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Δ^1_2-SETS OF REALS

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We consider and give a complete solution to, implications of the form: $(*)$ *Every X_1-set of reals has the property P_1 implies every X_2-set of reals has the property P_2*, for $X_1, X_2 \in \{\Delta^1_2, \Pi^1_1, \Sigma^1_2, \Pi^1_2\}$, and where P_1, P_2 are among 'to be Ramsey', 'K_σ-regular' and of course 'Lebesgue measurable' and 'Baire categoricity'. Naturally we are led to look for characterizations of such properties (by forcing). Not surprisingly, excepting the trivial implications, we get many consistency results, but 'fortunately' we get quite a number of theorems (= implications proved in ZFC), notably among the 'to be Ramsay' and 'K_σ-regular'.

Theorem 1. *The following are equivalent*:
(a) *Every Σ^1_2-set of reals is Ramsey.*
(b) *Every Δ^1_2-set of reals is Ramsey.*
(c) *For every $r \in \mathbb{R}$ there exists $s \in [\omega]^\omega$, s is $P(D_s[r])$-generic over $L[r][D_s]$.*

(Definitions are given in Section 0.) For this theorem we develop a forcing $P(D)$ (D an ultrafilter on ω) shooting a real 'through' the ultrafilter.

Theorem 2. *The following are equivalent*:
(a) *Every Σ^1_2-set of reals is K_σ-regular.*
(b) *Every Δ^1_2-set of reals is K_σ-regular.*
(c) *Every Π^1_1-set of reals is K_σ-regular.*
(d) *For every $r \in \mathbb{R}$, there exists $f \in \omega^\omega$, f is a σ-bound to $\omega^\omega \cap L[r]$.*

0. Introduction

The Borel sets were introduced by E. Borel in 1905. The analytic sets were introduced by Souslin who had proved the existence of an analytic non-Borel set. The Lebesque measurability of the analytic sets is due to Luzin in 1917 and the Barie categoricity of them is due to Luzin and Sierpinski in 1923. The projective sets were introduced by Luzin and Sierpinski, and Gödel showed that from the axiom of constructibility it is possible to prove that there exists a Δ^1_2-set of reals which is not Lebesque measurable and does not have the property of Baire.

Since forcing was born, much work on the Lebesgue measurability of the projective sets was done, and Martin–Solovay [9] proved that Martin's Axiom implies that every Σ^1_2-set of reals is Lebesgue measurable and has the property of

Baire. Moreover, the following characterization was given:

0.1. Theorem (Solovay). (a) *Every Σ^1_2-set of reals is Lebesgue measurable iff for every real r, the set of random reals over $L[r]$ is a measure one set.*

(b) *Every Σ^1_2-set of reals has the property of Baire iff for every real r, the set of Cohen reals over $L[r]$ is a comeager set.*

During the seventies, another two properties were studied, and Silver [17] proved that every analytic set is Ramsey and Kechris [7] proved that every analytic set is K_σ-regular (these properties will be stated below). Also it was proved that under Martin's Axiom every Σ^1_2-set of reals is Ramsey and K_σ-regular. From the work of Ihoda [2, 3], we will obtain the following.

0.2. Theorem. *The following assertions are equivalent:*

(a) *Every Σ^1_2-set of reals is K_σ-regular.*

(b) *Every Δ^1_2-set of reals is K_σ-regular.*

(c) *Every Π^1_1-set of reals is K_σ-regular.*

(d) *For every $r \in \mathbb{R}$ there exists $f \in {}^\omega\omega$ such that for every $g \in \omega^\omega \cap L[r]$ there exists $n \in \omega$ such that for every $m \geq n$, $g(m) < f(m)$.*

Looking at those two theorems, it seems natural to search for some forcing characterization of the proposition "Every Σ^1_2-set of reals is Ramsey".

In [3] the following was proved:

If "Every Σ^1_2-set of reals is Ramsey", then:

(i) *Every Σ^1_2-set of reals is K_σ-regular.*

(ii) *For every $r \in \mathbb{R}$, $[\omega]^\omega \cap L[r]$ is not a splitting family.*

Indeed, in [3] it was proved that (ii) follows from "Every Δ^1_2-set of reals is Ramsey". By making some minor changes in this proof, we begin Section 2 by showing:

0.3. Theorem. *If every Δ^1_2-set of reals is Ramsey, then for every $r \in \mathbb{R}$ there exists $a \in [\omega]^\omega$ such that for every π: $[\omega]^2 \to 2$, if $\pi \in L[r]$, then there exists $n \in \omega$ such that $a - n$ is homogeneous for π.*

It is not hard to deduce, from this thoerem, that if every Δ^1_2-set of reals is Ramsey, then every Σ^1_2-set of reals is K_σ-regular, and it seems plausible to find more intrinsic connections between the assertions "Every Σ^1_2-set of reals is Ramsey" and "Every Δ^1_2-set of reals is Ramsey". Our search gives the following:

0.4. Theorem. *The following assertions are equivalent:*

(a) *Every Σ^1_2-set of reals is Ramsey.*

(b) *Every Δ^1_2-set of reals is Ramsey.*

This theorem will be proved in Section 2, but prior to this we need to study the following forcing notion.

0.5. Definition. If D is an ultrafilter over ω, let $P(D)$ be the following partially ordered set:

(i) $p \in P(D)$ iff $p \subseteq \omega^{<\omega}$ is a tree and there exists $s \in p$, called the *stem* of p, such that for every $t \in p$, $t \subseteq s$ or $s \subseteq t$ and $\{n \in \omega : t\hat{\ }\langle n\rangle \in p\} \in D$.

(ii) If $p,q \in P(D)$, we say that $p \leqslant q$ if $q \subseteq p$.

(iii) Clearly we can identify a $P(D)$-generic object with an infinite subset of ω (= the generic branch $= \bigcap\{p : p \in G_{P(D)}\}$).

In Section 1 we prove the following facts about forcing with $P(D)$.

0.6. Theorem. (a) *If $x \in [\omega]^\omega$ is $P(D)$-generic over V, then for every $a \in D$, $x \subseteq^* a$ and for every $y \in [x]^\omega$, y is $P(D)$-generic over V. Where $x \subseteq^* a$ means* $(\exists n \in \omega)(x - n \subseteq a)$.

