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SHEVA–SHEVA–SHEVA: LARGE CREATURES

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ABSTRACT

We develop the theory of the forcing with trees and creatures for an inaccessible λ continuing Rosłanowski and Shelah [15], [17]. To make a real use of these forcing notions (that is to iterate them without collapsing cardinals) we need suitable iteration theorems, and those are proved as well. (In this aspect we continue Rosłanowski and Shelah [16] and Shelah [20], [21].)

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0. Introduction

The present paper has two themes.

The first is related to the quest for the right generalization of properness to higher cardinals (that is, for a property of forcing notions that would play in iterations with uncountable supports similar role to that of standard properness in CS iterations). The evidence that there is no straightforward generalization of properness to larger cardinals was given already in Shelah [18] (see [19, Appendix 3.6(2)]). Substantial progress has been achieved in Shelah [20], [21], but the properties there were tailored for generalizing the case *no new reals* of [19, Ch. V]. Then Rosłanowski and Shelah [16] gave an iterable condition for not collapsing λ^+ in λ-support iterations of $({<}\lambda)$-complete forcing notions (with possibly adding subsets of λ) and later Eisworth [6] gave another property preserved in λ-support iterations (and implying that λ^+ is not collapsed). At the moment it is not clear if the two properties (the one of [16] and that of [6]) are equivalent, though they have similar flavour. However, the existing iterable properties still do not cover many examples of natural forcing notions, specially those which come naturally in the context of λ-reals. This brings us to the second theme: developing the forcing for λ-reals.

A number of cardinal characteristics related to the Baire space ${}^{\omega}\omega$, the Cantor space ${}^{\omega}2$ and/or the combinatorial structure of $[\omega]^{\omega}$ can be extended to the spaces ${}^{\lambda}\lambda, {}^{\lambda}2$ and $[\lambda]^{\lambda}$ for any infinite cardinal λ. Following the tradition of Set Theory of the Reals we may call cardinal numbers defined this way for ${}^{\lambda}\lambda$ (and related spaces) **cardinal characteristics of λ-reals**. The menagerie of those characteristics seems to be much larger than the one for the continuum. But to decide if the various definitions lead to different (and interesting) cardinals we need a well developed forcing technology.

There has been a serious interest in cardinal characteristics of the λ-reals in literature. For example, Cummings and Shelah [5] investigated the natural generalizations $\mathfrak{b}_\lambda$, $\mathfrak{d}_\lambda$ of the unbounded number and the dominating number, respectively, giving simple constraints on the triple of cardinals $(\mathfrak{b}_\lambda, \mathfrak{d}_\lambda, 2^\lambda)$ and proving that any triple of cardinals obeying these constraints can be realized.

In a somewhat parallel work [22], Shelah and Spasojevič studied $\mathfrak{b}_\lambda$ and the generalization $\mathfrak{t}_\lambda$ of the tower number. Zapletal [23] investigated the splitting number $\mathfrak{s}_\lambda$ — here the situation is really complicated as the inequality $\mathfrak{s}_\lambda > \lambda^+$ needs large cardinals. One of the sources of interest in characteristics of the λ-reals is their relevance for our understanding of the club filter on λ (or the dual ideal of non-stationary subsets of λ) — see, e.g., Balcar and Simon [2, §5], Landver [10], Matet and Pawlikowski [11], Matet, Rosłanowski and Shelah [12]. First steps toward developing forcing for λ-reals have been done long time ago: in 1980 Kanamori [9] presented a systematic treatment of the λ-perfect–set forcing in products and iterations. Brown [3], [4] discussed the λ-superperfect forcing and other tree-like forcing notions.

Our aim in this paper is to provide tools for building forcing notions relevant for λ-reals (continuing in this Rosłanowski and Shelah [15], [17]) and give suitable iteration theorems (thus continuing Rosłanowski and Shelah [16]). However, we restrict our attention to the case when λ is a strongly inaccessible uncountable cardinal (after all, $\aleph_0$ is inaccessible), see 0.3 below.

The structure of the paper is as follows. It is divided into two parts, first one presents iteration theorems, the second one gives examples and applications. In Section A.1 we present some basic notions and methods relevant for iterating λ-complete forcing notions. The next section, A.2, gives preservation of λ-analogue of the Sacks property (in Theorem A.2.4) as well as preservation of being ${}^\lambda\lambda$-bounding (in Theorem A.2.7). Section A.3 introduces **fuzzy properness**, a more complicated variant of **properness over semi-diamonds** from [16]. Of course, we prove a suitable iteration theorem (see Theorem A.3.10). Then we give examples for the properties discussed in Part A. We start with showing that a forcing notion useful for uniformization is fuzzy proper (in Section B.4), and then we turn to developing forcing notions built with the use of **trees and creatures**. In Section B.5 we set the terminology and notation, and in the next section we discuss when the resulting forcing notions have the two bounding properties discussed in §A.2. Section B.7 shows how our methods result in fuzzy proper forcing notions, and the last section introduces some new characteristics of the λ-reals.

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Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [8]). In forcing we keep the older convention that **a stronger condition is the larger one**. Our main conventions are listed below.

Notation 0.1:

(1) For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$-name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$-names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below the letter (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$). The weakest element of $\mathbb{P}$ will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one, and that there is no other condition equivalent to it). We will also assume that all forcing notions under considerations are atomless.

By "λ-support iterations" we mean iterations in which domains of conditions are of size $\leq \lambda$. However, we will pretend that conditions in a λ-support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\zeta : \zeta < \zeta^* \rangle$ are total functions on ζ^* and for $p \in \lim(\bar{\mathbb{Q}})$ and $\alpha \in \zeta^* \setminus \mathrm{Dom}(p)$ we will let $p(\alpha) = \underset{\sim}{\emptyset}_{\underset{\sim}{\mathbb{Q}}_\alpha}$.

(2) For a filter D on λ, the family of all D-positive subsets of λ is called D^+. (So $A \in D^+$ if and only if $A \subseteq \lambda$ and $A \cap B \neq \emptyset$ for all $B \in D$.)

The club filter of λ is denoted by $\mathcal{D}_\lambda$.

(3) Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet ($\alpha, \beta, \gamma, \delta, \ldots$) and also by i, j (with possible sub- and superscripts).

Cardinal numbers will be called $\theta, \kappa, \lambda, \mu$ (with possible sub- and superscripts); λ is a fixed inaccessible cardinal (see 0.3).

(4) By χ we will denote a **sufficiently large** regular cardinal; $\mathcal{H}(\chi)$ is the family of all sets hereditarily of size less than χ. Moreover, we fix a well ordering $<^*_\chi$ of $\mathcal{H}(\chi)$.

(5) For regular cardinals $\lambda < \lambda^*$, $\mathcal{H}_{<\lambda}(\lambda^*)$ is the collection of all sets x which are hereditarily of size $< \lambda$ relatively to λ^*, i.e., such that $|\mathrm{Tc}^{\mathrm{ord}}(x)| < \lambda$ and $\mathrm{Tc}^{\mathrm{ord}}(x) \cap \mathrm{Ord} \subseteq \lambda^*$. Recall that $\mathrm{Tc}^{\mathrm{ord}}(x)$, the hereditary closure relative to the ordinals, is defined by induction on $\mathrm{rank}(x) = \gamma$ as follows:
   - if $\gamma = 0$ or x is an ordinal, then $\mathrm{Tc}^{\mathrm{ord}}(x) = \emptyset$,

- if $\gamma > 0$ and x is not an ordinal, then

$$\mathrm{Tc}^{\mathrm{ord}}(x) = \bigcup\{\mathrm{Tc}^{\mathrm{ord}}(y) : y \in x\} \cup x.$$

(6) A bar above a letter denotes that the object considered is a sequence; usually $\bar{X}$ will be $\langle X_i : i < \zeta\rangle$, where ζ is the length $\mathrm{lh}(\bar{X})$ of $\bar{X}$. Sometimes our sequences will be indexed by a set of ordinals, say $S \subseteq \lambda$, and then $\bar{X}$ will typically be $\langle X_\delta : \delta \in S\rangle$.

But also, η, ν and ρ (with possible sub- and superscripts) will denote sequences (nodes in quasi trees).

For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$.

(7) We will consider several games of two players. One player will be called **Generic** or **Complete** or just **I player**, and we will refer to this player as "she". Her opponent will be called **Antigeneric** or **Incomplete** or just **II player** and will be referred to as "he".

Definition 0.2:

(1) **A λ-quasi tree** is a set T of sequences of length $<\lambda$ with the $\vartriangleleft$-smallest element denoted by $\mathrm{root}(T)$.

(2) A λ-quasi tree T is **a λ-tree** if it is closed under initial segments longer then $\mathrm{lh}(\mathrm{root}(T))$.

(3) A λ-quasi tree is **complete** if the union of any $\vartriangleleft$-increasing sequence of length less than λ of members of T is in T.

(4) For a λ-quasi tree T and $\eta \in T$ we define **the successors of η in T**, **maximal points of T**, **the restriction of T to η**, and **the height of** T by:

$$\mathrm{succ}_T(\eta) = \{\nu \in T : \eta \vartriangleleft \nu \& \neg(\exists \rho \in T)(\eta \vartriangleleft \rho \vartriangleleft \nu)\},$$
$$\max(T) = \{\nu \in T : \text{ there is no } \rho \in T \text{ such that } \nu \vartriangleleft \rho\},$$
$$T^{[\eta]} = \{\nu \in T : \eta \trianglelefteq \nu\}, \quad \text{and} \quad \mathrm{ht}(T) = \sup\{\mathrm{lh}(\eta) : \eta \in T\}.$$

We put $\hat{T} = T \setminus \max(T)$.

(5) For $\delta < \lambda$ and a λ-quasi tree T we let

$$(T)_\delta = \{\eta \in T : \mathrm{lh}(\eta) = \delta\} \quad \text{and} \quad (T)_{<\delta} = \{\eta \in T : \mathrm{lh}(\eta) < \delta\}.$$

The set of all limit λ-branches through T is

$$\lim_\lambda(T) \stackrel{\mathrm{def}}{=} \{\eta : \eta \text{ is a } \lambda\text{-sequence} \quad \text{and} \quad (\forall \beta < \lambda)(\exists \alpha > \beta)(\eta{\restriction}\alpha \in T)\}.$$

(6) A subset F of a λ-quasi tree T is **a front** of T if no two distinct members of F are $\lhd$-comparable and

$$(\forall\eta \in \lim_\lambda(T) \cup \max(T))(\exists\alpha < \lambda)(\eta{\restriction}\alpha \in F).$$

Note that if T is a complete λ-quasi tree of height $< \lambda$, then $\max(T)$ is a front of T and every $\lhd$-increasing sequence of members of T has a $\lhd$-upper bound in $\max(T)$.

In the present paper we assume the following.

Context 0.3:
(a) λ is a strongly inaccessible cardinal,
(b) $\bar\lambda = \langle\lambda_\alpha : \alpha < \lambda\rangle$ is a strictly increasing sequence of uncountable regular cardinals, $\sup_{\alpha<\lambda}\lambda_\alpha = \lambda$,
(c) for each $\alpha < \lambda$,

$$\prod_{\beta<\alpha}\lambda_\beta < \lambda_\alpha \quad \text{and} \quad (\forall\xi < \lambda_\alpha)(|\xi|^\alpha < \lambda_\alpha).$$

A. Iteration theorems for λ-support iterations

A.1. Iterations of complete forcing notions and trees of conditions. In this section we recall some basic definitions and facts concerning complete forcing notions and λ-support iterations.

Definition A.1.1: Let $\mathbb{P}$ be a forcing notion.
(1) For a condition $r \in \mathbb{P}$ and a set $S \subseteq \lambda$, let $\Game_0^\lambda(\mathbb{P}, S, r)$ be the following game of two players, *Complete* and *Incomplete*:

> The game lasts at most λ moves and during a play the players construct a sequence $\langle(p_i, q_i) : i < \lambda\rangle$ of pairs of conditions from $\mathbb{P}$ in such a way that $(\forall j < i < \lambda)(r \le p_j \le q_j \le p_i)$ and at the stage $i < \lambda$ of the game: if $i \in S$, then Complete chooses p_i and Incomplete chooses q_i, and if $i \notin S$, then Incomplete chooses p_i and Complete chooses q_i.

Complete wins if and only if for every $i < \lambda$ there are legal moves for both players.
(2) We say that the forcing notion $\mathbb{P}$ is **(λ, S)-strategically complete** if Complete has a winning strategy in the game $\Game_0^\lambda(\mathbb{P}, S, r)$ for each condition $r \in \mathbb{P}$. We say that $\mathbb{P}$ is **strategically $({<}\lambda)$-complete** if it is $(\lambda, \emptyset)$-strategically complete.

(3) We say that $\mathbb{P}$ is $(<\lambda)$-**complete** if every $\leq_{\mathbb{P}}$-increasing chain of length less than λ has an upper bound in $\mathbb{P}$.

