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COLLECTIONWISE HAUSDORFF: INCOMPACTNESS AT SINGULARS

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Under set theoretic hypotheses, we construct a λ-collectionwise Hausdorff not λ^+-collectionwise Hausdorff space of character c for certain singular cardinals λ. For example if $V=L$, and $\mathrm{cf}(\lambda)$ is not weakly compact, or if there are no inner models with large cardinals, λ is singular strong limit, and $\mathrm{cf}(\lambda)$ is the successor of a singular strong limit. Moreover, after forcing collapsing c to ω these spaces retain their properties; thus we obtain first countable examples.

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1. Introduction

A common topic in set theoretic topology is sup = max problems—see [5, Chapter 3]. Given a closed discrete subset D of a topological space X, we say that D can be separated if there exists a disjoint family $\{U_d : d \in D\}$ of open subsets of X with $d \in U_d$. We can define

$$\mathrm{cwH}(X) = \sup\{\kappa \leq |X| : \text{each closed discrete } Y \in [X]^\kappa \text{ can be separated}\}.$$

If $\mathrm{cwH}(X)$ is a limit cardinal λ, we ask whether the sup is attained; i.e. is it the case that every closed discrete $Y \in [X]^\lambda$ can be separated?

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Similar questions are asked in algebra, set theory, and model theory. For example, let G be an Abelian group. If every subgroup $H \in [G]^{<\lambda}$ is free, is it true that every subgroup $K \in [G]^{\lambda}$ is free? Perhaps unfortunately, these questions are called compactness questions. (The unfortunate part is that to a topologist, a compactness property is something like "If $\mathcal{F}$ is a family of closed sets, and every $\mathcal{H} \in [\mathcal{F}]^{<\lambda}$ has nonempty intersection, then ... ". Questions about small subsets of a space have a nature different from questions about small subfamilies of sets.)

To avoid the ambiguity of the sup = max problem, we will use different terminology. We say that a space X is λ-cwH if every closed discrete $Y \in [X]^{<\lambda}$ can be separated. Now the sup = max problem translates to "For limit λ, does λ-cwH imply λ^{+}-cwH?" For reasons discussed in Section 7, we require the space to have small character. The character of a space X, $\chi(X)$, is the least cardinal κ such that every point $x \in X$ has a neighborhood base of cardinality at most κ. Thus "$\chi(X) = \omega$" and "X is first countable" have the same meaning.

The plan of the rest of this paper follows. In Section 2, we construct a regular, T_2, zero-dimensional space W, given some parameters. In Section 3 we discuss the set theoretic hypotheses used to construct our examples and use these hypotheses to define the parameters of W. In Section 4 we prove that W is not λ^{+}-cwH; in Section 5 we prove that W is λ-cwH. Forcing is used to get a first countable example in Section 6. In the last section we discuss the λ-cwH versus λ^{+}-cwH problem.

2. Construction of the space W

We construct W, given two index sets I and J, a countable partition $\{I_n : n \in \omega\}$ of I, and for each $i \in I$, a doubly indexed subset $\{j(i, k, l)\colon k, l \in \omega\}$ of J. We can view W as the union of three subsets: a top edge $X = \{x_i : i \in I\}$, a right edge $Y = \{y_j : j \in J\}$, and a rectangular array $Z = \{z_{ij} : i \in I, j \in J\}$.

The base for W also has three parts. Points of Z are isolated. For each $z \in Z$, $\{z\}$ is a basic open set. A basic open set for $y_j \in Y$ is indexed by $n \in \omega$ and can be viewed as a subset of a horizontal line:

$$B(j, n) = \{y_j\} \cup \{z_{ij} : (\exists m \geq n)(i \in I_m)\}.$$

A basic open set for $x_i \in X$ is indexed by $f \in {}^{\omega}\omega$ and can be viewed as a subset of a vertical line:

$$B(i, f) = \{x_i\} \cup \{z_{ij} : (\exists k \geq f(0))(\exists l \geq f(k))(j = j(i, k, l))\}.$$

It is easy to verify that W is T_2, that basic open sets are closed (hence W is regular and zero-dimensional), that each point has a neighborhood base of cardinality at most 2^{ω}, and that $X \cup Y$ is a closed discrete subset.