(b) *For every $P(D)$-sentence ϕ, and for every $p \in P(D)$ there exists $q \in P(D)$ such that*

$$p \leqslant q \Vdash \text{“}\phi\text{”} \quad \textit{or} \quad p \leqslant q \Vdash \text{“}\neg\phi\text{”}$$

and the stem of p is equal to the stem of q.

(c) *If P_D is the Silver forcing notion using a Ramsey ultrafilter D, then $a \in [\omega]^\omega$ is P_D-generic over V iff a is $P(D)$-generic over V.*

(d) *If $a \in [\omega]^\omega$ is $P(D)$-generic and P_D-generic over V, then D is a Ramsey ultrafilter.*

Because Mathias forcing is isomorphic to forcing a Ramsey ultrafilter with $\langle [\omega]^\omega/[\omega]^{<\omega}; \supseteq\rangle$ followed by Silver forcing with the generic ultrafilter, we can conclude that $P(D)$ is the natural forcing notion related to the property of ‘to be Ramsey’. Also it can be useful to remark that Ramsey ultrafilters do not necessarily exist and forcing with $\langle [\omega]^\omega/[\omega]^{<\omega}; \supseteq\rangle$ can collapse $2^{\aleph_0}$ (see [4]), but ultrafilters always exist in models of ZFC.

According to the above, we know that if every Δ^1_2-set of reals is Ramsey, then for ever $r \in \mathbb{R}$ there exists $a \in [\omega]^\omega$ such that

$$\{x \in [\omega]^\omega \cap L[r] : a \subseteq^* x\}$$

is an ultrafilter in the Boolean algebra $\mathcal{P}(\omega)^{L[r]}$.

Thinking about this, in Section 2 we define

0.7. Definition. (a) If $s \in [\omega]^\omega$, then $D_s = \{x \in [\omega]^\omega : s \subseteq^* x\}$.

(b) $D^s(r) = L[r][D_s] \cap D_s$.

(c) Clearly $D^s(r) \in L[r][D_s] = L[r][D^s(r)]$.

The proof of Theorem 0.4 is given by using Theorem 0.6 and the following

0.8. Theorem. *The following assertions are equivalent*:

(a) *Every Δ^1_2-set of reals is Ramsey.*

(b) *For every $r \in \mathbb{R}$ there exists $s \in {}^{\omega}[\omega]$ such that*

$$D^s(r) \text{ is an ultrafilter in } L[r][D_s], \quad \text{and}$$
$$s \text{ is } P(D_s(r))\text{-generic over } L[r][D_s].$$

Making small changes in this proof, in the last part of Section 2 we prove the following version of Theorem 0.8.

0.9. Theorem. *Let $r \in \mathbb{R}$. Then the following are equivalent*:

(a) *Every set, with a Δ^1_2-definition using only r as parameter, is Ramsey.*

(b) *Every set, with a Σ^1_2-definition using only r as parameter, is Ramsey.*

(c) *There exists $s \in {}^{\omega}[\omega]$ such that*

$$D_s(r) \text{ is an ultrafilter in } L[r][D_s], \quad \text{and}$$
$$s \text{ is } P(D_s(r))\text{-generic over } L[r][D_s].$$

From these theorems we can ask if the measurability of the Σ^1_2-sets of reals follows from the measurability of the Δ^1_2-sets of reals. In Section 3 we clear up this question by showing:

0.10. Theorem. (a) *Every Δ^1_2-set of reals is Lebesgue measurable iff for every real r there exists a random real over $L[r]$.*

(b) *Every Δ^1_2-set of reals has the property of Baire iff for every real r there exists a Cohen real over $L[r]$.*

We finish Section 3 by giving models for the following theorem.

0.11. Theorem. *Consistency of ZF implies*:

(a) Cons(ZFC + *every Δ^1_2-set of reals is Lebesgue measurable + there exist Δ^1_2-sets of reals which do not have the property of Baire, are not Ramsey and are not K_σ-regular*).

(b) Cons(ZFC + *every Δ^1_2-set of reals has the property of Baire + there exist Δ^1_2-sets of reals which are not Lebesgue measurable, are not Ramsey and are not K_σ-regular*).

(c) Cons(ZFC + *every Δ^1_2-set of reals is Ramsey + there exist Δ^1_2-sets which are not Lebesgue measurable and do not have the property of Baire*).

(d) Cons(ZFC + *every Δ^1_2-set of reals is k_σ-regular + there exists Δ^1_2-sets of reals which are not Lebesgue measurable, do not have the property of Baire, and are not Ramsey*).

From the theorems of this article and the results of [3] and [10], we have the following chart

Lebesgue *Baire*

$$\begin{tikzcd}
\Pi_1^1 \arrow[rr, leftrightarrow] & & \Sigma_1^1 & & \Sigma_2^1 \arrow[rr, leftrightarrow] & & \Pi_2^1 \\
& \Delta_2^1 \arrow[ul] \arrow[ur] \arrow[dl] & & & & \Delta_2^1 \arrow[ul] \arrow[ur] & \\
\Pi_2^1 \arrow[rr, leftrightarrow] & & \Sigma_2^1 \arrow[ul] \arrow[rr] & & \Sigma_2^1 \arrow[ur] \arrow[rr, leftrightarrow] \arrow[d] & & \Pi_1^1 \arrow[ul] \\
\Pi_2^1 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \Sigma_2^1 \arrow[dl, leftrightarrow] \arrow[rr] & & \Sigma_2^1 \arrow[dr, leftrightarrow] & & \Pi_2^1 \arrow[ll] \arrow[dl] \\
& \Delta_2^1 \arrow[dl] \arrow[dr] & & & & \Delta_2^1 \arrow[dl] \arrow[dr, leftrightarrow] & \\
\Pi_1^1 \arrow[rr, leftrightarrow] & & \Sigma_1^1 & & \Sigma_1^1 & & \Pi_1^1 \arrow[ll]
\end{tikzcd}$$

To be Ramsey *K_σ-regular*

Remarks. (1) All directions are proved in ZFC.

(2) There are no other possible implications between these propertics.

All our notation will be standard and can be found in [3]. We recall just the following:

(i) $[x]^\omega = \{a \subseteq x : |a| = \aleph_0\}$.

(ii) A subset $A \subseteq [\omega]^\omega$ is Ramsey iff there exists $a \in [\omega]^\omega$ such that $[a]^\omega \subseteq A$ or $[a]^\omega \subseteq \sim A$.