(4) Let $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be a model such that ${}^{<\lambda}N \subseteq N$, $|N| = \lambda$ and $\mathbb{P} \in N$. We say that a condition $p \in \mathbb{P}$ is $(N, \mathbb{P})$-**generic in the standard sense** (or just: $(N, \mathbb{P})$-**generic**) if for every $\mathbb{P}$-name $\underset{\sim}{\tau} \in N$ for an ordinal we have $p \Vdash$ "$\underset{\sim}{\tau} \in N$".

(5) $\mathbb{P}$ is λ-**proper in the standard sense** (or just: λ-**proper**) if there is $x \in \mathcal{H}(\chi)$ such that for every model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ satisfying

$$ {}^{<\lambda}N \subseteq N, \quad |N| = \lambda \quad \text{and} \quad \mathbb{P}, x \in N, $$

and every condition $q \in N \cap \mathbb{P}$ there is an $(N, \mathbb{P})$-generic condition $p \in \mathbb{P}$ stronger than q.

Remark A.1.2:

(1) Note that if $\mathbb{P}$ is strategically (λ, λ)-complete and D is a proper normal filter on λ, then in $\mathbf{V}^{\mathbb{P}}$ the normal filter on λ generated by D is also proper. (Abusing notation, we may call this filter also by D.)

(2) On strategic completeness (and variants) see [20, §A.1]; below we recall one result from there.

(3) As the referee pointed out, the idea of A.1.1(1) goes back to Foreman [7] where the extreme cases $S = \emptyset, \lambda$ were considered.

PROPOSITION A.1.3 (See [20, Proposition A.1.2]): *Suppose $\mathbb{P}$ is a forcing notion, $S \subseteq \lambda$.*

(1) *If $\mathbb{P}$ is $(<\lambda)$-complete, then it is (λ, S)-strategically complete.*

(2) *If $S' \subseteq S$ and $\mathbb{P}$ is (λ, S')-strategically complete, then it is (λ, S)-strategically complete.*

(3) *If $\mathbb{Q}$ is (λ, S)-strategically complete, then the forcing with $\mathbb{P}$ does not add new sequences of ordinals of length $< \lambda$.*

Thus the strategic $(<\lambda)$-completeness implies (λ, S)-strategic completeness for any $S \subseteq \lambda$. Also, (λ, λ)-strategic completeness is the weakest among those properties.

PROPOSITION A.1.4: *Suppose that $\mathbb{P}$ is a strategically $(<\lambda)$-complete (atomless) forcing notion, $\alpha^* < \lambda$ and $q_\alpha \in \mathbb{P}$ (for $\alpha < \alpha^*$). Then there are conditions $p_\alpha \in \mathbb{P}$ (for $\alpha < \alpha^*$) such that $q_\alpha \leq p_\alpha$ and for distinct $\alpha, \alpha' < \alpha^*$ the conditions $p_\alpha, p_{\alpha'}$ are incompatible.*

Proof: For $\alpha < \alpha^*$ let st_α be the winning strategy of Complete in the game $\eth^\lambda_0(\mathbb{P}, \emptyset, q_\alpha)$. By induction on $i < \alpha^*$ we define conditions q^i_α, p^i_α as follows:

$p^0_0 = q_0$, q^0_0 is the answer of Complete to $\langle p^0_0\rangle$ according to st_0, $q^0_\alpha = p^0_\alpha = q_\alpha$ for $\alpha > 0$.

Suppose that conditions p^j_α, q^j_α have been defined for $j < i$, $\alpha < \alpha^*$ (where $i < \alpha^*$) so that

(α) $(\forall \alpha < \alpha' < i)(q^{\alpha'}_\alpha, q^{\alpha'}_{\alpha'}$ are incompatible),

(β) for each $\alpha < i$, $\langle (p^j_\alpha, q^j_\alpha) : \alpha \leq j < i\rangle$ is a play of $\eth^\lambda_0(\mathbb{P}, \emptyset, q_\alpha)$ in which Complete uses the strategy st_α, and

(γ) $p^j_\alpha = q^j_\alpha = q_\alpha$ for $\alpha \geq i > j$.

For $\alpha < i$ let r_α be a condition stronger than all q^j_α for $j < i$ (there is one by (β)). If every r_α (for $\alpha < i$) is incompatible with q_i, then we let $p^i_\alpha = r_\alpha$ for $\alpha < i$, $p^i_\alpha = q_\alpha$ for $\alpha \geq i$. Otherwise, let $\alpha_0 < i$ be the first such that r_{α_0}, q_i are compatible. Then we may pick two incompatible conditions $p^i_{\alpha_0}, p^i_i$ above both r_{α_0} and q_i. Next we let $p^i_\alpha = r_\alpha$ for $\alpha < i$, $\alpha \neq \alpha_0$ and $p^i_\alpha = q_\alpha$ for $\alpha > i$. Finally, for $\alpha \leq i$, q^i_α is defined as the answer of Complete according to st_α to $\langle (p^j_\alpha, q^j_\alpha) : j < i\rangle^\frown \langle p^i_\alpha\rangle$, and $q^i_\alpha = q_\alpha$ for $\alpha > i$.

After the inductive definition is carried out we may pick upper bounds p_α to $\langle q^j_\alpha : j < \alpha^*\rangle$ (for $\alpha < \alpha^*$; exist by (β)). The conditions p_α are pairwise incompatible by (α), so we are done. ∎

Both completeness and strategic completeness are preserved in iterations:

PROPOSITION A.1.5: *Suppose that $\langle \mathbb{P}_\alpha, \mathop{\mathbb{Q}}\limits_{\sim}{}_\alpha : \alpha < \zeta^*\rangle$ is a λ-support iteration such that for each $\alpha < \zeta^*$*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\mathop{\mathbb{Q}}\limits_{\sim}{}_\alpha \text{ is } ({<}\lambda)\text{-complete.”}$$

Then the forcing $\mathbb{P}_{\zeta^*}$ is $({<}\lambda)$-complete.

PROPOSITION A.1.6: *Suppose $\bar{\mathbb{Q}} = \langle \mathbb{P}_\varepsilon, \mathop{\mathbb{Q}}\limits_{\sim}{}_\varepsilon : \varepsilon < \gamma\rangle$ is a λ-support iteration and for each $\varepsilon < \gamma$*

$$\Vdash_{\mathbb{P}_\varepsilon} \text{“}\mathop{\mathbb{Q}}\limits_{\sim}{}_\varepsilon \text{ is strategically } ({<}\lambda)\text{-complete”}.$$

Then:

(a) *$\mathbb{P}_\gamma$ is strategically $({<}\lambda)$-complete.*

(b) *Moreover, for each $\varepsilon \leq \gamma$ and $r \in \mathbb{P}_\varepsilon$ there is a winning strategy $\mathrm{st}(\varepsilon, r)$ of Complete in the game $\eth^\lambda_0(\mathbb{P}_\varepsilon, \emptyset, r)$ such that, whenever $\varepsilon_0 < \varepsilon_1 \leq \gamma$ and $r \in \mathbb{P}_{\varepsilon_1}$, we have:*

  (i) *if $\langle (p_i, q_i) : i < \lambda\rangle$ is a play of $\eth^\lambda_0(\mathbb{P}_{\varepsilon_0}, \emptyset, r{\restriction}\varepsilon_0)$ in which Complete follows the strategy $\mathrm{st}(\varepsilon_0, r{\restriction}\varepsilon_0)$,* *then* $\langle (p_i{}^\frown r{\restriction}[\varepsilon_0,\varepsilon_1), q_i{}^\frown r{\restriction}[\varepsilon_0,\varepsilon_1)) : i<\lambda\rangle$ *is a play of* $\eth^\lambda_0(\mathbb{P}_{\varepsilon_1},\emptyset,r)$ *in which Complete uses* $\mathrm{st}(\varepsilon_1,r)$*;*

(ii) *if* $\langle (p_i,q_i) : i<\lambda\rangle$ *is a play of* $\eth^\lambda_0(\mathbb{P}_{\varepsilon_1},\emptyset,r)$ *in which Complete plays according to the strategy* $\mathrm{st}(\varepsilon_1,r)$*,*
then $\langle (p_i{\restriction}\varepsilon_0, q_i{\restriction}\varepsilon_0) : i<\lambda\rangle$ *is a play of* $\eth^\lambda_0(\mathbb{P}_{\varepsilon_0},\emptyset,r{\restriction}\varepsilon_0)$ *in which Complete uses* $\mathrm{st}(\varepsilon_0,r)$*;*

(iii) *if* $\langle (p_i,q_i) : i<i^*\rangle$ *is a partial play of* $\eth^\lambda_0(\mathbb{P}_{\varepsilon_1},\emptyset,r)$ *in which Complete uses* $\mathrm{st}(\varepsilon_1,r)$ *and* $p'\in\mathbb{P}_{\varepsilon_0}$ *is stronger than all* $p_i{\restriction}\varepsilon_0$ *(for* $i<i^*$*), then there is* $p^*\in\mathbb{P}_{\varepsilon_1}$ *such that* $p'=p^*{\restriction}\varepsilon_0$ *and* $p^*\geq p_i$ *for* $i<i^*$*.*

Proof: Let $r\in\mathbb{P}_\gamma$. For each $\varepsilon<\gamma$ choose a $\mathbb{P}_\varepsilon$-name $\underset{\sim}{\mathrm{st}}_\varepsilon$ for a function such that in $\mathbf{V}^{\mathbb{P}_\varepsilon}$:

- the domain $\mathrm{Dom}(\underset{\sim}{\mathrm{st}}_\varepsilon)$ of $\underset{\sim}{\mathrm{st}}_\varepsilon$ consists of all sequences $\langle (p_i,q_i) : i<i^*\rangle^\frown\langle p_{i^*}\rangle$ such that $i^*<\lambda$, $p_i,q_j\in\underset{\sim}{\mathbb{Q}}_\varepsilon$ for $i\leq i^*$, $j<i^*$,
- if $\bar g=\langle p_i,q_i : i<i^*\rangle^\frown\langle p_{i^*}\rangle\in\mathrm{Dom}(\underset{\sim}{\mathrm{st}}_\varepsilon)$, then $\underset{\sim}{\mathrm{st}}_\varepsilon(\bar g)\in\underset{\sim}{\mathbb{Q}}_\varepsilon$ is stronger than p_{i^*},
- if $\bar g=\langle p_i,q_i : i<i^*\rangle^\frown\langle p_{i^*}\rangle\in\mathrm{Dom}(\underset{\sim}{\mathrm{st}}_\varepsilon)$ and $p^*_i=r(\varepsilon)$, then $\underset{\sim}{\mathrm{st}}_\varepsilon(\bar g)=r(\varepsilon)$,
- $\underset{\sim}{\mathrm{st}}_\varepsilon$ is a winning strategy of Complete in $\eth^\lambda_0(\underset{\sim}{\mathbb{Q}}_\varepsilon,\emptyset,r(\varepsilon))$ (when restricted to relevant sequences).

Now, for $\varepsilon_0\leq\gamma$, we define a strategy $\mathrm{st}(\varepsilon_0,r{\restriction}\varepsilon_0)$ of Complete in $\eth^\lambda_0(\mathbb{P}_{\varepsilon_0},\emptyset,r{\restriction}\varepsilon_0)$ as follows. Let $\langle (p_i,q_i) : i<i^*\rangle^\frown\langle p_{i^*}\rangle$ be a partial play of $\eth^\lambda_0(\mathbb{P}_{\varepsilon_0},\emptyset,r{\restriction}\varepsilon_0)$, $i^*<\lambda$. The answer q_{i^*} given to Complete by $\mathrm{st}(\varepsilon_0,r{\restriction}\varepsilon_0)$ is described by

- $\mathrm{Dom}(q_{i^*})=\mathrm{Dom}(p_{i^*})$, and for each $\varepsilon\in\mathrm{Dom}(q_{i^*})$:
- if $p_{i^*}(\varepsilon)=r(\varepsilon)$, then $q_{i^*}(\varepsilon)=r(\varepsilon)$, otherwise $q_{i^*}(\varepsilon)$ is the $<^*_\chi$-first $\mathbb{P}_\varepsilon$-name for a member of $\underset{\sim}{\mathbb{Q}}_\varepsilon$ such that

$$\Vdash_{\mathbb{P}_\varepsilon} q_{i^*}(\varepsilon)=\underset{\sim}{\mathrm{st}}_\varepsilon(\langle (p_i(\varepsilon),q_i(\varepsilon) : i<i^*\rangle^\frown\langle p_{i^*}(\varepsilon)\rangle). \qquad \blacksquare$$

Definition A.1.7 (Compare [20, A.3.3, A.3.2])*:*

(1) Let α,γ be ordinals, $\emptyset\neq w\subseteq\gamma$. **A standard** $(w,\alpha)^\gamma$**-tree** is a pair $\mathcal{T}=(T,\mathrm{rk})$ such that:
   - $\mathrm{rk}\colon T\longrightarrow w\cup\{\gamma\}$,
   - if $t\in T$ and $\mathrm{rk}(t)=\varepsilon$, then t is a sequence $\langle (t)_\zeta : \zeta\in w\cap\varepsilon\rangle$, where each $(t)_\zeta$ is a sequence of length α,
   - $(T,\vartriangleleft)$ is a tree with root $\langle\rangle$ and such that every chain in T has a $\vartriangleleft$-upper bound in T.

