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ScienceDirect
Annals of Pure and Applied Logic 149 (2007) 7–13
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

Increasing the groupwise density number by c.c.c. forcing

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Received 31 March 2004; received in revised form 6 July 2005; accepted 11 January 2007

Communicated by I. Moerdijk

This work is dedicated to James Baumgartner on the occasion of his 60th birthday.

Abstract

We show that $\aleph_2 \leq \mathfrak{b} < \mathfrak{g}$ is consistent.

Keywords: Cardinal characteristics; Forcing

0. Introduction

We show that for every regular cardinal with a definition in the ground model, the statement $\kappa = \mathfrak{b} < \mathfrak{b}^+ = \mathfrak{g}$ is consistent. In particular this holds for $\kappa = \aleph_2$. This answers a question of Andreas Blass.

We recall the definitions of the three cardinal characteristics $\mathfrak{b}$, $\mathfrak{g}$, $\mathfrak{u}$. The set of functions from ω to ω is written as ${}^\omega\omega$. For $f, g \in {}^\omega\omega$, we say g dominates f and write $f \leq^* g$ iff for all but finitely many n, $f(n) \leq g(n)$. A family $B \subseteq {}^\omega\omega$ is unbounded iff for every $g \in {}^\omega\omega$ there is some $f \in B$ such that $f \not\leq^* g$. The bounding number $\mathfrak{b}$ is the smallest cardinal of an unbounded family $B \subseteq {}^\omega\omega$.

For $X, Y \in [\omega]^\omega$ we write $Y \subseteq^* X$ to denote that $Y \setminus X$ is finite. A subset $\mathscr{G}$ of $[\omega]^\omega$ is called groupwise dense if $(\forall X \in \mathscr{G})(\forall Y \subseteq^* X)(Y \in \mathscr{G})$ and for every partition $\{[\pi_i, \pi_{i+1}) : i < \omega\}$ of ω into finite intervals there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) : i \in A\} \in \mathscr{G}$. The groupwise density number, $\mathfrak{g}$, is the smallest number of groupwise dense families with empty intersection.

By an ultrafilter we mean a non-principal ultrafilter on ω. Such an ultrafilter is called a P-point if for any $A_i \in \mathscr{U}$, $i < \omega$, there is an $A \in \mathscr{U}$, such that $A \subseteq^* A_i$ for $i < \omega$. Such an A is called a pseudointersection of A_i, $i < \omega$. An ultrafilter is called a Q-point if, given a strictly increasing sequence π_i, $i < \omega$, of natural numbers, there is some $A \in \mathscr{U}$ that for all $i < \omega$, $|A \cap [\pi_i, \pi_{i+1})| \leq 1$. For an ultrafilter $\mathscr{U}$ the cardinal

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0168-0072/$ - see front matter
doi:10.1016/j.apal.2007.07.001

$\chi(\mathscr{U}) = \min\{|\mathscr{B}| : \mathscr{B} \subseteq \mathscr{U} \wedge (\forall X \in \mathscr{U})(\exists Y \in \mathscr{B})(Y \subseteq X)\}$ is called the character of $\mathscr{U}$. The cardinal $\mathfrak{u}$, the ultrafilter characteristic, is defined as the minimal $\chi(\mathscr{U})$ for all non-principal ultrafilters $\mathscr{U}$ on ω.

The bounding number $\mathfrak{b}$ and groupwise density number $\mathfrak{g}$ can be in either order. For a regular $\kappa > \aleph_1$, we get the constellation $\aleph_1 = \mathfrak{g} < \mathfrak{b} = \kappa$ for example after adding uncountably (—their number does not matter, the continuum can be larger than κ—) many random reals over a model of MA and $2^\omega = \kappa$ [4] or in a finite support iteration of Hechler forcings of length κ [13].

Also $\aleph_1 < \mathfrak{g} < \mathfrak{b}$ is consistent. We sketch a proof given by the referee. Let $\kappa < \lambda$ be regular uncountable and assume CH. We take a finite support iteration $\langle \mathbb{P}_\beta, \tilde{\mathbb{Q}}_\alpha : \alpha < \lambda, \beta \le \lambda \rangle$ of length λ adding Hechler generics in the odd steps and going through all c.c.c. partial orders of size $< \kappa$ in the even steps. Then $\mathfrak{b} = 2^\omega = \lambda$ and book-keeping gives $\mathrm{MA}_{<\kappa}$, so that $\mathfrak{g} \ge \kappa$. The proof of $\mathfrak{g} \le \kappa$ is a standard modification of the argument for $\mathfrak{g} = \aleph_1$ in the Hechler model.

Recall the latter argument: if all iterands are Hechler forcing, then since Hechler forcing is Suslin, absoluteness gives us that $\mathbb{P}_A$ is completely embedded into $\mathbb{P}_\lambda$ for every $A \subseteq \lambda$, where $\mathbb{P}_A$ is defined as $\mathbb{P}_\lambda$ considering only coordinates from A and ignoring the others. Furthermore, when $\mathcal{A}$ is a directed family of subsets of λ such that for all countable subsets B of λ there is some $A \in \mathcal{A}$ with $B \subseteq A$, then $\mathbb{P}_\lambda$ is the direct limit of $\mathbb{P}_A$, $A \in \mathcal{A}$. This is so because the conditions in Hechler forcing are reals and hence arise in countable fragments of the iteration.

Now let $\mathcal{A}$ be a strictly increasing ω_1-chain of subsets of λ with $\bigcup \mathcal{A} = \lambda$. Then $V[G] \cap {}^\omega\omega = \bigcup_{A \in \mathcal{A}} V[G \cap \mathbb{P}_A] \cap {}^\omega\omega$, i.e., the reals arise in an ω_1-chain of intermediate models. By a standard argument, see [12,4], this yields $\mathfrak{g} \le \aleph_1$.

