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CHARACTERIZING AUTOMORPHISM GROUPS OF ORDERED ABELIAN GROUPS

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Abstract

A proof is given of the following theorem, which characterizes full automorphism groups of ordered abelian groups: *a group H is the automorphism group of some ordered abelian group if and only if H is right-orderable.*

In this short note we want to characterize the groups isomorphic to full automorphism groups of ordered abelian groups. The result will follow from classical theorems on ordered groups, adding an argument from proofs used to realize rings as endomorphism rings of abelian groups; see [**1**]. Recall that H is a right-ordered group (RO-group) if $(H, \cdot)$ is a group and $(H, <)$ is a linear order satisfying the following compatibility condition

$$\text{For all } h < g, k \text{ in } H, \text{ it follows that } hk < gk. \qquad \text{(RO)}$$

Similarly, we define (LO), the left compatibility condition. If $(H, \cdot, <)$ satisfies both (RO) and (LO), then H is an ordered group. Obviously, abelian RO-groups are ordered groups, in which case we often replace multiplication by '+'. A group H is right-orderable if it permits a linear order that makes it an RO-group. We do not distinguish between RO-groups and groups that are right-orderable. From the fact that cyclic ordered groups are infinite, it is clear that RO-groups are torsion-free. By an old theorem of Smirnov, a group is an RO-group if and only if it is (isomorphic to) a subgroup of $\mathrm{Aut}(A, +, <)$ of an ordered free abelian group A; see [**4**, Theorem 7.1.3, p. 129]. We shall use the obvious representation as a subgroup of $\mathrm{Aut}(A, +, <)$ below.

On the other hand, there are torsion-free groups (in fact, polycyclic groups) that are not RO-groups, a result due to Smirnov; see [**4**, p. 127]. Note that torsion-free polycyclic groups are even finitely generated, iterated extensions of $\mathbb{Z}$. Our main result then reads as follows.

Theorem 1. *For a group H, the following statements are equivalent.*

(1) *H is an RO-group.*

(2) *There is an ordered abelian group $G = (G, +, <)$ with* $\mathrm{Aut}(G, +, <) \cong H$.

(3) *If K is any ordered field, then it has an ordered extension field F such that* $\mathrm{Aut}(F) \cong H$.

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This result is in sharp contrast to an unpublished classification compiled in 1988 by A. L. S. Corner, giving all the finite groups that are automorphism groups of torsion-free abelian groups, in which many groups such as $\mathbb{Z}/7\mathbb{Z}$ are not listed. The equivalence between statements (1) and (3) is taken from [**2**], and (1) follows from (2) by the above theorem of Smirnov. It remains to show that statement (1) implies statement (2); in fact, we shall show a stronger implication, as follows.

(2^*) *There is an* $\aleph_1$*-free ordered abelian group* $G = (G, +, <)$ *with* $\mathrm{Aut}(G, +, <) \cong H$.

Here, G is $\aleph_1$-free if all its countable subgroups are free. Consider the group ring $R = \mathbb{Z}H$, and let $B = \bigoplus R$ be a 'large enough' free R-module. We shall follow the convention in [**1**] and view B as an abelian group as well as a right R-module, and we let $\mathbb{Z}$-endomorphisms act on the right. Hence $R \subseteq \mathrm{End}_{\mathbb{Z}} B$ by scalar multiplication on the right of B with elements from R; say, $\mathrm{End}_{\mathbb{Z}} G = \mathrm{End}\, G$. We shall construct a right R-module G such that $B \subseteq_* G \subseteq_* \widehat{B}$ and $\mathrm{End}\, G = R$. Here, $\widehat{B}$ is the S-adic completion of B with respect to some suitable, multiplicatively closed subset $S \subseteq \mathbb{N} \subseteq R$; for example, $S = \{p^n \mid n \in \omega\}$. Moreover, '$\subseteq_*$' denotes an S-pure submodule. It will be important that

$$G = \bigcup_{\alpha \in \lambda^*} G_\alpha$$

is the union of an ascending continuous chain of S-pure right R-submodules $G_\alpha \subseteq_* \widehat{B}$ with $G_0 = B$ and $G_{\alpha+1} = \langle G_\alpha, g_\alpha R \rangle_*$, such that $\mathrm{Ann}_R\, g_\alpha = 0$, $G_\alpha \cap g_\alpha R = 0$ and $G_{\alpha+1}/G_\alpha \cong S^{-1}R$ is the S-localization, which is S-divisible.