   We will keep the convention that $\mathcal{T}^x_y$ is (T^x_y,rk^x_y).

(2) Suppose that $w_0 \subseteq w_1 \subseteq \gamma$, $\alpha_0 \leq \alpha_1$, and $\mathcal{T}_1 = (T_1, \mathrm{rk}_1)$ is a standard $(w_1, \alpha_1)^\gamma$-tree. **The projection** $\mathrm{proj}^{(w_1,\alpha_1)}_{(w_0,\alpha_0)}(\mathcal{T}_1)$ **of** $\mathcal{T}_1$ **onto** (w_0, α_0) is defined as a standard $(w_0, \alpha_0)^\gamma$-tree $\mathcal{T}_0 = (T_0, \mathrm{rk}_0)$ such that

$$T_0 = \{\langle (t)_\zeta \restriction \alpha_0 : \zeta \in w_0 \cap \mathrm{rk}_1(t)\rangle : t = \langle (t)_\zeta : \zeta \in w_1 \cap \mathrm{rk}_1(t)\rangle \in T_1\}.$$

The mapping

$$T_1 \ni \langle (t)_\zeta : \zeta \in w_1 \cap \mathrm{rk}_1(t)\rangle \longmapsto \langle (t)_\zeta \restriction \alpha_0 : \zeta \in w_0 \cap \mathrm{rk}_1(t)\rangle \in T_0$$

will also be denoted by $\mathrm{proj}^{(w_1,\alpha_1)}_{(w_0,\alpha_0)}$.

(3) We say that $\bar{\mathcal{T}} = \langle \mathcal{T}_\alpha : \alpha < \alpha^*\rangle$ is **a legal sequence of γ-trees** if for some increasing continuous sequence $\bar{w} = \langle w_\alpha : \alpha < \alpha^*\rangle$ of subsets of γ we have
   (i) $\mathcal{T}_\alpha$ is a standard $(w_\alpha, \alpha)^\gamma$-tree (for $\alpha < \alpha^*$),
   (ii) if $\alpha < \beta < \alpha^*$, then $\mathcal{T}_\alpha = \mathrm{proj}^{(w_\beta,\beta)}_{(w_\alpha,\alpha)}(\mathcal{T}_\beta)$.

(4) Suppose that $\bar{\mathcal{T}} = \langle \mathcal{T}_\alpha : \alpha < \alpha^*\rangle$ is a legal sequence of γ-trees and α^* is a limit ordinal. Let $w_\alpha \subseteq \gamma$ be such that $\mathcal{T}_\alpha$ is a standard $(w_\alpha, \alpha)^\gamma$-tree (for $\alpha < \alpha^*$) and let $w = \bigcup_{\alpha<\alpha^*} w_\alpha$. **The inverse limit** $\overleftarrow{\lim}(\bar{\mathcal{T}})$ **of** $\bar{\mathcal{T}}$ is a standard $(w, \alpha^*)^\gamma$-tree $(T^{\lim}, \mathrm{rk}^{\lim})$ such that
   ($\otimes$) $T^{\lim}$ consists of all sequences t satisfying
      (i) $\mathrm{Dom}(t)$ is an initial segment of w (not necessarily proper);
      (ii) if $\zeta \in \mathrm{Dom}(t)$, then $(t)_\zeta$ is a sequence of length α^*;
      (iii) $\langle (t)_\zeta \restriction \alpha : \zeta \in w_\alpha \cap \mathrm{Dom}(t)\rangle \in T_\alpha$ for each $\alpha < \alpha^*$.

(5) A legal sequence $\bar{\mathcal{T}} = \langle \mathcal{T}_\alpha : \alpha < \alpha^*\rangle$ is **continuous** if for each limit ordinal $\beta < \alpha^*$, $\mathcal{T}_\beta = \overleftarrow{\lim}(\bar{\mathcal{T}} \restriction \beta)$.

(6) Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \mathop{\mathbb{Q}}\limits_{\sim}{}_i : i < \gamma\rangle$ be a λ-support iteration. **A standard tree of conditions in** $\bar{\mathbb{Q}}$ is a system $\bar{p} = \langle p_t : t \in T\rangle$ such that
   - (T, rk) is a standard $(w, \alpha)^\gamma$-tree for some $w \subseteq \gamma$ and an ordinal α,
   - $p_t \in \mathbb{P}_{\mathrm{rk}(t)}$ for $t \in T$, and
   - if $s, t \in T$, $s \vartriangleleft t$, then $p_s = p_t \restriction \mathrm{rk}(s)$.

(7) Let $\bar{p}^0, \bar{p}^1$ be standard trees of conditions in $\bar{\mathbb{Q}}$, $\bar{p}^i = \langle p^i_t : t \in T_i\rangle$, where $\mathcal{T}_0 = \mathrm{proj}^{(w_1,\alpha_1)}_{(w_0,\alpha_0)}(\mathcal{T}_1)$, $w_0 \subseteq w_1 \subseteq \gamma$, $\alpha_0 < \alpha_1$. We will write $\bar{p}^0 \leq^{w_1,\alpha_1}_{w_0,\alpha_0} \bar{p}^1$ (or just $\bar{p}^0 \leq \bar{p}^1$) whenever for each $t \in T_1$, letting $t' = \mathrm{proj}^{(w_1,\alpha_1)}_{(w_0,\alpha_0)}(t) \in T_0$, we have $p^0_{t'} \restriction \mathrm{rk}_1(t) \leq p^1_t$.

Remark A.1.8: Concerning Definition A.1.7(4), note that $T^{\lim}$ satisfies the requirements of A.1.7(1) (so $\lim^{\leftarrow}(\bar{\mathcal{T}})$ is indeed a standard $(w, \alpha^*)^\gamma$-tree). Also, if the sequence $\bar{\mathcal{T}}$ is continuous (and T_α's are not empty), then $T^{\lim} \neq \emptyset$.

Proposition A.1.9: *Assume that* $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \mathop{\mathbb{Q}}\limits_{\sim}{}_i : i < \gamma \rangle$ *is a* λ*-support iteration such that for all* $i < \gamma$ *we have*

$$\Vdash_{\mathbb{P}_i} \text{“}\mathop{\mathbb{Q}}\limits_{\sim}{}_i \textit{ is strategically } (<\lambda)\textit{-complete”}.$$

Suppose that $\bar{p} = \langle p_t : t \in T \rangle$ *is a standard tree of conditions in* $\bar{\mathbb{Q}}$, $|T| < \lambda$, *and* $\mathcal{I} \subseteq \mathbb{P}_\gamma$ *is open dense. Then there is a standard tree of conditions* $\bar{q} = \langle q_t : t \in T \rangle$ *such that* $\bar{p} \le \bar{q}$ *and* $(\forall t \in T)(\mathrm{rk}(t) = \gamma \Rightarrow q_t \in \mathcal{I})$.

Proof: For $\varepsilon \le \gamma$ and $r \in \mathbb{P}_\varepsilon$, let $\mathrm{st}(\varepsilon, r)$ be a winning strategy of Complete in $\Game_0^\lambda(\mathbb{P}_\varepsilon, \emptyset, r)$ as in A.1.6(b). Let

$$T^{\max} \overset{\text{def}}{=} \{t \in T : \neg(\exists t' \in T)(t \vartriangleleft t')\} = \{t_\zeta : \zeta < \kappa\}$$

(where $\kappa < \lambda$ is a cardinal). We construct partial plays $\langle (p_i^\zeta, q_i^\zeta) : i \le \kappa \rangle$ of $\Game_0^\lambda(\mathbb{P}_{\mathrm{rk}(p_{t_\zeta})}, \emptyset, p_{t_\zeta})$ (for $\zeta < \kappa$) in which Complete uses strategy $\mathrm{st}(\mathrm{rk}(p_{t_\zeta}), p_{t_\zeta})$ and such that

(α) if $\zeta < \kappa$ and $\mathrm{rk}(p_{t_\zeta}) = \gamma$, then $p_\zeta^\zeta \in \mathcal{I}$,

(β) if $t \vartriangleleft t_\zeta$, $t \vartriangleleft t_\xi$, $t \in T$, $\zeta, \xi < \kappa$, $i \le \kappa$, then $p_i^\zeta \restriction \mathrm{rk}(t) = p_i^\xi \restriction \mathrm{rk}(t)$ and $q_i^\zeta \restriction \mathrm{rk}(t) = q_i^\xi \restriction \mathrm{rk}(t)$.

So suppose we have defined p_j^ζ, q_j^ζ for $\zeta < \kappa$, $j < i < \kappa$. First we look at $\langle (p_j^i, q_j^i) : j < i \rangle$ — it is a play of $\Game_0^\lambda(\mathbb{P}_{\mathrm{rk}(p_{t_i})}, \emptyset, p_{t_i})$ in which Complete uses $\mathrm{st}(\mathrm{rk}(p_{t_i}), p_{t_i})$, so we may find a condition $p_i^i \in \mathbb{P}_{\mathrm{rk}(p_{t_i})}$ stronger than all p_j^i, q_j^i for $j < i$, and such that $\mathrm{rk}(p_{t_i}) = \gamma \Rightarrow p_i^i \in \mathcal{I}$. Next, for $\zeta < \kappa$, $\zeta \ne i$, we define p_i^ζ as follows: let $t \in T$ be such that $t \vartriangleleft t_\zeta$, $t \vartriangleleft t_i$ and $\mathrm{rk}(t)$ is the largest possible, we declare that

$$\mathrm{Dom}(p_i^\zeta) = (\mathrm{Dom}(p_i^i) \cap \mathrm{rk}(t)) \cup \bigcup_{j<i} \mathrm{Dom}(q_j^\zeta) \cup \mathrm{Dom}(p_{t_\zeta})$$

and $p_i^\zeta \restriction \mathrm{rk}(t) = p_i^i \restriction \mathrm{rk}(t)$, and for $\varepsilon \in [\mathrm{rk}(t), \gamma)$ we have that $p_i^\zeta(\varepsilon)$ is the $<_\chi^*$-first $\mathbb{P}_\varepsilon$-name for a member of $\mathop{\mathbb{Q}}\limits_{\sim}{}_\varepsilon$ such that

$$p_i^\zeta \restriction \varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}p_i^\zeta(\varepsilon) \text{ is an upper bound to } \{p_{t_\zeta}(\varepsilon)\} \cup \{q_j^\zeta(\varepsilon) : j < i\}\text{”}.$$

The definition of p_i^ζ's is correct by A.1.6(b)(iii+ii). Also, by the choice of “the $<_\chi^*$-first” names and clause (β) at earlier stages we get clause (β) for p_i^ζ's.

Finally we define q_i^ζ (for $\zeta < \kappa$) as the condition given to Complete by $\mathrm{st}(\mathrm{rk}(t_\zeta), p_{t_\zeta})$ in answer to $\langle (p_j^\zeta, q_j^\zeta) : j < i \rangle ^\frown \langle p_i^\zeta \rangle$. (Again, one easily verifies $(\alpha), (\beta)$.)

The conditions $p^\zeta_\kappa, q^\zeta_\kappa$ are chosen in a similar manner except that we do not have to worry about entering $\mathcal{I}$ anymore, so we may take p^0_κ to be any bound to the previously defined conditions p^0_i, q^0_i, and other $p^\zeta_\kappa, q^\zeta_\kappa$ are defined as earlier.

After the above construction is carried out, for $t \in T$ we let

$$q_t = p^\zeta_\kappa \restriction \mathrm{rk}(t) \text{ for some (equivalently, all) } \zeta < \kappa \text{ such that } t \trianglelefteq t_\zeta.$$

It should be clear that $\bar{q} = \langle q_t : t \in T\rangle$ is as required. ∎

Let us close this section by recalling an important result on easy ensuring that λ-support iteration satisfies the λ^{++}-cc. Its proof is a fairly straightforward modification of the proof of the respective result for CS iterations; see [19, Ch. III, Thm. 4.1], Abraham [1, §2] for the CS case, Eisworth [6, §3] for the general case of λ-support iterations.

Theorem A.1.10: *Assume $2^\lambda = \lambda^+$, $\lambda^{<\lambda} = \lambda$. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \lambda^{++}\rangle$ be λ-support iteration such that for all $i < \lambda^{++}$ we have*

- *$\mathbb{P}_i$ is λ-proper,*
- *$\Vdash_{\mathbb{P}_i}$ "$|\underset{\sim}{\mathbb{Q}}_i| \leq \lambda^+$".*

Then the limit $\mathbb{P}_{\lambda^{++}}$ satisfies the λ^{++}-cc.