(iii) $T \subset \omega^{<\omega}$ is a superperfect tree iff T is a tree and for every $t_1 \in T$ there exists $t_2 \in T$ such that $t_1 \subseteq t_2$ and $\{n \in \omega : t_2\hat{\ }\langle n \rangle \in T\} \in [\omega]^\omega$.

(iv) $A \subseteq \omega^\omega$ is K_σ-regular iff there exists a superfect tree $T \subseteq \omega^{<\omega}$ such that

$$\text{the set of branches of } T = [T] \subseteq A$$

or there exists $f \in \omega^\omega$ such that for every $g \in A$ there exists $n \in \omega$ such that $m > n$ implies $g(m) < f(m)$.

We say that $N \models \mathrm{ZFC}^*$ iff N is a transitive model of some part of ZFC, sufficiently rich in order to build the forcing framework. In many cases we need to use the fact that there is a Σ_2^1-good well order of $L[r]$. All this technology can be found in Jech [6, Part IV, p. 493]. We assume that the reader is familiar with forcing and the Boolean valued model notation, especially in the case of the Random Algebra and the Cohen Algebra. Finishing this section, we will present a modern proof of the Luzin theorem about the measurability of the analytic sets, this proof was inspired by [17]. The scheme of this proof is a central idea in this context. Let $\mu(A)$ be the Lebesgue measure of A.

0.12. Theorem (Luzin–Sierpinski). *Every analytic set is Lebesgue measurable.*

Proof. Let $\phi(x)$ be a Σ_1^1-formula. Let $N \prec H(\aleph_1, \in, \leqslant_{\aleph_1})$ be countable and such that the parameters of $\phi(x)$ belong to N. Therefore in N we can compute the Boolean value, related to random forcing, of $\phi(x)$.

Claim. $\mu(\{x: \phi(x)\}) = \mu(\|\phi(x)\|^N_{\text{random}}) = \mu(\|\phi(x)\|_{\text{random}})$.

Proof. Because N is countable,

$$\mu(\{x: x \text{ is random over } N\}) = 1.$$

Therefore it is sufficient to show that if x is random over N and $x \in \|\phi(x)\|_{\text{random}}$, then $\phi(x)$. Let x be random over N, $x \in \|\phi(x)\|_{\text{random}}$, then

$$N[x] \models \phi(x)$$

but Σ_1^1-formulas are absolute for countable models of ZFC*, thus $\phi(x)$ holds in the world. □

1. Between Silver and Laver reals

In this section D will denote a non-principal ultrafilter over ω.

1.1. Definition. (a) $P(D)$ will denote the following partially ordered set:

(i) $p \in P(D)$ iff p is a subtree $\omega^{<\omega}$ with the property that there exists $s \in p$ (denoted $s(p)$) so that $\forall t \in p$, $t \subseteq s$ or $s \subseteq t$, and if $s \subseteq t \in p$, then

$$\{n \in \omega: t\langle n\rangle \in p\} \in D,$$

for every $p \in P(D)$ and for every $s \in p$, s is an increasing function.

(ii) $p_1 \leqslant p_2$ iff $p_1 \supseteq p_2$.

(b) If $s \in p$, then $p_s = \{t \in p: t \subseteq s \text{ or } s \subseteq t\}$.

(c) $p_1 \leqslant^0 p_2$ iff $s(p_1) = s(p_2)$ and $p_1 \leqslant p_2$.

1.2. Fact. (a) *$P(D)$ is a σ-centered partially ordered set.*

(b) *If $G \subseteq P(D)$ is generic over V, then if*

$$g = \bigcup \{s \in \omega^{<\omega}: (\exists p \in G)(s(p) = s)\},$$

we have that $g \in \omega^\omega$ is increasing and $V[G] = V[g]$.

1.3. Definition. If $I \subseteq P(D)$ is a dense open subset of $(P(D), \leqslant)$ and $p \in P(D)$ we define $\text{rk}^I: p \to \text{ORD}$ by induction on the ordinals.

(i) $\text{rk}^I(s) = 0$ iff there exists $q \in I$, $p_s \leqslant^0 q$.

(ii) $\text{rk}^I(s) = \alpha > 0$ $\{n: s\char`^\langle n\rangle \in p$ and $\text{rk}^I(s\char`^\langle n\rangle)$ is well defined and less than $\alpha\} \in D$.

(iii) $\text{rk}^I(s) = \infty$ iff there does not exist $\alpha \in \text{ORD}$ such that $\text{rk}^I(s) = \alpha$.

1.4. Claim. *For every $s(p) \subseteq s \in p$ we have that* $\mathrm{rk}^I(s) < \infty$.

Proof. Let $s \in p$ be such that $\mathrm{rk}^I(s) = \infty$, $s(p) \subseteq s$. We define $p^*_s = \{t \in p : t \subseteq s$ or $s \subseteq t$ and for every $k \in [\lg(s), \lg(t))$, $\mathrm{rk}^I(t \restriction k) = \infty\}$. Clearly $p^*_s \subseteq p$, and by definition of rk^I, if $s \subseteq t \in p^*_s$, then $\{n : t\char`\^\langle n \rangle \in p^*_s\} \in D$. Therefore $p^*_s \in P(D)$ and as I is dense open in $P(D)$, there exists p^{**} such that $p^*_s \leqslant p^{**} \in I$. By hypothesis $s(p^{**}) \in p^*$, and this implies that $\mathrm{rk}^I(s(p^{**})) = \infty$, but clearly $\mathrm{rk}^I(s(p^{**})) = 0$. □

1.5. Definition. We say that $A \subseteq p \in P(D)$ is a front of p iff for every $s,t \in A$, $s \not\subseteq t$ and for every branch X of p there exists $k \in \omega$ such that $X \restriction k \in A$.

1.6. Lemma. (a) *If $I \subseteq P(D)$ is a dense open subset $P(D)$ and $p \in P(D)$, then there exists $q \in P(D)$ such that*

(i) $p \leqslant^0 q$,

(ii) $\{s \in q : q_s \in I\}$ *contains a front.*

(b) *If $\{I_n : n < \omega\}$ is a set of dense open subsets of $P(D)$, and $p \in P(D)$, then there exists $q \in P(D)$ such that*

(i) $p \leqslant^0 q$,

(ii) *for every $n \in \omega$, $\{t \in q : q_t \in I_n\}$ contains a front.*

Proof. (a) By induction on $\mathrm{rk}^I(s(p))$.

