \big\{y \in \mathrm{OB}_{i+1}^u : \big(\forall \rho \in {}^{n_*(i+1)}2\big)(\forall \alpha \in u)(g_x(\alpha)(\rho\restriction n_i^*)\\ &= (f_y(\alpha)(\rho))\restriction n_i^*) \quad \text{and} \quad (\forall \alpha \in u)(e_y(\alpha)(g_x(\alpha)) = (f_y(\alpha), g_y(\alpha)))\big\}.\end{aligned}$$

(D) For $j \leq \omega$ let

$$\mathbf{S}_{u,j} = \big\{\langle x_\ell : \ell < j\rangle : \big(\ell < j \Rightarrow x_\ell \in \mathrm{OB}_\ell^u\big) \quad \text{and} \quad (\ell + 1 < j \Rightarrow x_{\ell+1} \in \mathrm{suc}(x_\ell))\big\}.$$

(E) $\mathbf{S}_u = \bigcup\{\mathbf{S}_{u,\ell} : \ell < \omega\}$; we consider it a tree, ordered by $\triangleleft$.

(F) For $x \in \mathrm{OB}_i^u$ and $w \subseteq u$ let $x\restriction w = (f_x\restriction w, g_x\restriction w, e_x\restriction w)$.

(G) For $i \leq \omega$, $w \subseteq u$ and $\bar{x} = \langle x_j : j < i\rangle \in \mathbf{S}_{u,i}$ let $\bar{x} \restriction w = \langle x_j\restriction w : j < i\rangle$ and for $\alpha \in u$ let $\varkappa_{\bar{x}}^\alpha = \langle f_{x_j}(\alpha) : j < i\rangle$.

(H) For $\bar{x} \in \mathbf{S}_{u,\ell}$, $\ell \leq \omega$, and $\alpha \in u$ let $t_{\bar{x},\alpha} = t_{\bar{x}}^\alpha$ be the tree with $\ell g(\bar{x})$ levels, with the i-th level being ${}^{n_*(i)}2$ for $i < \ell g(\bar{x})$ and the order $<_{t_{\bar{x},\alpha}}$ defined by

$\eta <_{t_{\bar{x},\alpha}} \nu$ if and only if

for some $i < j < \ell g(\bar{x})$ we have $\eta \in {}^{n_*(i)}2, \nu \in {}^{n_*(j)}2$ *and* $f_{x_i}(\alpha)(\eta) \triangleleft f_{x_j}(\alpha)(\nu)$.

Since we are interested in getting "bounded branch intersections" we will need the following observation (part (5) is crucial in proving cone disjointness in some situation later).

Proposition 4.2 *Assume $\bar{x} \in \mathbf{S}_u$ and $\alpha \in u$.*

(1) *If $\rho \in {}^{n_*(j)}2$ and $j < \ell g(\bar{x})$, then $\langle g_{x_i}(\alpha)(\rho\restriction n_*(i)) : i \leq j\rangle$ is $\triangleleft$–increasing noting $g_{x_i}(\alpha)(\rho\restriction n_*(i)) \in {}^{n_*(i)}2$.*

(2) *$\varkappa_{\bar{x}}^\alpha \in \mathbf{T}_{\ell g(\bar{x})}$ and $t_{\varkappa_{\bar{x}}^\alpha} = t_{\bar{x}}^\alpha$, on $t_{\varkappa_{\bar{x}}^\alpha}$ see Definition* 3.2(3).

(3) *If $i < j < \ell g(\bar{x})$ and $\nu \in {}^{n_*(j)}2$, then $(f_{x_j}(\alpha)(\nu))\restriction n_i^*$ depends just on $\bar{x}\restriction(i+1)$, actually just on g_{x_i}, i.e., it is equal to $g_{x_i}(\alpha)(\nu\restriction n_i^*)$.*

(4) *The sequence $\langle g_{x_j}(\alpha), f_{x_j}(\alpha) : j < \ell g(\bar{x})\rangle$ is fully determined by $\langle e_{x_j}(\alpha) : j < \ell g(\bar{x})\rangle$.*

(5) *Assume $\alpha_1 \neq \alpha_2$ are from u and $i < \ell g(\bar{x})$ and $\eta_1, \eta_2 \in {}^{\eta_*(i)}2$ but*

$$(g_{x_i}(\alpha_1))^{-1} \circ f_{x_i}(\alpha_1))(\eta_1) \neq ((g_{x_i}(\alpha_2))^{-1} \circ f_{x_i}(\alpha_2))(\eta_2).$$

Then the sets $\{\rho : \eta_1 <_{t_{\bar{x},\alpha_1}} \rho\}$ and $\{\rho : \eta_2 <_{t_{\bar{x},\alpha_2}} \rho\}$ are disjoint.

P r o o f. (1), (2), (3) and (4) can be shown by straightforward induction on j.

(5) Assume towards contradiction that

$(*)_1$ $\quad \eta_1 <_{t_{\bar{x},\alpha_1}} \rho \quad$ *and* $\quad \eta_2 <_{t_{\bar{x},\alpha_2}} \rho.$

So $\rho \in t_{\bar{x},\alpha_2}$ and hence $\rho \in {}^{n_*(j)}2$ for some $j < \ell g(\bar{x})$. Since $\eta_1 <_{t_{\bar{x},\alpha_1}} \rho$, necessarily $i < j < \ell g(\bar{x})$ and by the definition of $<_{t_{\bar{x},\alpha_1}}$ and $<_{t_{\bar{x},\alpha_2}}$:

$$(*)_2 \qquad f_{x_i}(\alpha_1)(\eta_1) \vartriangleleft f_{x_j}(\alpha_1)(\rho) \quad \text{and} \quad f_{x_i}(\alpha_2)(\eta_2) \vartriangleleft f_{x_j}(\alpha_2)(\rho).$$

This means that

$$(*)_3 \qquad f_{x_i}(\alpha_1)(\eta_1) = (f_{x_j}(\alpha_1)(\rho)){\restriction}n_i^* \quad \text{and} \quad f_{x_j}(\alpha_2)(\eta_2) = (f_{x_j}(\alpha_2)(\rho)){\restriction}n_i^*.$$

Consequently, by part (3), letting $\rho' = \rho{\restriction}n_i^*$:

$$(*)_4 \qquad f_{x_i}(\alpha_1)(\eta_1) = g_{x_i}(\alpha_1)(\rho') \quad \text{and} \quad f_{x_i}(\alpha_2)(\eta_2) = g_{x_i}(\alpha_2)(\rho'),$$

and therefore

$$(*)_5 \qquad ((g_{x_i}(\alpha_1))^{-1} \circ f_{x_i}(\alpha_1))(\eta_1) = \rho' = ((g_{x_i}(\alpha_2))^{-1} \circ f_{x_i}(\alpha_2))(\eta_2),$$

contradicting our assumptions. □

Below we may replace the role of D_i^u by $\{\langle (f_{x_j}(\alpha), g_{x_j}(\alpha)) : j < i\rangle : \bar{x} \in \mathbf{S}_{u,i}\}$.

Definition 4.3 For a finite set $u \subseteq \mathrm{Ord}$ and an integer $i < \omega$ we let

(I) (α) $D_i^u = \{(\alpha, g) : \alpha \in u$ and $g \in \mathrm{Per}({}^{n_*(i-1)}2)$ if $i > 0$, $g \in \mathrm{Per}({}^{0}2)$ if $i = 0\}$; if $\bar{x} \in \mathbf{S}_{u,i}$ and $\alpha \in u$, then stipulate $g_{x_{-1}}(\alpha)$ is the unique $g \in \mathrm{Per}({}^{0}2)$.

(β) pos_i^u is the set of all functions h with domain D_i^u such that $h(\alpha, g)$ is a pair $(h_1(\alpha, g), h_2(\alpha, g))$ satisfying
- $h_1(\alpha, g), h_2(\alpha, g) \in \mathrm{Per}({}^{n_*(i)}2)$, and
- $(h_\ell(\alpha, g)(\rho)){\restriction}n_*(i-1) = g(\rho{\restriction}n_*(i-1))$ for $\ell \in \{1, 2\}$, $i > 0$ and $\rho \in {}^{n_*(i)}2$.

Also, for $h \in \mathrm{pos}_i^u$ and $w \subseteq u$ we let $h \upharpoonright w = h{\restriction}D_i^w$.

(γ) wpos_i^u is the family of all functions $\mathcal{F} : \mathrm{pos}_i^u \longrightarrow [0, 1]$ which are not constantly zero, and

$$\mathrm{vpos}_i^u = \Big\{\mathcal{F} \in \mathrm{wpos}_i^u : \mathrm{range}(\mathcal{F}) \subseteq \Big\{\frac{m}{2^{n_*(i)}} : m = 0, 1, \dots 2^{n_*(i)}\Big\}\Big\}.$$

If above we allow the constantly zero function instead of wpos_i^u, vpos_i^u we get ypos_i^u, xpos_i^u, respectively. A set $A \subseteq \mathrm{pos}_i^u$ will be identified with its characteristic function $\chi_A \in \mathrm{vpos}_i^u$.

(δ) For $\mathcal{F} \in \mathrm{wpos}_i^u$ we let

$$\mathrm{set}(\mathcal{F}) = \{h \in \mathrm{pos}_i^u : \mathcal{F}(h) > 0\} \quad \text{and} \quad \|\mathcal{F}\| = \sum \{\mathcal{F}(h) : h \in \mathrm{pos}_i^u\}.$$

If $|\mathrm{pos}_i^u| \geq \|\mathcal{F}\| \cdot (k_i^*)^{3^{k_i^*}-1}$, then we put $\mathrm{nor}_i^0(\mathcal{F}) = 0$; otherwise we let

$$\mathrm{nor}_i^0(\mathcal{F}) = k_i^* - \log_3\left(\log_{k_i^*}\left(\frac{k_i^* \cdot |\mathrm{pos}_i^u|}{\|\mathcal{F}\|}\right)\right).$$

(ε) For $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{wpos}_i^u$ we let
- $\mathcal{F}_1 \leq \mathcal{F}_2$ if and only if $(\forall h \in \mathrm{pos}_i^u)(\mathcal{F}_1(h) \leq \mathcal{F}_2(h))$;
- $(\mathcal{F}_1 + \mathcal{F}_2)(h) = \mathcal{F}_1(h) + \mathcal{F}_2(h)$ and $(\mathcal{F}_1 \cdot \mathcal{F}_2)(h) = \mathcal{F}_1(h) \cdot \mathcal{F}_2(h)$ for $h \in \mathrm{pos}_i^u$;
- $[\mathcal{F}_1]$ is the function from pos_i^u to $\{\frac{m}{2^{n_*(i)}} : m = 0, 1, \dots, 2^{n_*(i)}\}$ given by

$$[\mathcal{F}_1](h) = \lfloor \mathcal{F}_1(h) \cdot 2^{n_*(i)} \rfloor \cdot 2^{-n_*(i)} \qquad \text{for } h \in \mathrm{pos}_i^u.$$

(ζ) For $\bar{x} \in \mathbf{S}_{u,i}$ and $h \in \mathrm{pos}_i^u$ we let $\mathrm{suc}_{\bar{x}}(h)$ be $\bar{x}^\frown\langle y\rangle$ where $y \in \mathrm{OB}_i^u$ is defined by:
- $(f_y(\alpha), g_y(\alpha)) = h(\alpha, g_{x_{i-1}}(\alpha))$ for $\alpha \in u$,
- $e_y(\alpha)(\pi) = h(\alpha, \pi)$ for $\alpha \in u$ and $\pi \in \mathrm{Per}({}^{n_*(i-1)}2)$.

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(J) (α) $\underline{\mathrm{CR}}^u_i$ is the set of all pairs $\mathfrak{c} = (\mathcal{F}, m) = (\mathcal{F}_\mathfrak{c}, m_\mathfrak{c})$ such that m is a non-negative real and $\mathcal{F} \in \mathrm{wpos}^u_i$ and $\mathrm{nor}^0_i(\mathcal{F}) \geq m$. We also let $\mathrm{CR}^u_i = \{\mathfrak{c} \in \underline{\mathrm{CR}}^u_i : \mathcal{F}_\mathfrak{c} \in \mathrm{vpos}^u_i\}$.

(β) For $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$, we let $\mathrm{nor}^1_i(\mathfrak{c}) = \big(\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) - m_\mathfrak{c}\big)$ and $\mathrm{nor}^2_i(\mathfrak{c}) = \log_{\ell^*_i}\big(\mathrm{nor}^1_i(\mathfrak{c})\big)$ if non-negative and well defined, and it is zero otherwise. (Remember that $\ell^*_i = \beth(30i + 10)$.) We will write $\mathrm{nor}_i(\mathfrak{c}) = \mathrm{nor}^2_i(\mathfrak{c})$.

(γ) For $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$ let $\underline{\Sigma}(\mathfrak{c})$ be the set of all $\mathfrak{d} \in \underline{\mathrm{CR}}^u_i$ such that $\mathcal{F}_\mathfrak{d} \leq \mathcal{F}_\mathfrak{c}$ and $m_\mathfrak{d} \geq m_\mathfrak{c}$. For $\mathfrak{c} \in \mathrm{CR}^u_i$ we let $\Sigma(\mathfrak{c}) = \underline{\Sigma}(\mathfrak{c}) \cap \mathrm{CR}^u_i$.

(K) $\mathbb{Q}_u = (\mathbb{Q}_u, \leq_{\mathbb{Q}_u})$ is defined by

(α) conditions in $\mathbb{Q}_u$ are pairs $p = (\bar{x}, \bar{\mathfrak{c}}) = (\bar{x}_p, \bar{\mathfrak{c}}_p)$ such that

(a) $\bar{x} \in \mathbf{S}_{u,i}$ for some $i = \mathbf{i}(p) < \omega$, so $\bar{x}_p = \langle x_{p,j} : j < \mathbf{i}(p)\rangle$,

(b) $\bar{\mathfrak{c}} = \langle \mathfrak{c}_j : j \in [\mathbf{i}(p), \omega)\rangle$, so $\mathfrak{c}_j = \mathfrak{c}^p_j$, and $\mathfrak{c}_j \in \mathrm{CR}^u_j$,

(c) the sequence $\langle \mathrm{nor}_j(\mathfrak{c}_j) : j \in [\mathbf{i}(p), \omega)\rangle$ diverges to ∞;

(β) $p \leq_{\mathbb{Q}_u} q$ if and only if (both are from $\mathbb{Q}_u$ and)

(a) $\bar{x}_p \trianglelefteq \bar{x}_q$, and

(b) if $\mathbf{i}(p) \leq j < \mathbf{i}(q)$, then for some $h \in \mathrm{set}\left(\mathcal{F}_{\mathfrak{c}^p_j}\right)$ we have $\bar{x}_q{\restriction}(j+1) = \mathrm{suc}_{\bar{x}_q \restriction j}(h)$ (see clause (I)(ζ) above),

(c) if $i \in [\mathbf{i}(q), \omega)$, then $\mathfrak{c}^q_i \in \Sigma\big(\mathfrak{c}^p_i\big)$.

$\underline{\mathbb{Q}}_u = (\underline{\mathbb{Q}}_u, \leq_{\underline{\mathbb{Q}}_u})$ is defined similarly, replacing CR^u_j, Σ by $\underline{\mathrm{CR}}^u_j$, $\underline{\Sigma}$, respectively.