We will use set theoretic hypotheses to define I and J, and to carefully define the partition $\{I_n : n \in \omega\}$ and the sets $\{j(i, k, l)\colon k, l \in \omega\}$. The plan is to use $\diamond$ to show that for every open U containing Y, there is an $x_i \in X$ with $x_i \in \mathrm{cl}(U)$; thus, W is not λ^+-cwH. We will use nonreflecting stationary sets to prove that W is λ-cwH.

3. Set theoretic axioms

The exact set theoretic hypothesis we use is detailed, so we begin by asserting that it holds in two particular situations

(i) if $V=L$ and κ is regular not weakly compact, and

(ii) if there are no inner models with large cardinals, and $\kappa=\mu^+$, where μ (and λ) are singular strong limit cardinals of uncountable cofinality. Let HYP be the axiom asserting that the situation of the next paragraph holds.

HYP. Let $\kappa=\mathrm{cf}(\lambda)<\lambda$. Let $\{\lambda_\alpha: \alpha<\kappa\}$ be a continuous increasing sequence of cardinals cofinal in λ satisfying for all $\alpha<\kappa$, $2^{\lambda_\alpha}<\lambda_{\alpha+1}$. Let $A\subset\{\delta\in\kappa: \mathrm{cf}(\delta)=\omega\}$ be stationary such that for all $\gamma<\kappa$, $A\cap\gamma$ is not stationary in γ. Let B be the set of successor cardinals of κ. For $\beta\in B$ let $T_\beta\subset\{\delta\in\lambda_\beta^+: \mathrm{cf}(\delta)=\omega\}$ be stationary such that for all $\gamma<\lambda_\beta^+$, $T_\beta\cap\gamma$ is not stationary in γ. Partition A and each T_β into countably many stationary pieces: $A=\bigcup\{A_n: n\in\omega\}$ and $T_\beta=\bigcup\{T_{\beta n}: n\in\omega\}$. Assume that for each $n\in\omega$, $\diamondsuit(A_n)$ holds; to wit, there is $\{S_\alpha: \alpha\in A\}$ such that for all $X\subset\kappa$, for all club $C\subset\kappa$, for all $n\in\omega$ there is $\alpha\in A_n\cap C$ such that $X\cap\alpha=S_\alpha$.

Assume $V=L$. We know that GCH holds (Gödel), that $\diamondsuit$ holds for all stationary sets (Jensen), and that for every regular, not weakly compact κ, there are nonreflecting stationary subsets of $\{\delta\in\kappa: \mathrm{cf}(\delta)=\omega\}$. Thus every singular cardinal is strong limit, and A, T_β, $\{S_\alpha: \alpha\in A\}$ as in HYP exist.

Assume that there are no inner models with large cardinals. Then Jensen's Covering Lemma holds. We know then that if μ is a singular strong limit cardinal, $2^\mu=\mu^+$, and that a nonreflecting stationary subset of μ^+ exists. So let $\kappa=\mu^+$ where μ is singular strong limit and $\mathrm{cf}(\mu)>\omega$, and let λ be a singular strong limit of cofinality κ. Choose $\{\lambda_\gamma: \gamma<\kappa\}$ so that λ_γ is a singular strong limit cardinal. The covering lemma gives us A and T_β: since $\mu^\omega=\mu$ and $2^\mu=\mu^+=\kappa$, Gregory's argument gives us $\diamondsuit^*\{\delta\in\kappa: \mathrm{cf}(\delta)=\omega\}$; then Kunen's argument gives us $\diamondsuit(A_n)$ (see [4] or [8] Theorem 32).

Of course, HYP holds in many other situations, but we will not pursue this further.

Having discussed our hypotheses, we now use them to give the parameters used to define W in Section 2.

Let $A'=\{\alpha\in A: S_\alpha\subset B$ and $\sup S_\alpha=\alpha\}$. Set $A'_n=A'\cap A_n$. For $\alpha\in A'$, set $I_\alpha=[\lambda_\alpha, 2^{\lambda_\alpha})$; for $\beta\in B$, set $J_\beta=[\lambda_\beta, \lambda_\beta^+)$. Set $I_n=\bigcup\{I_\alpha: \alpha\in A'_n\}$, $J=\bigcup\{J_\beta: \beta\in B\}$.