Now return to the above situation: Say $A \subseteq \lambda$ is closed if for all even $\alpha \in A$, $\mathrm{supp}(\tilde{\mathbb{Q}}_\alpha) \subseteq A$, where $\mathrm{supp}(\tilde{\mathbb{Q}}_\alpha)$ is the union of the supports of the conditions determining what the order $\tilde{\mathbb{Q}}_\alpha$ is. By the countable chain condition and since the supports of the conditions are finite, $|\mathrm{supp}(\tilde{\mathbb{Q}}_\alpha)| < \kappa$ for all even α. Then for each $B \subseteq \lambda$ of size $< \kappa$ there is some closed $A \supseteq B$ of size $< \kappa$. If A is closed then $\mathbb{P}_A$ is completely embedded into $\mathbb{P}_\lambda$. Furthermore, when $\mathcal{A}$ is a directed family of closed subsets of λ such that for all $B \subseteq \lambda$ of size $< \kappa$ there is some $A \in \mathcal{A}$ with $B \subseteq A$, then $\mathbb{P}_\lambda$ is the direct limit of the $\mathbb{P}_A$, $A \in \mathcal{A}$. Now there is a strictly increasing κ-chain $\mathcal{A}$ of closed subsets of λ with $\bigcup \mathcal{A} = \lambda$. Again we get $V[G] \cap {}^\omega\omega = \bigcup_{A \in \mathcal{A}} V[G \cap \mathbb{P}_A] \cap {}^\omega\omega$ and $\mathfrak{g} \le \kappa$.

In all models so far known of the reverse inequality $\mathfrak{b} < \mathfrak{g}$ we have had $\aleph_1 = \mathfrak{b} < \mathfrak{g} = 2^\omega = \aleph_2$. The models given by a countable support iteration of Blass–Shelah, Miller or Matet forcing over a ground model satisfying CH fulfil even $\aleph_1 = \mathfrak{u} < \mathfrak{g} = 2^\omega = \aleph_2$. Since $\mathfrak{b} \le \mathfrak{u}$ [11], the latter is stronger than $\mathfrak{b} < \mathfrak{g}$. For the constellation $\mathfrak{b} < \mathfrak{g} \le \mathfrak{u}$ one can for example interweave random reals at the odd steps of a countable support iteration of Miller forcings, see [2, Model 7.5.5].

The main part of this work is to show that the inequality $\mathfrak{b} < \mathfrak{b}^+ = \mathfrak{g}$ can hold above $\aleph_2$. There is nothing special about $\aleph_2$; any regular cardinal that is definable without parameters can serve. Our construction yields $\aleph_2 = \mathfrak{b} < \mathfrak{g} = \mathfrak{u} = 2^\omega = \aleph_3$ and it is open how to keep $\mathfrak{u}$ small. Moreover, our construction does not allow to push $\mathfrak{g}$ strictly above $\mathfrak{b}^+$. In the last section of this work we show that $\mathfrak{g} \le \mathfrak{d}_\mathfrak{b}$, and this is possibly a partial explanation for the obstacles in getting $\mathfrak{g} > \mathfrak{b}^+$.

The main part of this paper will be the proof of

Theorem 0.1. $\aleph_2 \le \mathfrak{b} < \mathfrak{g}$ *is consistent relative to ZFC.*

Here is an outline: In Section 1 we state and prove some properties of Matet forcing with stable ordered-union ultrafilters and prove a key lemma. In Section 2 we finish the proof of Theorem 0.1. In Section 3 we show $\mathfrak{g} \le \mathfrak{d}_\mathfrak{b}$.

1. A variant of Matet forcing

We shall define a variant of Matet forcing. For this purpose, we first introduce some notation about ordered-union ultrafilters. Our nomenclature follows Blass [3] and Eisworth [8].

We let $\mathbb{F}$ be the collection of all finite subsets of ω. For $a, b \in \mathbb{F}$ we write $a < b$ if $(\forall n \in a)(\forall m \in b)(n < m)$. We shall work with filters on $\mathbb{F}$, i.e. subsets of $\mathscr{P}(\mathbb{F})$ that are closed under intersections and supersets. A sequence $\bar{a} = \langle a_n : n \in \omega \rangle$ of members of $\mathbb{F}$ is called unmeshed if for all n, $a_n < a_{n+1}$. The set $(\mathbb{F})^\omega$ denotes the collection of all infinite unmeshed sequences in $\mathbb{F}$. If X is a subset of $\mathbb{F}$, we write $FU(X)$ for the set of all finite unions of members of X. We write $FU(\bar{a})$ instead of $FU(\{a_n : n \in \omega\})$. We let $\mathbb{P} \lessdot \mathbb{Q}$ denote that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$.

Definition 1.1. Given $\bar{a}$ and $\bar{b}$ in $(\mathbb{F})^\omega$, we say that $\bar{b}$ is a condensation of $\bar{a}$ and we write $\bar{b} \sqsubseteq \bar{a}$ if $\bar{b} \subseteq FU(\bar{a})$. We say $\bar{b}$ is almost a condensation of $\bar{a}$ and we write $\bar{b} \sqsubseteq^* \bar{a}$ iff there is an n such that $\langle b_t : t \geq n \rangle$ is a condensation of $\bar{a}$.

Definition 1.2. In the Matet forcing, $\mathbb{M}$, the conditions are pairs $(a, \bar{c})$ such that $a \in \mathbb{F}$ and $\bar{c} \in (\mathbb{F})^\omega$ and $a < c_0$. The forcing order is $(b, \bar{d}) \leq (a, \bar{c})$ (the stronger condition is the smaller one) iff $a \subseteq b$ and $b \setminus a$ is a union of finitely many of the c_n and $\bar{d}$ is a condensation of $\bar{c}$.

Definition 1.3. A filter $\mathscr{F}$ on $\mathbb{F}$ is said to be an ordered-union filter if it has a basis of sets of the form $FU(\bar{d})$ for $\bar{d} \in (\mathbb{F})^\omega$. An ordered-union filter is said to be stable if, whenever it contains $FU(\bar{d}_n)$ for $\bar{d}_n \in (\mathbb{F})^\omega$, $n < \omega$, then it also contains some $FU(\bar{e})$ for some $\bar{e}$ that is almost a condensation of each $\bar{d}_n$.