(L) If $u_1, u_2 \subseteq \mathrm{Ord}$ are finite, $|u_1| = |u_2|$ and $h : u_1 \longrightarrow u_2$ is the order preserving bijection, then $\hat{h}$ is the isomorphism from $\mathbb{Q}_{u_1}$ onto $\mathbb{Q}_{u_2}$ induced by h in a natural way.

Proposition 4.4 *Let $u \subseteq \mathrm{Ord}$ be a finite non-empty set, $i \in (1, \omega)$ and $|u| \leq n_*(i-1)$. Then*

(a) $\big|\mathrm{pos}^u_{i-1}\big| < \beth(30i+3)$, $\big|\mathrm{vpos}^u_{i-1}\big| < \beth(30i+4)$, $\mathrm{nor}^0_i\big(\mathrm{pos}^u_i\big) = k^*_i$ *and* $\mathrm{nor}_i\big(\mathfrak{c}^{\max}_{u,i}\big) = \beth(30i+19)/\beth(30i+9)$ *and* $\mathrm{CR}^u_i = \Sigma\big(\mathfrak{c}^{\max}_{u,i}\big)$, *where* $\mathfrak{c}^{\max}_{u,i} = \big(\mathrm{pos}^u_i, 0\big)$.

(b) $|\mathbf{S}_{u,i}| < \ell^*_i$ *and if* $\bar{x} \in \mathbf{S}_{u,i}$ *and* $h \in \mathrm{pos}^u_i$, *then* $\mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$.

(c) *If* $\mathcal{F}_1 \leq \mathcal{F}_2$ *are from* wpos^u_i, *then* $0 \leq \mathrm{nor}^0_i(\mathcal{F}_1) \leq \mathrm{nor}^0_i(\mathcal{F}_2)$.

(d) *If* $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$ *and* $\mathrm{nor}^1_i(\mathfrak{c}) \geq 1$, *then* $\mathfrak{c}$ *has* k^*_i*–bigness with respect to* nor^1_i, *which means that: if* $\mathcal{F}_\mathfrak{c} = \sum\{\mathcal{Y}_k : k < k^*_i\}$ *then* $\mathrm{nor}^1_i(\mathfrak{c}) \leq \max\big\{\mathrm{nor}^1_i(\mathcal{Y}_m, m_\mathfrak{c}) + 1 : k < k^*_i\big\}$; *moreover, if* $\mathcal{F}' \leq \mathcal{F}_\mathfrak{c}$, $\|\mathcal{F}'\| \geq \|\mathcal{F}_\mathfrak{c}\|/k^*_i$ *then* $\mathrm{nor}^0_i(\mathcal{F}') \geq \mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) - 1$.

(e) *Both* CR^u_i *and* $\underline{\mathrm{CR}}^u_i$ *have halving with respect to* nor^1_i, *that is*

(α) *if* $\mathfrak{c} = (\mathcal{F}_\mathfrak{c}, m_\mathfrak{c})$, $m_1 = \big(\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) + m_\mathfrak{c}\big)/2$, $\mathfrak{d} = (\mathcal{F}_\mathfrak{c}, m_1)$, *then* $\mathrm{nor}^1_i(\mathfrak{d}) \geq \mathrm{nor}^1_i(\mathfrak{c})/2$, *and*

(β) *if* $\mathfrak{d}' \in \Sigma(\mathfrak{d})$ *is such that* $\mathrm{nor}^1_i(\mathfrak{d}') \geq 1$, *then* $\mathfrak{d}'' := (\mathcal{F}_{\mathfrak{d}'}, m_\mathfrak{c})$ *satisfies*

$$\mathfrak{d}'' \in \Sigma(\mathfrak{c}), \quad \mathrm{nor}^1_i(\mathfrak{d}'') \geq \mathrm{nor}^1_i(\mathfrak{c})/2 \quad \textit{and} \quad \mathcal{F}_{\mathfrak{d}''} = \mathcal{F}_{\mathfrak{d}'}.$$

Proof. *Clause (a)*: Clearly by the definition $\mathfrak{c}^{\max}_{u,i} = \big(\mathrm{pos}^u_i, 0\big) \in \mathrm{CR}^u_i = \Sigma\big(\mathfrak{c}^{\max}_{u,i}\big)$ and

$$\mathrm{nor}^0_i\big(\mathrm{pos}^u_i\big) = k^*_i - \log_3\Big(\log_{k^*_i}\big(k^*_i\big)\Big) = k^*_i,$$

so $\mathrm{nor}^1_i\big(\mathfrak{c}^{\max}_{u,i}\big) = k^*_i - 0 = k^*_i$ and $\mathrm{nor}_i\big(\mathfrak{c}^{\max}_{u,i}\big) = \log_{\ell^*_i}(k^*_i) = \log_{\beth(30i+10)}\big(\beth(30i+20)\big) = \log_2\big(\beth(30i+20)\big)/\log_2\big(\beth(30i+10)\big) = \beth(30i+19)/\beth(30i+9)$. Now, for every $j > 0$, letting $A_j = \mathrm{Per}({}^{n_*(j)}2) \times \mathrm{Per}({}^{n_*(j)}2)$ and recalling Definition 4.3(I)(α), we have

$$\big|D^u_j\big| \leq \big(2^{n_*(j-1)}!\big) \times |u| \leq 2^{(2^{n_*(j-1)})^2} \times |u| \quad \text{and} \quad |A_j| \leq \big(2^{n_*(j)}!\big)^2 \leq 2^{2^{2n_*(j)+1}} \leq 2^{2^{3n_*(j)}}.$$

Since $|u| \leq n_*(i-1)$, we get $\big|D^u_j\big| \leq 2^{2^{2n_*(j-1)}} \times n_*(i-1)$. Since $2^{2^{2n_*(i-2)}} \leq n_*(i-1)$, $n_*(i-1)^2 \leq 2^{n_*(i-1)}$ and $4n_*(i-1) + 1 \leq 2^{n_*(i-1)}$, we conclude now that

$$\big|\mathrm{pos}^u_{i-1}\big| \leq |A_{i-1}|^{|D^u_{i-1}|} \leq \Big(2^{2^{3n_*(i-1)}}\Big)^{|D^u_{i-1}|} \leq 2^{2^{3n_*(i-1)} \times 2^{2^{2n_*(i-2)}} \times n_*(i-1)} \leq 2^{2^{4n_*(i-1)}} < \beth(30i+3)$$

and

$$\left|\mathrm{vpos}^u_{i-1}\right| = \left(2^{n_*(i-1)}+1\right)^{|\mathrm{pos}^u_{i-1}|} < 2^{(n_*(i-1)+1)\times 2^{2^{4n_*(i-1)}}} < 2^{2^{2^{4n_*(i-1)+1}}} < \beth(30i+4).$$

Clause (b): Let B_j be the set of all functions from $\mathrm{Per}({}^{n_*(j-1)}2)$ to $\mathrm{Per}({}^{n_*(j)}2)\times\mathrm{Per}({}^{n_*(j)}2)$. Then we have

$$|B_j| = \left(2^{n_*(j)}!\right)^{2\cdot(2^{n_*(j-1)}!)} \le 2^{2^{2n_*(j)}\cdot 2\cdot(2^{n_*(j-1)}!)} \le 2^{2^{4n_*(j)}}$$

and hence for $j<i$:

$$\begin{aligned}\left|\mathrm{OB}^u_j\right| \;&\le\; |{}^u\mathrm{Per}({}^{n_*(j)}2)|\cdot|{}^u\mathrm{Per}({}^{n_*(j)}2)|\cdot|{}^uB_j| \le \left(2^{n_*(j)}!\right)^{2|u|}\cdot 2^{2^{4n_*(j)}\cdot|u|}\\ &\le\; 2^{2^{2n_*(j)+1}\cdot|u|+2^{4n_*(j)}\cdot|u|} \le 2^{2^{7n_*(j)}\cdot n_*(i-1)} \le 2^{2^{8n_*(i-1)}}.\end{aligned}$$

Therefore,

$$|\mathbf{S}_{u,i}| \le \prod_{j<i}\left|\mathrm{OB}^u_j\right| \le \left(2^{2^{8n_*(i-1)}}\right)^i < 2^{2^{9n_*(i-1)}} < \ell^*_i.$$

Clause (d): Assume $\mathfrak{c}\in\underline{\mathrm{CR}}^u_i$ and $\mathcal{F}_\mathfrak{c} = \sum\{\mathcal{Y}_k : k<k^*_i\}$, hence $\|\mathcal{F}_\mathfrak{c}\| = \sum\{\|\mathcal{Y}_k\| : k<k^*_i\}$. Let $k(*)<k^*_i$ be such that $\|\mathcal{Y}_{k(*)}\|$ is maximal. Plainly $\|\mathcal{F}_\mathfrak{c}\|\le k^*_i\times\|\mathcal{Y}_{k(*)}\|$ and therefore it suffices to prove the "moreover" part. So assume $\mathcal{Y}\le\mathcal{F}_\mathfrak{c}$, $\|\mathcal{F}_\mathfrak{c}\|\le k^*_i\times\|\mathcal{Y}\|$. Then

$$\begin{aligned}\mathrm{nor}^0_i(\mathcal{Y}) \;&=\; k^*_i - \log_3\left(\log_{k^*_i}\left(\frac{k^*_i\cdot|\mathrm{pos}^u_i|}{\|\mathcal{Y}\|}\right)\right) \ge k^*_i - \log_3\left(\log_{k^*_i}\left(\frac{k^*_i\cdot|\mathrm{pos}^u_i|}{\|\mathcal{F}_\mathfrak{c}\|}\cdot k^*_i\right)\right)\\ &\ge\; k^*_i - \log_3\left(3\log_{k^*_i}\left(\frac{k^*_i\cdot|\mathrm{pos}^u_i|}{\|\mathcal{F}_\mathfrak{c}\|}\right)\right) = \mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) - 1,\end{aligned}$$

so we are done.
Clauses (c) *and* (e): Obvious. □

Observation 4.5

(1) $\mathbb{Q}_u$, $\underset{\sim}{\mathbb{Q}}_u$ *are non-trivial partial orders.*
(2) $\mathbb{Q}_u$ *is a dense subset of* $\underset{\sim}{\mathbb{Q}}_u$.

Proof. (1) Should be clear.
(2) For $\mathfrak{c}\in\underline{\mathrm{CR}}^u_i$ such that $\mathrm{nor}^1_i(\mathfrak{c})>1$ we set $[\mathfrak{c}] = ([\mathcal{F}_\mathfrak{c}], m_\mathfrak{c})$ (see Definition 4.3(I)(ε)). Note that $\frac{\|[\mathcal{F}_\mathfrak{c}]\|}{|\mathrm{pos}^u_i|}\ge\frac{\|\mathcal{F}_\mathfrak{c}\|}{|\mathrm{pos}^u_i|}-\frac{1}{2^{n_*(i)}}$ and hence (as $(k^*_i)^{3^{k^*_i}}<2^{n_*(i)}$ and $\frac{\|\mathcal{F}_\mathfrak{c}\|}{|\mathrm{pos}^u_i|}>(k^*_i)^{1-3^{k^*_i}}$) we have $\frac{\|[\mathcal{F}_\mathfrak{c}]\|}{|\mathrm{pos}^u_i|}\ge\frac{\|\mathcal{F}_\mathfrak{c}\|}{|\mathrm{pos}^u_i|}^3\cdot\frac{1}{2}$ and hence easily $\mathrm{nor}^0_i([\mathcal{F}_\mathfrak{c}])\ge\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c})-1$. Consequently, $[\mathfrak{c}]\in\mathrm{CR}^u_i$ and $\mathrm{nor}^1_i([\mathfrak{c}])\ge\mathrm{nor}^1_i(\mathfrak{c})-1$.
Now suppose that $p\in\underset{\sim}{\mathbb{Q}}_u$. We may assume that $\mathrm{nor}_i\left(\mathfrak{c}^p_i\right)>1$ for all $i\ge\mathbf{i}(p)$. Put $\mathbf{i}(q)=\mathbf{i}(p)$, $\mathfrak{c}^q_i=\left[\mathfrak{c}^p_i\right]$ for $i\ge\mathbf{i}(q)$ and $\bar{x}_q=\bar{x}_p$. Then $q=(\bar{x}_q,\langle\mathfrak{c}^q_i : i\ge\mathbf{i}(q)\rangle)\in\mathbb{Q}_u$ is a condition stronger than p. □

Definition 4.6 Let $u\subseteq$ Ord be a finite non-empty set.

(1) Let $\underset{\sim}{\bar{x}}$ and $\underset{\sim}{\varkappa}_\alpha, \underset{\sim}{t}_\alpha$ for $\alpha\in u$ be the following $\mathbb{Q}_u$-names:
 (a) $\underset{\sim}{\bar{x}} = \underset{\sim}{\bar{x}}_u = \bigcup\{\bar{x}_p : p\in\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_u}\}$ and $\underset{\sim}{\varkappa}_\alpha = \langle\underset{\sim}{\pi}_{\alpha,i} : i<\omega\rangle$, where

$$\underset{\sim}{\pi}_{\alpha,i}[\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_u}] = \pi \quad\text{ if and only if }\quad \text{for some } p\in\underset{\sim}{\mathbf{G}} \text{ we have } \ell g(\bar{x}_p)>i \text{ and } f_{x_{p,i}}(\alpha)=\pi.$$

 (b) $\underset{\sim}{t}_\alpha = t^*_{\underset{\sim}{\varkappa}_\alpha}$, i.e., it is a tree (see Definition 3.2(4)).
(2) For $p\in\mathbb{Q}_u$ let $\mathrm{pos}(p)=\{\bar{x}_q : p\le_{\mathbb{Q}_u} q\}$ and for $\bar{x}\in\mathrm{pos}(p)$ let $p^{[\bar{x}]}=\left(\bar{x},\left\langle\mathfrak{c}^p_i : i\in[\ell g(\bar{x}),\omega)\right\rangle\right)$.

Observation 4.7 *Let $u \subseteq$ Ord be a finite non-empty set, $\alpha \in u$. Then:*

(1) $\Vdash_{\mathbb{Q}_u}$ " $\underset{\sim}{\bar{x}} \in \mathbf{S}_{u,\omega}$ ".

(2) *We can reconstruct $\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_u}$ from $\underset{\sim}{\bar{x}}$. As a matter of fact, $\langle e_{\underset{\sim}{\bar{x}}_i} : i < \omega\rangle$ determines $\langle f_{\underset{\sim}{\bar{x}}_i}, g_{\underset{\sim}{\bar{x}}_i} : i < \omega\rangle$ (and also $\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_u}$).*

(3) $\underset{\sim}{\varkappa}_\alpha = \bigcup \{\varkappa^\alpha_{\bar{x}} : \bar{x} = \bar{x}_p$ *and* $p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_u}\}$.

(4) $\Vdash_{\mathbb{Q}_u}$ " $\underset{\sim}{\varkappa}_\alpha \in \mathbf{T}_\omega$."

(5) *If $h : u \longrightarrow$ Ord is one-to-one, then $\hat{h}$ (see Definition 4.3(L)) maps $\underset{\sim}{\bar{x}}_u$ to $\underset{\sim}{\bar{x}}_{h[u]}$, $(\bar{x}_u)_i$ to $(\underset{\sim}{\bar{x}}_{h[u]})_i$, etc.*

Observation 4.8

(1) $p^{[\bar{x}]} \in \mathbb{Q}_u$ *and* $p \leq_{\mathbb{Q}_u} p^{[\bar{x}]}$ *for every* $\bar{x} \in \mathrm{pos}(p)$.