Note that B is S-dense in $\widehat{B}$ and $G_{\alpha+1}$ is S-pure in $\widehat{B}$, so G_α is also S-dense in $G_{\alpha+1}$; hence $G_{\alpha+1}/G_\alpha = \langle G_\alpha, g_\alpha R \rangle_*/G_\alpha$ is S-divisible of rank 1, and is thus isomorphic to $S^{-1}R$.

Here, $\langle G_\alpha, g_\alpha R \rangle_*$ denotes the smallest subgroup of $\widehat{B}$ which is S-pure and contains $G_\alpha, g_\alpha R$. This part will follow by arguments that we have used in several earlier papers; for example, in [**1**]. As H is assumed to be an RO-group by statement (1) of Theorem 1, we shall turn $R = \mathbb{Z}H$ into a linear order satisfying the compatibility condition (RO) for multiplication with positive elements in the group ring.

PROPOSITION 2. *If H is an RO-group, then the group ring $R = \mathbb{Z}H$ has a natural linear order satisfying* (RO) *for multiplication with positive elements. The monoid of positive elements of R will be denoted by $R^{>0}$.*

We postpone the proof of Proposition 2, and assume here that it holds. It follows that $(R^+, <)$ is an ordered free abelian group and $R^{>0} \subseteq \mathrm{End}(R, +, <)$; thus also

$$R^{>0} \subseteq \mathrm{End}(B, +, <),$$

where the linear order will be extended and B becomes an ordered abelian group.

We want to extend the order inductively on to G. If $P_\alpha = \{g \in G_\alpha, 0 < g\}$ denotes the positive cone of G_α, then we want to define the positive cone $P_{\alpha+1}$ of $G_{\alpha+1} = \langle G_\alpha, g_\alpha R \rangle_*$.

If $y \in G_{\alpha+1}$, there is $s \in S$ such that $ys = x + g_\alpha r$ for some $r \in R$ and $x \in G_\alpha$. Thus we define

$$y \in P_{\alpha+1} \iff \begin{cases} r > 0, & \text{or} \\ r = 0 \ \text{and} \ x \in P_\alpha. \end{cases}$$

It is easy to see that $G_{\alpha+1} = -P_{\alpha+1} \cup P_{\alpha+1} \cup \{0\}$, and $P_{\alpha+1}$ is well defined. If also $ys' = x' + g_\alpha r'$, then $xs' + g_\alpha rs' = x's + g_\alpha r's$. Hence

$$xs' - x's = g_\alpha(r's - rs') \in G_\alpha \cap g_\alpha R = 0.$$

From $\mathrm{Ann}_R\, g_\alpha = 0$, it follows that $r's = rs'$, and $s, s' > 0$ implies that $r' > 0$ if and only if $r > 0$. Moreover, $r = 0$ if and only if $r' = 0$, and similarly $xs' = x's$ implies that $x \in P_\alpha$ if and only if $x' \in P_\alpha$. Note that $g_\alpha > 0$ follows from the fact that $1 > 0$. If $r' \in R^{>0}$ and $y \in P_{\alpha+1}$ as above, then $yr's = xr' + g_\alpha rr'$; hence either $r = 0$ and $xr' \in P_\alpha$ by the induction hypothesis, or else $rr' > 0$ by Proposition 2, so $yr' \in P_{\alpha+1}$ and

$$R^{>0} \subseteq \mathrm{End}(G_{\alpha+1}, +, <)$$

follows from $R^{>0} \subseteq \mathrm{End}(G_\alpha, +, <)$. At limit ordinals $\beta < \lambda^*$ we take unions; thus $P_\beta = \bigcup_{\alpha<\beta} P_\alpha$ and it follows that $(G, +, <)$ is an ordered abelian group with

$$R^{>0} \subseteq \mathrm{End}(G, +, <).$$

If $r < 0$, then the action of r on B (and hence multiplication on a summand Re of B) shows that $0e < 1e$ turns into $re < 0e$. Together with $R = \mathrm{End}\, G$, we obtain

$$R^{>0} = \mathrm{End}(G, +, <).$$

We derive the following result, assuming that Proposition 2 holds and that the presentation of $G = \bigcup_{\alpha<\lambda} G_\alpha$, as used above, follows.