A.2. Bounding properties. The results on preservation in CS iterations of properties like the Sacks property and ${}^\omega\omega$-bounding property were among the earliest in the theory of proper forcing. Here we introduce relatives of these two properties for λ-reals and we show suitable iteration theorems. For both properties, the properness is "built into the property".

Recall that $\lambda, \bar{\lambda}$ are assumed to be as specified in Context 0.3.

Definition A.2.1: Let $\mathbb{P}$ be a forcing notion.

(1) For a condition $p \in \mathbb{P}$ and an ordinal $i_0 < \lambda$ we define a game $\Game^{\rm Sacks}_{\bar{\lambda}}(i_0, p, \mathbb{P})$ of two players, **Generic** and **Antigeneric**. A play lasts at most λ moves indexed by ordinals from the interval $[i_0, \lambda)$, and during it the players construct a sequence $\langle (s_i, \bar{q}^i, \bar{p}^i) : i_0 \leq i < \lambda\rangle$ as follows. At stage i of the play (where $i_0 \leq i < \lambda$), first Generic chooses $s_i \subseteq {}^{\leq i+1}\lambda$ and a system $\bar{q}^i = \langle q^i_\eta : \eta \in s_i \cap {}^{i+1}\lambda\rangle$ such that

(α) s_i is a complete λ-tree of height $i+1$ and

$$(\forall \eta \in s_i)(\exists \nu \in s_i)(\eta \trianglelefteq \nu \,\&\, \mathrm{lh}(\nu) = i+1);$$

and $\mathrm{lh}(\mathrm{root}(s_i)) = i_0$,

(β) for all j such that $i_0 \leq j < i$ we have $s_j = s_i \cap {}^{\leq j+1}\lambda$,

(γ) $q^i_\eta \in \mathbb{P}$ for all $\eta \in s_i \cap {}^{i+1}\lambda$, and

(δ) if $i_0 \le j < i$, $\nu \in s_i \cap {}^{j+1}\lambda$ and $\nu \lhd \eta \in s_i \cap {}^{i+1}\lambda$, then $p^j_\nu \le q^i_\eta$ and $p \le q^i_\eta$,

(ε) $|s_i \cap {}^{i+1}\lambda| < \lambda_i$.

Then Antigeneric answers choosing a system $\bar{p}^i = \langle p^i_\eta : \eta \in s_i \cap {}^{i+1}\lambda\rangle$ of conditions in $\mathbb{P}$ such that $q^i_\eta \le p^i_\eta$ for each $\eta \in s_i \cap {}^{i+1}\lambda$.

Generic wins a play if she always has legal moves (so the play lasts λ steps) and there are a condition $q \ge p$ and a $\mathbb{P}$-name $\underset{\sim}{\rho}$ such that

($\circledast$) $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\rho} \in {}^\lambda\lambda \,\&\, (\forall i \in [i_0, \lambda))(\underset{\sim}{\rho}{\restriction}(i+1) \in s_i \,\&\, q^i_{\underset{\sim}{\rho}\restriction(i+1)} \in \Gamma_{\mathbb{P}})$".

(2) We say that $\mathbb{P}$ has the **strong $\bar{\lambda}$-Sacks property** whenever

(a) $\mathbb{P}$ is strategically $(<\lambda)$-complete, and

(b) Generic has a winning strategy in the game $\Game^{\rm Sacks}_{\bar{\lambda}}(i_0, p, \mathbb{P})$ for any $i_0 < \lambda$ and $p \in \mathbb{P}$.

(3) We say that $\mathbb{P}$ has the **$\bar{\lambda}$-Sacks property** if for every $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\underset{\sim}{\tau}$ such that $p \Vdash \underset{\sim}{\tau} : \lambda \longrightarrow \mathbf{V}$, there are a condition $q \ge p$ and a sequence $\langle a_\alpha : \alpha < \lambda\rangle$ such that $|a_\alpha| < \lambda_\alpha$ (for $\alpha < \lambda$) and $q \Vdash$ "$(\forall \alpha < \lambda)(\underset{\sim}{\tau}(\alpha) \in a_\alpha)$".

Remark A.2.2:

(1) At a stage $i < \lambda$ of a play of $\Game^{\rm Sacks}_{\bar{\lambda}}(i_0, p, \mathbb{P})$, the Antigeneric player may play stronger conditions, and using A.1.4 we may require that if $\bar{p}^i = \langle p^i_\eta : \eta \in s_i \cap {}^{i+1}\lambda\rangle$ is his move, then the conditions p^i_η are pairwise incompatible. Thus the winning criterion ($\circledast$) could be replaced by

($\circledast$)$^-$ $q \Vdash_{\mathbb{P}}$ "$(\forall i \in [i_0, \lambda))(\exists \eta \in s_i \cap {}^{i+1}\lambda)(q^i_\eta \in \Gamma_{\mathbb{P}})$"

(thus eliminating the use of $\underset{\sim}{\rho}$). However, the λ-branch along which the conditions are from the generic filter will be new (so we cannot replace the name $\underset{\sim}{\rho}$ by an object $\rho \in {}^\lambda\lambda$).

(2) Note that if Generic has a winning strategy in $\Game^{\rm Sacks}_{\bar{\lambda}}(0, p, \mathbb{P})$, then she has one in $\Game^{\rm Sacks}_{\bar{\lambda}}(i_0, p, \mathbb{P})$ for all $i_0 < \lambda$. (Remember: the sequence $\bar{\lambda}$ is increasing.) The reason why we have i_0 as a parameter is a notational convenience.

(3) Plainly, if Generic has a winning strategy in $\Game^{\rm Sacks}_{\bar{\lambda}}(i_0, p, \mathbb{P})$, then she has one with the following property:

($\boxtimes_{\rm nice}$) if $s_i, \bar{q}_i$ are given to Generic as a move at a stage $i \in [i_0, \lambda)$, then for every $\eta \in s_i \cap {}^i\lambda$, the set $\{\alpha < \lambda : \eta^\frown\langle\alpha\rangle \in s_i\}$ is an initial segment of λ_i and $\eta(j) = 0$ for all $j < i_0$.

Strategies satisfying the condition ($\boxtimes_{\rm nice}$) will be called **nice**.

(4) Easily, if $\mathbb{P}$ has the strong $\bar{\lambda}$-Sacks property, then it has the $\bar{\lambda}$-Sacks property.

Let us note that the demand A.2.1(2b) already implies a large amount of completeness.

PROPOSITION A.2.3: *If Generic has a winning strategy in the game*

$$\partial^{\rm Sacks}_{\bar{\lambda}}(0,p,\mathbb{P}),\quad \textit{for any } p\in\mathbb{P},$$

then $\mathbb{P}$ is (λ,λ)-strategically complete.

Proof: The main point is that the trees s_i played by Generic are complete, so no branches "die" at limit levels (see 0.2(3)). So when playing a game of $\partial^\lambda_0(\mathbb{P},\lambda,r)$, Complete may construct aside a play $\langle (s_i,\bar{q}^i,\bar{p}^i): i<\lambda\rangle$ of $\partial^{\rm Sacks}_{\bar{\lambda}}(0,r,\mathbb{P})$ and decide her moves in $\partial^\lambda_0(\mathbb{P},\lambda,r)$ as follows. Let st be a winning strategy of Generic in $\partial^{\rm Sacks}_{\bar{\lambda}}(0,r,\mathbb{P})$.

At the beginning of the game of $\partial^\lambda_0(\mathbb{P},\lambda,r)$, Complete writes aside the first move $(s_0,\bar{q}^0)$ given by st and she picks a node $\eta_0\in s_0\setminus\{\langle\rangle\}$. Then (in $\partial^\lambda_0(\mathbb{P},\lambda,r)$) she plays $p_0=q^0_{\eta_0}$. If q_0 is the answer of Incomplete to this move, Complete writes aside $p^0_{\eta_0}=q_0$, $p^0_\eta=q^0_\eta$ for $\eta\in s_0\setminus\{\eta_0,\langle\rangle\}$, thus creating a move of Antigeneric in $\partial^{\rm Sacks}_{\bar{\lambda}}(0,r,\mathbb{P})$.

Suppose that the players have arrived to a stage $i<\lambda$ of $\partial^\lambda_0(\mathbb{P},\lambda,r)$ and

- they have played $\langle p_j,q_j: j<i\rangle$, and
- Generic has written aside a partial play $\langle (s_j,\bar{q}^j,\bar{p}^j): j<i\rangle$ of $\partial^{\rm Sacks}_{\bar{\lambda}}(0,r,\mathbb{P})$ in which st has been used, and
- Generic has chosen a $\vartriangleleft$-increasing sequence $\langle \eta_j : j<i\rangle$ of nodes $\eta_j\in s_j\cap {}^{j+1}\lambda$.

Now Complete applies the strategy st to the play of $\partial^{\rm Sacks}_{\bar{\lambda}}(0,r,\mathbb{P})$ she has written aside, getting $(s_i,\bar{q}^i)$. The tree s_i is complete and it extends all the trees s_j (for $j<i$), so there is a node $\eta_i\in s_i\cap {}^{i+1}\lambda$ such that $\eta_j\vartriangleleft\eta_i$ (for $j<i$). Now, in the play of $\partial^\lambda_0(\mathbb{P},\lambda,r)$ she puts $p_i=q^i_{\eta_i}$. If q_i is the answer of Incomplete, she writes aside a move of Antigeneric in $\partial^{\rm Sacks}_{\bar{\lambda}}(0,r,\mathbb{P})$ as follows: $p^i_{\eta_i}=q_i$, $p^i_\eta=q^i_\eta$ for $\eta\in s_i\cap {}^{i+1}\lambda\setminus\{\eta_i\}$.

Easily, the procedure described above gives a winning strategy of Complete in $\partial^\lambda_0(\mathbb{P},\lambda,r)$. ∎

THEOREM A.2.4: *Suppose that $\bar{\mathbb{Q}}=\langle \mathbb{P}_\alpha,\underset{\sim}{\mathbb{Q}}_\alpha:\alpha<\gamma\rangle$ is a λ-support iteration such that for all $\alpha<\gamma$:*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\underset{\sim}{\mathbb{Q}}_\alpha \textit{ has the strong } \bar{\lambda}\textit{-Sacks property”}.$$

Then:

(a) $\mathbb{P}_\gamma$ *has the* $\bar{\lambda}$*-Sacks property.*

(b) *If* $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$ *and* $\bar{\lambda}, \lambda, p, \bar{\mathbb{Q}}, \mathbb{P}_\gamma, \ldots \in N$, $p \in \mathbb{P}_\gamma$, *then there is an* $(N, \mathbb{P}_\gamma)$*-generic condition* $r \in \mathbb{P}_\gamma$ *stronger than* p.

Proof: (a) First note that each $\mathbb{P}_\alpha$ is strategically $({<}\lambda)$-complete (by A.1.6; remember A.2.1(2a)), so our assumptions on $\lambda, \bar{\lambda}$ hold in intermediate universes $\mathbf{V}^{\mathbb{P}_\alpha}$.

For $\alpha < \gamma$ and $i_0 < \lambda$ and a $\mathbb{P}_\alpha$-name $\underset{\sim}{q}$ for a condition in $\underset{\sim}{\mathbb{Q}}_\alpha$, let $\underset{\sim}{\mathrm{st}}_\alpha(i_0, \underset{\sim}{q})$ be the $<^*_\chi$-first $\mathbb{P}_\alpha$-name for a nice (see A.2.2(3)) winning strategy of Generic in the game $\Game^{\mathrm{Sacks}}_{\bar{\lambda}}(i_0, \underset{\sim}{q}, \underset{\sim}{\mathbb{Q}}_\alpha)$.

Let $\underset{\sim}{\tau}$ be a $\mathbb{P}_\gamma$-name for a function from λ to $\mathbf{V}$, $p \in \mathbb{P}_\gamma$. Pick a model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ such that

$$\bar{\lambda}, \lambda, \underset{\sim}{\tau}, p, \bar{\mathbb{Q}}, \mathbb{P}_\gamma, \ldots \in N, \quad \text{and} \quad |N| = \lambda \quad \text{and} \quad {}^{<\lambda}N \subseteq N.$$

Note that if $i_0 < \lambda$, $\alpha \in N \cap \gamma$, and $\underset{\sim}{q} \in N$ is a $\mathbb{P}_\alpha$-name for a member of $\underset{\sim}{\mathbb{Q}}_\alpha$, then $\underset{\sim}{\mathrm{st}}_\alpha(i_0, \underset{\sim}{q}) \in N$. Also, as $\bar{\mathbb{Q}}$ is a λ-support iteration of $({<}\lambda)$-strategically complete forcing notions, we may use A.1.6 inside N and for each $\varepsilon \in N \cap (\gamma+1)$ and $r \in \mathbb{P}_\varepsilon \cap N$ fix a winning strategy $\mathrm{st}^*(\varepsilon, r) \in N$ of Complete in the game $\Game^\lambda_0(\mathbb{P}_\varepsilon, \emptyset, r)$ so that conditions (i)–(iii) of A.1.6(b) hold.