(2) *If $p \in \mathbb{Q}_u$ and $i \in [\ell g(\bar{x}_p), \omega)$, then the set $\mathcal{I}_{p,i} := \{p^{[\bar{x}]} : \bar{x} \in \mathrm{pos}(p) \cap \mathbf{S}_{u,i}\}$ is predense above p in $\mathbb{Q}_u$.*

Proposition 4.9 *$\mathbb{Q}_u$ is a proper ${}^\omega\omega$–bounding forcing notion with rapid continuous reading of names, i.e., if $p \in \mathbb{Q}_u$ and $p \Vdash$ " $\underset{\sim}{h}$ is a function from ω to $\mathbf{V}$," then for some $q \in \mathbb{Q}_u$ we have:*

(a) $p \leq q$ *and* $\mathbf{i}(p) = \mathbf{i}(q)$,

(b) *for every $i < \omega$ the set $\{y : q \nVdash_{\mathbb{Q}_u}$ " $\underset{\sim}{h}(i) \neq y$"$\}$ is finite, moreover, for some $j \in [\ell g(\bar{x}_q), \omega)$, for each $\bar{x} \in \mathrm{pos}(q) \cap \mathbf{S}_{u,j}$ the condition $q^{[\bar{x}]}$ forces a value to $\underset{\sim}{h}(i)$,*

(c) *if $p \Vdash_{\mathbb{Q}_u}$ "$(\forall i < \omega)(\underset{\sim}{h}(i) < k^*_i)$," then:*

($\circledast$) *if $\bar{x} \in \mathrm{pos}(q)$ has length $i > \mathbf{i}(q)$, then $q^{[\bar{x}]}$ forces a value to $\underset{\sim}{h}(i)$.*

Proof. The proof follows from work in [6]; therefore, we will use definitions and notations from [6] in the proof. First note that we may assume $|u| < \mathbf{i}(p)$ (as otherwise we fix $i > |u|$ and we carry out the construction successively for all $\bar{x} \in \mathrm{pos}(p)$ of length i).

For $i < \mathbf{i}(p)$ let $\mathbf{H}(i) = \{x_{p,i}\}$ and for $i \geq \mathbf{i}(p)$ let $\mathbf{H}(i) = \mathrm{pos}^u_i$. Let K^* consists of all creatures $t = (\mathrm{nor}[t], \mathrm{val}[t], \mathrm{dis}[t])$ such that

- for some $i \geq \mathbf{i}(p)$ and $\mathfrak{c} \in \mathrm{CR}^u_i$ we have $\mathrm{dis}[t] = (\mathfrak{c}, i)$ and $\mathrm{nor}[t] = \mathrm{nor}^1_i(\mathfrak{c})$, and
- $\mathrm{val}[t] = \{(\bar{w}, \bar{w}^\frown\langle h\rangle) : \bar{w} \in \prod\limits_{j<i} \mathbf{H}(j)\ \&\ h \in \mathrm{set}(\mathcal{F}_\mathfrak{c})\}$.

(Note the use of nor^1_i and not nor^2_i above.) For $t \in K^*$ with $\mathrm{dis}[t] = (\mathfrak{c}, i)$ we let

$$\Sigma^*(t) = \{s \in K : \mathrm{dis}[s] = (\mathfrak{d}, i)\ \&\ \mathfrak{d} \in \Sigma(\mathfrak{c})\}.$$

Then (K^*, Σ^*) is a local finitary big creating pair (for $\mathbf{H}$) with the Halving Property (remember Proposition 4.4(d,e)). Now define $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ by $f(j,i) = (\ell^*_i)^{j+1}$. Let $p^* \in \mathbb{Q}^*_f(K^*, \Sigma^*)$ be a condition such that $w^{p^*} = \bar{x}_p$ and $\mathrm{dis}\big[t^{p^*}_i\big] = \big(\mathfrak{c}^p_{i+\mathbf{i}(p)}, i + \mathbf{i}(p)\big)$ for $i < \omega$. Note that $\mathbb{Q}_u$ above p is essentially the same as $\mathbb{Q}^*_f(K^*, \Sigma^*)$ above p^* (compare Observation 4.7(2)). It should be clear that it is enough to find a condition $q^* \geq p^*$ with the properties (a)–(c) restated for $\mathbb{Q}^*_f(K, \Sigma)$.

Let $\varphi_\mathbf{H}(i) = \Big|\prod\limits_{j<i} \mathbf{H}(j)\Big|$. It follows from Proposition 4.4(a) that $\varphi_\mathbf{H}(i) \leq \big|\mathrm{pos}^u_{i-1}\big|^i < (\beth(30i+3))^i < \beth(30i+4)$ and $2^{\varphi_\mathbf{H}(i)} < \beth(30i+5)$. Therefore,

$$\begin{aligned} 2^{\varphi_\mathbf{H}(i)} \cdot (f(j,i) + \varphi_\mathbf{H}(i) + 2) &\leq \beth(30i+5) \cdot ((\beth(30i+10))^{j+1} + \beth(30i+4) + 2) \\ &< \beth(30i+7) \cdot (\beth(30i+10))^{j+1} < (\beth(30i+10))^{j+2} = f(j+1, i). \end{aligned}$$

Since plainly $f(j,i) \leq f(j, i+1)$, we conclude that the function f is $\mathbf{H}$-fast. Therefore [6, Theorem 2.2.11] gives us a condition q^* satisfying (a)+(b) (restated for $\mathbb{Q}^*_f(K^*, \Sigma^*)$). Proceeding as in [6, Theorem 5.1.12] but using the large amount of bigness here (see Proposition 4.4(d)) we may find a stronger condition saisfying also demand (c).

Note that to claim just properness of $\mathbb{Q}_u$ one could use the quite strong halving of nor_i and [7]. □

Observation 4.10

(1) $D_i^{u_1 \cup u_2} = D_i^{u_1} \cup D_i^{u_2}$.

(2) $h \in \mathrm{pos}_i^{u_1 \cup u_2}$ *if and only if* h *is a function with domain* $D_i^{u_1 \cup u_2}$ *and* $h{\restriction}D_i^{u_\ell} \in \mathrm{pos}_i^{u_\ell}$ *for* $\ell = 1, 2$.

Definition 4.11 Assume that $\emptyset \neq w \subseteq u \subseteq \mathrm{Ord}$ are finite, $v = u \setminus w \neq \emptyset$. Let $\mathcal{F} \in \mathrm{wpos}_i^u$. We define $\mathcal{F} \upharpoonleft w : \mathrm{pos}_i^w \longrightarrow [0,1]$ by

$$(\mathcal{F} \upharpoonleft w)(h) = \frac{\sum\{\mathcal{F}(e) : h \subseteq e \in \mathrm{pos}_i^u\}}{|\mathrm{pos}_i^v|} \qquad \text{for } h \in \mathrm{pos}_i^w.$$

We will also keep the convention that if $u \subseteq \mathrm{Ord}$ and $\mathcal{F} \in \mathrm{pos}_i^u$, then $\mathcal{F} \upharpoonleft u = \mathcal{F}$.

Proposition 4.12 *Assume that* $\emptyset \neq u_0 \subseteq u_1 \subseteq \mathrm{Ord}$ *are finite,* $u_0 \neq u_1$ *and* $\mathcal{F}_1 \in \mathrm{wpos}_i^{u_1}$. *Let* $\mathcal{F}_0 := \mathcal{F}_1 \upharpoonleft u_0$. *Then*

(1) $\mathcal{F}_0 \in \mathrm{wpos}_i^{u_0}$ *and* $\frac{\|\mathcal{F}_0\|}{|\mathrm{pos}_i^{u_0}|} = \frac{\|\mathcal{F}_1\|}{|\mathrm{pos}_i^{u_1}|}$.

(2) *If* $\mathcal{F}_2 \in \mathrm{wpos}_i^{u_0}$, $\mathcal{F}_2 \leq \mathcal{F}_0$, *then there is* $\mathcal{F}_3 \in \mathrm{wpos}_i^{u_1}$ *such that* $\mathcal{F}_3 \leq \mathcal{F}_1$ *and* $\mathcal{F}_3 \upharpoonleft u_0 = \mathcal{F}_2$.

Proof. Let $v = u_1 \setminus u_0$.

(1) Plainly, $\mathcal{F}_0 \in \mathrm{wpos}_i^{u_0}$. Also

$$\|\mathcal{F}_0\| = \frac{1}{|\mathrm{pos}_i^v|} \sum \left\{ \sum \left\{ \mathcal{F}_1(e) : h \subseteq e \in \mathrm{pos}_i^{u_1} \right\} : h \in \mathrm{pos}_i^{u_0} \right\} = \frac{\|\mathcal{F}_1\|}{|\mathrm{pos}_i^v|} = \frac{|\mathrm{pos}_i^{u_0}|}{|\mathrm{pos}_i^{u_1}|} \cdot \|\mathcal{F}_1\|.$$

(2) Suppose $\mathcal{F}_2 \in \mathrm{wpos}_i^{u_0}$, $\mathcal{F}_2 \leq \mathcal{F}_0$. For $e \in \mathrm{pos}_i^{u_1}$ such that $\mathcal{F}_0(e \upharpoonleft u_0) > 0$ we put

$$\mathcal{F}_3(e) = \mathcal{F}_1(e) \cdot \frac{\mathcal{F}_2(e \upharpoonleft u_0)}{\mathcal{F}_0(e \upharpoonleft u_0)},$$

and for $e \in \mathrm{pos}_i^{u_1}$ such that $\mathcal{F}_0(e \upharpoonleft u_0) = 0$ we let $\mathcal{F}_3(e) = 0$. Then clearly $\mathcal{F}_3 \in \mathrm{wpos}_i^{u_1}$, $\mathcal{F}_3 \leq \mathcal{F}_1$ and for $h \in \mathrm{pos}_i^{u_0}$ we have (assuming $\mathcal{F}_0(h) \neq 0$; otherwise it is even easier):

$$(\mathcal{F}_3 \upharpoonleft u_0)(h) = \frac{\sum \left\{ \mathcal{F}_3(e) : h \subseteq e \in \mathrm{pos}_i^{u_1} \right\}}{|\mathrm{pos}_i^v|} = \frac{\mathcal{F}_2(h)}{\mathcal{F}_0(h)} \cdot \frac{\sum \left\{ \mathcal{F}_1(e) : h \subseteq e \in \mathrm{pos}_i^{u_1} \right\}}{|\mathrm{pos}_i^v|} = \mathcal{F}_2(h).$$

□

Definition 4.13

(1) We say that a pair $(\mathcal{F}_1, \mathcal{F}_2)$ is *balanced* when for some $i < \omega$ and finite non-empty sets $u_1, u_2 \subseteq \mathrm{Ord}$ we have $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ for $\ell = 1, 2$ and $\|\mathcal{F}_1\| / |\mathrm{pos}_i^{u_1}| = \|\mathcal{F}_2\| / |\mathrm{pos}_i^{u_2}|$ and, moreover, if $u_1 \cap u_2 \neq \emptyset$ then also $\mathcal{F}_1 \upharpoonleft (u_1 \cap u_2) = \mathcal{F}_2 \upharpoonleft (u_1 \cap u_2)$.

(2) A pair $(\mathcal{F}_1, \mathcal{F}_2)$ is *strongly balanced* if it is balanced and $0 \neq |u_1 \setminus u_2| = |u_2 \setminus u_1|$ (where $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ for $\ell = 1, 2$).

(3) Assume $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ (for $\ell = 1, 2$). Let $u = u_1 \cup u_2$. We define $\mathcal{F} = \mathcal{F}_1 * \mathcal{F}_2 \in \mathrm{ypos}_i^{u_1 \cup u_2}$ (see Definition 4.3(I)(γ)) by putting for $h \in \mathrm{pos}_i^{u_1 \cup u_2}$

$$\mathcal{F}(h) = \mathcal{F}_1(h \upharpoonleft u_1) \cdot \mathcal{F}_2(h \upharpoonleft u_2).$$

Remark 4.14

(1) Note that $\mathcal{F}_1 * \mathcal{F}_2$ can be constantly zero, so it does not have to be a member of wpos. However, below we will apply to it our notation and definitions formulated for wpos.

(2) If $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ ($\ell = 1, 2$), $u_0 = u_1 \cap u_2 \neq \emptyset$, and $\mathcal{F}_3 = \mathcal{F}_1 * \mathcal{F}_2$, then $\mathcal{F}_3 \upharpoonleft u_0 = (\mathcal{F}_1 \upharpoonleft u_0) \cdot (\mathcal{F}_2 \upharpoonleft u_0)$.

(3) If $u_1 \cap u_2 = \emptyset$, $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$, then $\|\mathcal{F}_1 * \mathcal{F}_2\| = \|\mathcal{F}_1\| \cdot \|\mathcal{F}_2\|$.

(4) Suppose $(\mathcal{F}_1, \mathcal{F}_2)$ is balanced, $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ (for $\ell = 1, 2$). Choose finite $u_1', u_2' \subseteq \mathrm{Ord}$ such that $u_1 \subseteq u_1'$, $u_2 \subseteq u_2'$, $u_1 \cap u_2 = u_1' \cap u_2'$ and $|u_1' \setminus u_2'| = |u_2' \setminus u_1'| \neq 0$. For $\ell = 1, 2$ and $h \in \mathrm{pos}_i^{u_\ell'}$ put $\mathcal{F}_\ell'(h) = \mathcal{F}\ell(H{\restriction}u_\ell) \times |\mathrm{pos}_i^{u_\ell}| / \mathrm{pos}_i^{u_\ell'}|$. Then

Proposition 4.15

(1) *If* (u_1, u_2) *is a* Δ*-system pair,* $u_1 \neq u_2 \neq \emptyset$, $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ *for* $\ell = 1, 2$, *and* $\mathcal{F}_2 = \mathrm{OP}_{u_2,u_1}(\mathcal{F}_1)$, *then the pair* $(\mathcal{F}_1, \mathcal{F}_2)$ *is strongly balanced.*

(2) *If* $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ *for* $\ell = 1, 2$ *and* $\|\mathcal{F}_\ell\|/\left|\mathrm{pos}_i^{u_\ell}\right| \geq a > 0$, *the pair* $(\mathcal{F}_1, \mathcal{F}_2)$ *is balanced,* $u_3 = u_1 \cup u_2$ *and* $\mathcal{F} =: \mathcal{F}_1 * \mathcal{F}_2$, *then* $\|\mathcal{F}\|/\left|\mathrm{pos}_i^{u_3}\right| \geq \frac{a^3}{8}$.

P r o o f. (1) Straightforward.

(2) Let $u_0 = u_1 \cap u_2$. We may assume $u_0 \neq \emptyset$ (see 4.14(3)). Let $\mathcal{F}_3 := \mathcal{F}$ and $\mathcal{F}_0 = \mathcal{F}_1 \restriction u_0 = \mathcal{F}_2 \restriction u_0$. For $h \in \mathrm{pos}_i^{u_0}$ and $\ell \leq 3$ let $\mathcal{F}_\ell^{[h]} : \mathrm{pos}_i^{u_\ell} \longrightarrow [0,1]$ be defined by

$$\mathcal{F}_\ell^{[h]}(e) = \begin{cases} \mathcal{F}_\ell(e) & \text{if } \quad h \subseteq e, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$(*)_0$ $k_\ell = \left|\left\{e \in \mathrm{pos}_i^{u_\ell} : h \subseteq e\right\}\right|$ for $h \in \mathrm{pos}_i^{u_0}$, $\ell = 1, 2$, i.e., this number does not depend on h.