Ordered-union ultrafilters need not exist, as their existence implies the existence of Q-points [3] and there are models without Q-points [10]. Under MA(σ-centred) stable (even $< 2^\omega$-stable) ordered-union ultrafilters exist [3].

It is well known [9,4] that the forcing $\mathbb{M}$ can be decomposed into two steps $\mathbb{P} * \mathbb{M}(\underset{\sim}{\mathscr{U}})$, such that $\mathbb{P}$ is ω_1-closed (that is, every descending sequence of conditions of countable length has a lower bound) and adds a stable ordered-union ultrafilter $\mathscr{U}$ on the set $\mathbb{F}$, and that $\mathbb{M}(\mathscr{U})$ is the Matet forcing with sequences from the ultrafilter (and hence it is σ-centred).

Definition 1.4. Given a $\sqsubseteq^*$-descending sequence $\bar{a}^\alpha$, $\alpha < \beta$, the notion of forcing $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$ consists of all pairs $(s, \bar{a})$, such that $s \in \mathbb{F}$ and $\bar{a}$ is an end segment of one of the $\bar{a}^\alpha$'s and $s < \min(a_0)$. The forcing order is the same as in the Matet forcing.

We shall use $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$ for $\sqsubseteq^*$-descending sequences of length 1, of length $< \kappa$ and of length κ. The forcing $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$ diagonalises (“shoots a real through”) $\bigcup\{a_n^\alpha : n < \omega\}$, $\alpha < \beta$.

Note that for a $\sqsubseteq^*$-descending sequence with a last element, $\mathbb{M}(\bar{a}^\alpha : \alpha \leq \beta)$ is equivalent to $\mathbb{M}(\bar{a}^\beta)$ and this is in turn equivalent to Cohen forcing. However, $\mathbb{M}(\bar{a}^\gamma)$ is not a complete suborder of $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$.

We shall show that given a set of κ groupwise dense families, there are $\bar{a}^\alpha$, $\alpha < \kappa$, such that $\mathbb{M}(\bar{a}^\alpha : \alpha < \kappa)$ adds a real through all the families. This is similar to the fact shown by Blass [4], that the original Matet forcing $\mathbb{M}$ adds a real that lies in all groupwise dense families from the ground model. By unpublished results of Blass and Laflamme [4], Matet forcing preserves P-points and hence, by the iteration theorem for preserving P-points [7], it preserves $\mathfrak{u}$. However, our finite support iteration of iterands of the form $\mathbb{M}(\bar{a}^\alpha : \alpha < \kappa)$ and other iterands will not preserve $\mathfrak{u}$, as the iteration adds Cohen reals in limit steps and also at some successor steps that force a part of $\mathrm{MA}_{<\kappa}$. We shall only keep $\mathfrak{b}$ small.

We write names for reals in c.c.c. forcings $\mathbb{P}$ in a standardised form $\underset{\sim}{g} = \mathrm{Name}(\bar{k}, \bar{p}) = \{\langle (n, k_{n,m}), p_{n,m} \rangle : n, m \in \omega\}$, such that $\{p_{n,m} : m \in \omega\}$ is predense in $\mathbb{P}$ and $p_{n,m} \Vdash_{\mathbb{P}} \underset{\sim}{g}(n) = k_{n,m}$ and such that $k_{n,m} = k_{n,m'}$ if $p_{n,m}$ and $p_{n,m'}$ are compatible.

Lemma 1.5. *Let $\bar{a}^\alpha$, $\alpha < \delta$, be a $\sqsubseteq^*$-descending sequence. Assume $\mathbb{Q} = \mathbb{M}(\bar{a}^\alpha : \alpha < \delta)$ and $\mathrm{cf}(\delta) > \aleph_0$ and $\underset{\sim}{g}$ is a $\mathbb{Q}$-name for a member of ${}^\omega\omega$. Then we can find an $\alpha_0 < \delta$ such that for every $\alpha \in [\alpha_0, \delta)$ there are $p_{n,m} \in \mathbb{M}(\bar{a}^{\alpha})$ and $k_{n,m} \in \omega$ such that $\{p_{n,m} : m < \omega\}$ is predense in $\mathbb{Q}$ and $p_{n,m} \Vdash_{\mathbb{Q}} \underset{\sim}{g}(n) = k_{n,m}$.*

Proof. We assume that $\underset{\sim}{g} = \{\langle (n, h_{n,m}), q_{n,m} \rangle : m, n < \omega\}$. Since $\mathrm{cf}(\delta) > \omega$, there is some $\alpha_0 < \kappa$ such that all $q_{n,m}$ are in $\mathbb{M}(\bar{a}^\beta : \beta \leq \alpha)$. Now, given $\alpha \in [\alpha_0, \delta)$, we take

$$I_n = \{q \in \mathbb{M}(\bar{a}^\alpha) : (\exists m)(q \leq_{\mathbb{Q}} q_{n,m})\}.$$

Then I_n is predense in $\mathbb{Q}$. Now let $p_{n,m}$, $m < \omega$, list I_n and choose $k_{n,m}$ such that $p_{n,m} \Vdash_{\mathbb{Q}} \underset{\sim}{g}(n) = k_{n,m}$. Then $\bar{k}$, $\bar{p}$ describe $\underset{\sim}{g}$ as desired. $\square$

The following lemma will be used in those successor steps of our planned iterated forcing in which we want to add an infinite set that is in κ groupwise dense sets at the same time.