For $\alpha\in A'$, choose $\{\beta(\alpha,k): k\in\omega\}\subset S_\alpha$, increasing and cofinal in α. Let $\{G_i: i\in I_\alpha\}$ enumerate sequences of sets of the form $G_i=\{G_{ik}: k\in\omega\}$ where

$$G_{ik}\subset J_{\beta(\alpha,k)} \quad\text{and}\quad |G_{ik}|=\lambda^+_{\beta(\alpha,k)}.$$

For $i\in I_\alpha$, $\alpha\in A'_n$, and $k\in\omega$ define $\{j(i,k,l): l\in\omega\}$ and $\sigma(i,k)$ so that

$$\{j(i,k,l): l\in\omega\}\subset G_{ik} \text{ is increasing, and } \sigma(i,k)=\sup\{j(i,k,l): l\in\omega\}\in T_{\beta n}.$$

4. W is not λ^+-cwH

W is not λ^+-cwH because $X \cup Y$ is a closed discrete subset which cannot be separated. In fact, we will show that whenever $Y \subset U$ open, then $X \cap \text{cl}(U) \neq \emptyset$.

Suppose that $Y \subset U$ open. For each $y_j \in Y$, there is $n(j) \in \omega$ with $y_j \in B(j, n(j)) \subset U$. For $\beta \in B$, $n \in \omega$, set $H(\beta, n) = \{j \in J_\beta: n(j) = n\}$. Because λ_β^+ is regular, there is $n(\beta) \in \omega$ so that $|H(\beta, n(\beta))| = \lambda_\beta^+$. Because κ is regular there is $\bar{n} \in \omega$ so that $D = \{\beta \in B: n(\beta) = \bar{n}\}$ is cofinal in κ. Then $C = \{\gamma \in \kappa: \sup(D \cap \gamma) = \gamma\}$ is club in κ. By HYP, there is $\alpha \in A_{\bar{n}} \cap C$ with $S_\alpha = D \cap \alpha$. Because $D \subset B$ and $\alpha \in C$, $\alpha \in A'$. Thus $\{\beta(\alpha, k): k \in \omega\}$ is defined. For $k \in \omega$, set $H_k = H(\beta(\alpha, k), \bar{n})$. Note that $H_k \subset J_{\beta(\alpha, k)}$ and $|H_k| = \lambda_{\beta(\alpha, k)}^+$. Thus, for some $i \in I_\alpha$, $G_i = \{H_k: k \in \omega\}$. Now we see that for all $k, l \in \omega$, $j(i, k, l) \in H_k = H(\beta(\alpha, k)\bar{n})$; that is, $n(j(i, k, l)) = \bar{n}$, or in terms of the topology, $z_{ij(i,k,l)} \in B(j(i, k, l), \bar{n}) \subset U$. Hence $x_i \in \text{cl}(U)$ and W is not λ^+-cwH.

5. W is λ-cwH

In the proof that W is λ-cwH, we will use the following lemma. The proof is by induction on γ (or, viewed topologically, it is a corollary of Engelking-Lutzer [1].

Lemma. *Let T be a set of limit ordinals such that for all $\gamma \leq \sup T$, $T \cap \gamma$ is not stationary in γ. There is a function $p: T \to \sup T$ so that $p(\alpha) < \alpha$ and the family of intervals, $\{(p(\alpha), \alpha]: \alpha \in T\}$, is point-finite.*

Let $D \subset W$ be closed discrete, $|D| < \lambda$. The isolated points are not a problem, so we assume $D \subset X \cup Y$. We will define $\{f_i: i \in I \cap D\}$ satisfying the following claim.

Claim. *For all $j \in J \cap D$, $\theta(j) = \{n \in \omega: (\exists i \in I_n)(z_{ij} \in B(i, f_i))\}$ is finite.*

Given the claim, define $n_j = \sup \theta(j) + 1$. Then $\{B(i, f_i): i \in I \cap D\} \cup \{B(j, n_j): j \in J \cap D\}$ is a disjoint family of open sets separating D.

Because $|D| < \lambda$ there is $\delta \in B$ so that $|D| < \lambda_\delta$.

Recall from Section 3 that $\{\beta(\alpha, k): k \in \omega\}$ is an increasing sequence cofinal in α, and that for $i \in I_\alpha$, $k \in \omega$, $\sup\{j(i, k, l): l \in \omega\} = \sigma(i, k) \in T_\beta \subset \lambda_\beta^+$. Thus we make the

Observation. If $B(j, n) \cap B(i, f) \neq 0$, then $\lambda_{\beta(\alpha, f(0))} < j < i$.