Fix a list $\bar{\mathcal{I}} = \langle \mathcal{I}_\xi : \xi < \lambda \rangle$ of all open dense subsets of $\mathbb{P}_\gamma$ from N and a bijection $\pi : N \cap \gamma \longrightarrow \lambda$ (we may assume that $\gamma \geq \lambda$). For $i < \lambda$ let $w_i = \pi^{-1}[i]$ (thus $\bar{w} = \langle w_i : i < \lambda \rangle$ is an increasing continuous sequence of subsets of $N \cap \gamma$, each of size $< \lambda$, and $\bigcup_{i<\lambda} w_i = N \cap \gamma$).

By induction on $i < \lambda$ we define sequences

$$\langle \mathcal{T}_i : i < \lambda \rangle \quad \text{and} \quad \langle \bar{p}^i, \bar{p}^i_* : i < \lambda \text{ is not a limit ordinal} \rangle$$

such that the following requirements are satisfied.

(α) $\langle \mathcal{T}_i : i < \lambda \rangle$ is a continuous legal sequence of γ-trees; $\mathcal{T}_i \in N$ is a standard $(w_i, i)^\gamma$-tree, $|T_{i+1}| < \lambda_i$, and $(\forall t \in T_i)(\exists t' \in T_i)(t \trianglelefteq t' \;\&\; \mathrm{rk}_i(t') = \gamma)$.

(β) For $i < \lambda$ and $t \in T_i$ such that $\mathrm{rk}_i(t) < \gamma$ let $\psi_i(t) = \{(s)_{\mathrm{rk}_i(t)} : t \vartriangleleft s \in T_i\}$. Then (for each i, t as above) $\emptyset \neq \psi_i(t) \subseteq \prod_{j<i} \lambda_j$ and for each $\eta \in \psi_i(t)$ and $i' < \pi(\mathrm{rk}_i(t))$ we have $\eta(i') = 0$.

(γ) If $\xi \in N \cap \gamma$, $\pi(\xi) < i < j < \lambda$, $t \in T_j$, $\mathrm{rk}_j(t) = \xi$ and $t' = \mathrm{proj}^{w_j, j}_{w_i, i}(t) \in T_i$ (so $\mathrm{rk}_i(t') = \xi$), then $\psi_i(t') = \{\eta{\restriction}i : \eta \in \psi_j(t)\}$.

(δ) $T_0 = \{\langle\rangle\}$, $\bar{p}^0 = \langle p^0_{\langle\rangle} \rangle$, $p^0_{\langle\rangle} = p$, and for $i < \lambda$, $\bar{p}^{i+1} = \langle p^{i+1}_t : t \in T_{i+1} \rangle$ and $\bar{p}^{i+1}_* = \langle p^{i+1}_{*,t} : t \in T_{i+1} \rangle$ are standard trees of conditions in $\bar{\mathbb{Q}}$, both belonging to N and such that $\bar{p}^{i+1}_* \leq \bar{p}^{i+1}$.

(ε) If $i<j<\lambda$, then $\bar{p}^{i+1} \leq^{w_{j+1},j+1}_{w_{i+1},i+1} \bar{p}^{j+1}$.

(ζ) If $t_{i+1} \in T_{i+1}$ (for $i<\lambda$) are such that $\mathrm{rk}_{i+1}(t_{i+1})=\gamma$ and $t_{i+1} = \mathrm{proj}^{w_{j+1},j+1}_{w_{i+1},i+1}(t_{j+1})$ (for $i<j$), then $\langle p^{i+1}_{*,t_{i+1}}, p^{i+1}_{t_{i+1}} : i<\lambda\rangle$ is a play of the game $\Game^\lambda_0(\mathbb{P}_\gamma,\emptyset,p)$ in which Complete uses the strategy $\mathrm{st}^*(\gamma,p)$.

(η) If $t\in T_{i+1}$, $\mathrm{rk}_{i+1}(t)=\gamma$, then $p^{i+1}_t \in \mathcal{I}_\xi$ for all $\xi\leq i$ and p^{i+1}_t forces a value to $\underset{\sim}{\tau}(i)$.

(θ) Assume that $\xi\in N\cap\gamma$, $\pi(\xi)=i_0\leq i$ and $t\in T_{i+1}$ is such that $\mathrm{rk}_{i+1}(t)=\xi$. Let, for $j\leq i$, $t_j=\mathrm{proj}^{w_{i+1},i+1}_{w_j,j}(t)$ and let $\underset{\sim}{r}$ be the $<^*_\chi$-first $\mathbb{P}_\xi$-name for a member of $\underset{\sim}{\mathbb{Q}}_\xi$ such that

$$\Vdash_{\mathbb{P}_\xi} \text{"if there is a common upper bound to } \{p^j_{t_j}(\xi) : j\leq i_0 \text{ is non-limit}\}, \text{ then } \underset{\sim}{r} \text{ is such an upper bound, else } \underset{\sim}{r}=p(\xi)\text{"}.$$

Furthermore, for $i_0\leq j\leq i$ and $\eta\in\psi_{j+1}(t_{j+1})$, fix $s^{j+1}_\eta\in T_{j+1}$ such that $\mathrm{rk}_{j+1}(s^{j+1}_\eta)>\xi$ and $(s^{j+1}_\eta)_\xi=\eta$, $t_{j+1}\lhd s^{j+1}_\eta$, and put $\underset{\sim}{r}^j_\eta = p^{j+1}_{s^{j+1}_\eta}(\xi)$. *Then* the condition p^{i+1}_t forces in $\mathbb{P}_\xi$ the following:

there is a partial play $\langle s_j,\bar{q}^j,\bar{r}^j : i_0\leq j\leq i\rangle$ of the game $\Game^{\mathrm{Sacks}}_\lambda(i_0,\underset{\sim}{r},\underset{\sim}{\mathbb{Q}}_\xi)$ in which the Generic player uses the strategy $\underset{\sim}{\mathrm{st}}_\xi(i_0,\underset{\sim}{r})$ and, for $i_0\leq j\leq i$,

$$s_j\cap{}^{j+1}\lambda=\psi_{j+1}(t_{j+1}) \quad\text{and}\quad \bar{r}^j=\langle \underset{\sim}{r}^j_\eta : \eta\in s_j\cap{}^{j+1}\lambda\rangle.$$

Concerning the choice of s^{j+1}_η (and $\underset{\sim}{r}^j_\eta$) in clause ($\theta$) above, note that (for t,η,j as above):

if $s^+_\eta, s^*_\eta\in T_{j+1}$ are such that $\mathrm{rk}_{j+1}(s^x_\eta)>\xi$, $(s^x_\eta)_\xi=\eta$ and $t_{j+1}\lhd s^x_\eta$ (for $x\in\{*,+\}$), then $p^{j+1}_{s^+_\eta}(\xi)=p^{j+1}_{s^*_\eta}(\xi)=p^{j+1}_s(\xi)$, where $s=t_{j+1}\cup\{(\xi,\eta)\}=s^+_\eta{\restriction}(\xi+1)=s^*_\eta{\restriction}(\xi+1)$

(remember $\bar{p}^{j+1}$ is a standard tree of conditions; see the last demand in A.1.7(6)).

Let us describe how the construction of $\langle\mathcal{T}_i : i<\lambda\rangle$ and $\langle\bar{p}^{i+1} : i<\lambda\rangle$ is carried out. We start with letting $T_0=\{\langle\rangle\}$, $p^0_{\langle\rangle}=p$ (as in (δ)). Now suppose that we have defined $\mathcal{T}_j,\bar{p}^j,\bar{p}^j_*$ for $j<i<\lambda$ so that clauses (α)–(θ) are satisfied. If i is a limit ordinal, then we let $\mathcal{T}_i=\overleftarrow{\lim}(\langle\mathcal{T}_j : j<i\rangle)\in N$ ($\bar{p}^i,\bar{p}^i_*$ are not defined). It is straightforward to verify conditions (α)–(γ) (use the inductive hypothesis), clauses (δ)–(θ) are not relevant.

So suppose now that i is a successor ordinal, say $i=i_0+1$. First we let $\mathcal{T}^*$ be the largest standard $(w_i,i)^\gamma$-tree such that $\mathrm{proj}^{w_i,i}_{w_{i_0},i_0}(\mathcal{T}^*)=\mathcal{T}_{i_0}$, if $t=\langle(t)_\zeta : \zeta\in w_i\cap\mathrm{rk}^*(t)\rangle\in T^*$, then $(t)_\zeta(i_0)<\lambda_{i_0}$, if $\pi(\zeta)=i_0$, then $(t)_\zeta{\restriction}i_0\equiv 0$. (Plainly $\mathcal{T}^*\in N$ and $|T^*|<\lambda$.) Next, for each $t\in T^*$ we define a condition

$q_t \in \mathbb{P}_{\mathrm{rk}^*(t)} \cap N$ and names $\underset{\sim}{\alpha}^t(\xi)$ for ordinals (for $\xi \in w_i \cap \mathrm{rk}^*(t)$). For this let us fix $t \in T^*$ and let $t_j = \mathrm{proj}^{w_i,i}_{w_j,j}(t) \in T_j$ for $j < i$. Put

$$\mathrm{Dom}(q_t) = \Big(w_i \cup \bigcup\{\mathrm{Dom}(p^j_{t_j}) : j < i \text{ is not a limit }\}\Big) \cap \mathrm{rk}^*(t),$$

and for $\zeta \in \mathrm{Dom}(q_t)$ let $q_t(\zeta)$ be a $\mathbb{P}_\zeta$-name for a member of $\underset{\sim}{\mathbb{Q}}_\zeta$ chosen as follows. If $\zeta \in \mathrm{Dom}(q_t) \setminus w_i$, then $q_t(\zeta)$ is the $<^*_\chi$-first $\mathbb{P}_\zeta$-name such that

$$\Vdash_{\mathbb{P}_\zeta} \text{“if possible, then } q_t(\zeta) \text{ is an upper bound to } \{p^j_{t_j}(\zeta) : j < i \text{ is non-limit}\}\text{”}.$$

If $\zeta \in \mathrm{Dom}(q_t) \cap w_i$, then $\underset{\sim}{\alpha}^t(\zeta) \in N$ is a $\mathbb{P}_\zeta$-name for an element of λ_i and $q_t(\zeta)$ is the $<^*_\chi$-first $\mathbb{P}_\zeta$-name for a condition in $\underset{\sim}{\mathbb{Q}}_\zeta$ with the following property.

Let $\underset{\sim}{r}$ be the $<^*_\chi$-first $\mathbb{P}_\zeta$-name for a member of $\underset{\sim}{\mathbb{Q}}_\zeta$ such that

$$\Vdash_{\mathbb{P}_\zeta} \text{“if possible, then } \underset{\sim}{r} \text{ is an upper bound to } \{p^j_{t_j}(\zeta) : j \le \pi(\zeta) \text{ is non-limit}\}, \text{ else } \underset{\sim}{r} = p(\zeta)\text{”}.$$

Now, suppose that $G_\zeta \subseteq \mathbb{P}_\zeta$ is a generic filter over $\mathbf{V}$ and $p^{j+1}_{t_{j+1}} \restriction \zeta \in G_\zeta$ for all $j < i_0$, and work in $\mathbf{V}[G_\zeta]$. Then, by clause (θ), there is a partial play $\langle s_j, \bar{q}^j, \bar{r}^j : \pi(\zeta) \le j < i_0\rangle$ of the game $\eth^{\mathrm{Sacks}}_{\bar\lambda}(\pi(\zeta), \underset{\sim}{r}^G, \underset{\sim}{\mathbb{Q}}^{G_\zeta}_\zeta)$ in which Generic uses $\underset{\sim}{\mathrm{st}}_\zeta(\pi(\zeta), \underset{\sim}{r})^{G_\zeta}$, and $s_j \cap {}^{j+1}\lambda = \psi_{j+1}(t_{j+1}\restriction\zeta)$ and $\bar{r}^j = \langle r^j_\eta : \eta \in s_j \cap {}^{j+1}\lambda\rangle$, where $r^j_\eta = (p^{j+1}_{s^{j+1}_\eta}(\zeta))^{G_\zeta}$ for $s^{j+1}_\eta \in T_{j+1}$ such that $t_{j+1}\restriction\zeta \lhd s^{j+1}_\eta$, $(s^{j+1}_\eta)_\zeta = \eta$ and $\mathrm{rk}_{j+1}(s^{j+1}) = \gamma$. So we may look at the answer s_{i_0}, $\bar{q}^{i_0} = \langle q^{i_0}_\nu : \nu \in s_{i_0} \cap {}^{i_0+1}\lambda\rangle$ to this play according to the strategy $\underset{\sim}{\mathrm{st}}_\zeta(\pi(\zeta), \underset{\sim}{r})^{G_\zeta}$. Then, $q_t(\zeta)^{G_\zeta}$ is a condition stronger than all $r^j_{(t_{j+1})_\zeta}$ for $\pi(\zeta) \le j < i_0$, and such that

$$\text{if } (t)_\zeta \in s_{i_0}, \text{ then } q_t(\zeta)^{G_\zeta} = q^{i_0}_{(t)_\zeta}.$$

Also, $\underset{\sim}{\alpha}^t(\zeta)^{G_\zeta} = \{\alpha < \lambda_i : (t_{i_0})_\zeta{}^\frown\langle\alpha\rangle \in s_{i_0}\}$.
(If $\pi(\zeta) = i_0$, then we do not have the partial play we started with — the game just begins and we look at the first move of Generic, requiring that $q_t(\zeta)^{G_\zeta}$ is stronger than $\underset{\sim}{r}^{G_\zeta}$ and if $(t)_\zeta \in s_{i_0}$ then $q_t(\zeta)^{G_\zeta} = q^{i_0}_{(t)_\zeta}$.)