Lemma 1.6. *Assume that κ is a regular uncountable cardinal, $2^\omega = \kappa$, $\mathrm{MA}_{<\kappa}$(σ-centred), $\{\mathscr{G}_\alpha : \alpha < \kappa\}$ is a set of groupwise dense subsets and that $\bar{f} = \langle f_\alpha : \alpha < \kappa \rangle$ is a $\leq^*$-increasing and -unbounded sequence of functions in ${}^\omega\omega$. Then there is a σ-centred forcing notion $\mathbb{Q}$ of size κ such that*

$$\Vdash_{\mathbb{Q}} \text{“}\bar{f} \text{ is unbounded } \wedge \exists X \in [\omega]^\omega \bigwedge_{\alpha < \kappa} X \in \mathscr{G}_\alpha\text{”}.$$

Proof. We shall build $\mathbb{Q} = \mathbb{M}(\bar{a}^\alpha : \alpha < \kappa)$ by choosing $\bar{a}^\alpha \in (\mathbb{F})^\omega$ by induction on $\alpha < \kappa$ such that $\bar{a}^\beta \sqsubseteq^* \bar{a}^\alpha$ for $\alpha < \beta$. Since $\mathrm{cf}(\kappa) > \omega$, each $\mathbb{Q}$-name for a real has an equivalent $\mathbb{M}(\bar{a}^\beta)$-name for all sufficiently large β. We shall show that we can choose $\mathbb{Q}$ carefully, with a sealing argument, such that in the end there will be no name for a new function dominating all the $f_\alpha, \alpha < \kappa$.

Now we carry out the construction. Let $\langle \bar{b}^\alpha, \underset{\sim}{g}^\alpha : \alpha < \kappa\rangle$ list the pairs $(\bar{b}, \underset{\sim}{g})$ such that $\bar{b} \in (\mathbb{F})^\omega$ and $\underset{\sim}{g} = \{\langle (n, k_{n,m}), p_{n,m}\rangle : m, n \in \omega\}$ is an $\mathbb{M}(\bar{b})$-name for a function in ${}^\omega\omega$ such that each pair $(\bar{b}, \underset{\sim}{g})$ appears κ many times.

Now we shall choose by induction on $\alpha < \kappa$ some $\bar{a}^\alpha \in (\mathbb{F})^\omega$ with the following properties:

(a) If $\beta < \alpha$ then $\bar{a}^\alpha \sqsubseteq^* \bar{a}^\beta$.
(b) If $\alpha = 2\beta + 1$, then $\bigcup_{n<\omega} a_n^\alpha \in \mathscr{G}_\beta$.
(c) If $\alpha = 2\beta + 2$ and for some $\gamma < 2\beta + 2$, $\bar{b}^\beta = \bar{a}^\gamma$ and $\underset{\sim}{g}^\beta$ is a $\mathbb{M}(\bar{b}^\beta)$-name of a member of ${}^\omega\omega$ that can be construed as an $\mathbb{M}(\bar{a}^{2\beta+1})$-name, then $\bar{a}^\alpha$ guarantees that for some $\zeta_\alpha < \kappa$,

$$\Vdash_{\mathbb{Q}} \underset{\sim}{g}^\beta \not\geq^* f_{\zeta_\alpha}.$$

For $\alpha = 0$ we let $\bar{a}^0 = \langle \{n\} : n < \omega\rangle$.

Let $\alpha < \kappa$ be a limit ordinal. We apply $\mathrm{MA}_{<\kappa}(\sigma\text{-centred})$ to the σ-centred forcing notion $\{(\bar{a}, n, F) : \bar{a}$ is a finite unmeshed sequence of subsets of n and F is a finite subset of $\alpha\}$, ordered by $(\bar{b}, n', F') \le (\bar{a}, n, F)$ iff $n' \ge n$, $F' \supseteq F$, and $\bar{b} = \bar{a}\hat{\ }\bar{c}$ with $c_i \cap n = \emptyset$ and $(\forall \gamma \in F)(\forall k)(b_k \subseteq [n, n') \to b_k \in FU(\bar{a}^\gamma))$, and the dense sets $\mathscr{I}_{\beta,n} = \{(\bar{a}, m, F) : \bigcup \bar{a} \setminus n \ne \emptyset \wedge \beta \in F \wedge m \ge n\}$, $\beta < \alpha, n < \omega$, and thus we get a filter G intersecting all the $\mathscr{I}_{\beta,n}$ and set $\bar{a}^\alpha = \bigcup\{\bar{a} : (\exists n, F)((\bar{a}, n, F) \in G)\}$. Then $\bar{a}^\alpha$ is as desired.

Step $\alpha = 2\beta + 1$. We show that, given $\mathscr{G}_\beta$ and $\bar{a}^{2\beta}$, there is some condensation $\bar{a}^{2\beta+1} \sqsubseteq^* \bar{a}^{2\beta}$ such that $\bigcup_n a_n^{2\beta+1} \in \mathscr{G}_\beta$: We apply the definition of groupwise density to the partition $\{[\min(a_n^{2\beta}), \min(a_{n+1}^{2\beta})) : n < \omega\}$ and get an infinite set I such that $\bigcup\{[\min(a_i^{2\beta}), \min(a_{i+1}^{2\beta})) : i \in I\} \in \mathscr{G}_\beta$. Then also $\bigcup\{a_i^{2\beta} : i \in I\} \in \mathscr{G}_\beta$. Then we re-index the sequence $\langle a_i^{2\beta} : i \in I\rangle$ by the natural numbers, so $a_n^{2\beta+1} = a_{i_n}^{2\beta}$ for the increasing enumeration $\langle i_n : n < \omega\rangle$ of I.

Step $\alpha = 2\beta + 2$. We assume that for some $\gamma < 2\beta + 2$, $\bar{b}^\beta = \bar{a}^\gamma$ and $\underset{\sim}{g}^\beta$ is a $\mathbb{M}(\bar{b}^\beta)$-name of a member of ${}^\omega\omega$ that has an equivalent $\mathbb{M}(\bar{a}^{2\beta+1})$-name. Otherwise we can take $\bar{a}^{2\beta+2} = \bar{a}^{2\beta+1}$.