[Why? By the definition of $\mathrm{pos}_i^{u_\ell}$ and Observation 4.10.]

$(*)_1$ $\mathcal{F}_\ell$ is the disjoint sum of $\left\langle \mathcal{F}_\ell^{[h]} : h \in \mathrm{pos}_i^{u_0} \right\rangle$ for $\ell = 1, 2, 3$; the "disjoint" means that $\left\langle \mathrm{set}\left(\mathcal{F}_\ell^{[h]}\right) : h \in \mathrm{pos}_i^{u_0} \right\rangle$ are pairwise disjoint. Hence $\|\mathcal{F}_\ell\| = \sum\left\{\left\|\mathcal{F}_\ell^{[h]}\right\| : h \in \mathrm{pos}_i^{u_0}\right\}$.

[Why? By the definition of $\mathrm{pos}_i^{u_\ell}$ and $\mathcal{F}_\ell^{[h]}$.]

$(*)_2$ $k_\ell \geq \left\|\mathcal{F}_\ell^{[h]}\right\| = \mathcal{F}_0(h) \cdot k_\ell$ for $\ell = 1, 2$.

[Why? By Defintion 4.11.]

$(*)_3$ $\left\|\mathcal{F}_3^{[h]}\right\| = \left\|\mathcal{F}_2^{[h]}\right\| \times \left\|\mathcal{F}_1^{[h]}\right\|$.

[Why? By the choice of $\mathcal{F}_3^{[h]}$.]

Let (noting that $0 < a \leq 1$)

$(*)_4$ $A_0 = \left\{h \in \mathrm{pos}_i^{u_0} : \mathcal{F}_0(h) \geq \frac{a}{2}\right\}$.

Now

$(*)_5$ $|A_0| \geq \frac{a}{2-a} \times \left|\mathrm{pos}_i^{u_0}\right|$.

[Why? Letting $d = |A_0|/\left|\mathrm{pos}_i^{u_0}\right|$ and $b = \frac{a}{2}$ $\left(\text{so } 0 < b \leq \frac{1}{2}\right)$ we have

$$h \in \mathrm{pos}_i^{u_0} \setminus A_0 \quad \Rightarrow \quad \left\|\mathcal{F}_1^{[h]}\right\| \leq \frac{a}{2} k_1 = b k_1$$

(remember $(*)_2$). Also $\left\|\mathcal{F}_1^{[h]}\right\| \leq k_1$ for all $h \in \mathrm{pos}_i^{u_0}$ and $k_1 \cdot \left|\mathrm{pos}_i^{u_0}\right| = \left|\mathrm{pos}_i^{u_1}\right|$. Hence

$$\begin{aligned} a \times \left|\mathrm{pos}_i^{u_1}\right| \leq \|\mathcal{F}_1\| &= \sum\left\{\left\|\mathcal{F}_1^{[h]}\right\| : h \in \mathrm{pos}_i^{u_0}\right\} \\ &= \sum\left\{\left\|\mathcal{F}_1^{[h]}\right\| : h \in \mathrm{pos}_i^{u_0} \setminus A_0\right\} \\ &\quad + \sum\left\{\left\|\mathcal{F}_1^{[h]}\right\| : h \in A_0\right\} \leq b k_1 \cdot \left(\left|\mathrm{pos}_i^{u_0}\right| - |A_0|\right) + k_1 |A_0| \\ &= b k_1 (1-d)\left|\mathrm{pos}_i^{u_0}\right| + k_1 d \left|\mathrm{pos}_i^{u_0}\right| \\ &= k_1 \cdot \left|\mathrm{pos}_i^{u_0}\right| \cdot (b(1-d) + d) = \left|\mathrm{pos}_i^{u_1}\right|(b + (1-b)d). \end{aligned}$$

Hence $a \leq b + (1-b)d$ and $\frac{a-b}{1-b} \leq d$. So, as $b = a/2$, we have $d \geq \frac{a/2}{1-a/2} = \frac{a}{2-a}$. By the choice of d we conclude $|A_0| = d \times \left|\mathrm{pos}_i^{u_0}\right| \geq \frac{a}{2-a} \times \left|\mathrm{pos}_i^{u_0}\right|$, i.e., $(*)_5$ holds.]

Now

$(*)_6$ $\|\mathcal{F}_3\| \geq \frac{a^2}{4} \times k_1 \times k_2 \times |A_0|$.

[Why? By $(*)_3$, $\|\mathcal{F}_3^{[h]}\| = \|\mathcal{F}_1^{[h]}\| \times \|\mathcal{F}_2^{[h]}\|$ for all $h \in \mathrm{pos}_i^{u_0}$ and hence

$$\begin{aligned} \|\mathcal{F}_3\| &= \sum \{\|\mathcal{F}_3^{[h]}\| : h \in \mathrm{pos}_i^{u_0}\} \\ &= \sum \{\|\mathcal{F}_1^{[h]}\| \times \|\mathcal{F}_2^{[h]}\| : h \in \mathrm{pos}_i^{u_0}\} \\ &\geq \sum \{\|\mathcal{F}_1^{[h]}\| \times \|\mathcal{F}_2^{[h]}\| : h \in A_0\} \\ &\geq \sum \left\{\frac{a^2}{4} \cdot k_1 \cdot k_2 : h \in A_0\right\} \\ &= \frac{a^2}{4} \cdot k_1 \cdot k_2 \cdot |A_0|. \end{aligned}$$

So $(*)_6$ holds.]

Lastly,

$(*)_7$ $\|\mathcal{F}_3\| \geq \frac{a^3}{8} |\mathrm{pos}_i^{u_3}|$.

Why? Note that $k_1 \cdot k_2 \cdot |\mathrm{pos}_i^{u_0}| = |\mathrm{pos}_i^{u_3}|$ and hence

$$\begin{aligned} \|\mathcal{F}_3\| &\geq \frac{a^2}{4} \times k_1 \times k_2 \times |A_0| \\ &= \frac{a^2}{4} \left(|A_0|/|\mathrm{pos}_i^{u_0}|\right)\left(k_1 \times k_2 \times |\mathrm{pos}_i^{u_0}|\right) \\ &= \frac{a^2}{4} \times \left(|A_0|/|\mathrm{pos}_i^{u_0}|\right) \times |\mathrm{pos}_i^{u_3}| \\ &\geq \frac{a^2}{4} \times \frac{a}{2-a} \times |\mathrm{pos}_i^{u_3}| \geq \frac{a^3}{8} |\mathrm{pos}_i^{u_3}|. \end{aligned}$$

□

So $(*)_7$ holds and we are done.

In Proposition 4.15(2) we can get a better bound, the proof gives $\frac{a^4}{4(2-a)^2}$ and we can point out the minimal value (when all are equal).

Definition 4.16 Let $\mathbb{P}, \mathbb{Q}$ be forcing notions.

(1) A mapping $\mathbf{j} : \mathbb{P} \longrightarrow \mathbb{Q}$ is called *a projection of* $\mathbb{P}$ *onto* $\mathbb{Q}$ when:
  (a) $\mathbf{j}$ is "onto" $\mathbb{Q}$ and
  (b) $p_1 \leq_{\mathbb{P}} p_2 \quad \Rightarrow \quad \mathbf{j}(p_1) \leq_{\mathbb{Q}} \mathbf{j}(p_2)$.
(2) A projection $\mathbf{j} : \mathbb{P} \longrightarrow \mathbb{Q}$ is $\lessdot$*–complete* if (in addition to (a), (b) above):
  (c) if $\mathbb{Q} \models$" $\mathbf{j}(p) \leq q$," then some p_1 satisfies $p \leq_{\mathbb{P}} p_1$ and $q \leq_{\mathbb{Q}} \mathbf{j}(p_1)$.

Definition 4.17 If $\emptyset \neq u \subseteq v \subset \mathrm{Ord}$ are finite, then $\mathbf{j}_{u,v}$ is a function from $\underline{\mathbb{Q}}_v$ onto $\underline{\mathbb{Q}}_u$ defined by: for $q \in \underline{\mathbb{Q}}_v$ we have $\mathbf{j}_{u,v}(q) = p \in \underline{\mathbb{Q}}_u$ if and only if

(α) $\mathbf{i}(p) = \mathbf{i}(q)$ and $\bar{x}_p = \bar{x}_q \restriction u$, and
(β) for $i \in [\mathbf{i}(p), \omega)$ we have $\mathfrak{c}_i^p := \mathrm{proj}_u(\mathfrak{c}_i^q)$ which means $\mathfrak{c}_i^p = \left(\mathcal{F}_{\mathfrak{c}_i^q} \restriction u, m_{\mathfrak{c}_i^p}\right)$.

Proposition 4.18 *If* $u \subseteq v \in \mathrm{Ord}^{<\aleph_0}$, *then* $\mathbf{j}_{u,v}$ *is a* (*well defined*) $\lessdot$*–complete projection from* $\underline{\mathbb{Q}}_v$ *onto* $\underline{\mathbb{Q}}_u$.

P r o o f. It follows from Proposition 4.12 that

$(*)_1$ if $\mathfrak{c} \in \underline{\mathrm{CR}}_i^v$, then $\mathrm{proj}_u(\mathfrak{c}) \in \underline{\mathrm{CR}}_i^u$ and $\mathrm{nor}_i(\mathrm{proj}_u(\mathfrak{c})) = \mathrm{nor}_i(\mathfrak{c})$.

Also, by the definition of proj_u and Definition 4.11, we easily get

$(*)_2$ if $\mathfrak{c} \in \underline{\mathrm{CR}}^v_i$, $\mathfrak{d} \in \underline{\Sigma}(\mathfrak{c})$, then $\mathrm{proj}_u(\mathfrak{d}) \in \underline{\Sigma}(\mathrm{proj}_u(\mathfrak{c}))$, and

$(*)_3$ if $\mathfrak{d} \in \underline{\mathrm{CR}}^u_i$, $\mathcal{F} : \mathrm{pos}^v_i \longrightarrow [0,1]$ is defined by $\mathcal{F}(h) = \mathcal{F}_{\mathfrak{d}}(h \restriction u)$, then $(\mathcal{F}, m_{\mathfrak{d}}) \in \underline{\mathrm{CR}}^v_i$, $\mathrm{nor}_i\big((\mathcal{F}, m_{\mathfrak{d}})\big) = \mathrm{nor}_i(\mathfrak{d})$ and $\mathrm{proj}_u\big((\mathcal{F}, m_{\mathfrak{d}})\big) = \mathfrak{d}$.

Therefore $\mathbf{j}_{u,v}$ is a projection from $\underline{\mathbb{Q}}_v$ onto $\underline{\mathbb{Q}}_u$. To show that it is $\lessdot$–complete we note that, by 4.12(2),

$(*)_4$ if $\mathfrak{c}_1 \in \underline{\mathrm{CR}}^v_i$, $\mathfrak{c}_0 = \mathrm{proj}_u(\mathfrak{c}_1)$ and $\mathfrak{c}_2 \in \underline{\Sigma}(\mathfrak{c}_0)$, then some $\mathfrak{c}_3 \in \underline{\mathrm{CR}}^v_i$ satisfies $\mathfrak{c}_3 \in \underline{\Sigma}(\mathfrak{c}_1)$ and $\mathrm{proj}_u(\mathfrak{c}_3) = \mathfrak{c}_2$.

The rest should be clear. □

Proposition 4.19 *Assume (u_1, u_2) is a Δ–system pair, i.e., $u_1, u_2 \subseteq \mathrm{Ord}$, $|u_1| = |u_2| < \aleph_0$ and so OP_{u_2,u_1} (the order isomorphism from u_1 onto u_2, see Notation 2.1(10)) is the identity on $u_1 \cap u_2$. Let $u = u_1 \cup u_2$. Further assume that $p_\ell \in \underline{\mathbb{Q}}_{u_\ell}$ for $\ell = 1, 2$, $\mathrm{nor}^1_i\big(\mathfrak{c}^{p_\ell}_i\big) \geq 1$ for all $i \geq \mathbf{i}(p_\ell)$ and OP_{u_1,u_2} maps p_1 to p_2. Then there is a condition $q \in \underline{\mathbb{Q}}_u$ such that:*

(a) $\mathbf{i}(q) = \mathbf{i}(p_1)$ *and* $p_\ell \leq_{\underline{\mathbb{Q}}_{u_\ell}} \mathbf{j}_{u_\ell,u}(q)$ *for* $\ell = 1, 2$, *and*

(b) $\mathrm{nor}^1_i\big(\mathfrak{c}^q_i\big) \geq \mathrm{nor}^1_i\big(\mathfrak{c}^{p_1}_i\big) - 1$ *for* $i \in [\mathbf{i}(q), \omega)$.

Proof. We shall mainly use Proposition 4.15(2).
First, we set $\mathbf{i}(q) = \mathbf{i}(p_1)$ and we let $\bar{x} = \langle x_i : i < \mathbf{i}(q)\rangle$, where $x_i = (f_{x_i}, g_{x_i}, e_{x_i})$ is defined by

- $f_{x_i} = f_{x^{p_1}_i} \cup f_{x^{p_2}_i}$, it is well defined function because $f_{x^{p_\ell}_i} \in {}^{u_\ell}\big(\mathrm{Per}\big({}^{n_*(i)}2\big)\big)$ for $\ell = 1, 2$ are well defined functions, with the same restriction to $u_0 = u_1 \cap u_2$;
- $g_{x_i} = g_{x^{p_1}_i} \cup g_{x^{p_2}_i}$ (similarly well defined);
- $e_{x_i} = e_{x^{p_1}_i} \cup e_{x^{p_2}_i}$ (again, it is well defined).

Easily,

- $\bar{x} \in \mathbf{S}_{u,\mathbf{i}(q)}$.

Second, we let $\bar{\mathfrak{c}} = \langle \mathfrak{c}_i : i \in [\mathbf{i}(q), \omega)\rangle$ where for $i \in [\mathbf{i}(q), \omega)$ we let $\mathfrak{c}_i = (\mathcal{F}_i, m_i)$, where

- $\mathcal{F}_i = \mathcal{F}_{\mathfrak{c}^{p_1}_i} * \mathcal{F}_{\mathfrak{c}^{p_2}_i}$,
- $m_i = m_{\mathfrak{c}^{p_\ell}_i}$ for $\ell = 1, 2$.