This finishes the definition of $\bar{q} = \langle q_t : t \in T^*\rangle$. One easily checks that $\bar{q} \in N$ is a tree of conditions (remember the choice of “the $<^*_\chi$-first names”). Also, by induction on $\zeta \in \mathrm{Dom}(q_t)$, one verifies that $\bar{p}^j \le^{w_i,i}_{w_j,j} \bar{q}$ for all non-limit $j \le i_0$. (Note that if $\pi(\zeta) = i_0$, $t \in T^*$, and $\mathrm{rk}^*(t) > \zeta$, then in the inductive process we know that by clause (ζ)

$$q_t\restriction\zeta \Vdash_{\mathbb{P}_\zeta} \text{“there is a common upper bound to } \{p^j_{t_j}(\zeta) : j \le \pi(\zeta) \text{ is non-limit}\}\text{”}$$

and thus $q_t{\restriction}\zeta$ forces that the respective condition $\underset{\sim}{r}$ is stronger than all $p^j_{t_j}(\zeta)$ (for non-limit $j \leq \pi(\zeta)$).)

Next, we use A.1.9 to pick a standard tree of conditions $\bar{p}^* = \langle p^*_t : t \in T^*\rangle \in N$ such that $\bar{q} \leq \bar{p}^*$ and for each $t \in T^*$ with $\mathrm{rk}^*(t) = \gamma$ the condition p^*_t decides the values of all names $\underset{\sim}{\alpha}^{t'}(\zeta)$ for $t' \in T^*$, $\zeta \in w_i \cap \mathrm{rk}(t')$ and the value of $\underset{\sim}{\tau}(i_0)$ (and let $p^*_t \Vdash$ "$\underset{\sim}{\tau}(i_0) = \tau^t_{i_0}$"), and such that $p^*_t \in \mathcal{I}_\xi$ for all $\xi \leq i_0$. For $t \in T^*$ with $\mathrm{rk}^*(t) = \gamma$ and for $\zeta \in w_i$ let $\alpha^t(\zeta)$ be the value forced to $\underset{\sim}{\alpha}^t(\zeta)$ by p^*_t. Since $\underset{\sim}{\alpha}^t(\zeta)$ is a $\mathbb{P}_\zeta$-name, we have that

$$t_0 \vartriangleleft t_1 \in T^* \;\&\; \mathrm{rk}^*(t_1) = \gamma \;\&\; \zeta \in w_i \cap \mathrm{rk}^*(t_0) \Rightarrow p^*_{t_0} \Vdash \underset{\sim}{\alpha}^{t_0}(\zeta) = \alpha^{t_1}(\zeta).$$

So we may naturally define $\alpha^t(\zeta)$ also for $t \in T^*$ with $\mathrm{rk}(t) < \gamma$. Now we let

$$T_i = T_{i_0+1} = \{t \in T^* : (\forall \zeta \in w_i \cap \mathrm{rk}^*(t))((t)_\zeta(i_0) < \alpha^t(\zeta))\}$$

and $p^i_{*,t} = p^*_t$ for $t \in T_i$ (thus defining $\bar{p}^i_*$). Plainly, $T_i \in N$ is a standard $(w_i, i)^\gamma$-tree satisfying (α)-(γ), $\bar{p}^i_* \in N$. Finally, using the properties of the strategies st^* stated in A.1.6(b) (and the clause (ζ) from earlier stages) we may pick a standard tree of conditions $\bar{p}^i = \langle p^i_t : t \in T_i\rangle$ such that $\bar{p}^* \leq \bar{p}^i$ and

if $t \in T_i$, $\mathrm{rk}_i(t) = \gamma$, $t_j = \mathrm{proj}^{w_i,i}_{w_j,j}(t)$ for non-limit $j \leq i$,
then $\langle p^{j+1}_{*,t_{j+1}}, p^{j+1}_{t_{j+1}} : j < i\rangle$ is a partial play of $\Game^\lambda_0(\mathbb{P}_\gamma, \emptyset, p)$ in which Complete uses the winning strategy $\mathrm{st}^*(\gamma, p)$.

Now one easily verifies that $\mathcal{T}_i, \bar{p}^i, \bar{p}^i_*$ satisfy requirements (α)–(θ), thus the construction is complete.

Let $\mathcal{T}_\lambda = \overleftarrow{\lim}(\langle \mathcal{T}_j : j < \lambda\rangle)$. We will consider this standard $(N \cap \gamma, \lambda)^\gamma$-tree in universes $\mathbf{V}^{\mathbb{P}_\xi}$ (for $\xi \leq \gamma$), so let us note that forcings $\mathbb{P}_\xi$ may add new branches in $\mathcal{T}_\lambda$. But if (in $\mathbf{V}^{\mathbb{P}_\xi}$) $t \in \mathcal{T}_\lambda$ and $i < \lambda$, then

$$t|i \stackrel{\mathrm{def}}{=} \langle (t)_\zeta{\restriction} i : \zeta \in w_i \cap \mathrm{rk}_\lambda(t)\rangle = \mathrm{proj}^{N\cap\gamma,\lambda}_{w_i,i}(t) \in \mathbf{V}.$$

Also if $i < \lambda$ is limit, then the equality $\mathcal{T}_i = \overleftarrow{\lim}(\langle \mathcal{T}_j : j < i\rangle)$ holds in $\mathbf{V}^{\mathbb{P}_\xi}$ as well.

Let us stress it again, the tree $\mathcal{T}_\lambda$ will be considered in the universes after forcing extensions; each of the forcing notions does not add new branches (nodes) to the trees $\mathcal{T}_j$ (for $j < \lambda$) but adds new nodes to $\mathcal{T}_\lambda$ (the forcings involved do not add new sequences of ordinals of length $< \lambda$, but they typically do add λ-sequences). Now, $t|i$ (for $t \in \mathcal{T}_\lambda$ and $i < \lambda$) is the restriction of t to level i; the domain of t is restricted to w_i and the values, which are λ-sequences, are restricted to i. In other words we take the projection of t to the tree $\mathcal{T}_i$. Thus, in $(\mathrm{ii})_\alpha$ below, the index $\underset{\sim}{t}^\alpha|(i+1)$ is a node in the tree T_{i+1}! What may be somewhat confusing here is that we have $\underset{\sim}{t}_\alpha$ and $\underset{\sim}{t}^\alpha$— the former is a name for a λ-sequence, the latter is a sequence of such names. Thus $\underset{\sim}{t}^\alpha$ may be (and actually is) a name for a member of T_λ and we may look at its projection on T_{i+1} which is $\underset{\sim}{t}^\alpha|(i+1)$.

We are going to define a condition $r \in \mathbb{P}_\gamma$ such that $\mathrm{Dom}(r) = N \cap \gamma$ and the names $r(\alpha)$ are defined by induction on $\alpha \in N \cap \gamma$. For $\alpha \in N \cap \gamma$ we will also choose $\mathbb{P}_{\alpha+1}$-names $\underset{\sim}{t}_\alpha$ for functions in ${}^\lambda\lambda$, and we will put $\underset{\sim}{t}^\alpha = \langle \underset{\sim}{t}_\beta : \beta < \alpha \,\&\, \beta \in N\rangle$. The construction will be carried out so that (for each $\alpha \in N \cap (\gamma+1)$):

$(\mathrm{i})_\alpha$ $r\restriction\alpha \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{t}^\alpha \in \mathcal{T}_\lambda$",

$(\mathrm{ii})_\alpha$ $r\restriction\alpha \Vdash_{\mathbb{P}_\alpha}$ "$(\forall i < \lambda)((p^{i+1}_{\underset{\sim}{t}^\alpha|i+1})\restriction\alpha \in \Gamma_{\mathbb{P}_\alpha})$".

Arriving at a limit stage $\alpha \in N \cap (\gamma+1)$, we have defined $r\restriction\alpha$ and $\underset{\sim}{t}^\alpha$, and we should only check that conditions $(\mathrm{i})_\alpha$, $(\mathrm{ii})_\alpha$ hold (assuming $(\mathrm{i})_\beta$, $(\mathrm{ii})_\beta$ hold for $\beta < \alpha$, $\beta \in N$).

RE: $(\mathrm{i})_\alpha$: $\mathcal{T}_\lambda$ is a standard tree, so every chain in T_λ has a $\lhd$-bound. Now the first condition follows immediately from the inductive hypothesis.

RE: $(\mathrm{ii})_\alpha$: Suppose that $G_\alpha \subseteq \mathbb{P}_\alpha$ is generic over $\mathbf{V}$ and $r\restriction\alpha \in G_\alpha$. Let $i < \lambda$ and for $\beta \leq \alpha$ let $t^\beta_i = (\underset{\sim}{t}^\beta|i)^{G_\alpha \cap \mathbb{P}_\beta} \in T_i$. Then, by $(\mathrm{ii})_\beta$, we know that $p^{i+1}_{t^\beta_{i+1}}\restriction\beta \in G_\alpha \cap \mathbb{P}_\beta$ (for each $\beta \in \alpha \cap N$). But $p^{i+1}_{t^\alpha_{i+1}}\restriction\beta = p^{i+1}_{t^\beta_{i+1}}\restriction\beta$ (as $t^\beta_{i+1} \lhd t^\alpha_{i+1}$), so remembering that $p^{i+1}_{t^\alpha_{i+1}} \in N$ we conclude $p^{i+1}_{t^\alpha_{i+1}}\restriction\alpha \in G_\alpha$.

Now suppose that we arrived at stage $\alpha+1 \in N \cap (\gamma+1)$ and we have defined $r\restriction\alpha$, $\underset{\sim}{t}^\alpha$ so that $(\mathrm{i})_\alpha$+ $(\mathrm{ii})_\alpha$ hold. Let $G_\alpha \subseteq \mathbb{P}_\alpha$ be generic over $\mathbf{V}$, $r\restriction\alpha \in G_\alpha$. For $i < \lambda$ let $t^\alpha_i = (\underset{\sim}{t}^\alpha|i)^{G_\alpha} \in T_i$ (remember $(\mathrm{i})_\alpha$). Plainly, $t^\alpha_j = \mathrm{proj}^{w_i,i}_{w_j,j}(t^\alpha_i)$ for $j < i < \lambda$. By $(\mathrm{ii})_\alpha$ we get $p^{i+1}_{t^\alpha_{i+1}}\restriction\alpha \in G_\alpha$ for all $i < \lambda$.

$(\boxplus)_\alpha$ Let $i_0 = \pi(\alpha)$ and let $\underset{\sim}{r}$ be the $<^*_\chi$-first $\mathbb{P}_\alpha$-name for an element of $\underset{\sim}{\mathbb{Q}}_\alpha$ such that ($\underset{\sim}{r} \in \mathbf{V}$, of course, and)

$$\Vdash_{\mathbb{P}_\alpha} \text{"if there is a common upper bound to } \{p^j_{t^\alpha_j}(\alpha) : j \leq i_0 \text{ is non-limit}\} \text{ then } \underset{\sim}{r} \text{ is such an upper bound, else } \underset{\sim}{r} = p(\alpha)\text{"}.$$

(Note: for each $j^* < \lambda$ the sequence $\langle t^\alpha_j : j < j^*\rangle$ belongs to the ground model $\mathbf{V}$, and even to N.)

Fix $j^* < \lambda$, $j^* > i_0$ for a moment. In $\mathbf{V}$, for each $i \in [i_0, j^*]$ and $\eta \in \psi_{i+1}(t^\alpha_{i+1})$ let us choose $s^{i+1}_\eta \in T_{i+1}$ such that $t^\alpha_{i+1} \lhd s^{i+1}_\eta$, $(s^{i+1}_\eta)_\alpha = \eta$. Now work in $\mathbf{V}[G_\alpha]$. Since $p^{j^*+1}_{t^\alpha_{j^*+1}} \in G_\alpha$, we may use clause (θ) of the construction to claim that there is a partial play $\bar{\sigma}^{j^*} = \langle s_i, \bar{q}^i, \bar{r}^i : i_0 \le i \le j^* \rangle$ of the game $\eth^{\rm Sacks}_{\bar{\lambda}}(i_0, \underset{\sim}{r}^{G_\alpha}, (\underset{\sim}{\mathbb{Q}}_\alpha)^{G_\alpha})$ in which Generic uses $\underset{\sim}{\rm st}_\alpha(i_0, \underset{\sim}{r}^{G_\alpha})$ and $s_i \cap {}^{i+1}\lambda = \psi_{i+1}(t^\alpha_{i+1})$ and $\bar{r}^i = \langle (p^{i+1}_{s^{i+1}_\eta}(\alpha))^{G_\alpha} : \eta \in s_i \cap {}^{i+1}\lambda \rangle$.