(b) We will give q^n by induction on n using (a). Let q^0 be such that $p \leqslant^0 q^0$ and $\{s \in q^0 : q^0_s \in I_0\}$ contains a front, say, A_0. Suppose we have defined q^n, A_n satisfying: A_n is a front for q^n and $q^n_s \in I_n$ for every $s \in A_n$. Every member of A_n extends some member of A_{n-1}. $q_n \subseteq q_{n-1}$. For each $s \in A_n$, let $q(s)$, $A(s)$ be such that $q^n_s \leqslant^0 q(s)$, $A(s)$ is a front of q^n_s and for every $t \in A(s)$, $q(s)_t \in I_{n+1}$. Then let $A_{n+1} = \bigcup\{A(s) : s \in A_n\}$ and $q^{n+1} = \{t : (\exists s \in A_n)(t \subseteq s \vee t \in q(s))]$. Then we define $q = \bigcap q_n$. □

1.7. Theorem. *If ϕ is a $P(D)$-sentence and $p \in P(D)$, then there exists $q \in P(D)$, such that $p \leqslant^0 q$ and*

$$q \Vdash \text{“}\phi\text{”} \quad \text{or} \quad q \Vdash \text{“}\neg\phi\text{”}.$$

Proof. Let $I = \{q : q \Vdash \text{“}\phi\text{”}$ or $q \Vdash \text{“}\neg\phi\text{”}\}$. Clearly I is a dense open subset of $\langle P(D), \leqslant \rangle$. We will prove the theorem by induction on $\mathrm{rk}^I(t)$ for $s \subseteq t \in p$. If $\mathrm{rk}^I(t) = 0$, then there exists $q^0 \geqslant p_t$ such that

$$q \Vdash \text{“}\phi\text{”} \quad \text{or} \quad q \Vdash \text{“}\neg\phi\text{”}.$$

If $\mathrm{rk}^I(t) = \alpha$, then $\{n : \mathrm{rk}^I(t\char`\^\langle n \rangle) < \alpha\} = a \in D$ for each $n \in a$. Let $q^0_n \geqslant p_{t\char`\^\langle n \rangle}$ such that

$$q_n \Vdash \text{“}\phi\text{”} \quad \text{or} \quad q_n \Vdash \text{“}\neg\phi\text{”}.$$

Let $a_\phi = \{n \in a : q_n \Vdash \text{“}\phi\text{”}\}$, $a_{\neg\phi} = \{n \in a : \phi_n \Vdash \text{“}\neg\phi\text{”}\}$. Therefore, without loss of generality, $a_\phi \in D$, and we define $q \in P(D)$ by $q = \bigcup_{n \in a_\phi} q_n^0 \geqslant p_t$. Clearly

$$q \Vdash \text{“}\phi\text{”}.$$

This finishes the proof of the theorem. $\square$

1.8. Definition. Let $x \in [\omega]^\omega$. Then $fx \in \omega^\omega$ will be the unique increasing and onto function satisfying $\mathrm{Range}(fx) = x$. Let $x \in [\omega]^\omega$ and $p \in P(D)$, then we define:

(1) x satisfies p iff fx is a branch of p. In this case we write $x \,\mathrm{S}\, p$.

(2) x strongly satisfies p iff for every $n \in \omega$, $(x - n) \,\mathrm{S}\, p$. In this case we write

$x \,\mathrm{ST}\, p$.

(3) x almost strongly satisfies p iff there exists $n \in \omega$ such that $(x - n) \,\mathrm{ST}\, p$. In this case we write $x \,\mathrm{AST}\, p$.

1.9. Fact. *If $G \subseteq P(D)$ is generic and $g \in \omega^\omega$ is defined as in* 1.2(b), *then for every maximal antichain $I \subseteq P(D)$ there exists $p \in I$ such that* $\mathrm{Range}(g) \,\mathrm{S}\, p$.

1.10. Definition. (a) We say that $p \in P(D)$ is *simple* iff there exists $\langle A^p_{n-1} : n < \omega \rangle$ such that for every $s \in p$ the following conditions hold:

(i) If $s = \langle\ \rangle = s(p)$, then $s\hat{\ }\langle m \rangle \in p$ iff $m \in A^p_{-1}$.

(ii) If $s \neq \langle\ \rangle$ and $n = \max(\mathrm{Range}(s))$, then $s\hat{\ }\langle m \rangle \in p$ iff $m \in A^p_n$.

(iii) If $n = \max(\mathrm{Range}(s(p))$, then $A^p_n \supseteq A^p_{n+1} \supseteq A^p_{n+2} \cdots$.

(b) $P'(D) = \{p \in P(D) : p \text{ is simple}\}$.

1.10. Fact. (a) *$P'(D)$ is a dense subset of $P(D)$.*

(b) *If $p \in P'(D)$ and $x \,\mathrm{S}\, p$ and $y \in [x - \mathrm{Range}(s(p))]^\omega$, then* $\mathrm{range}(s(p)) \cup y \,\mathrm{S}\, p$.

(c) *If $\langle\ \rangle = s(p)$ and $x \,\mathrm{S}\, p$, then $x \,\mathrm{ST}\, p$.*

Proof. (a) Without loss of generality, $s(p) = \langle\ \rangle$. By induction on ω we will build $\langle A_{n-1} : n < \omega \rangle$. For every $s \in p$, we define $A_s = \{n \in \omega : s\hat{\ }\langle n \rangle \in p\}$. Set $A_{-1} = A_{\langle\ \rangle}$. Suppose we have defined $A_{-1} \supseteq A_0 \supseteq \cdots \supseteq A_n$. We try to define A_{n+1}: Let $\langle t_k : k < l \rangle$, $l \leqslant (n+1)^{n+1}$ such that $t_k \in p$ and $t_k(|t_k| - 1| = n + 1$, and $\forall m \in \mathrm{Range}(t_k)$ and $m = t_k(h)$: then $m \in A_{t_k(h)-1}$. Now we define $A^*_{n+1} = \bigcap_l A_{t_k}$. If $l \neq 0$, then $A_{n+1} = A_n \cap A^*_{n+1}$ otherwise $A_{n+1} = A_n$. Now let p' be such that $\langle A_{n-1} : n < \omega \rangle$ witness $p' \in P'(D)$. Clearly $p \leqslant p'$. Parts (b) and (c) are easy. $\square$

1.11. Lemma. *If $p \in P(D)$ and $s(p) = \langle\ \rangle$, then*

$$\Vdash_{P(D)} \text{“}\mathrm{Range}(g)\ \mathrm{AST}\ p\text{”}.$$

Proof. Let $p \in P(D)$ be given, clearly there exists $p' \in P'(D)$ such that $p \leqslant^0 p'$. Let $q \in P$ be given, we define $q \cap p' \in P(D)$ by