For each $n < \omega$ we choose a finite set $a_n^{\alpha+}$ such that $a_n^{2\beta+1}$ is an initial segment of $a_n^{\alpha+}$ and there is some $u_n \subseteq \{n, n+1, \dots, \ell_n - 1\}$ such that $n \in u_n$ and

$$a_n^{\alpha+} = \bigcup\{a_\ell^{2\beta+1} : \ell \in u_n\}$$

and such that for every $w \subseteq \{0, 1, \dots, \min(a_n^{2\beta+1}) - 1\}$ there is some $m_n^\beta(w)$ such that

$$p^\beta_{n, m_n^\beta(w)} \ge (w \cup a_n^{\alpha+}, \bar{a}^{2\beta+1} \restriction [\ell_n, \omega)).$$

Since there are only finitely many $w \subseteq \min a_n^{2\beta+1}$, there is such an $a_n^{\alpha+}$.

Now in order to be able to concatenate the $a_n^{\alpha+}$ and in order to ensure that $\underset{\sim}{g}^\beta$ will not be a dominating function we thin out: Let $k(w, n)$ be one $k^\beta_{n, m_n^\beta(w)}$ that is in $\underset{\sim}{g}^\beta$ together with $p^\beta_{n, m_n^\beta(w)} \ge (w \cup a_n^{\alpha+}, \bar{a}^{2\beta+1} \restriction [\ell_n, \omega))$. Now we take $h(n) = \max\{k(w, n) : w \subseteq \min(a_n^{2\beta+1})\}$. By our premise on $\bar{f}$ there is some $\zeta_\alpha < \kappa$ such that $X = \{n \in \omega : h(n) < f_{\zeta_\alpha}(n)\}$ is infinite. Now we choose an infinite $Y \subseteq X$ such that $(\forall n \in Y)(\ell_n < \min(Y \setminus (n+1)))$. Let n_i^β, $i \in \omega$, enumerate Y. Then we set $\bar{a}^\alpha = \langle a^{\alpha+}_{n_i^\beta} : i < \omega\rangle$.

For every $n \in Y$ and $w \subseteq \min(a_n^{2\beta+1})$ we have that $(w \cup a_n^\alpha, \bar{a}^\alpha \restriction [n+1, \omega)) \le_{\mathbb{Q}} (w \cup a_n^\alpha, \bar{a}^{2\beta+1} \restriction [\ell_n, \omega))$.

Now we show that $\mathbb{Q} = \mathbb{M}(\bar{a}^\alpha : \alpha < \kappa)$ is as desired. It is σ-centred, because for every $w \in \mathbb{F}$, $\mathbb{Q}_w = \{(w, \bar{a}^\beta \restriction [\ell, \omega)) : \ell \in \omega, w < a_\ell^\beta, \beta \in \kappa\}$ is centred.

Then the generic $W = \bigcup\{w : \exists \bar{a}(w, \bar{a}) \in G\}$ is an infinite subset of ω and since every $(w, \bar{a}) \in \mathbb{Q}$ forces in $\mathbb{Q}$ that $w \subseteq W \subseteq w \cup \bigcup\{a_n : n < \omega\}$, we have by the choice of the $\bar{a}^\alpha$ in the odd steps, that the generic W is in each $\mathscr{G}_\alpha, \alpha < \kappa$.

Now we show that

$$\Vdash_{\mathbb{Q}} \bar{f} \text{ is unbounded.}$$

Assume towards a contradiction that there is a $\mathbb{Q}$-name $\underset{\sim}{g}$ for a real and there is $p \in \mathbb{Q}$ such that $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{g}$ dominates $\bar{f}$". By Lemma 1.5 there is some $\gamma < \kappa$ such that $\underset{\sim}{g}$ is an $\mathbb{M}(\bar{a}^\gamma)$-name. Then for some $\beta \geq \gamma$ we have $(\bar{b}^\beta, \underset{\sim}{g}^\beta) = (\bar{a}^\gamma, \underset{\sim}{g})$. So at stage $\alpha = 2\beta + 2$ in our construction we take care of $\underset{\sim}{g}$'s equivalent $\mathbb{M}(\bar{a}^{2\beta+1})$-name $\mathrm{Name}(\bar{k}^\beta, \bar{p}^\beta)$. Let ζ_α and $\bar{a}^\alpha$ be as in this step. Assume that there are some $p \geq q$ and some $n(*)$ such that $q \Vdash_{\mathbb{Q}} (\forall n \geq n(*))(\underset{\sim}{g}(n) \geq^* f_{\zeta_\alpha}(n))$. By the form of $\mathbb{Q}$, $q = (s, \bar{a}^{\alpha(1)})$ for some $\alpha(1) \geq \alpha$ and some s, such that $\bar{a}^{\alpha(1)}$ is a condensation of $\bar{a}^\alpha$. So there is some $n_i^\beta \geq n(*)$ such that there are r_i, r_{i+1} and j such that $a_j^{\alpha(1)} \subseteq r_{i+1}$ and $a_j^{\alpha(1)} \cap [r_i, r_{i+1}) = a^{\alpha+}_{n_i^\beta} = a_i^\alpha$. Then we set $s' = s \cup (\bigcup \bar{a}^{\alpha(1)} \cap [0, r_i))$, and we set $q' = (s' \cup a_i^\alpha, a_{j+1}^{\alpha(1)}, \ldots)$.

We set $m_{n_i^\beta}^\beta(s') = m$. Then q' witnesses that q and $p^\beta_{n_i^\beta, m}$ are compatible, because $q \geq q'$ and $p^\beta_{n_i^\beta, m} \geq q'$. However, our choice of m yields $p^\beta_{n_i^\beta, m} \Vdash_{\mathbb{Q}} \underset{\sim}{g}(n_i^\beta) = k^\beta_{n_i^\beta, m} < f_{\zeta_\alpha}(n_i^\beta)$. Contradiction. $\square$

2. A finite support iteration

Now we describe a finite support iteration.