Let $i \in [\mathbf{i}(q), \omega)$. By Proposition 4.15(1) we know that the pair $\big(\mathcal{F}_{\mathfrak{c}^{p_1}_i}, \mathcal{F}_{\mathfrak{c}^{p_2}_i}\big)$ is (strongly) balanced. Let $a = \frac{\|\mathcal{F}_{\mathfrak{c}^{p_1}_i}\|}{|\mathrm{pos}^{u_1}_i|} = \frac{\|\mathcal{F}_{\mathfrak{c}^{p_2}_i}\|}{|\mathrm{pos}^{u_2}_i|}$. Then, by Proposition 4.15(2) we have $\|\mathcal{F}_i\| \geq \frac{a^3}{8} \times \big|\mathrm{pos}^u_i\big|$. Hence, recalling $k^*_i \geq 3$,

$$\begin{aligned}
\mathrm{nor}^0_i(\mathcal{F}_i) &= k^*_i - \log_3\left(\log_{k^*_i}\left(\frac{k^*_i \cdot |\mathrm{pos}^u_i|}{\|\mathcal{F}_i\|}\right)\right)\\
&\geq k^*_i - \log_3\left(\log_{k^*_i}\left(\frac{8k^*_i}{a^3}\right)\right)\\
&\geq k^*_i - \log_3\left(3\log_{k^*_i}\left(\frac{k^*_i}{a}\right)\right)\\
&= k^*_i - \log_3\left(\log_{k^*_i}\left(\frac{k^*_i \cdot |\mathrm{pos}^{u_1}_i|}{\|\mathcal{F}_{\mathfrak{c}^{p_1}_i}\|}\right)\right) - 1\\
&= \mathrm{nor}^0_i\big(\mathcal{F}_{\mathfrak{c}^{p_1}_i}\big) - 1 = \mathrm{nor}^0_i\big(\mathcal{F}_{\mathfrak{c}^{p_2}_i}\big) - 1.
\end{aligned}$$

Now clearly $q := (\bar{x}, \bar{\mathfrak{c}})$ is as required. □

5 Definable branches and disjoint cones

Now we come to the claim on creatures specifically to deal with the bounded intersection of branches. We think below of H_ℓ as part of a name of a branch of the αth tree.

Lemma 5.1 *Assume that* $u = u_1 \cup u_2$ *are finite non-empty sets of ordinals,* $|u_2 \setminus u_1| = |u_1 \setminus u_2| \neq 0$, $w = u_1 \cap u_2$. *Suppose also that* $i = j + 1 < \omega$, $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$ *(for* $\ell = 1, 2$*) and the pair* $(\mathcal{F}_1, \mathcal{F}_2)$ *is balanced (hence strongly balanced). Let* S *be a finite set (e.g.,* ${}^{n_*(i)}2$*) and* $H_\ell : \mathrm{pos}_i^{u_\ell} \longrightarrow S$. *Then there are* $\mathcal{F}'_1, \mathcal{F}'_2, \mathcal{F}$ *such that:*

(a) $\mathcal{F} \in \mathrm{wpos}_i^u$,
(b) $\mathcal{F}'_\ell \leq \mathcal{F}_\ell$ *for* $\ell = 1, 2$ *and* $\mathcal{F} = \mathcal{F}'_1 * \mathcal{F}'_2$,
(c) *the pair* $(\mathcal{F}'_1, \mathcal{F}'_2)$ *is balanced,*
(d) $\|\mathcal{F}'_\ell\| \geq \frac{1}{8}\|\mathcal{F}_\ell\|$ *for* $\ell = 1, 2$,
(e) *one of the following occurs:*
 (α) *if* $h \in \mathrm{set}(\mathcal{F})$ *then* $H_1(h \restriction u_1) \neq H_2(h \restriction u_2)$,
 (β) (Case 1) $u_1 \cap u_2 = \emptyset$*: for some* $s \in S$ *we have* $h \in \mathrm{set}(\mathcal{F}) \Rightarrow H_1(h \restriction u_1) = s = H_2(h \restriction u_2)$*;*
 (Case 2) *general: for some function* H' *from* pos_i^w *to* S *we have:*

$$h \in \mathrm{set}(\mathcal{F}) \quad \Rightarrow \quad H_1(h \restriction u_1) = H'(h \restriction (u_1 \cap u_2)) = H_2(h \restriction u_2).$$

P r o o f. Let $\langle s_m : m < m_* \rangle$ list of all members of S. Let $g \in \mathcal{G} := \mathrm{pos}_i^w$. Now for every $m \leq m_*$ we define

($\oplus_1$) (a) $\mathcal{F}_{\ell,g} : \mathrm{pos}_i^{u_\ell} \longrightarrow [0, 1]$ is given by $\mathcal{F}_{\ell,g}(h) = \mathcal{F}_\ell(h)$ if $g \subseteq h$ and $\mathcal{F}_{\ell,g}(h) = 0$ otherwise,
 (b) $k_{\ell,g} := \|\mathcal{F}_{\ell,g}\|$,
 (c) $k^{<}_{\ell,m,g} := \sum \{\mathcal{F}_{\ell,g}(h) : g \subseteq h \in \mathrm{pos}_i^{u_\ell} \ \&\ H_\ell(h) \in \{s_{m_1} : m_1 < m\}\}$,
 (d) $k^{=}_{\ell,m,g} := \sum \{\mathcal{F}_{\ell,g}(h) : g \subseteq h \in \mathrm{pos}_i^{u_\ell} \ \&\ H_\ell(h) = s_m\}$,
 (e) $k^{\geq}_{\ell,m,g} := \sum \{\mathcal{F}_{\ell,g}(h) : g \subseteq h \in \mathrm{pos}_i^{u_\ell} \ \&\ H_\ell(h) \in \{s_{m_1} : m \leq m_1 < m_*\}\}$.

Since we are assuming that $(\mathcal{F}_1, \mathcal{F}_2)$ is strongly balanced, we have

($\oplus_2$) $k_{1,g} = k_{2,g}$, call it k_g.

Plainly, $k^{<}_{\ell,m,g}, k^{=}_{\ell,m,g}, k^{\geq}_{\ell,m,g}, k_g$ are non-negative reals and

$(*)_1$ $k^{<}_{\ell,m,g} + k^{\geq}_{\ell,m,g} = k_g$.

Hence

$(*)_2$ $\max\{k^{<}_{\ell,m,g}, k^{\geq}_{\ell,m,g}\} \geq k_g/2$.

Also,

$(*)_3$ $k^{<}_{\ell,m,g} \leq k^{<}_{\ell,m+1,g}$ and $k^{\geq}_{\ell,m,g} \geq k^{\geq}_{\ell,m+1,g}$, in fact $k^{<}_{\ell,m,g} + k^{=}_{\ell,m,g} = k^{<}_{\ell,m+1,g}$ and $k^{\geq}_{\ell,m+1,g} + k^{=}_{\ell,m,g} = k^{\geq}_{\ell,m,g}$, and

$(*)_4$ $k^{<}_{\ell,0,g} = 0 = k^{\geq}_{\ell,m_*,g}$.

Hence for some $m_{\ell,g}$ we have

$(*)_5$ $k^{<}_{\ell,m_{\ell,g}+1,g} \geq k_g/2$ and $k^{\geq}_{\ell,m_{\ell,g},g} \geq k_g/2$.

Therefore:

$(*)_6$ one of the following possibilities holds:
 (a) both $k^{<}_{\ell,m_{\ell,g},g}$ and $k^{\geq}_{\ell,m_{\ell,g}+1,g}$ are greater than or equal to $k_g/4$, or
 (b) $k^{=}_{\ell,m_{\ell,g},g} \geq k_g/4$.

[Why? If clause (b) fails then by $(*)_5$ we get clause (a).]

Choose $(\iota_g, \mathcal{F}^*_{1,g}, \mathcal{F}^*_{2,g})$ as follows.

$(*)_7$ *Case 1:* $k^{=}_{1,m_{1,g},g} \geq k_g/4$ and $k^{=}_{2,m_{2,g},g} \geq k_g/4$.

Let $\iota_g = 1$, and $\mathcal{F}^*_{\ell,g} : \text{pos}^{u_\ell}_i \longrightarrow [0,1]$ be such that $\mathcal{F}^*_{\ell,g}(h) = \mathcal{F}_{\ell,g}(h)$ if $g \subseteq h$ and $H_\ell(h) = s_{m_{\ell,g}}$, and $\mathcal{F}^*_{\ell,g}(h) = 0$ otherwise (for $\ell = 1, 2$).
Case 2: $k^{=}_{1,m_{1,g},g} \geq k_g/4$ and $k^{=}_{2,m_{2,g},g} < k_g/4$.
Let $\iota_g = 2$ and $\mathcal{F}^*_{\ell,g} : \text{pos}^{u_\ell}_i \longrightarrow [0,1]$ (for $\ell = 1, 2$) be defined by:
$\mathcal{F}^*_{1,g}(h) = \mathcal{F}_{1,g}(h)$ if $g \subseteq h$ and $H_1(h) = s_{m_{1,g}}$, and $\mathcal{F}^*_{1,g}(h) = 0$ otherwise;
$\mathcal{F}^*_{2,g}(h) = \mathcal{F}_{2,g}(h)$ if $g \subseteq h$ and $H_2(h) \neq s_{m_{1,g}}$, and $\mathcal{F}^*_{2,g}(h) = 0$ otherwise.
Case 3: $k^{=}_{1,m_{1,g},g} < k_g/4$ and $k^{=}_{2,m_{2,g},g} \geq k_g/4$.
Let $\iota_g = 3$ and $\mathcal{F}^*_{\ell,g} : \text{pos}^{u_\ell}_i \longrightarrow [0,1]$ (for $\ell = 1, 2$) be defined by:
$\mathcal{F}^*_{1,g}(h) = \mathcal{F}_{1,g}(h)$ if $g \subseteq h$ and $H_1(h) \neq s_{m_{2,g}}$, and $\mathcal{F}^*_{1,g}(h) = 0$ otherwise;
$\mathcal{F}^*_{2,g}(h) = \mathcal{F}_{2,g}(h)$ if $g \subseteq h$ and $H_2(h) = s_{m_{2,g}}$, and $\mathcal{F}^*_{2,g}(h) = 0$ otherwise.
Case 4: $k^{=}_{1,m_{1,g},g} < k_g/4$, $k^{=}_{2,m_{2,g},g} < k_g/4$ and $m_{1,g} \leq m_{2,g}$.
Let $\iota_g = 4$ and $\mathcal{F}^*_{\ell,g} : \text{pos}^{u_\ell}_i \longrightarrow [0,1]$ (for $\ell = 1, 2$) be defined by:
$\mathcal{F}^*_{1,g}(h) = \mathcal{F}_{1,g}(h)$ if $g \subseteq h$ and $H_1(h) \in \{s_0, \ldots, s_{m_{1,g}-1}\}$, and $\mathcal{F}^*_{1,g}(h) = 0$ otherwise;
$\mathcal{F}^*_{2,g}(h) = \mathcal{F}_{2,g}(h)$ if $g \subseteq h$ and $H_2(h) \in \{s_{m_{1,g}}, \ldots, s_{m_*-1}\}$, and $\mathcal{F}^*_{2,g}(h) = 0$ otherwise.
Case 5: $k^{=}_{1,m_{1,g},g} < k_g/4$, $k^{=}_{2,m_{2,g},g} < k_g/4$ and $m_{1,g} > m_{2,g}$.
Let $\iota_g = 5$ and $\mathcal{F}^*_{\ell,g} : \text{pos}^{u_\ell}_i \longrightarrow [0,1]$ (for $\ell = 1, 2$) be defined by:
$\mathcal{F}^*_{1,g}(h) = \mathcal{F}_{1,g}(h)$ if $g \subseteq h$ and $H_1(h) \in \{s_{m_{2,g}}, \ldots, s_{m_*-1}\}$, and $\mathcal{F}^*_{1,g}(h) = 0$ otherwise;
$\mathcal{F}^*_{2,g}(h) = \mathcal{F}_{2,g}(h)$ if $g \subseteq h$ and $H_2(h) \in \{s_0, \ldots, s_{m_{2,g}-1}\}$, and $\mathcal{F}^*_{2,g}(h) = 0$ otherwise.

Now:

$(*)_8$ $\|\mathcal{F}^*_{\ell,g}\| \geq \frac{1}{4}\|\mathcal{F}_{\ell,g}\| = \frac{1}{4}k_g$ for $\ell = 1, 2$.

[Why? By $(\oplus_2)$ and $(*)_7$: check each case.]
Finally choose $\mathcal{F}^{**}_{\ell,g}$ (for $\ell = 1, 2$ and $g \in \mathcal{G}$) such that:

$(*)_9$ (a) $\mathcal{F}^{**}_{\ell,g} \leq \mathcal{F}^*_{\ell,g}$, $\|\mathcal{F}^{**}_{\ell,g}\| \geq \frac{1}{4}k_g$, and $\|\mathcal{F}^{**}_{1,g}\| = \|\mathcal{F}^{**}_{2,g}\|$,
(b) *if* $(\iota_g = 1 \wedge m_{1,g} = m_{2,g})$ *then* for some $s = s(g) \in S$

$$h_1 \in \text{set}(\mathcal{F}^{**}_{1,g}) \wedge h_2 \in \text{set}(\mathcal{F}^{**}_{2,g}) \quad \Rightarrow \quad H_1(h_1) = H_2(h_2) = s,$$

(c) *if* $(\iota_g \neq 1 \vee m_{1,g} \neq m_{2,g})$ *then*

$$h_1 \in \text{set}(\mathcal{F}^{**}_{1,g}) \wedge h_2 \in \text{set}(\mathcal{F}^{**}_{2,g}) \quad \Rightarrow \quad H_1(h_1) \neq H_2(h_2).$$

[Why is this possible? We can choose them to satisfy clause (a) by $(*)_8$ and clauses (b) and (c) follow: look at the choices inside $(*)_7$.]
Now we stop fixing $g \in \mathcal{G}$. Put

$$\mathcal{G}^1 = \{g \in \mathcal{G} : \iota_g = 1 \text{ and } m_{1,g} = m_{2,g}\} \quad \text{and} \quad \mathcal{G}^2 = \{g \in \mathcal{G} : \iota_g \neq 1 \text{ or } m_{1,g} \neq m_{2,g}\}.$$

When we vary $g \in \mathcal{G}$, obviously

$(\circledast_1)$ $\mathcal{F}_\ell$ is the disjoint sum of $\langle \mathcal{F}_{\ell,g} : g \in \mathcal{G}\rangle$,

and hence

$(\circledast_2)$ $\|\mathcal{F}_\ell\| = \sum\{k_g : g \in \mathcal{G}\}$.