(i) $s(q \cap p') = s(q)$,

(ii) $s(q) \subseteq t \in q \cap p'$ iff $t \in q$ and the unique $s \in \omega^{<\omega}$ such that $s(q)\hat{\ }s = t$ belongs to p'. Clearly

$$q \cap p' \Vdash \text{“Range(g)} - \text{Range}(s(q)))\ \text{ST}\ p\text{”}$$

and this implies that the set of conditions $q \in P(D)$ such that $q \Vdash$ “Range(g) AST p” is dense in $\langle P(D), \leqslant \rangle$. □

1.12. Lemma. *If $V \subseteq V'$ are models of* ZFC *and $x \in [\omega]^\omega \cap V'$, and D is an ultrafilter in V, then fx is $P(D)$-generic over V iff for every $p \in P(D)$, if $s(p) = \langle\ \rangle$ then x* AST *p.*

Proof. ($\Rightarrow$) is Lemma 1.11.

($\Leftarrow$) For every $n \in \omega$, $w \subset n$, $p \in P(D)$ we define $p^n_w \in P(D)$ by

$$\text{range}(s(p^n_w)) = w \quad \text{and} \quad s(p^n_w) \subseteq t \in p^n_w$$

iff there exists $t' \in p$ such that

$$\text{range}(t') - n = \text{range}(t) - n.$$

Let I be a dense open subset of $P(D)$, $I \in V$. Then we define

$$I^n_w = \{p \in P(D) \colon (\exists q \in I)(q \leqslant p^n_w)\}.$$

Clearly I^n_w is a dense open subset of $P(D)$. Using Lemma 1.6(b) we can find $p \in P(D)$ such that for every $n \in \omega$, $w \subseteq n$, $\{s \in p \colon p_s \in I^n_w\}$ contains a front. By hypothesis, there exists $n \in \omega$ such that

$$x - n\ \text{ST}\ p$$

Therefore there exists $k \in \omega$ such that if $s = f_{x-n} \restriction k$, then

$$p_s \in I^n_{x \cap n}.$$

This implies that there exist $q \in I$ such that

$$q \leqslant (p_s)^n_{x \cap n}$$

and it is not hard to verify that x S q. This concludes the proof of the lemma. □

1.13. Lemma. *In the notation of* 1.12, *fx is $P(D)$-generic over V iff for every $p \in P'(D)$, if $s(p) = \langle\ \rangle$, then there exists $n \in w$ such that*

$$x - n\ \text{S}\ p.$$

Proof. ($\Rightarrow$) Clear from 1.12.

($\Leftarrow$) Let $p \in P(D)$, $s(p) = \langle\ \rangle$, there exists $p' \in P'(D)$ such that $p \leq^0 p'$. Therefore there exists $n \in \omega$ such that

$$x - n \, S \, p'.$$

Therefore, by 1.10(c), $x - n$ ST P' and this implies that x AST p. $\square$

1.14. Theorem. *If $V \subseteq V' \subseteq V''$ are models of ZFC and $D \in V$ is an ultrafilter over ω, and $x \in V'$ is such that fx is $P(D)$-generic over V, then for every $y \in [x]^\omega \cap V''$, fy is $P(D)$-generic over V.*

Proof. By 1.13. $\square$

1.15. Theorem. *Let $D \in V$ be an ultrafilter. Let ϕ be a Σ^1_2-formula with parameters in V, let g be generic over V. Then*

$$V[g] \models (\exists x \in [\omega]^\omega)([x]^\omega \subseteq \{x: \phi(x)\} \vee [x]^\omega \subseteq \{x: \neg\phi(x)\}).$$

Proof. By 1.7, there exists $p \in P(D)$, $s(p) = \langle\ \rangle$ and

$$p \Vdash \text{“}\phi(\text{Range}(g))\text{”} \vee p \Vdash \text{“}\neg\phi(\text{Range}(g))\text{”},$$

without loss of generality $p \Vdash$ “$\phi(\text{Range}(g))$”. Let $p' \geq^0 p$ such that $p' \in P'(D)$, therefore $p' \Vdash$ “$\phi(\text{Range}(g))$”. g is generic over V. Therefore there exists $n \in \omega$ such that $g - n \, S \, p'$, hence p' belongs to the generic filter generated by every $y \in [g - n]^\omega$, and for such y

$$V[y] \models \phi(y),$$

but $V[y] \subseteq V[g]$ and ϕ is Σ^1_2 so

$$V[g] \models \text{“}[g - n]^\omega \subseteq \{y: \phi(y)\}\text{”}. \qquad \square$$

1.16. Theorem. *Let D be an ultrafilter in V. Then $r \in [\omega]^\omega$ is $P(D)$-generic over V iff $(\forall a \in D)$ $(r \subseteq^* a)$ and $(\forall \pi: [\omega]^2 \to 2)$ $(\pi \in V)$ $(\exists n \in \omega)$ $(\text{card}(\pi''[r - n]^2) = 1)$.*

Proof. ($\Rightarrow$) The first part is clear. For the second part, fix $\pi: [\omega]^2 \to 2$. We define $\phi(x)$ by $\phi(x)$ iff $\forall l, k, n, m \in x$

$$\pi\langle l, k\rangle = \pi\langle n, m\rangle.$$

Then clearly $\phi(x)$ is Σ^1_2-formula. Then $\exists n \in \omega$ such that

$$V[r] \models \text{“}[r - n]^\omega \subseteq \{x: \phi(x)\} \vee [r - n]^\omega \subseteq \{x: \neg\phi(x)\}\text{”}.$$

If the first possibility holds, then we are done. If the second possibility holds, take $y \in [r - n]^\omega$ such that $|\pi''[y]| = 1$, then $\phi(y)$, a contradiction.

($\Leftarrow$) It is sufficient to show that for every $p \in P'(D)$, $(\exists n \in \omega)$ $(r - n\, S\, p)$. Fix such a p and let $\langle A_i^p\colon -1 \leq i \leq \omega\rangle$ be witnessing $p \in P'(D)$.