Theorem 2.1. *Let $\kappa = \mathrm{cf}(\kappa) > \aleph_1$ and assume $\kappa^{<\kappa} = \kappa$ and assume that $\diamondsuit(S)$ holds for some stationary $S \subseteq \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \kappa\}$. There is some finite support iteration $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \kappa^+, \beta \leq \kappa^+\rangle$ such that*

$$\Vdash_{\mathbb{P}_{\kappa^+}} \mathrm{MA}_{<\kappa} \wedge 2^\omega = \kappa^+ \wedge \mathfrak{g} = \kappa^+ \wedge \mathfrak{b} = \kappa.$$

Proof. By $\diamondsuit(S)$ there is $\bar{Y} = \langle Y_\delta : \delta \in S\rangle$, such that $Y_\delta \subseteq \delta$ and for all $Y \subseteq \kappa^+$ the set $\{\delta \in S : Y_\delta = Y \cap \delta\}$ is a stationary subset of κ.

As the ground model has $\kappa^{<\kappa} = \kappa$, we can fix an enumeration $\underset{\sim}{\mathbb{Q}}'_\beta$, $\beta \in \kappa^+ \setminus (S \cup \kappa)$ of all c.c.c. names of partial orders on all ordinals $< \kappa$, such that each name appears cofinally often before each $\alpha \in \kappa^+$ of cofinality κ.

We choose $\underset{\sim}{\mathbb{Q}}_\beta$ by induction on $\beta < \kappa^+$. In the first κ steps we add κ Hechler reals f_α, $\alpha < \kappa$, and these will be the $\leq^*$-increasing unbounded sequence whose unboundedness will be preserved through the rest of the iteration.

In the following steps we distinguish two cases: First case: If $\beta \in S$ and $\Vdash_{\mathbb{P}_\beta}$ "Y_β is a code for a $\mathbb{P}_\beta$-name of a family $\{\mathscr{G}_\zeta : \zeta < \kappa\}$ of κ groupwise dense subsets of $[\omega]^\omega$". Then we take $\underset{\sim}{\mathbb{Q}}_\beta$ such that $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is as in Lemma 1.6", and we get $\Vdash_{\mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta}$ "there is an infinite subset of ω that is in each $\underset{\sim}{\mathscr{G}}_\zeta$, $\zeta < \kappa$".

Second case: Not all the criteria from the first case are fulfilled. Then, as in the usual iteration for Martin's axiom, $\underset{\sim}{\mathbb{Q}}_\beta$ will be $\underset{\sim}{\mathbb{Q}}'_\beta$ with weights p, where we have $p \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{Q}}'_\beta$ is a c.c.c. forcing of cardinality less than κ", and $\underset{\sim}{\mathbb{Q}}_\beta$ will be the trivial partial order with orthogonal weight.

As $\kappa^{<\kappa} = \kappa$ also in the final model we have $\mathrm{MA}_{<\kappa}$, because if $\mathbb{P}$ is a c.c.c.-notion of forcing of cardinality $< \kappa$ in $\mathbf{V}^{\mathbb{P}_{\kappa^+}}$ and if $\gamma < \kappa$ and D_α, $\alpha < \gamma$, is a sequence of predense subsets of $\mathbb{P}$, then this holds in an initial segment $\mathbf{V}^{\mathbb{P}_\delta}$ for some $\delta \in \kappa^+ \setminus S$ and hence by what we did in successor steps for $\delta \notin S$, there is a directed $G \subseteq \mathbb{P}$ such that $\bigwedge_{\alpha < \gamma} G \cap D_\alpha \neq \emptyset$.

By Lemma 1.6, in each Matet step of the iteration the unbounded family f_α, $\alpha < \kappa$, is preserved. By [1, 2.1] also in each extension by $\mathbb{Q}$ of size $< \kappa$ the unbounded family is preserved. By the preservation theorem for finite support iterations from [2, 6.5.3], the unbounded well-ordered family f_α, $\alpha < \kappa$, is preserved in all limit steps of the iteration. Thus we have $\mathfrak{b} = \kappa$ in the extension.

Let $\mathscr{G}_\alpha$, $\alpha < \kappa$, be a family of groupwise dense sets in $V^{\mathbb{P}}$. As $\langle Y_\delta : \delta \in S\rangle$ is a diamond sequence and as being κ groupwise dense families reflects down into a κ-club set in κ^+ (a proof for the countable support iteration of proper forcings is given by [6], and a simpler form thereof works for finite support iteration of c.c.c. forcings), at stationarily many steps Y_δ guesses a name for $\mathscr{G}_\alpha \cap \mathbf{V}^{\mathbb{P}_\delta}$, $\alpha < \kappa$, and by the choice of $\mathbb{P}_{\delta+1}$ in the first case, the forcing adds a real that is in all the $\mathscr{G}_\alpha$. Hence $\mathfrak{g} = \kappa^+$. $\square$

Corollary 2.2. $\aleph_2 \leq \mathfrak{b} < \mathfrak{g}$ *is consistent relative to* ZFC.

Proof. We take a ground model of GCH and then we force $\diamondsuit(S)$ for some stationary $S \subseteq \{\alpha < \aleph_3 : \mathrm{cf}(\alpha) = \aleph_2\}$. Then we apply the previous theorem with $\kappa = \aleph_2$. $\square$

3. An upper bound on $\mathfrak{g}$

Definition 3.1. Let κ be a regular cardinal. On ${}^{\kappa}\kappa$ we define the almost order $f \le^* g$ iff there is some $\alpha < \kappa$ such that for all $\beta \ge \alpha$, $f(\beta) \le g(\beta)$. A set $D \subseteq {}^{\kappa}\kappa$ is called dominating in $({}^{\kappa}\kappa, \le^*)$ iff for every $f \in {}^{\kappa}\kappa$ there is some $g \in D$ such that $g \ge^* f$. So we have the dominating number $\mathfrak{d}_\kappa$ which is the smallest size of a dominating set.

Theorem 3.2. $\mathfrak{g} \le \mathfrak{d}_{\mathfrak{b}}$.