As $\mathcal{G} = \text{pos}^w_i$ is the disjoint union of $\mathcal{G}^1, \mathcal{G}^2$, plainly

$(\circledast_3)$ for some $\mathcal{G}' \in \{\mathcal{G}^1, \mathcal{G}^2\}$ the following occurs:
$$\sum\{k_g : g \in \mathcal{G}'\} \geq \|\mathcal{F}_1\|/2 = \|\mathcal{F}_2\|/2.$$

Lastly, we put $\mathcal{F}'_\ell = \sum\{\mathcal{F}^{**}_{\ell,g} : g \in \mathcal{G}'\}$ (for $\ell = 1, 2$). We note that

$$\|\mathcal{F}'_\ell\| = \sum\{\|\mathcal{F}^{**}_{\ell,g}\| : g \in \mathcal{G}'\} \geq \sum\left\{\frac{1}{4}k_g : g \in \mathcal{G}'\right\} \geq \frac{1}{4}(\|\mathcal{F}_\ell\|/2) = \frac{1}{8}\|\mathcal{F}_\ell\|.$$

Now it should be clear that $\mathcal{F}'_1, \mathcal{F}'_2$ and $\mathcal{F} = \mathcal{F}'_1 * \mathcal{F}'_2$ are as required. □

Crucial Lemma 5.2 *Assume that*

(a) u_1, u_2 *are finite subsets of* Ord, $|u_1 \setminus u_2| = |u_2 \setminus u_1| \neq 0$,
(b) $\mathcal{F}_\ell \in \mathrm{wpos}_i^{u_\ell}$, $i < \omega$ *and* $\|\mathcal{F}_\ell\| \geq a \times |\mathrm{pos}_i^{u_\ell}| > 0$,
(c) H_ℓ *is a function from* $\mathbf{S}_{u_\ell, i+1}$ *to* ${}^{n_*(i)}2$,
(d) *the pair* $(\mathcal{F}_1, \mathcal{F}_2)$ *is balanced.*

Let $u = u_1 \cup u_2$ *and* $w = u_1 \cap u_2$ *and* $|u| < n_*(i-1)$. *Then we can find* $\mathcal{F}'_\ell \in \mathrm{wpos}_i^{u_\ell}$ *and partial functions* $\mathbf{h}_\ell$ *from* $\mathbf{S}_{u_\ell, i} \times \mathbf{S}_{w, i+1}$ *into* ${}^{n_*(i)}2$ *for* $\ell = 1, 2$ *and* $\mathcal{F} \in \mathrm{wpos}_i^u$ *such that:*

(α) $\mathcal{F}'_\ell \leq \mathcal{F}_\ell$, $\|\mathcal{F}'_\ell\| \geq 8^{-k_*}\|\mathcal{F}_\ell\|$, *where* $k_* = |\mathbf{S}_{u,i}| < \ell_i^*$, *and the pair* $(\mathcal{F}'_1, \mathcal{F}'_2)$ *is balanced,*
(β) $\mathcal{F} = \mathcal{F}'_1 * \mathcal{F}'_2$ *and so* $\mathcal{F} \restriction u_\ell \leq \mathcal{F}_\ell$ *for* $\ell = 1, 2$ *and* $\|\mathcal{F}\| / |\mathrm{pos}_i^u| \geq \frac{a^3}{2^{9k_*+3}}$,
(γ) *if* $h \in \mathrm{set}(\mathcal{F})$, $\bar{x} \in \mathbf{S}_{u,i}$ *(so* $\ell g(\bar{x}) = i$*) and* $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, *then*

$$H_1(\bar{y} \restriction u_1) = H_2(\bar{y} \restriction u_2) \;\Rightarrow\; \mathbf{h}_1(\bar{x} \restriction u_1, \bar{y} \restriction w) = \mathbf{h}_2(\bar{x} \restriction u_2, \bar{y} \restriction w) = H_1(\bar{y} \restriction u_1) = H_2(\bar{y} \restriction u_2).$$

(δ) *moreover, for each* $\bar{x} \in \mathbf{S}_{u,i}$ *the truth value of the equality* $H_1(\bar{y} \restriction u_1) = H_2(\bar{y} \restriction u_2)$ *in clause* (γ) *is the same for all* $h \in \mathrm{set}(\mathcal{F})$.

Proof. Let $\langle \bar{x}_k : k < k_* \rangle$ list $\mathbf{S}_{u,i}$ (without repetitions). We choose $(\mathcal{F}_k, \mathcal{F}_{1,k}, \mathcal{F}_{2,k})$ by induction on $k \leq k_*$ such that:

(i) $\mathcal{F}_{\ell,k} \in \mathrm{wpos}_i^{u_\ell}$ for $\ell = 1, 2$,
(ii) if $k = 0$, then $\mathcal{F}_{\ell,k} = \mathcal{F}_\ell$,
(iii) $\mathcal{F}_{\ell,k}$ is $\leq$-decreasing with k, i.e., $\mathcal{F}_{\ell,k+1} \leq \mathcal{F}_{\ell,k}$,
(iv) $\|\mathcal{F}_{\ell,k}\| \geq \frac{1}{8^k}\|\mathcal{F}_\ell\|$,
(v) $(\mathcal{F}_{1,k}, \mathcal{F}_{2,k})$ is balanced,
(vi) $\mathcal{F}_k = \mathcal{F}_{1,k} * \mathcal{F}_{2,k}$, so also $\leq$-decreasing with k,
(vii) for each k one of the following occurs:
  (α) *if* $h \in \mathrm{set}(\mathcal{F}_{k+1})$ and $\bar{y} = \mathrm{suc}_{\bar{x}_k}(h) \in \mathbf{S}_{u,i+1}$, *then* $H_1(\bar{y} \restriction u_1) \neq H_2(\bar{y} \restriction u_2)$;
  (β) *if* $h', h'' \in \mathrm{set}(\mathcal{F}_{k+1})$ and $h' \restriction w = h'' \restriction w$, $\bar{y}' = \mathrm{suc}_{\bar{x}_k}(h')$, $\bar{y}'' = \mathrm{suc}_{\bar{x}_k}(h'')$, *then*

$$H_1(\bar{y}' \restriction u_1) = H_1(\bar{y}'' \restriction u_1) = H_2(\bar{y}' \restriction u_2) = H_2(\bar{y}'' \restriction u_2).$$

If we carry out the definition then $\mathcal{F} = \mathcal{F}_{k_*}$ is as required. Note that $\|\mathcal{F}_{\ell,k_*}\| \geq \frac{\|\mathcal{F}_\ell\|}{8^{k_*}}$, hence the bound on $\|\mathcal{F}\|$, i.e., clause (β) of Crucial Lemma 5.2 holds by Proposition 4.15; that is we choose $8^{-k_*}a$ here for a there and $\frac{a^3}{8}$ there means $\frac{(8^{-k_*}a)^3}{8} = \frac{a^3}{2^{9k_*+3}}$ here.

The initial step of $k = 0$ is obvious. For the inductive step, for $k+1$ we define $H_{\ell,k}$ as follows: for $h \in \mathrm{pos}_i^{u_\ell}$ we put $H_{\ell,k}(h) = H_\ell\big(\mathrm{suc}_{\bar{x}_k \restriction u_\ell}(h)\big)$ and we apply Lemma 5.1 to $\mathcal{F}_{1,k}, \mathcal{F}_{2,k}, H_{1,k}, H_{2,k}$ here standing for $\mathcal{F}_1, \mathcal{F}_2, H_1, H_2$ there. This way we obtain $\mathcal{F}_{1,k+1}, \mathcal{F}_{2,k+1}$ and we set $\mathcal{F}_{k+1} = \mathcal{F}_{1,k+1} * \mathcal{F}_{2,k+1}$. If in clause 5.1(e) subclause (α) holds, then the demand in (vii)(α) is satisfied. Otherwise, we get a function H' such that for each $h \in \mathrm{set}(\mathcal{F}_{k+1})$ we have

$$H_{1,k}(h \restriction u_1) = H'(h \restriction w) = H_{2,k}(h \restriction u_2).$$

Consequently, the demand in (vii)(β) is fulfilled. Moreover this choice works for any $\mathcal{F}' \subseteq \mathcal{F}_{k+1}$, so we are done. $\square$

Lemma 5.3

(1) *Assume that* $u \subseteq \mathrm{Ord}$ *is finite,* $\alpha \in u$ *and* $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$, $i > 0$. *Suppose also that there are* $\bar{x} \in \mathbf{S}_{u,i}$ *and functions* $\mathbf{h}_1, \mathbf{h}_2$ *such that*
if $h \in \mathrm{set}(\mathcal{F}_\mathfrak{c})$ *and* $\bar{y} = \mathrm{suc}_{\bar{x}}(h) = \bar{x}^\frown\langle y\rangle$ (*see Definition* 4.3(I)(ζ)),
then $\eta_\ell := \mathbf{h}_\ell(h \restriction (u \setminus \{\alpha\})) \in {}^{n_*(i)}2$ *is well defined for* $\ell = 1, 2$ *and* $(g_y(\alpha)^{-1} \circ f_y(\alpha))(\eta_1) = \eta_2$.
Then $\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) = 0$.

(2) *Assume that* $w \subseteq u \subseteq \mathrm{Ord}$ *are finite,* $\alpha_1, \alpha_2 \in u \setminus w$, $\alpha_1 \neq \alpha_2$ *and* $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$, $i > 0$. *Suppose also that* $\bar{x} \in \mathbf{S}_{u,i}$ *and there are functions* $\mathbf{h}_1, \mathbf{h}_2$ *such that*
if $h \in \mathrm{set}(\mathcal{F}_\mathfrak{c})$ *and* $\bar{y} = \mathrm{suc}_{\bar{x}}(h) = \bar{x}^\frown\langle y\rangle$,
then $\eta_\ell := \mathbf{h}_\ell(\bar{x}, \bar{y} \restriction w) \in {}^{n_*(i)}2$ *is well defined for* $\ell = 1, 2$ *and*

$$(g_y(\alpha_1)^{-1} \circ f_y(\alpha_1))(\eta_1) = (g_y(\alpha_2)^{-1} \circ f_y(\alpha_2))(\eta_2).$$

Then $\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) = 0$.

P r o o f. (1) First we try to give an upper bound to $|\mathrm{set}(\mathcal{F}_\mathfrak{c})|/|\mathrm{pos}^u_i|$. Thinking of "randomly drawing" $h_0 \in \mathrm{pos}^{u\setminus\{\alpha\}}_i$ with equal probability, we get an upper bound to the fraction of $h \in \mathrm{pos}^u_i$, $h \restriction (u \setminus \{\alpha\}) = h_0$ such that if $\mathrm{suc}_{\bar{x}}(h) = \bar{x}^\frown\langle y\rangle$, then
$\eta_\ell := \mathbf{h}_\ell(h \restriction (u \setminus \{\alpha\})) \in {}^{n_*(i)}2$ is well defined for $\ell = 1, 2$ and $\big(g^{-1}_y(\alpha) \circ f_y(\alpha)\big)(\eta_1) = \eta_2$.
Since

$$g_y(\alpha)(\nu){\restriction}n_*(i-1) = g_{x_{i-1}}(\alpha)(\nu{\restriction}n_*(i-1)) = f_y(\alpha)(\nu){\restriction}n_*(i-1)) \quad \text{for all } \nu \in {}^{n_*(i)}2,$$

clearly it is $\leq 1/2^{n_*(i)-n_*(i-1)}$. So $\|\mathcal{F}_\mathfrak{c}\|/|\mathrm{pos}^u_i| \leq |\mathrm{set}(\mathcal{F}_\mathfrak{c})|/|\mathrm{pos}^u_i| \leq 1/2^{n_*(i)-n_*(i-1)} < (k^*_i)^{1-3^{k^*_i}}$ and consequently $\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) = 0$.
(2) For $e \in \mathrm{pos}^{u\setminus\{\alpha_1\}}_i$ let $\bar{y}_e = \mathrm{suc}_{\bar{x}\restriction(u\setminus\{\alpha_1\})}(e) = (\bar{x} \restriction (u \setminus \{\alpha_1\}))^\frown\langle y_e\rangle$, $\mathbf{h}'_1(e) = \mathbf{h}_1(\bar{x}, \bar{y}_e \restriction w)$ and $\mathbf{h}'_2(e) = (g_{y_e}(\alpha_2)^{-1} \circ f_{y_e}(\alpha_2))(\mathbf{h}_2(\bar{x}, \bar{y}_e \restriction w))$. Since $\alpha_1, \alpha_2 \notin w$ and $\alpha_2 \in u \setminus \{\alpha_1\}$, for each $h \in \mathrm{set}(\mathcal{F}_\mathfrak{c})$ the values $\mathbf{h}'_1(h \restriction (u \setminus \{\alpha_1\})), \mathbf{h}'_2(h \restriction (u \setminus \{\alpha_1\}))$ are well defined and, letting $\bar{y} = \mathrm{suc}_{\bar{x}}(h) = \bar{x}^\frown\langle y\rangle$,

$$(g_y(\alpha_1)^{-1} \circ f_y(\alpha_1))(\mathbf{h}'_1(h \restriction (u \setminus \{\alpha_1\}))) = \mathbf{h}'_2(h \restriction (u \setminus \{\alpha_1\})).$$

Therefore clause (1) applies and $\mathrm{nor}^0_i(\mathcal{F}_\mathfrak{c}) = 0$. □

Before we state the main corollary to Crucial Lemma 5.2, let us recall that if $\emptyset \neq w \subseteq u$, $\mathfrak{c} \in \underline{\mathrm{CR}}^u_i$, then $\mathrm{proj}_w(\mathfrak{c}) = (\mathcal{F}_\mathfrak{c} \restriction w, m_\mathfrak{c}) \in \underline{\mathrm{CR}}^w_i$ (see Definition 4.17(β)). Also, if $\emptyset = w = u_1 \cap u_2$ and $\mathfrak{c}_\ell \in \underline{\mathrm{CR}}^{u_\ell}_i$, then $\mathrm{proj}_w(\mathfrak{c}_1) = \mathrm{proj}_w(\mathfrak{c}_2)$ will mean that $\mathrm{nor}_i(\mathfrak{c}_1) = \mathrm{nor}_i(\mathfrak{c}_2)$ and $m_{\mathfrak{c}_1} = m_{\mathfrak{c}_2}$.

Crucial Corollary 5.4 *Assume that*

(a) u_1, u_2 *are finite subsets of* Ord, $|u_1 \setminus u_2| = |u_2 \setminus u_1|$, $u = u_1 \cup u_2$, $w = u_1 \cap u_2$, $\alpha_1 \in u_1 \setminus u_2$ *and* $\alpha_2 \in u_2 \setminus u_1$, $1 < i < \omega$, $|u| < n_*(i-1)$,
(b) $\mathfrak{c}_\ell \in \underline{\mathrm{CR}}^{u_\ell}_i$ *and* $\mathrm{nor}_i(\mathfrak{c}_\ell) > 2$ (*for* $\ell = 1, 2$), *and* $\mathrm{proj}_w(\mathfrak{c}_1) = \mathrm{proj}_w(\mathfrak{c}_2)$,
(c) $H_\ell : \mathbf{S}_{u_\ell,i+1} \longrightarrow {}^{n_*(i)}2$.