THEOREM 3. *If H is an RO-group, if λ is any cardinal with $|H| \leqslant \lambda$, and if $R^{>0}$ is the monoid of positive elements of the group ring $R = \mathbb{Z}H$, then there is an $\aleph_1$-free, ordered abelian group $(G, +, <)$ of cardinality $\lambda^{\aleph_0}$ with $R^{>0} = \mathrm{End}(G, +, <)$.*

Proof. First, we want to establish the last claim. We note that R^+ is a free abelian group; in particular, R^+ is cotorsion-free (that is, $\mathrm{Hom}(\widehat{\mathbb{Z}}, R) = 0$), which is needed to apply [**1**, Theorem 6.3, p. 465]. We need a very special case of that theorem, putting $\mathfrak{N} = \{0\}$ and $J_1 = J = \emptyset$. Thus $\mathrm{End}\, G = R$ is immediate. It is easy to check that G is $\aleph_1$-free. The group G is obtained by transfinite induction as $\bigcup_{\alpha<\lambda^*} G_\alpha = G$ over $\alpha < \lambda^*$ with $|\lambda^*| = \lambda^{\aleph_0}$ by using a weak version of Shelah's Black Box (see [**1**, Appendix]). Conditions (II_0) (II_μ) and ($\mathrm{III}_{\alpha+1}$) given there show that $G_{\alpha+1}$ is of the right form (replacing A by R), and [**1**, Lemma 3.4, p. 456] for $N_\alpha^k = 0$, together with Condition (III_α), implies that $G_\alpha \cap g_\alpha R = 0$, and $\mathrm{Ann}_R\, g_\alpha = 0$. □

It remains to show Proposition 2.

Proof of Proposition 2. If $r \in R$, write $r = \sum_{h\in H} r_h h$ with $r_h \in \mathbb{Z}$; similarly, $r' = \sum_{h\in H} r'_h h$. We say that

$$r < r' \iff \begin{cases} \exists\, h^* \in H, & r_{h^*} < r'_{h^*}, \quad \text{and} \\ \forall h > h^*, & r_h = r'_h. \end{cases}$$

Let $[r] = \{h : r_h \neq 0\}$; then the positive cone of R is

$$R^{>0} = \{r \in R : \exists \text{ maximal } h^* \in [r] \text{ and } r_{h^*} > 0\}.$$

It is easy to check that this is a linear ordering on R. From $R^{>0} \cdot R^{>0} \subseteq R^{>0}$, it follows that multiplication with elements from $R^{>0}$ satisfies condition (RO); thus

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$R^{>0} \subseteq \mathrm{End}(R,+,<)$. The ordering extends naturally to direct sums (see, for example, [**4**, Theorem 2.1.1]), and thus $R^{>0} \subseteq \mathrm{End}(B,+,<)$. □

As in the case of polynomial rings $\mathbb{R}[x]$, we can show the following proposition.

PROPOSITION 4. *If H is an RO-group, if $R = \mathbb{Z}H$ is the group ring, and if $U(R)$ are its units, then $U(R) = \pm H$.*

Proof. If

$$r = \sum_{h \in H} r_h h \in R \quad \text{and} \quad r' = \sum_{h \in H} r'_h h \in R$$

are as above, with $rr' = r'r = 1$, then the product of the maximal coefficients r_{h^*} and $r'_{h'^*}$ must be 1. This is possible only if $h^* h'^* = 1$ and all other coefficients are 0. It follows that $r = r_{h^*} h^*$ and $r' = r_{h^*}^{-1} h_*^{-1}$; also, r_{h^*} and $r_{h'^*}$ are units of the coefficient ring $\mathbb{Z}$. Hence $r, r' \in \pm H$. □

REMARK. Proposition 4 also follows from a more general result of Strojnowski on unique product groups; see [**3**, Corollary 8.4.8, p. 272].

From Proposition 4, it follows that the units of $R^{>0}$ are $U(R^{>0}) = H$. From Theorem 3 and $\mathrm{Aut}(G,<) = U(\mathrm{End}(G,+,<))$ our main result follows, which immediately proves the implication '(1) $\Longrightarrow$ (2)' of Theorem 1.

COROLLARY 5. *If H is an RO-group of cardinality $|H| \leqslant \lambda$, then there is an $\aleph_1$-free, ordered abelian group $(G,<)$ of cardinality $\lambda^{\aleph_0}$ with $\mathrm{Aut}(G,+,<) = H$.*

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