In our definition of f_i, we will require that if $i \in I_\alpha$, $i > \lambda_\delta$, then $\beta(\alpha, f_i(0)) > \delta$. When we consider this requirement and the observations, we see that to verify the claim, we need to consider only two cases: (i) $j < i < \lambda_\delta$; (ii) $\lambda_\delta < j < i$.

We are now ready to start defining f_i, which we will do in two cases (i) $i<\lambda_\delta$, and (ii) $\lambda_\delta<i$. Towards the first case, note that $\delta<\kappa$, so we may apply the lemma to $A\cap\delta$ to get a function $p:A\cap\delta\to\delta$. For $i\in I_\alpha$, $\alpha\in A\cap\delta$, define $f_i\in{}^\omega\omega$ so that $\beta(\alpha,f_i(0))>p(\alpha)$. (The values of $f_i(k)$ for $k>0$ are irrelevent.) We now verify the first case of the claim. If $j\in J_\beta$, $j<\lambda_\delta$, then by the conclusion of the lemma there are only finitely many α's with $p(\alpha)<\beta(\alpha,f_i(0))\leq\beta<\alpha$. Each α is in one I_n, so there are at most finitely many n.

Towards defining f_i in case $\lambda_\delta<i$, we consider $\beta\in B$, $\beta\geq\delta$. Because $|D|=\lambda_\delta<\lambda^+_\beta$, $\eta_\beta=\sup(D\cap J_\beta)<\lambda^+_\beta$. We apply the lemma to $T_\beta\cap\eta_\beta$ to get a function $q_\beta: T_\beta\cap\eta_\beta\to\eta_\beta$.

Now let $i\in I_\alpha\cap D$ with $a>\delta$. In defining $f_i(k)$ we consider whether $\sigma(i,k)$ is less than $\eta_{\beta(\alpha,k)}$ (and we remember the requirement $\beta(\kappa,f_i(0))>\delta$). If $\sigma(i,k)>\eta_{\beta(\alpha,k)}$, define $f_i(k)$ so that $j(i,k,f_i(k))>\eta_{\beta(\alpha,k)}$. If $\sigma(i,k)\leq\eta_{\beta(\alpha,k)}$, define $f_i(k)$ so that $j(i,k,f_i(k))>q_{\beta(\alpha,k)}(\sigma(i,k))$.

Finally, we verify case (ii) of the claim. Suppose $\lambda_\delta<j<i$. Then $j\in J_\beta$ with $\delta\leq\beta$. Since $j\in J_\beta\cap D$, $j\leq\eta_\beta$, so by the conclusion of the lemma, there are at most finitely σ's $\in T_\beta$ such that $q_\beta(\sigma)<j\leq\sigma$. Although σ might be $\sigma(i,k)$ for many i's from many I_α's, σ is in only one $T_{\beta n}$, so all these i's are in the same I_n. Finitely many σ's yields finitely many n's. This completes verifying the claim and the proof that W is λ-cwH.

6. Forcing to get first countable

Assume that $\kappa>2^\omega$, and that the space W with base $B=\{\{z_{ij}\}: i\in I, j\in J\}\cup\{B(j,n): j\in J, n\in\omega\}\cup\{B(i,f): i\in I, f\in{}^\omega\omega\}$ was constructed as in the previous sections. Extend the universe by forcing with finite pieces of a function from ω onto 2^ω. (In the notation of [5], let $P=\mathrm{Fn}(\omega,2^\omega)$). In this section we verify that in the extension the space with point set W and base B has become first countable, but remains regular, T_2, zero-dimensional, λ-cwH, not λ^+-cwH. Regular, T_2, zero-dimensional in the extension follow directly from regular, T_2, zero-dimensional in the ground model. First countable follows from the generic map ω onto $(2^\omega)^V$.

In $V[G]$, let $D\subset W$, $|D|\leq\lambda_\gamma<\lambda$. There is a name $\dot D$ for D in V. Set $E=\{w\in W: (\exists p\in P)(p\Vdash|\dot D|\leq\lambda_\gamma$ and $p\Vdash w\in\dot D)\}$. $E\in V$, and because $|P|=2^\omega<\lambda_\gamma$, $|E|\leq\lambda_\gamma$. Thus there is a separation of E in V, and because in $V[G]$, $D\subset E$, this gives a separation of D. Thus W is λ-cwH in $V[G]$.