It should be clear that (in $\mathbf{V}[G_\alpha]$) $\bar{\sigma}^{j^*} \vartriangleleft \bar{\sigma}^{j^{**}}$ for $i_0 < j^* < j^{**} < \lambda$, so we have a play $\bar{\sigma} = \bigcup_{i_0 < j^* < \lambda} \bar{\sigma}^{j^*} = \langle s_i, \bar{q}^i, \bar{r}^i : i_0 \le i < \lambda \rangle$ of the game $\eth^{\rm Sacks}_{\bar{\lambda}}(i_0, \underset{\sim}{r}^{G_\alpha}, (\underset{\sim}{\mathbb{Q}}_\alpha)^{G_\alpha})$ with the respective properties. This play is won by Generic, so there are a condition $q \in (\underset{\sim}{\mathbb{Q}}_\alpha)^{G_\alpha}$ and a $(\underset{\sim}{\mathbb{Q}}_\alpha)^{G_\alpha}$-name $\underset{\sim}{\rho}$ for a member of ${}^\lambda\lambda$ such that $q \ge \underset{\sim}{r}^{G_\alpha}$ and

($\otimes$)

$$q \Vdash_{(\underset{\sim}{\mathbb{Q}}_\alpha)^{G_\alpha}} \text{“}(\forall i \in [i_0, \lambda))(\underset{\sim}{\rho}{\restriction}(i+1) \in \psi_{i+1}(t^\alpha_{i+1})\ \&\ p^{i+1}_{s^{i+1}_{\underset{\sim}{\rho}\restriction(i+1)}}(\alpha)^{G_\alpha} \in \Gamma_{(\underset{\sim}{\mathbb{Q}}_\alpha)^{G_\alpha}})\text{”}.$$

Let $r(\alpha), \underset{\sim}{t}_\alpha$ be names for the $q, \underset{\sim}{\rho}$ as above (i.e., $r(\alpha)$ is a $\mathbb{P}_\alpha$-name of a member of $\underset{\sim}{\mathbb{Q}}_\alpha$ and $\underset{\sim}{t}_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name of a member of ${}^\lambda\lambda$ and $r{\restriction}\alpha$ forces that they have the property stated in ($\otimes$)). It follows from our choices that $(\mathrm{i})_{\alpha+1}$ + $(\mathrm{ii})_{\alpha+1}$ hold, finishing the inductive construction of $r \in \mathbb{P}_\gamma$ and $\underset{\sim}{t}_\alpha$'s.

For $\alpha < \lambda$ let $a_\alpha = \{\tau^t_\alpha : t \in T_{\alpha+1}\ \&\ \mathrm{rk}_{\alpha+1}(t) = \gamma\}$ (remember: τ^t_α is the value forced to $\underset{\sim}{\tau}(\alpha)$ by $p^{\alpha+1}_t$). Plainly, $|a_\alpha| < \lambda_\alpha$ for each $\alpha < \lambda$.

The proof of the iteration theorem will be complete once we show the following

Claim A.2.4.1: *The condition $r \in \mathbb{P}_\gamma$ (defined earlier) is stronger than p, it is $(N, \mathbb{P}_\gamma)$-generic and $r \Vdash_{\mathbb{P}_\gamma}$ “$(\forall \alpha < \lambda)(\underset{\sim}{\tau}(\alpha) \in a_\alpha)$”.*

Proof of the Claim: First, by induction on $\alpha \in N \cap (\gamma+1)$ we are showing that $p{\restriction}\alpha \le r{\restriction}\alpha$. There is nothing to do at limit stages, so let us deal with non-limit ones. Assume we have shown that $p{\restriction}\alpha \le r{\restriction}\alpha$.

Suppose that $G_\alpha \subseteq \mathbb{P}_\alpha$ is generic over $\mathbf{V}$, $r{\restriction}\alpha \in G_\alpha$. Let $t^\alpha_j = (\underset{\sim}{t}^\alpha|j)^{G_\alpha} \in T_j$, and let $i_0, \underset{\sim}{r}$ be defined as in $(\boxplus)_\alpha$. Since, by $(\mathrm{ii})_\alpha$, $p^j_{t^\alpha_j}{\restriction}\alpha \in G_\alpha$ (for non-limit $j \le i_0$) and by the clause (ζ) of the construction, we get

$$\mathbf{V}[G_\alpha] \models \text{“there is a common upper bound to } \{p^j_{t^\alpha_j}(\alpha)^{G_\alpha} : j \le i_0 \text{is non-limit}\}\text{”},$$

and thus

$$\mathbf{V}[G_\alpha] \models \text{“}(\forall j < i_0)(p^{j+1}_{t^\alpha_{j+1}}(\alpha)^{G_\alpha} \le \underset{\sim}{r}^{G_\alpha})\text{”}.$$

By the choice of $r(\alpha)$ we have $r(\alpha)^{G_\alpha} \ge \underset{\sim}{r}^{G_\alpha} \ge p(\alpha)^{G_\alpha}$.

Hence $r{\restriction}\alpha \Vdash p(\alpha) \le r(\alpha)$, as needed.

Now, let $G \subseteq \mathbb{P}_\gamma$ be generic over $\mathbf{V}$, $r \in G$. For $i < \lambda$ let $t_i = (t^\gamma|i)^G \in T_i$. By $(\mathrm{ii})_\gamma$ we know that $p^{i+1}_{t_{i+1}} \in G$. By clause (η) we have $p^{i+1}_{t_{i+1}} \in \mathcal{I}_i$ and (by the definition of a_i) $p^{i+1}_{t_{i+1}} \Vdash \underset{\sim}{\tau} \in a_i$. The former implies that G intersects $\mathcal{I} \cap N$ for each open dense subset $\mathcal{I}$ of $\mathbb{P}_\gamma$ from N, the latter gives $\underset{\sim}{\tau}^G(i) \in a_i$. ∎

(b) Included in the proof of (a). ∎

Definition A.2.5: Let $\mathbb{P}$ be a forcing notion.

(1) For a condition $p \in \mathbb{P}$ and an ordinal $i_0 < \lambda$ we define a game $\Game^{\rm bd}_{\lambda}(i_0, p, \mathbb{P})$ like $\Game^{\rm Sacks}_{\bar\lambda}(i_0, p, \mathbb{P})$, but demand A.2.1(1($\varepsilon$)) is replaced by
$(\varepsilon)^-$ $|s_i \cap {}^{i+1}\lambda| < \lambda$.

(2) $\mathbb{P}$ has **the strong λ-bounding property** if
   (a) $\mathbb{P}$ is strategically $(<\lambda)$-complete, and
   (b) Generic has a winning strategy in the game $\Game^{\rm bd}_{\lambda}(i_0, p, \mathbb{P})$ for every $i_0 < \lambda$, $p \in \mathbb{P}$.

(3) $\mathbb{P}$ has **the λ-bounding property** if for every $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\underset{\sim}{\tau}$ such that $p \Vdash$ "$\underset{\sim}{\tau} \to \mathbf{V}$", there are a condition $q \geq p$ and a sequence $\langle a_\alpha : \alpha < \lambda\rangle$ such that $|a_\alpha| < \lambda$ (for $\alpha < \lambda$) and $q \Vdash$"$(\forall \alpha < \lambda)(\underset{\sim}{\tau}(\alpha) \in a_\alpha)$".

Remark A.2.6:

(1) All the remarks stated in A.2.2, A.2.3 have their (obvious) parallels for the λ-bounding properties.

(2) Clearly, (strong) $\bar\lambda$-Sacks property implies (strong, respectively) λ-bounding property.

THEOREM A.2.7: *Suppose that $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma\rangle$ is a λ-support iteration such that for all $\alpha < \lambda$:*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\underset{\sim}{\mathbb{Q}}_\alpha \text{ has the strong } \lambda\text{-bounding property”}.$$

Then:

(a) *$\mathbb{P}_\gamma$ has the λ-bounding property.*

(b) *If $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$ and $\lambda, p, \bar{\mathbb{Q}}, \mathbb{P}_\gamma, \ldots \in N$, $p \in \mathbb{P}_\gamma$, then there is an $(N, \mathbb{P}_\gamma)$-generic condition $r \in \mathbb{P}_\gamma$ stronger than p.*

Proof: Basically the same as for A.2.4, just replacing each occurrence of λ_i by λ. ∎

The results of this section will be improved, simplified and generalized in [14].

A.3. Fuzzy properness over λ. A properness-type property preserved in λ-support iterations, so called **properness over semi-diamonds**, was introduced in Rosłanowski and Shelah [16]. That property worked for any uncountable regular cardinal λ satisfying $\lambda^{<\lambda} = \lambda$ (not necessarily strongly inaccessible), so because of the known ZFC limitations a number of natural forcing notions were not covered. For the context considered in this paper we may do much better: **fuzzy properness** introduced in this section captures more examples. Even though we do not prove a real preservation in λ-support iterations, our iteration theorem A.3.10 is satisfactory for most applications (see sections B.4 and B.8 later).

In this section we fix λ^*, A, W and D such that

CONTEXT A.3.1:

(1) $\lambda^* > \lambda$ is a regular cardinal, $A \subseteq \mathcal{H}_{<\lambda}(\lambda^*)$ (see 0.1(5)), $W \subseteq [A]^\lambda$, and if $a \in W$, $w \in [a]^{<\lambda}$, $f\colon w \longrightarrow a$, then $f \in a$ (hence also $0 \in a$ for $a \in W$),

(2) for every $x \in \mathcal{H}(\chi)$ there is a model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ such that $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$, $x \in N$ and $N \cap A \in W$,

(3) D is a normal filter on λ such that there is a D-diamond (see A.3.2).

Definition A.3.2:

(1) We say that $\bar{F} = \langle F_\delta : \delta \in S\rangle$ is **a D-pre-diamond sequence** if
   - $S \in D^+$ contains all successor ordinals below λ, $\lambda \setminus S$ is unbounded in λ, $0 \notin S$, and
   - $F_\delta\colon \delta \longrightarrow \lambda$ for all $\delta \in S$.

(2) **A convenient D-diamond** is a D-pre-diamond $\bar{F} = \langle F_\delta : \delta \in S\rangle$ such that

$$(\forall f \in {}^\lambda\lambda)(\{\delta \in S : F_\delta \subseteq f\} \in D^+).$$

Definition A.3.3: Let $\mathbb{P}$ be a forcing notion. **A λ-base for $\mathbb{P}$ over W** is a pair $(\mathfrak{R}, \bar{\mathfrak{Y}})$ such that

(a) $\mathfrak{R} \subseteq \mathbb{P} \times \lambda \times A$ is a relation such that

$$\text{if } (p, \delta, x) \in \mathfrak{R} \text{ and } p \leq_{\mathbb{P}} p', \text{ then } (p', \delta, x) \in \mathfrak{R},$$

(b) $\bar{\mathfrak{Y}} = \langle \mathfrak{Y}_a : a \in W\rangle$ where, for each $a \in W$, $\mathfrak{Y}_a\colon \lambda \longrightarrow [a]^{<\lambda}$,

(c) if $q \in \mathbb{P}$, $a \in W$, and $\delta < \lambda$ is a limit ordinal, then there are $p \geq_{\mathbb{P}} q$ and $x \in \mathfrak{Y}_a(\delta)$ such that $(p, \delta, x) \in \mathfrak{R}$.

If $\mathfrak{R}$ is understood and $(p, \delta, x) \in \mathfrak{R}$, then we may say p *obeys* x *at* δ.

Definition A.3.4: Let $\mathbb{P}$ be a forcing notion and let $(\mathfrak{R}, \bar{\mathfrak{Y}})$ be a λ-base for $\mathbb{P}$ over W. Also let a model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be such that $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$, $a \stackrel{\rm def}{=} N \cap A \in W$ and $\{\lambda, \mathbb{P}, D, \mathfrak{R}\} \in N$. Furthermore, let $h\colon \lambda \longrightarrow N$ be such that the range $\mathrm{Rng}(h)$ of the function h includes $\mathbb{P} \cap N$ and let $\bar{F} = \langle F_\delta : \delta \in S\rangle$ be a D-pre-diamond sequence.