Then we can find $\mathfrak{d}_\ell \in \underline{\Sigma}(\mathfrak{c}_\ell)$, $\ell = 1, 2$, *such that:*

(α) $\mathrm{proj}_w(\mathfrak{d}_1) = \mathrm{proj}_w(\mathfrak{d}_2)$,
(β) $\mathrm{nor}_i(\mathfrak{d}_\ell) \geq \mathrm{nor}_i(\mathfrak{c}_\ell) - 1$,
(γ) *if* $h \in \mathrm{set}(\mathcal{F}_{\mathfrak{d}_1} * \mathcal{F}_{\mathfrak{d}_2})$, $\bar{x} \in \mathbf{S}_{u,i}$ *and* $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, *and* $\eta_\ell = H_\ell(\bar{y} \restriction u_\ell) \in {}^{n_*(i)}2$ (*for* $\ell = 1, 2$), *then*

$$\eta_1 = \eta_2 \quad \Rightarrow \quad (g_{y_i}(\alpha_1)^{-1} \circ f_{y_i}(\alpha_1))(\eta_1) \neq (g_{y_i}(\alpha_2)^{-1} \circ f_{y_i}(\alpha_2))(\eta_2).$$

Proof. Let $\mathcal{F}_\ell = \mathcal{F}_{\mathfrak{c}_\ell}$. By assumptions (a,b), the pair $(\mathcal{F}_1, \mathcal{F}_2)$ is strongly balanced and $\mathrm{nor}_i^0(\mathcal{F}_\ell) > (\ell_i^*)^2$. Apply Crucial Lemma 5.2 to choose $\mathcal{F}_1', \mathcal{F}_2', \mathbf{h}_1, \mathbf{h}_2$ such that

$(*)_1$ $\mathcal{F}_\ell' \in \mathrm{wpos}_i^{u_\ell}$, $\mathcal{F}_\ell' \leq \mathcal{F}_\ell$, $\|\mathcal{F}_\ell'\| \geq 8^{-k_*} \cdot \|\mathcal{F}_\ell\|$ (where $k_* = |\mathbf{S}_{u,i}|$), and the pair $(\mathcal{F}_1', \mathcal{F}_2')$ is balanced,

$(*)_2$ $\mathbf{h}_\ell : \mathbf{S}_{u_\ell,i} \times \mathbf{S}_{w,i+1} \longrightarrow {}^{n_*(i)}2$,

$(*)_3$ if $h \in \mathrm{set}(\mathcal{F}_1' * \mathcal{F}_2')$, $\bar{x} \in \mathbf{S}_{u,i}$ and $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, then

$$H_1(\bar{y} \upharpoonright u_1) = H_2(\bar{y} \upharpoonright u_2) \quad \Rightarrow \quad \mathbf{h}_1(\bar{x} \upharpoonright u_1, \bar{y} \upharpoonright w) = \mathbf{h}_2(\bar{x} \upharpoonright u_2, \bar{y} \upharpoonright w) = H_1(\bar{y} \upharpoonright u_1) = H_2(\bar{y} \upharpoonright u_2).$$

Next, for $\bar{y} \in \mathbf{S}_{u_\ell,i+1}$, $\ell = 1, 2$, put

$$H_\ell'(\bar{y}) = (g_{y_i}(\alpha_\ell)^{-1} \circ f_{y_i}(\alpha_\ell))(\mathbf{h}_\ell(\bar{y}{\restriction}i, \bar{y} \upharpoonright w)) \in {}^{n_*(i)}2.$$

Apply Crucial Lemma 5.2 again (this time using clause (δ) there too) to choose $\mathcal{F}_1'', \mathcal{F}_2'', \mathbf{h}_1'', \mathbf{h}_2''$ such that

$(*)_4$ $\mathcal{F}_\ell'' \in \mathrm{wpos}_i^{u_\ell}$, $\mathcal{F}_\ell'' \leq \mathcal{F}_\ell'$, $\|\mathcal{F}_\ell''\| \geq 8^{-k_*} \cdot \|\mathcal{F}_\ell'\|$, and the pair $(\mathcal{F}_1'', \mathcal{F}_2'')$ is balanced,

$(*)_5$ $\mathbf{h}_\ell'' : \mathbf{S}_{u_\ell,i} \times \mathbf{S}_{w,i+1} \longrightarrow {}^{n_*(i)}2$,

$(*)_6$ for each $\bar{x} \in \mathbf{S}_{u,i}$ one of the following occurs:
$(\alpha)_{\bar{x}}$ if $h \in \mathrm{set}(\mathcal{F}_1'' * \mathcal{F}_2'')$ and $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, then $H_1'(\bar{y} \upharpoonright u_1) \neq H_2'(\bar{y} \upharpoonright u_2)$, or
$(\beta)_{\bar{x}}$ if $h \in \mathrm{set}(\mathcal{F}_1'' * \mathcal{F}_2'')$ and $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, then

$$\mathbf{h}_1''(\bar{x} \upharpoonright u_1, \bar{y} \upharpoonright w) = \mathbf{h}_2''(\bar{x} \upharpoonright u_2, \bar{y} \upharpoonright w) = H_1'(\bar{y} \upharpoonright u_1) = H_2'(\bar{y} \upharpoonright u_2).$$

It follows from $(*)_1 + (*)_4$ that $\frac{|\mathrm{pos}_i^{u_\ell}|}{\|\mathcal{F}_\ell''\|} \leq 64^{k_*} \cdot \frac{|\mathrm{pos}_i^{u_\ell}|}{\|\mathcal{F}_\ell\|} < 64^{\ell_i^*} \cdot \frac{|\mathrm{pos}_i^{u_\ell}|}{\|\mathcal{F}_\ell\|}$ and hence (remembering that $\mathrm{nor}_i^0(\mathcal{F}_\ell) > (\ell_i^*)^2$) we have

$$\begin{aligned}
\mathrm{nor}_i^0(\mathcal{F}_\ell'') &\geq k_i^* - \log_3\left(\log_{k_i^*}\left(\frac{k_i^* \cdot |\mathrm{pos}_i^{u_\ell}|}{\|\mathcal{F}_\ell\|} \cdot 64^{\ell_i^*}\right)\right) \\
&\geq k_i^* - \log_3\left(\log_{k_i^*}\left(\frac{k_i^* \cdot |\mathrm{pos}_i^{u_\ell}|}{\|\mathcal{F}_\ell\|} \cdot k_i^*\right)\right) \\
&\geq k_i^* - \log_3\left(\log_{k_i^*}\left(\left(\frac{k_i^* \cdot |\mathrm{pos}_i^{u_\ell}|}{\|\mathcal{F}_\ell\|}\right)^3\right)\right) \\
&= k_i^* - \log_3\left(3\log_{k_i^*}\left(\frac{k_i^* \cdot |\mathrm{pos}_i^{u_\ell}|}{\|\mathcal{F}_\ell\|}\right)\right) \\
&= \mathrm{nor}_i^0(\mathcal{F}_\ell) - 1 > \ell_i^*.
\end{aligned}$$

In particular, $\|\mathcal{F}_\ell''\|/\big|\mathrm{pos}_i^{u_\ell}\big| > (k_i^*)^{1-3^{k_i^*-\ell_i^*}}$ and by 4.15(2) we get

$$\frac{\|\mathcal{F}_1'' * \mathcal{F}_2''\|}{|\mathrm{pos}_i^u|} \geq \left(\frac{1}{2}(k_i^*)^{1-3^{k_i^*-\ell_i^*}}\right)^3,$$

so

$(*)_7$ $\mathrm{nor}_i^0(\mathcal{F}_1'' * \mathcal{F}_2'') \geq k_i^* - \log_3\big(\log_{k_i^*}\big(k_i^* \cdot \big(2(k_i^*)^{3^{k_i^*-\ell_i^*}-1}\big)^3\big) > \ell_i^* - 2 > 0.$

Now we claim that

$(*)_8$ in clause $(*)_6$ before, the possibility $(\beta)_{\bar{x}}$ cannot occur.

Suppose towards contradiction that for some $\bar{x} \in \mathbf{S}_{u,i}$ the statement in $(\beta)_{\bar{x}}$ holds true. Then, remembering $\mathbf{h}_\ell : \mathbf{S}_{u_\ell,i} \times \mathbf{S}_{w,i+1} \longrightarrow {}^{n_*(i)}2$, we have

$(\circledast)$ if $h \in \mathrm{set}(\mathcal{F}_1'' * \mathcal{F}_2'')$ and $\bar{y} = \mathrm{suc}_{\bar{x}}(h)$ and $\eta_\ell = \mathbf{h}_\ell(\bar{x} \upharpoonright u_\ell, \bar{y} \upharpoonright w)$ (for $\ell = 1, 2$), then $\big(g_{y_i}(\alpha_1)^{-1} \circ f_{y_i}(\alpha_1)\big)(\eta_1) = \big(g_{y_i}(\alpha_2)^{-1} \circ f_{y_i}(\alpha_2)\big)(\eta_2)$.

Since $\alpha_1 \neq \alpha_2$ are in $u \setminus w$ we may apply Lemma 5.3(2) to get that $\mathrm{nor}_i^0(\mathcal{F}_1'' * \mathcal{F}_2'') = 0$, contradicting $(*)_7$.

Thus, putting together $(*)_3$ and $(*)_6 + (*)_8$ we conclude that

$(*)_9$ if $h \in \mathrm{set}(\mathcal{F}_1'' * \mathcal{F}_2'')$, $\bar{x} \in \mathbf{S}_{u,i}$ and $\bar{y} = \mathrm{suc}_{\bar{x}}(h)$, $\eta_\ell = H_\ell(\bar{y} \restriction u_\ell)$ (for $\ell = 1, 2$), then

$$\eta_1 = \eta_2 \quad \Rightarrow \quad (g_{y_i}(\alpha_1)^{-1} \circ f_{y_i}(\alpha_1))(\eta_1) \neq (g_{y_i}(\alpha_2)^{-1} \circ f_{y_i}(\alpha_2))(\eta_2).$$

Now we set $\mathfrak{d}_\ell = (\mathcal{F}_\ell'', m_{\mathfrak{c}_\ell})$ (for $\ell = 1, 2$). Since $\mathcal{F}_\ell'' \leq \mathcal{F}_\ell' \leq \mathcal{F}_\ell$ and $\mathrm{nor}_i^0(\mathcal{F}_\ell'') \geq \mathrm{nor}_i^0(\mathcal{F}_\ell) - 1 > m_{\mathfrak{c}_\ell}$, we know that $\mathfrak{d}_\ell \in \underline{\Sigma}(\mathfrak{c}_\ell)$, and since $(\mathcal{F}_1'', \mathcal{F}_2'')$ is balanced we conclude $\mathrm{proj}_w(\mathfrak{d}_1) = \mathrm{proj}_w(\mathfrak{d}_2)$. Also $\mathrm{nor}_i(\mathfrak{d}_\ell) \geq \mathrm{nor}_i(\mathfrak{c}_\ell) - 1$ and thus $\mathfrak{d}_1, \mathfrak{d}_2$ are as required in $(\alpha), (\beta)$. Finally, the demand (γ) is given by $(*)_9$. □

Lemma 5.5 *Assume that*

(a) $u_1, u_2 \subseteq \mathrm{Ord}$ *are finite non-empty sets of the same size,* $|u_1 \setminus u_2| = |u_2 \setminus u_1|$,

(b) $w = u_1 \cap u_2$, $u = u_1 \cup u_2$, *and for* $\ell = 1, 2$*:*

(c) $p_\ell \in \underline{\mathbb{Q}}_{u_\ell}$ *and* $\alpha_{\ell,k} \in u_\ell \setminus w$ *and* $\underset{\sim}{\rho}_{\ell,k}$ *is a* $\underline{\mathbb{Q}}_{u_\ell}$*–name for a branch of* $\underset{\sim}{t}_{\alpha_{\ell,k}}$ *(i.e., this is forced) for* $k < \omega$, *and*

(d) $\mathbf{j}_{w,u_1}(p_1), \mathbf{j}_{w,u_2}(p_2)$ *are compatible in* $\underline{\mathbb{Q}}_w$ *(see* 4.17, 4.18*).*

Then there is $q \in \underline{\mathbb{Q}}_u$ *such that* $p_\ell \leq_{\underline{\mathbb{Q}}_{u_\ell}} \mathbf{j}_{u_\ell,u}(q)$ *for* $\ell = 1, 2$ *and*

$$q \Vdash_{\underline{\mathbb{Q}}_u} \text{“}\, \underset{\sim}{\rho}_{1,k}, \underset{\sim}{\rho}_{2,k} \textit{ have bounded intersection.”}$$

P r o o f. Without loss of generality

(⊛) for $\underline{\mathbb{Q}}_{u_\ell}$, for each $j < \omega$ the sequence $\underset{\sim}{\rho}_{\ell,j}$ can be read continuously above p_ℓ; moreover for every large enough i, say $i \geq i_\ell(j)$ the sequence $\underset{\sim}{\rho}_{\ell,j} \restriction i$ can be read from $\underset{\sim}{\bar{x}}_{u_\ell} \restriction i$.

[Why? First by Proposition 4.18 there is q_1 such that $p_1 \leq_{\underline{\mathbb{Q}}_{u_1}} q_1$ and

$$(\forall q)[q_1 \leq_{\underline{\mathbb{Q}}_{u_1}} q \;\Rightarrow\; \mathbf{j}_{w,u_1}(q), \mathbf{j}_{w,u_2}(p_2) \text{ are compatible in } \underline{\mathbb{Q}}_w].$$

Second, by Observation 4.5 and Proposition 4.9, there is $p_1' \in \underline{\mathbb{Q}}_{u_1}$ satisfying (⊛) and such that $q_1 \leq_{\underline{\mathbb{Q}}_{u_1}} p_1'$. Third, we may choose $q_2 \geq_{\underline{\mathbb{Q}}_{u_2}} p_2$ such that

$$(\forall q)[q_2 \leq_{\underline{\mathbb{Q}}_{u_2}} q \Rightarrow \mathbf{j}_{w,u_1}(p_1'), \mathbf{j}_{w,u_2}(q) \text{ are compatible in } \underline{\mathbb{Q}}_w].$$

Fourth, by Proposition 4.9, there is $p_2' \in \underline{\mathbb{Q}}_{u_2}$ satisfying (⊛) and such that $q_2 \leq_{\underline{\mathbb{Q}}_{u_2}} p_2'$. Clearly (p_1', p_2') are as required.] Passing to stronger conditions if needed we may also require that $\mathbf{i}(p_1) = \mathbf{i}(p_2) = \mathbf{i}$, $\mathbf{j}_{w,u_1}(p_1) = \mathbf{j}_{w,u_2}(p_2)$ (note $(*)_4$ from the proof of Proposition 4.18), $|u| < n_*(\mathbf{i} - 1)$ and $\mathrm{nor}_i(\mathfrak{c}_i^{p_\ell}) > 100$ for $i \geq \mathbf{i}$. Without loss of generality, letting $i(j) = \max\{i_1(j), i_2(j)\}$, it satisfies $i(0) = \mathbf{i}$, $i(j+1) > i(j) + 10$ and

$$\mathrm{nor}_i(\mathfrak{c}_i^{p_1}) = \mathrm{nor}_i(\mathfrak{c}_i^{p_2}) > 2j + 2 \quad \text{ for } i \geq i(j).$$

Fix $i \geq \mathbf{i}$ for a moment. Let k be such that $i(k) \leq i < i(k+1)$. We shall shrink $\mathfrak{c}_i^{p_1}, \mathfrak{c}_i^{p_2}$ in order to take care of $(\alpha_{1,m}, \underset{\sim}{\rho}_{1,m}, \alpha_{2,m}, \underset{\sim}{\rho}_{2,m})$ for $m \leq k$. By (⊛) from the beginning of the proof we know that

(i) if $\bar{y} \in \mathbf{S}_{u_\ell,i+1} \cap \mathrm{pos}(p_\ell)$,
then the condition $(p_\ell)^{[\bar{y}]} \in \underline{\mathbb{Q}}_{u_\ell}$ decides $\underset{\sim}{\rho}_{\ell,m}(i)$ for $m \leq k$, say $(p_\ell)^{[\bar{y}]} \Vdash_{\underline{\mathbb{Q}}_{u_\ell}}$ “$\underset{\sim}{\rho}_{\ell,m}(i) = H_{\ell,m}(\bar{y})$,” where $H_{\ell,m} : \mathbf{S}_{u_\ell,i+1} \longrightarrow {}^{n_*(i)}2$.