We define $\pi_p\colon [\omega]^2 \to 2$ by

$$\pi_p\langle n, m\rangle = 1 \quad \text{iff} \quad m \in A_n^p.$$

There exists $n \in \omega$ such that $|\pi_p''[r-n]^2| = 1$ because $r \subseteq^* a$ $\forall a \in D$, $\pi_p[r-n]^2 = \{1\}$ and this implies that f_{r-n} is a branch of p'. $\square$

1.17. Definition. r is a *Ramsey real over* V iff $\exists D \in V$, D is an ultrafilter and $(\forall a \in D)(\exists n)(r - n \subseteq a)$ and $\forall \pi\colon [\omega]^2 \to 2$, $\pi \in V$, $(\exists n \in \omega)$ $(|\pi''[r-n]^2| = 1)$. $\square$

1.18. Definition (Silver forcing). If D is an ultrafilter over ω, we define P_D by

$$\langle s, a\rangle \in P_D \quad \text{iff} \quad s \in [\omega]^{<\omega},\ a \in D \text{ and } \sup(s) < \inf(a),$$
$$\langle s, a\rangle \leq \langle t, b\rangle \quad \text{iff} \quad s \subseteq t,\ b \subseteq a \text{ and } t - s \subseteq a.$$

1.19. Theorem (Mathias). *If D is a Ramsey ultrafilter, then $r \in [\omega]^\omega$ defines a generic object to P_D iff for every $a \in D$, there exists $n \in \omega$ such that $r - n \subseteq a$.*

Proof. If $\langle s, a\rangle \in P_D$, let $p_{\langle s,a\rangle} \in P'(D)$ be defined by (without loss of generality, $s = \emptyset$) $\langle A_{n-1}^p\colon n < \omega\rangle$ where

$$A_{-1}^p = a, \qquad A_{n-1}^p = a - n.$$

Then this proves that P_D can be seen as a dense subset of $P(D)$. Then use the fact that r is $P(D)$-generic. $\square$

1.20. Theorem. (i) *If D is Ramsey ultrafilter, then forcing with P_D is equivalent to forcing with $P(D)$.*

(ii) *If there exists $r \in [\omega]^\omega$ such that r is P_D-generic over V iff r is $P(D)$-generic over V, then D is a Ramsey ultrafilter.*

Proof. (i) Clearly by 1.19 and 1.16.

(ii) Suppose there exists $r \in {}^\omega[\omega]$ such that r is P_D and $P(D)$-generic, and $\pi\colon [\omega]^2 \to 2$ belonging to V and such that for every $a \in D$, there exist n_1, n_2, m_1, $m_2 \in a$ such that $\pi\langle n_1, n_2\rangle \neq \pi\langle m_1, m_2\rangle$. It is not hard to show that

$$\Vdash_{P_D} \text{“}(\forall n \in \omega)(\exists n_1 n_2 m_1 m_2 \in r - n)(\pi\langle n_1, n_2\rangle \neq \pi\langle m_1, m_2\rangle)\text{”}.$$

But, using (1.16),

$$\Vdash_{P(D)} \text{“}(\exists n \in \omega)(\pi \restriction [r-n]^2 \text{ is constant})\text{”}$$

and this is a contradiction. $\square$

In a forthcoming work we will prove the following facts about $P(D)$ and P_D.

1.21. Theorem. (a) *If there exists an x which is $P(D)$-generic over V, then there exists a, b, both $P(D)$-generic over V such that $b \notin V[a]$ and $a \in V[b]$.*

(b) *If x is P_D-generic over V, then there exists $a \in [\omega]^\omega \cap V[x]$ such that $x \notin V[a]$.*

(c) *There exists a forcing notion P such that if G is P-generic over V, then:*

(i) $(\forall a \in [\omega]^\omega \cap V[G])(V[a] = V$ *or* $V[a] = V[G])$.

(ii) *There exists $a \in [\omega]^\omega$ such that $(\forall x \in [\omega]^\omega \cap V)(a \subseteq^* x$ or $a \subseteq^* \sim x)$.*

2. On Δ^1_2(Ramsey)

For a real $r \in [\omega]^\omega$ we define $D_r = \{a \in [\omega]^\omega : r \subseteq^* a\}$. Clearly D_r is a filter over ω. Also we can consider $L[D_r]$ the class of all sets which are constructible from D_r. (Definition 0.7 is more general, but all of our results can be translated from L to $L[a]$.) The following facts can be checked by the reader.

2.1. Facts. (i) $D_r \cap L[D_r] \subseteq [\omega]^\omega \cap L_{\omega_1}[D_r] = [\omega]^\omega \cap L[D_r]$.

(ii) $L[D_r] = L[D_r \cap L[D_r]]$, *and we write* $D^r = D_r \cap L[D_r]$.

(iii) $L[D^r] \subseteq L[r]$.

(iv) *If $s \subseteq^* r$ and D^r is an ultrafilter over ω in $L[D_r]$, then $D^s = D^r$.*

(v) *There exists a Σ^1_2-formula $\phi_1(r, x, y)$ such that for every $a,b \in \mathbb{R}$ we have that: $\phi_1(r, a, b)$ iff $a = \langle a_i : i < \omega\rangle$ and for every $i < \omega$, $a_i \leqslant_{L[D^r]} b$ and for every $c \in \mathbb{R}$ if $c \leqslant_{L[D^r]} b$ then there exists $i < \omega$, $c = a_i$, and if $r_1 =^* r_2$ then $\phi_1(r_1, a, b)$ iff $\phi_1(r_2, a, b)$ ($r_1 =^* r_2$ iff $r_1 \subseteq^* r_2$ and $r_2 \subseteq^* r_1$).*

(The proof of these facts is similar to the proof of the corresponding facts for L and references can be found in [6].)

2.2. Lemma. Δ^1_2(Ramsey) *implies that for every $r \in [\omega]^\omega$ there exists $s \in [r]^\omega$ such that for every $\pi : [\omega]^2 \to 2$ if $\pi \in L[D^s]$, then there exists $n \in \omega$ and $s - n$ is homogeneous for π.*

Proof. If this does not hold, we fix such an r and $\langle\ ,\ \rangle : [\omega]^2 \to \omega$ the canonical correspondence between $[\omega]^2$ and ω. Now we define the following function $C : [r]^\omega \to 2$:

$C(s) = 1$ iff there exists $\pi : [\omega]^2 \to 2$ such that:

(i) $\pi \in L[D^s]$ (Σ^1_2).

(ii) For every $\pi' <_{L[D^s]} \pi$, there exists $n \in \omega$, $s - n$ is homogeneous for π' (Σ^1_2).