Proof. Let $D = \{f_\varepsilon : \varepsilon < \mathfrak{d}_{\mathfrak{b}}\}$ be a dominating family. We shall build groupwise dense families $\mathscr{G}_f$, $f \in D$, such that their intersection is empty. First we introduce some notation and present a characterisation of $\mathfrak{b}$ in terms of a slightly different ordering than $\le^*$ on ${}^{\omega}\omega$. $\square$

Definition 3.3. (1) $\mathrm{Inc}(\omega) = \{\bar{n} : \bar{n} = \langle n_i : i < \omega\rangle$ is increasing$\}$.
(2) ([5, Def. 2.9]) $\bar{m} \le^{**} \bar{n}$ iff $(\forall^\infty i)(|\{j : m_j \in [n_i, n_{i+1}]\}| \ge 2)$.

We thank Boaz Tsaban for telling us that the following lemma was originally proved by Blass. We nevertheless let our proof stand, since it is self-contained and in contrast to Blass' elegant proof, does not speak about morphisms and duality.

Lemma 3.4. *([5, Theorem 2.10])*

(1) $\le^{**}$ *is a partial order.*
(2) $(\mathrm{Inc}(\omega), \le^{**})$ *is* $\mathfrak{b}$*-directed.*
(3) *There is an* $\le^{**}$*-increasing sequence of length* $\mathfrak{b}$ *with no upper bound.*

Proof. (1) is easy. (2) Let $\gamma < \mathfrak{b}$ and $\bar{n}_\alpha$, $\alpha < \gamma$, be given. We first need the twofold iteration operation. For a strictly increasing function $f : \omega \to \omega$ we define $\tilde{f}$ by $\tilde{f}(0) = 0$, $\tilde{f}(n+1) = f(f(\tilde{f}(n)))$. We take $f \ge^* \bar{n}_\alpha$ for all $\alpha < \gamma$.

Now we have $(\forall \alpha < \gamma)(\forall^\infty i)(f(i) \ge n_\alpha(i))$. We show that $\tilde{f} \ge^{**} \bar{n}_\alpha$ for all $\alpha < \gamma$. We fix α and take i_0 so that $(\forall i \ge i_0)(f(i) \ge n_\alpha(i) \wedge f(\tilde{f}(i)) - \tilde{f}(i) \ge 2)$. Then for $i \ge i_0$ we get: $\tilde{f}(i+1) = f(f(\tilde{f}(i)))$ and $f(\tilde{f}(i)) \ge n_\alpha(\tilde{f}(i)) \ge \tilde{f}(i)$ and $f(f(\tilde{f}(i))) \ge n_\alpha(f(\tilde{f}(i)))$, so at least $n_\alpha(\tilde{f}(i)), n_\alpha(\tilde{f}(i)+1), \dots, n_\alpha(f(\tilde{f}(i)))$ are in the interval $[\tilde{f}(i), \tilde{f}(i+1)]$, so at least 2 elements.

(3) Let f_α, $\alpha < \mathfrak{b}$, be an unbounded family of strictly increasing functions. We let $n_{\alpha,i} = f_\alpha(i)$. There is no $\bar{n} \ge^{**} \bar{n}_\alpha$ for all $\alpha < \mathfrak{b}$ as otherwise $\bar{n} \ge^* f_\alpha$ for all $\alpha < \mathfrak{b}$. Now we use (2) to choose by induction on $\alpha < \mathfrak{b}$ an $\le^{**}$-increasing sequence $\langle \bar{m}_\alpha : \alpha < \mathfrak{b}\rangle$ by taking for each $\alpha < \mathfrak{b}$ some $\bar{m}_\alpha \ge^{**} \bar{n}_\alpha$ such that $\bar{m}_\alpha \ge^{**} \bar{m}_\beta$ for all $\beta < \alpha$. $\square$

Definition 3.5. Let $\langle \bar{n}_\alpha : \alpha < \mathfrak{b}\rangle$ be a $\le^{**}$-increasing and -unbounded sequence in $\mathrm{Inc}(\omega)$.

(1) Let $A \in [\omega]^\omega$ and $\bar{n} \in \mathrm{Inc}(\omega)$. We let $\mathrm{In}(A, \bar{n}) = \{i : A \cap [n_i, n_{i+1}) \ne \emptyset\}$.
(2)

$$\mathscr{G}(\langle \bar{n}_\alpha : \alpha < \mathfrak{b}\rangle) = \{A \in [\omega]^\omega : (\exists \alpha)\langle n_{\alpha,i} : i \in \mathrm{In}(A, \bar{n}_\alpha)\rangle \ge^{**} \bar{n}_{\alpha+1}\}.$$

Remark. Since $\bar{n}_\alpha$, $\alpha < \mathfrak{b}$, is increasing and unbounded, there is some minimal $\beta \ge \alpha$ such that $\langle n_{\alpha,i} : i \in \mathrm{In}(A, \bar{n}_\alpha)\rangle \not\ge^{**} \bar{n}_\beta$. The requirement for $\bar{n}_\beta$ in the definition of $\mathscr{G}(\langle \bar{n}_\alpha : \alpha < \mathfrak{b}\rangle)$ goes in the opposite direction: $\bar{n}_\alpha \le^{**} \bar{n}_\beta \le^{**} \langle n_{\alpha,i} : i \in \mathrm{In}(A, \bar{n}_\alpha)\rangle$ and hence A has to be sufficiently small.

Lemma 3.6. *If* $\langle \bar{n}_\alpha : \alpha < \mathfrak{b}\rangle$ *is* $\le^{**}$*-unbounded and* $\alpha_0 < \mathfrak{b}$*, then* $\mathscr{G}(\langle \bar{n}_\alpha : \alpha_0 < \alpha < \mathfrak{b}\rangle)$ *is groupwise dense.*

Proof. We have that $\mathrm{In}(B, \bar{n}_\alpha) \subseteq^* \mathrm{In}(A, \bar{n}_\alpha)$ if $B \subseteq^* A$ and thus $\mathscr{G}(\langle \bar{n}_\alpha : \alpha_0 < \alpha < \mathfrak{b}\rangle)$ is closed under infinite almost subsets. Now let a partition $\{[\pi_i, \pi_{i+1}) : i < \omega\}$ be given and set $\bar{\pi} = \langle \pi_{2i} : i < \omega\rangle$. Then take $\alpha \ge \alpha_0$ such that $\bar{n}_\alpha \not\le^{**} \bar{\pi}$. So there are infinitely many i such that there is at most one element j such that $n_{\alpha,j} \in [\pi_{2i}, \pi_{2i+2}]$.