Use Crucial Corollary 5.4 $(k+1)$ times to choose $\mathfrak{d}_i^1 \in \underline{\Sigma}(\mathfrak{c}_i^{p_1})$ and $\mathfrak{d}_i^2 \in \underline{\Sigma}(\mathfrak{c}_i^{p_2})$ such that:

(ii) $\mathrm{proj}_w(\mathfrak{d}_i^1) = \mathrm{proj}_w(\mathfrak{d}_i^2)$,

(iii) $\mathrm{nor}_i(\mathfrak{d}_i^\ell) \geq \mathrm{nor}_i(\mathfrak{c}_i^{p_\ell}) - (k+1)$ (for $\ell = 1, 2$),

(iv) *if* $h \in \mathrm{set}(\mathcal{F}_{\mathfrak{d}^1_i} * \mathcal{F}_{\mathfrak{d}^2_i})$, $\bar{x} \in \mathbf{S}_{u,i}$, $\bar{y} = \mathrm{suc}_{\bar{x}}(h) \in \mathbf{S}_{u,i+1}$, $m \le k$, $\ell = 1,2$ and $\eta_\ell = H_{\ell,m}(\bar{y} \restriction u_\ell) \in {}^{n_*(i)}2$, *then*

$$\eta_{1,m} = \eta_{2,m} \quad \Rightarrow \quad \big(g_{y_i}(\alpha_{1,m})^{-1} \circ f_{y_i}(\alpha_{1,m})\big)(\eta_{1,m}) \neq \big(g_{y_i}(\alpha_{2,m})^{-1} \circ f_{y_i}(\alpha_{2,m})\big)(\eta_{2,m}).$$

After this construction is carried out for every $i \ge \mathbf{i}$ we define

- $q_\ell = (\bar{x}_{p_\ell}, \bar{\mathfrak{d}}^\ell)$, where $\bar{\mathfrak{d}}^\ell = \langle \mathfrak{d}^\ell_i : i \in [\mathbf{i}, \omega) \rangle$, $\ell = 1, 2$,
- $q = (\bar{x}_{p_1} \cup \bar{x}_{p_2}, \bar{\mathfrak{d}})$, where $\bar{\mathfrak{d}} = \langle \mathfrak{d}_i : i \in [\mathbf{i}, \omega) \rangle$, $\mathcal{F}_{\mathfrak{d}_i} = \mathcal{F}_{\mathfrak{d}^1_i} * \mathcal{F}_{\mathfrak{d}^2_i}$, $m_{\mathfrak{d}_i} = m_{\mathfrak{d}^1_i} = m_{\mathfrak{d}^2_i}$.

It follows from (iii) (and the choice of $i(j)$) that $q_\ell \in \underset{\sim}{\mathbb{Q}}_{u_\ell}$ and, by Proposition 4.15(2), $q \in \underset{\sim}{\mathbb{Q}}_u$. Plainly $p_\ell \le_{\underset{\sim}{\mathbb{Q}}_{u_\ell}} q_\ell \le_{\underset{\sim}{\mathbb{Q}}_{u_\ell}} \mathbf{j}_{u_\ell,u}(q)$.

Now, let $k < \omega$ and consider $i \ge i(k)$. It follows from (iv) and Proposition 4.2(5) that for each $\bar{x} \in \mathbf{S}_{u,i} \cap \mathrm{pos}(q)$ and $h \in \mathrm{set}(\mathcal{F}_{\mathfrak{d}_i})$, if $\bar{y} = \mathrm{suc}_{\bar{x}}(h)$ and $\eta_{\ell,k} = H_{\ell,k}(\bar{y} \restriction u_\ell)$, then

$$\eta_{1,k} = \eta_{2,k} \quad \Rightarrow \quad q^{[\bar{y}]} \Vdash_{\underset{\sim}{\mathbb{Q}}_u} \text{“} \{\rho : \eta_{1,k} <_{t_{\varkappa_{\alpha_{1,k}}}} \rho\} \cap \{\rho : \eta_{2,k} <_{t_{\varkappa_{\alpha_{2,k}}}} \rho\} = \emptyset \text{”}.$$

Since $q^{[\bar{y}]} \Vdash_{\underset{\sim}{\mathbb{Q}}_u}$ “$\underset{\sim}{\rho}_{\ell,k}(i) = \eta_{\ell,k}$” (for $\ell = 1, 2$) we may conclude that

$$q^{[\bar{y}]} \Vdash_{\underset{\sim}{\mathbb{Q}}_u} \text{“either } \underset{\sim}{\rho}_{1,k}(i) \neq \underset{\sim}{\rho}_{2,k}(i) \text{ or } (\forall j > i)(\underset{\sim}{\rho}_{1,k}(j) \neq \underset{\sim}{\rho}_{2,k}(j)).\text{”}$$

Hence immediately we see that q is as required in the assertion of the lemma. □

(1) If we can deal only with one case (i.e., one k in Lemma 5.5(c)), we have to use $\mathcal{A} = \mathbf{T}^*_\omega$, not “any uncountable” $\mathcal{A} \subseteq \mathbf{T}^*_\omega$. But actually it is enough in Lemma 5.5 to deal with finitely many pairs.

(2) We can prove in Lemma 5.5 that there is a pair (p'_1, p'_2) such that:
   (a) $p_\ell \le_{\mathbb{Q}_{u_\ell}} p'_\ell$ for $\ell = 1, 2$,
   (b) $\mathbf{j}_{w,u_1}(p'_1), \mathbf{j}_{w,u_2}(p'_2)$ are compatible,
   (c) if $p \in \mathbb{Q}_u$ satisfies $p'_\ell \le_{\mathbb{Q}_{u_\ell}} \mathbf{j}_{u,u_\ell}(p)$, then p is as required.

If $u = \{\alpha\}$ is a singleton, then considering $\mathrm{OB}^u_i, \mathbf{S}_{u,i}, \mathbf{S}_u, \mathrm{pos}^u_i, \mathrm{wpos}^u_i, \underset{\sim}{\mathbb{Q}}_u$ we may ignore u (and α) in a natural way arriving to the definitions of $\mathrm{OB}_i, \mathbf{S}_i, \mathbf{S}, \mathrm{pos}_i, \mathrm{wpos}_i, \underset{\sim}{\mathbb{Q}}$, respectively. Let $\varkappa : \mathbf{S}_\omega \longrightarrow \mathbf{T}_\omega$ be the mapping given by $\varkappa(\bar{x}) = \langle f_{x_i} : i < \omega \rangle$ (on $\mathbf{T}$ see Definition 3.2(2), concerning $\varkappa$ compare Definition 4.1(G)).

The following proposition finishes the proof of Theorem 3.1.

Proposition 5.6 *Let $N_* \prec (\mathcal{H}(\beth^+_7), \theta)$ be countable.*

(1) *There is a perfect subtree $\mathbf{S}^* \subseteq \mathbf{S}$ (so $\mathbf{S}^*_\omega = \lim_\omega(\mathbf{S}^*) \subseteq \mathbf{S}_\omega$) such that:*
if $n < \omega$, $\bar{x}_\ell \in \mathbf{S}^*_\omega$ for $\ell < n$ are pairwise distinct then $(\bar{x}_0, \ldots, \bar{x}_{n-1})$ is a generic for $\underset{\sim}{\mathbb{Q}}_n$ over N_*.

(2) *Moreover, $\varkappa[\mathbf{S}^*_\omega] \subseteq \mathbf{T}_\omega$ is strongly pbd (see Definition 3.4(3)) and* ar-cl$\{A_{\varkappa(\bar{x})} : \bar{x} \in \mathbf{S}^*_\omega\}$ *is Borel.*

P r o o f. By Propositions 4.9 and 4.19 and (for part (2)) by Lemma 5.5. In details, let $\mathcal{T}$ be a perfect subtree of ${}^{\omega>}2$ such that in each level only in one node we have splitting and let $\mathcal{T}_i = \{\eta \in \mathcal{T} : \eta \text{ of the } i\text{-th level}\}$.

Let $h_i : |\mathcal{T}_i| \longrightarrow \mathcal{T}_i$ be a bijection such that

$$m' < m'' < n_i \ \Leftrightarrow \ h_i(m') <_{\mathrm{lex}} h_i(m''),$$

where $n_i = |\mathcal{T}_i|$. Let $\langle (m_j, k_j, \underset{\sim}{\rho}_j) : j < \omega \rangle$ list all the triples $(m, k, \underset{\sim}{\rho})$ satisfying: $m < \omega$, $k < m$ and $\underset{\sim}{\rho}$ is a $\mathbb{Q}_m$–name of a branch of $\underset{\sim}{t}_k$ such that $\underset{\sim}{\rho}$ belongs to N_*.

Let η_i be the unique member of $\mathcal{T}_i$ such that $\{\eta_i{}^\frown\langle 0 \rangle, \eta_i{}^\frown\langle 1 \rangle\} \in \mathcal{T}_{i+1}$. For $\ell = 0, 1$ let $f_{i,\ell} : \mathcal{T}_i \longrightarrow \mathcal{T}_{i+1}$ be such that

$$[\eta \in \mathcal{T}_i \setminus \{\eta_i\} \quad \Rightarrow \quad f_{i,\ell}(\eta) \restriction i = \eta] \quad \text{and} \quad f_{i,\ell}(\eta_i) = \eta_i{}^\frown\langle \ell \rangle.$$

Let $u_{i,\ell} =$ Rang$(g_{i,\ell})$ where $g_{i,\ell} = h^{-1}_{i+1} \circ f_{i,\ell} \circ h_i$. For an order preserving function g from the finite $u \subset$ Ord into Ord let $\hat{g}$ be the isomorphism from $\mathbb{Q}_u$ onto $\mathbb{Q}_{g[u]}$ induced by g.

Let $\langle \mathcal{I}_{n,i} : i < \omega \rangle$ list all the dense open subsets of $\mathbb{Q}_n$ which belong to N_*. By induction on $i < \omega$ choose p_i such that if $\ell \in \{1, 2\}$ then (recalling $\mathbf{j}_{u_{i,\ell},n_j}$ is a complete projection from $\mathbb{Q}_{n_j}$ onto $\mathbb{Q}_{u_{i,\ell}}$) we have

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(i) $p_i \in \mathbb{Q}_{n_i}$, $\hat{g}_{i,\ell}(p_i) \leq_{\mathbb{Q}_{u_{i,\ell}}} \mathbf{j}_{u_{i,\ell},n_{i+1}},(p_{i+1})$ for $\ell = 0, 1$.

(ii) If $u \subseteq n_i$ and h^*_u is $\mathrm{OP}_{u,|u|}$, i.e., the order preserving function from $\{0, \ldots, |u| - 1\}$ onto u, and $\hat{h}^*_u$ is defined as above and $k < i$, then $\mathbf{j}_{u,n_i}(p_i) \in \mathbb{Q}_u$ belongs to $\hat{h}^*_u(\mathcal{I}_{|u|,k})$.

(iii) Assume that for $\ell = 0, 1$ the objects $j_\ell < \omega$, $u_\ell \subseteq \mathcal{T}_i$ satisfy

$$\eta_i \in u_\ell,\ |u_\ell| = m_{j_\ell},\ h^*_{u_\ell}(k_{j_\ell}) = h_i^{-1}(\eta_i)$$

and let $\underset{\sim}{\rho}_\ell = \hat{g}_{i,\ell}\big(\hat{h}^*_{u_\ell}(\underset{\sim}{\rho}_{j_\ell})\big)$ $\big($so it is a $\mathbb{Q}_{n_{i+1}}$–name for a branch of $\underset{\sim}{t}_{g_{i,\ell}(h^*_u(\eta_i))}\big)$. *Then* $\Vdash_{\mathbb{Q}_{n_{i+1}}}$ "the branches $\underset{\sim}{\rho}_0$ of $\underset{\sim}{t}_{f_{i,0}(\eta_i)}$ and $\underset{\sim}{\rho}_1$ of $\underset{\sim}{t}_{f_{i,1}(k_{\eta_i})}$ have bounded intersection."

This is straightforward. □

Theorem 5.7

(1) *There is a Borel arithmetically closed set $\mathbf{B} \subseteq \mathcal{P}(\omega)$ such that there is no arithmetically closed 2-Ramsey ultrafilter on it.*

(2) *Moreover, there is a Borel*[1] *$\mathcal{A}_* \subseteq \mathcal{B}$ such that for every uncountable $\mathcal{A}' \subseteq A$, there is no definably closed minimal ultrafilter on the arithmetic closure of* ar-cl$(\mathcal{A}')$ *of $\mathcal{A}'$.*

(3) *We can demand that above each* ar-cl$(\mathcal{A}')$ *is a standard system.*

P r o o f. (1) and (2) Let $\mathcal{A} = \mathbf{T}^*_\omega$ be as in the proof of Proposition 5.6 and let $\mathcal{B}$ be the arithmetic closure ar-cl$(\mathcal{A})$ of $\mathcal{A}$. For every $A_t \in \mathcal{A}$ there towards contradiction assume D is a $\mathbf{B}$–minimal ultrafilter where $\underset{\sim}{B} =$ ar-cl$(\mathcal{A}')$, $\mathcal{A}' \subseteq \mathcal{A}$ is uncountable.

Now for every $A_t \in \mathcal{A}'$, $(\mathbb{N}, <_t)$ is a tree with finite levels (hence finite splittings), a root and the set of levels is $\mathbb{N}$. For every $i < \omega$ the set $\{n < \omega\colon$ in $<^*_t$ the level of n is $< i\}$ is finite and hence its compliment belongs to D. The rest is divided to $\langle\{m : b \leq^*_t m\} : b$ is of level exactly i for $<^*_t\rangle$. This is a finite division hence for some unique $b = b^t_i$ of level i such that $\{m : b \leq^*_t m\} \in D$. As D is a 2-Ramsey ultrafilter

(i) $\big\langle b^t_i : i < \omega\big\rangle$ is definable in $\mathbb{N}_{\mathcal{A}'}$.

We define a function g_t on $\mathbb{N}$ by $g_t(c) = \max\big\{i : b^t_i \leq_t c\big\}$. Again

(ii) g_t is definable in $\mathbb{N}_{\mathcal{A}'}$.

As D is minimal there is $C_t \subseteq \mathbb{N}$ definable in $\mathbb{N}_{\mathcal{A}'}$ and such that

(iii) $g_t{\restriction}C_t$ is one-to-one.

Let C_t be the first order definable in $\mathbb{N}_{\mathcal{A}_t}$ where $\mathcal{A}_t \subseteq \mathcal{A}'$ is finite, $t \in \mathcal{A}_t$ for simplicity and so is the set $\big\{b^t_i : i < \omega\big\}$. As each $\mathbb{Q}_u$ is ${}^\omega\omega$–bounding and we can further shred c_t below there is $h_* \in N_*$ [recall we are forcing over the countable $N_* \prec (H(\chi), \in)$, so our $\mathcal{B}$ is $\bigcup\{\mathcal{P}(\omega) \cap N[t_0, \ldots, t_{n-1}] : t_\ell \in T^*_\omega\}$] such that

(iv) $h_* \in {}^\omega\omega$ is increasing, $h_*(0) = 0$, and

(v) if $c \in C_t$ and $g_t(c) <_t h_*(i)$ then $c <_{\mathbb{N}} h_*(i + 1)$.

Without loss of generality now by the infinite Δ-system for finite sets for some $t_1 \neq t_2$ we have $\{t_1, t_2\} \cap (\mathcal{A}_{t_1} \cap \mathcal{A}_{t_2}) = \emptyset$, etc.

Moreover, replacing $\mathcal{A}_{t_1} \cup A_2, \mathcal{A}_1, \mathcal{A}_2, t_1, t_2$ by $u = u_1 \cup u_2, u_1, u_0, \alpha_1 \in u_1 \setminus u_2, \alpha_2 \in u_2 \setminus u_1$ we have the situation from §2 by a similar proof. We get $C_{t_2} \cap C_{t_2}$ is finite, but both are in an ultrafilter, so we are done.

(3) We let $\mathbb{Q}$ be as in [8] for $\lambda \geq \beth_{\omega_1}$, use what is proved there. □

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[1] In order to eliminate this, we have to force over $\mathbb{N}$.

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