(iii) For every $n \in \omega$, $s - n$ is not homogeneous for π (Σ^1_2).

(iv) There exists n_1, $m_1 \in s$ such that $\pi(n_1, m_1) = 1$ and for every n_2, $m_2 \in s$ if $\pi(n_2, m_2) = 0$, then $\langle n_1, m_1\rangle < \langle n_2, m_2\rangle$.

The following facts are easily checked:

(1) $\{s : C(s) = 0\}$ is a Σ^1_2-set. ((iv) is an arithmetical relation.)

(2) For every $s \in [r]^\omega$ there exists $n_1, \ldots, n_m \in s$ such that $C(s)=1$ iff $C(s-\{n_1, \ldots, n_m\})=0$.

We define $A_0=\{s: C(s)=0\}$, $A_1=\{s: C(s)=1\}$. Then clearly A_0 is a Δ^1_2-set and A_0 is not Ramsey. $\square$

2.3. Corollary. Δ^1_2(Ramsey) *implies that for every* $r \in [\omega]^\omega$ *there exists* $s \in [r]^\omega$ *such that*

$$L[D^s] \models \text{“}D^s \textit{ is an ultrafilter over } \omega\text{”}.$$

2.4. Corollary. Δ^1_2(Ramsey) *implies that for every* $r \in \mathbb{R}$, *there exists* $f \in {}^\omega\omega$, *such that* f *is a* σ*-bound to* $\omega^\omega \cap L[r]$. $\square$

2.5. Definition. If $r \in [\omega]^\omega$ and D_r is an ultrafilter over ω in $L[D^r]$, then without loss of generality we can denote $P(D^r)$ as P_r, and if $s \subseteq^* r$, then $P_s = P_r$.

2.6. Theorem. Δ^1_2(Ramsey) *implies that for every* $r \in [\omega]^\omega$ *there exists* $s \in [r]^\omega$ *such that* s *is* P_s*-generic over* $L[D^s]$.

Proof. By 2.2 and 1.16. $\square$

2.7. Theorem. Σ^1_2(Ramsey) *iff* Δ^1_2(Ramsey).

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose Δ^1_2(Ramsey). Let $\phi(x)$ be a Σ^1_2-formula. Without loss of generality, the parameters of $\phi(x)$ belong to L. Let $s \in [\omega]^\omega$ be such that:

(i) D^s is an ultrafilter over ω in $L[D^s]$.

(ii) s is P_s-generic over $L[D^s]$.

In $L[D^s]$ there exists $p \in P'_s$, $s(p)=\langle\ \rangle$ and

$$p \Vdash_{P_s} \text{“}\phi(\mathbf{r}_G)\text{”} \quad \text{or} \quad p \Vdash_{P_s} \text{“}\neg\phi(\mathbf{r}_G)\text{”}.$$

As s is P_s-generic over $L[D^s]$, we know that there exists $n \in \omega$ such that if $t \in [s-n]^\omega$, then

$$t\, \mathrm{S}\, p$$

and as we know that t is P_s-generic over $L[D^s]$, we have that for every $t \in [s-n]^\omega$

$$L[D^s][t] \models \phi(t) \quad \text{or} \quad L[D^s][t] \models \neg\phi(t).$$

And this implies that $\{x: \phi(x)\}$ is Ramsey. $\square$

2.8. Theorem. Δ^1_2(Ramsey) *iff* $\{r \in [\omega]^\omega: r$ *is Ramsey real over* $L[D^r]\}$ *is an open dense set in the topology introduced in* [1] *to* $[\omega]^\omega$.

Proof. By 2.2 and 1.17. $\square$

2.9. Definition. We say $\Delta^1_2(r_0)$(Ramsey) iff every Δ^1_2 set with the parameters of its definition in $L[r_0]$ is Ramsey. Similarly we define $\Sigma^1_2(r_0)$(Ramsey).

2.10. Theorem. $\Delta^1_2(r_0)$(Ramsey) *iff* $\Sigma^1_2(r_0)$(Ramsey).

Proof. The set given in Proof 2.2 is $\Delta^1_2(r_0)$. □

By this last fact we have the following characterization.

2.11. Theorem. $\Delta^1_2(r_0)$(Ramsey) *iff for every* $r \in L[r_0]$ *there exists* $s \in [r]^\omega$ *such that* s *is* P_s*-generic over* $L[r_0][D^s]$. □

3. Δ^1_2(Lebesgue), Δ^1_2(Baire), Δ^1_2(K_σ-regular), Δ^1_2(Ramsey)

3.1. Theorem. (i) $\Delta^1_2(r_0)$(Lebesgue) *iff* $(\exists r \in \mathbb{R})$ *(r is random real over* $L[r_0]$*).*

(ii) Δ^1_2(Lebesgue) *iff* $(\forall r_0 \in \mathbb{R})$ $(\exists r \in \mathbb{R})$ *(r is random real over* $L[r_0]$*).*

(iii) $\Delta^1_2(r_0)$(Baire) *iff* $(\exists r \in \mathbb{R})$ *(r is Cohen real over* $L[r_0]$*).*

(iv) Δ^1_2(Baire) *iff* $(\forall r_0 \in \mathbb{R})(\exists r \in \mathbb{R})$ *(r is Cohen real over* $L[r_0]$*).*

Proof. We will prove only (iii).

($\Rightarrow$) Suppose that $V \models$ "$\Delta^1_2(r_0)$(Baire)". If there is no Cohen real over $L[r_0]$, let $\langle B_\alpha : \alpha < \omega_1 \rangle$ be a $\Sigma^1_2(r_0)$-good well order of the Borel meager set coded in $L[r_0]$. Therefore $\mathbb{R} = \bigcup_\alpha B_\alpha$. We define the following order on the members of $\mathbb{R}$:

$$x < y \quad \text{iff} \quad (\exists \alpha)\Big(x \in B_\alpha \wedge y \notin \bigcup_{\beta \leq \alpha} B_\beta\Big),$$

$$x \leq y \quad \text{iff} \quad (\exists \alpha)\Big(x \in B_\alpha \wedge y \notin \bigcup_{\beta < \alpha} B_\beta\Big).$$

Then we define the following set $A \subseteq \mathbb{R}^2$, $A = \{\langle x, y \rangle : x < y\}$. Then clearly A is a $\Sigma^1_2(r_0)$ set of pairs of reals and using the Kuratowski–Ulam theorem we have that if A has the property of Baire, then A is meager. But $\sim A = \{\langle x, y \rangle : y \leq x\}$ also is a $\Sigma^1_2(r_0)$-set of pairs of reals. Therefore A, as well as $\sim A$, is a $\Delta^1_2(r_0)$-set of reals and this implies that A is meager. But

$$\mathbb{R} \times \mathbb{R} = A \cup \sim A$$

and this is a contradiction.