Now we inductively choose increasing sequences i_n, j_n, j'_n, $n \in \omega$ and $u_n \in 2$. We take i_0 such that there is at most one $n_{\alpha,j} \in [\pi_{2i_0}, \pi_{2i_0+2}]$ and such that there is some $n_{\alpha,j} \le \pi_{2i_0+2}$. We name the largest j such that $n_{\alpha,j} \le \pi_{2i_0+2}$ to be j_0. If $n_{\alpha,j_0} \le \pi_{2i_0+1}$, then let $j'_0 = j_0$, otherwise let $j'_0 = j_0 - 1$.

Now let i_n and j_n be defined. Then we take $i_{n+1} > i_n$ such that there is at most one $n_{\alpha,j}$ in $[\pi_{2i_{n+1}}, \pi_{2i_{n+1}+2}]$ and again we let $j_{n+1} > j_n$ be so that $n_{\alpha,j_{n+1}}$ is the largest $n_{\alpha,j} \le \pi_{2i_{n+1}+2}$. If $n_{\alpha,j_{n+1}} \le \pi_{2i_{n+1}+1}$, then let $j'_{n+1} = j_{n+1}$, otherwise let $j'_{n+1} = j_{n+1} - 1$. In addition we take i_{n+1} so large such that $[n_{\alpha,j'_n}, n_{\alpha,j'_{n+1}}]$ contains at least two different

$n_{\alpha+1,j}$. We let $u_n = 1 - (j_n - j'_n)$ and finally we let $A = \bigcup\{[\pi_{2i_n+u_n}, \pi_{2i_n+u_n+1}) : n \in \omega\}$. By the construction, $\mathrm{In}(A, \bar{n}_\alpha)$ is infinite and $\langle n_{\alpha,i} : i \in \mathrm{In}(A, \bar{n}_\alpha)\rangle = \langle n_{\alpha,j'_n} : n \in \omega\rangle \geq^{**} \bar{n}_{\alpha+1}$. $\square$

Proof of Theorem 3.2. Suppose that $\{f_\varepsilon : \varepsilon < \mathfrak{d}_\mathfrak{b}\}$ is a dominating family. We take some fixed $\leq^{**}$-increasing and -unbounded sequence $\langle \bar{n}_\gamma : \gamma < \mathfrak{b}\rangle$. For each $\varepsilon < \mathfrak{d}_\mathfrak{b}$ let

$$E_\varepsilon = \{\alpha < \mathfrak{b} : (\forall \beta < \alpha)(f_\varepsilon(\beta) < \alpha)\}.$$

This is a club in the regular cardinal $\mathfrak{b}$, and let $\langle \xi_{\varepsilon,\alpha} : \alpha < \mathfrak{b}\rangle$ be the increasing continuous enumeration of it. We show that

$$\bigcap_{\varepsilon \in \mathfrak{d}_\mathfrak{b}, \alpha_0 < \mathfrak{b}} \mathscr{G}(\langle \bar{n}_{\xi_{\varepsilon,\alpha}} : \alpha_0 < \alpha < \mathfrak{b}\rangle) = \emptyset.$$

Assume towards a contradiction that A is infinite and in this intersection. We define $f_A : \mathfrak{b} \to \mathfrak{b}$ by

$$f_A(\alpha) = \min\{\gamma : \gamma \geq \alpha \wedge \langle n_{\alpha,i} : i \in \mathrm{In}(A, \bar{n}_\alpha)\rangle \not\geq^{**} \bar{n}_\gamma\}.$$

Since f_ε, $\varepsilon < \mathfrak{d}_\mathfrak{b}$, is a dominating family, there is some ε and some α_0 such that for all $\alpha \geq \alpha_0$, $f_A(\alpha) \leq f_\varepsilon(\alpha)$. Since $A \in \mathscr{G}(\langle \bar{n}_{\xi_{\varepsilon,\beta}} : \alpha_0 < \beta < \kappa\rangle)$, there is some $\alpha_0 < \xi_{\varepsilon,\beta} \in E_\varepsilon$ such that $\langle n_{\xi_{\varepsilon,\beta},i} : i \in \mathrm{In}(A, \bar{n}_{\xi_{\varepsilon,\beta}})\rangle \geq^{**} \bar{n}_{\xi_{\varepsilon,\beta+1}}$.

Hence $\xi_{\varepsilon,\beta+1} < f_A(\xi_{\varepsilon,\beta})$. But $\xi_{\varepsilon,\beta+1} \in E_\varepsilon$, that means $f_\varepsilon(\xi_{\varepsilon,\beta}) < \xi_{\varepsilon,\beta+1} < f_A(\xi_{\varepsilon,\beta})$, which contradicts the choice of ε and α_0. $\square$

Remark. So Theorem 3.2 shows that c.c.c. forcing of any length over a model of GCH will give $\mathfrak{g} \leq \mathfrak{d}_\mathfrak{b} = \mathfrak{b}^+$, since c.c.c. forcing does not increase the value of $\mathfrak{d}_\mathfrak{b}$ if it preserves the value of $\mathfrak{b}$.

Acknowledgements

We thank the referee for finding a serious mistake in an earlier version of this work and for giving valuable suggestions.

The first author was supported by the Austrian "Fonds zur wissenschaftlichen Förderung", grants no. 13983 and 16334, and by travel support by the Landau Center.

The second author's research is supported by the United States–Israel Binational Science Foundation (Grant no. 2002323). This is his Publication 843.

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