To show that W is not λ^+-cwH, we follow the argument of Section 4. In $V[G]$, let $Y\subset U$ open. For each $y_j\in Y$ there is $n(j)\in\omega$ with $y_j\in B(j,n(j))\subset U$. For $\beta\in B$, $n\in\omega$, set $H(\beta,n)=\{j\in J_\beta: n(j)=n\}$. Because λ^+_β is regular there is $n(\beta)\in\omega$ so that $|H(\beta,n(\beta))|=\lambda^+_\beta$. Now we move down to V. If in $V[G]$, $j\in H(\beta,n(\beta))$, then there is $p\in G$, $p\Vdash j\in\dot H(\beta,n(\beta))$. For $p\in P$, $\beta\in B-\{B': \lambda_{\beta'}\leq 2^\omega\}$, set $H(\beta,p)=\{j\in J_\beta: p\Vdash j\in\dot H(\beta,n(\beta))\}$. Since $\lambda^+_\beta>2^\omega$ there is $p(\beta)\in G$ with $|H(\beta,p(\beta)|=\lambda^+_\beta$.

Because κ is regular and greater than 2^ω, there are $\bar{n} \in \omega$ and $\bar{p} \in G$ so that $D = \{\beta \in B: n(\beta) = \bar{n}$ and $p(\beta) = \bar{p}\}$ is cofinal in κ. The point is that D and $\{H(\beta, \bar{p}): \beta \in D\}$ are in V. We may continue the argument in V to find $x_i \in I$ such that $x_i \in \mathrm{cl}(\bigcup\{B(j, \bar{n}): (\exists \beta \in D)(j \in H(\beta, \bar{p}))\}) \subset \mathrm{cl}(U)$. We conclude that in $V[G]$, W is not λ^+-cwH.

7. Specific cases of "Does λ-cwH imply λ^+-cwH?"

The question "Does λ-cwH imply λ^+-cwH?" is not interesting without additional hypotheses. We ask that the space be at least T_2 and regular because there is a nonregular, T_2 space which is ω-cwH and not ω^+-cwH (see [5, 0.20]). Even in this generality, the question is too easy. For every uncountable λ, there is a T_2, regular, zero-dimensional space which is λ-cwH, not λ^+-cwH. For $\lambda = \omega_1$, Bing's Example G is an example. In fact, Bing's Example G is the prototype for most examples in this area. For $\lambda > \omega_1$, for each infinite $a \in [\lambda]^{<\lambda}$, let $K(a)$ be the one-point compactification of the set a with the discrete topology. In the product space $X_\lambda = \prod\{K(a): a \subset \lambda, \omega \leq |a| < \lambda\}$ define special points, y_α, $\alpha \in \lambda$, and isolate the rest of the points in the manner of Bing's G. Then X_λ is λ-cwH, not λ^+-cwH. (For more details, see [2, pp. 447–448]; the construction works for all $\lambda > \omega_1$, not just cf $\lambda = \omega$, the hardest case.)

What additional hypotheses should be added? Interest in the problem began with the early work of Tall and Fleissner on the normal Moore space conjecture. They showed that in certain situations, normal, λ-cwH spaces of small character are λ^+-cwH, and they wondered whether stronger set theoretic hypotheses could make the topological hypothesis normal superfluous.

Many of the usual compactness results hold in the small character, cwH case. (See [2] and [7].) For example, if λ is weakly compact, a regular, T_2, λ-cwH space of character less than λ is λ^+-cwH. In the other direction, if there is a nonreflecting stationary subset of $\{\alpha \in \lambda: \mathrm{cf}(\alpha) = \omega\}$, then there is a regular, T_2, first countable, locally countable (i.e., every point has a countable neighborhood) λ-cwH, not λ^+-cwH space. To avoid nonreflecting stationary sets, one collapses large cardinals; and similar techniques give CON($\exists$ strongly compact cardinal)$\rightarrow$CON(regular, T_2, ω_2-cwH, locally countable spaces are cwH). These techniques seem not to extend to first countable spaces, so we wonder whether ZFC implies the existence of a regular, T_2, ω_2-cwH, not cwH space.

For λ singular the techniques of [6] don't seem to apply, so perhaps the best implication is in [3], while the best counterexamples are in this paper.

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