($\Leftarrow$) Let $\phi_1(x)$, $\phi_2(x)$ with parameters in $L[r_0]$ be Σ^1_2-formulas. Let P be the Cohen real forcing and in $L[r_0]$ let $B_i \subseteq P$ be a maximal antichain satisfying

$$s \in B_i \Rightarrow s \Vdash_P \text{"}\phi(\mathrm{r})\text{"}.$$

Claim 1. *If* $V \models (\forall x)(\phi_1(x) \Leftrightarrow \neg\phi_2(X))$, *then* $B_1 \cap B_2 = \emptyset$ *and* $B_1 \cup B_2$ *is a maximal antichain of* P.

Proof. Suppose that $s \in B_1 \cap B_2$. Then

$$s \Vdash_P \text{“}\phi_1(\mathbf{r}) \wedge \phi_2(r)\text{”}.$$

Pick in V a Cohen real r over $L[r_0]$, such that $s \subseteq r$. Then

$$L[r_0][r] \models \phi_1(r) \wedge \varphi_2(r).$$

By the Schoenfield absoluteness lemma,

$$V \models \phi_1(r) \wedge \phi_2(r)$$

a contradiction.

The second assertion is clear using the fact that the Cohen reals over $L[r_0]$ are dense in $\mathbb{R}$.

Now working in $L[r_0]$, we pick $N \prec H(\aleph_1, \in, \leqslant)$ such that $\|N\| = \aleph_0$, $B_i \in N$, $\phi_i \in N$, $i = 1,2$.

Claim 2. *If $s \in B_i$ and $s \subseteq r$ is a Cohen real over N, then $V \models$ “$\phi_i(r)$”.*

Clearly on proving the claim we have finished.

Proof of the Claim. As $N \prec H(\aleph_1, \in, \leqslant)$, in N we have that

$$s \Vdash_{\text{Cohen}} \text{“}\phi_i(\mathbf{r}_G)\text{”}.$$

Thus $N[r] \models$ “$\phi_i(r)$” but Σ^1_2-formulas are up-absolute, therefore $V \models$ “$\phi_i(r)$”. □

The following theorem was essentially proved in [3]:

3.2. Theorem. *The following assertions are equivalent*:

(i) $\Sigma^1_2(r_0)$(K_σ-regular),

(ii) $\Delta^1_2(r_0)$(K_σ-regular),

(iii) $\Pi^1_1(r_0)$(K_σ-regular),

(iv) *$L[r_0] \cap {}^\omega\omega$ is σ-bounded.*

Proof. (i)$\rightarrow$(ii)$\rightarrow$(iii) are clear. (iv)$\rightarrow$(i), see [3].

(iii)$\rightarrow$(iv). If $L[r_0] \cap {}^\omega\omega$ is not σ-bounded, then in [2] it was proved that there exists a non-K_σ-regular Π^1_1-set of reals. □

3.3. Theorem. (i) $\Delta^1_2(r_0)$(Lebesgue)$\Rightarrow L[r_0] \cap \mathbb{R}$ *is meager.*

(ii) $\Delta^1_2(r_0)$(Baire)$\Rightarrow L[r_0] \cap \mathbb{R}$ *has measure zero*

(iii) $\Delta^1_2(r_0)$(K_σ-regular)$\Rightarrow L[r_0] \cap \mathbb{R}$ *is meager.*

(iv) $\Delta^1_2(r_0)$(Ramsey)$\Rightarrow L[r_0] \cap \mathbb{R}$ *is a meager-measure zero set.*

Proof. (iv), see [3].

3.4. Theorem. $\Delta^1_2(r_0)$(Ramsey)$\Rightarrow \Sigma^1_2(r_0)$($K_\sigma$-regular).

Proof. It is sufficient to show that

$$\Sigma^1_2(r_0)(\text{Ramsey}) \quad \Rightarrow \quad L[r_0] \cap {}^\omega\omega \text{ is } \sigma\text{-bounded},$$

but this was proved in [3]. $\square$

This is the sole implication between these properties. More information can be found in [3].

Now we describe pure models for each of these properties.

3.5. Theorem. *If* ZFC *is consistent, then:*

(i) *There exists a model V such that*

$$V \models \text{“}\Delta^1_2(\text{Lebesgue}) + \neg\Delta^1_2(\text{Baire}) + \neg\Delta^1_2(\text{Ramsey}) + \neg\Delta^1_2(K_\sigma\text{-regular})\text{”}.$$

(ii) *There exists a model V such that*

$$V \models \text{“}\Delta^1_2(\text{Baire}) + \neg\Delta^1_2(\text{Lebesgue}) + \neg\Delta^1_2(\text{Ramsey}) + \neg\Delta^1_2(K_\sigma\text{-regular})\text{”}.$$

(iii) *There exists a model V such that*

$$B \models \text{“}\Delta^1_2(K_\sigma\text{-regular}) + \neg\Delta^1_2(\text{Lebesgue}) + \neg\Delta^1_2(\text{Baire}) + \neg\Delta^1_2(\text{Ramsey})\text{”}.$$

(iv) *There exists a model V such that*

$$V \models \text{“}\Delta^1_2(\text{Ramsey}) + \neg\Delta^1_2(\text{Lebesgue}) + \neg\Delta^1_2(\text{Baire})\text{”}.$$

Proof. (i) Force with a product of κ-many random reals, $\kappa \geqslant \aleph_1$. Here every new function from ω to ω is bounded by an old function.

(ii) Force κ-many Cohen reals, $\kappa \geqslant \aleph_1$.

(iii) Force, from L, ω_1-Laver's reals with countable support. By [5] in this model,

$$L \cap \mathbb{R} \text{ has outer measure one.}$$

This says that $\neg\Delta^1_2(\text{Ramsey})$ holds. It is well known that in this model no real is random or Cohen over L. This model answers the question of [2]:

$$\Sigma^1_2(K_\sigma\text{-regular}) \quad \Rightarrow \quad L \cap \mathbb{R} \text{ has measure zero.}$$

(iv) Force ω_1-Mathias reals with countable support. $\square$

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