(1) Let $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda\rangle \subseteq N$ list all open dense subsets of $\mathbb{P}$ from N. A sequence $\bar{p} = \langle p_\alpha : \alpha < \delta\rangle$ of conditions from $\mathbb{P} \cap N$ of length $\delta \leq \lambda$ is

called $\bar{\mathcal{I}}$**-exact** if

$$(\forall \xi < \delta)(\exists \alpha < \delta)(p_\alpha \in \mathcal{I}_\xi).$$

(2) We say that $\bar{F}$ is **a quasi D-diamond sequence for** $(N, h, \mathbb{P})$ if for some (equivalently, all) list $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda \rangle$ of all open dense subsets of $\mathbb{P}$ from N, for every $\leq_{\mathbb{P}}$-increasing sequence $\bar{p} = \langle p_\alpha : \alpha < \lambda \rangle \subseteq \mathbb{P} \cap N$ such that $\bar{p}$ is $\bar{\mathcal{I}}$-exact, or equivalently

$$E \stackrel{\rm def}{=} \{\delta < \lambda : \langle p_\alpha : \alpha < \delta \rangle \text{ is } \bar{\mathcal{I}}\text{-exact}\} \in D,$$

we have

$$\{\delta \in E \cap S : (\forall \alpha < \delta)(h \circ F_\delta(\alpha) = p_\alpha)\} \in D^+.$$

(3) For a limit ordinal $\delta \in S$ we define $\mathcal{Y}(\delta) = \mathcal{Y}(N, \mathbb{P}, h, \bar{F}, \mathfrak{R}, \bar{\mathfrak{Y}}, \delta)$ as the set

$$\begin{aligned}\{x \in \mathfrak{Y}_a(\delta) : \ & \text{if } \langle h \circ F_\delta(\alpha) : \alpha < \delta \rangle \text{ is a } \leq_{\mathbb{P}}\text{-increasing sequence} \\ & \text{of conditions from } \mathbb{P}, \\ & \text{then there is a condition } p \in \mathbb{P} \text{ such that} \\ & (\forall \alpha < \delta)(h \circ F_\delta(\alpha) \leq_{\mathbb{P}} p) \text{ and } (p, \delta, x) \in \mathfrak{R}\}\end{aligned}$$

(Note: if $x \in \mathcal{Y}(\delta)$, then there is $p \in N$ witnessing this.)

(4) Let $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda \rangle \subseteq N$ list all open dense subsets of $\mathbb{P}$ from N. A sequence $\bar{q} = \langle q_{\delta,x} : \delta \in S \text{ limit } \& \, x \in \mathcal{X}_\delta \rangle \subseteq N \cap \mathbb{P}$ is called **a weak fuzzy candidate over $\bar{F}$ for** $(N, h, \mathbb{P}, \mathfrak{R}, \bar{\mathfrak{Y}}, \bar{\mathcal{I}})$ whenever $\emptyset \neq X_\delta \subseteq \mathcal{Y}(\delta)$ (for limit $\delta \in S$) and

(α) $\{\delta \in S : (\forall x \in \mathcal{X}_\delta)(q_{\delta,x} \in \mathcal{I}_\alpha)\} = S \bmod D$ for each $\alpha < \lambda$, and

(β) if $\delta \in S$ is a limit ordinal, $x \in \mathcal{X}_\delta$, and $\langle h \circ F_\delta(\alpha) : \alpha < \delta \rangle$ is a $\leq_{\mathbb{P}}$-increasing $\bar{\mathcal{I}}$-exact sequence of members of $\mathbb{P} \cap N$, then $(\forall \alpha < \delta)(h \circ F_\delta(\alpha) \leq_{\mathbb{P}} q_{\delta,x})$ and $(q_{\delta,x}, \delta, x) \in \mathfrak{R}$.

If above $X_\delta = \mathcal{Y}(\delta)$ for each limit $\delta \in S$, then $\bar{q}$ is called **a fuzzy candidate over $\bar{F}$ for** $(N, h, \mathbb{P}, \mathfrak{R}, \bar{\mathfrak{Y}}, \bar{\mathcal{I}})$.

Omitting $\bar{\mathcal{I}}$ means "for some $\bar{\mathcal{I}}$".

(5) Let $\bar{q} = \langle q_{\delta,x} : \delta \in S \text{ limit } \& \, x \in \mathcal{X}_\delta \rangle$ be a weak fuzzy candidate over $\bar{F}$ for $(N, h, \mathbb{P}, \mathfrak{R}, \bar{\mathfrak{Y}}, \bar{\mathcal{I}})$, and $r \in \mathbb{P}$. We define a game $\eth^{\rm fuzzy}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{P}, \bar{F}, \bar{q})$ of two players, the **Generic player** and the **Antigeneric player**, as follows. A play lasts λ moves, in the $i^{\rm th}$ move a condition $r_i \in \mathbb{P}$ and a set $C_i \in D$ are chosen such that $(\forall j < i)(r \leq r_j \leq r_i)$, and Generic chooses r_i, C_i if $i \in S = \mathrm{Dom}(\bar{F})$, and Antigeneric chooses r_i, C_i if $i \notin S$. In the end Generic wins the play if (there were always legal moves for both players and)

(α) $(\forall\alpha<\lambda)(\exists i<\lambda)(\exists p\in\mathbb{P}\cap N)(p\in\mathcal{I}_\alpha \,\&\, p\leq r_i)$, and

(β) if $\delta\in S\cap\bigcap_{i<\delta}C_i$ is a limit ordinal, $\langle h\circ F_\delta(\alpha):\alpha<\delta\rangle$ is a $\leq_{\mathbb{P}}$-increasing $\bar{\mathcal{I}}$-exact sequence and $(\forall\alpha<\delta)(\exists i<\delta)(h\circ F_\delta(\alpha)\leq r_i)$, then for some $x\in\mathcal{X}_\delta$ we have $q_{\delta,x}\leq r_\delta$.

(6) Let $\bar{q}$ be a weak fuzzy candidate over $\bar{F}$ for $(N,h,\mathbb{P},\mathfrak{R},\bar{\mathfrak{Y}},\bar{\mathcal{I}})$. We say that a condition $r\in\mathbb{P}$ is **$(\mathfrak{R},\bar{\mathfrak{Y}})$-fuzzy generic for $\bar{q}$ (over $(N,\bar{\mathcal{I}},h,\mathbb{P},\bar{F})$)** if Generic has a winning strategy in the game $\eth^{\rm fuzzy}_\lambda(r,N,\bar{\mathcal{I}},h,\mathbb{P},\bar{F},\bar{q})$.

Remark A.3.5:

(1) For any two lists $\bar{\mathcal{I}}^1,\bar{\mathcal{I}}^2$ of open dense subsets of $\mathbb{P}$ from N, on a club E of λ we have

$$\{\mathcal{I}^1_\xi:\xi<\delta\}=\{\mathcal{I}^2_\xi:\xi<\delta\}$$

for $\delta\in E$. Thus the corresponding notions of exactness agree for $\delta\in E$. As Generic can choose $C_i\subseteq E$, in A.3.4(4,5,6) we may not mention $\bar{\mathcal{I}}$ as a parameter.

(2) Plainly, every fuzzy candidate is a weak fuzzy candidate.

Definition A.3.6: Let $\mathbb{P}$ be a $(<\lambda)$-complete forcing notion.

(1) We say that $\mathbb{P}$ is **fuzzy proper over quasi D-diamonds for W** whenever for some λ-base $(\mathfrak{R},\bar{\mathfrak{Y}})$ for $\mathbb{P}$ over W and for some $c\in\mathcal{H}(\chi)$,

($\circledast$) *if*
- $N\prec(\mathcal{H}(\chi),\in,<^*_\chi)$, $|N|=\lambda$, ${}^{<\lambda}N\subseteq N$, $\lambda,\mathbb{P},c,\mathfrak{R}\in N$, and $a\stackrel{\rm def}{=}N\cap A\in W$, $p\in\mathbb{P}\cap N$,
- $h\colon\lambda\longrightarrow N$ satisfies $\mathbb{P}\cap N\subseteq\mathrm{Rng}(h)$, and
- $\bar{F}$ is a quasi D-diamond for $(N,h,\mathbb{P})$ and $\bar{q}$ is a fuzzy candidate over $\bar{F}$,

then there is $r\in\mathbb{P}$ stronger than p and such that r is $(\mathfrak{R},\bar{\mathfrak{Y}})$-fuzzy generic for $\bar{q}$.

(We may call $(\mathfrak{R},\bar{\mathfrak{Y}})$ and c **witnesses for fuzzy properness**.)

(2) $\mathbb{P}$ is **strongly fuzzy proper over quasi D-diamonds** whenever for some λ-base $(\mathfrak{R},\bar{\mathfrak{Y}})$ for $\mathbb{P}$ over W and for some $c\in\mathcal{H}(\chi)$,

($\circledast$)$^+$ *if*
- $N\prec(\mathcal{H}(\chi),\in,<^*_\chi)$, $|N|=\lambda$, ${}^{<\lambda}N\subseteq N$, $\lambda,\mathbb{P},c,\mathfrak{R}\in N$, and $a\stackrel{\rm def}{=}N\cap A\in W$, $p\in\mathbb{P}\cap N$,
- $h\colon\lambda\longrightarrow N$ satisfies $\mathbb{P}\cap N\subseteq\mathrm{Rng}(h)$,
- $\bar{F}$ is a quasi D-diamond for $(N,h,\mathbb{P})$ and $\bar{q}$ is a weak fuzzy candidate over $\bar{F}$,

then there is a condition $r\in\mathbb{P}$ stronger than p such that r is $(\mathfrak{R},\bar{\mathfrak{Y}})$-fuzzy generic for $\bar{q}$.

(3) $\mathbb{P}$ is **weakly fuzzy proper over quasi D-diamonds** whenever for some λ-base $(\mathfrak{R}, \bar{\mathfrak{Y}})$ for $\mathbb{P}$ over W and for some $c \in \mathcal{H}(\chi)$,

$(\circledast)^-$ *if* • $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$, $\lambda, \mathbb{P}, c, \mathfrak{R} \in N$, and $a \stackrel{\rm def}{=} N \cap A \in W$, $p \in \mathbb{P} \cap N$,

• $h\colon \lambda \longrightarrow N$ satisfies $\mathbb{P} \cap N \subseteq \mathrm{Rng}(h)$,

then for some quasi D-diamond $\bar{F}$ for $(N, h, \mathbb{P})$ and a weak fuzzy candidate $\bar{q}$ over $\bar{F}$, there is a condition $r \in \mathbb{P}$ stronger than p such that r is $(\mathfrak{R}, \bar{\mathfrak{Y}})$-fuzzy generic for $\bar{q}$.

(4) $\mathbb{P}$ is **fuzzy proper for W** if it is fuzzy proper over quasi D'-diamonds for every normal filter D' on λ (which has diamonds). Similarly for **strongly fuzzy** and **weakly fuzzy proper**.

Remark A.3.7: Strong fuzzy properness is very close to **properness over semi-diamonds** of Rosłanowski and Shelah [16] and even closer to **properness over diamonds** introduced by Eisworth [6]. (Note that considering the condition A.3.6$(\circledast)^+$ we may assume that the weak fuzzy candidate $\bar{q} = \langle q_{\delta,x} : \delta \in S \text{ is limit } \& \ x \in \mathcal{X}_\delta\rangle$ is such that $|\mathcal{X}_\delta| = 1$ for each relevant δ, so one may treat it as $\bar{q} = \langle q_\delta : \delta \in S \text{ is limit } \rangle$.) Thus fuzzy properness has a flavour of a weaker property. However, the differences in technical details of the conditions introduced in this section and those in [16] and/or [6] make it unclear if there are any implications between the "properness conditions" in this section and those in the other two papers.

Proposition A.3.8: *Let $N, \mathbb{P}, h, \bar{\mathcal{I}}, \mathfrak{R}, \bar{\mathfrak{Y}}$ be as in A.3.4, $\bar{F} = \langle F_\delta : \delta \in S\rangle$ be a D-pre-diamond. Assume also that the forcing notion $\mathbb{P}$ is $(<\lambda)$-complete.*

(1) *There exists a fuzzy candidate $\bar{q}$ over $\bar{F}$ for $(N, h, \mathbb{P}, \mathfrak{R}, \bar{\mathfrak{Y}}, \bar{\mathcal{I}})$. In fact we can even demand:*

 (+) *for every $\alpha < \lambda$, for every large enough $\delta \in S$, $q_{\delta,x} \in \mathcal{I}_\alpha$ for all $x \in \mathcal{Y}(\delta)$.*

(2) *If r is $(\mathfrak{R}, \bar{\mathfrak{Y}})$-fuzzy generic for some weak fuzzy candidate $\bar{q}$, then r is $(N, \mathbb{P})$-generic (in the standard sense).*

(3) *Assume that a condition r is $(N, \mathbb{P})$-generic (in the standard sense), $\bar{F}$ is a quasi D-diamond and $\bar{q}$ is a weak fuzzy candidate over $(N, \bar{\mathcal{I}}, h, \mathbb{P}, \bar{F})$. Suppose that Generic has a strategy in the game $\Game^{\rm fuzzy}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{P}, \bar{F}, \bar{q})$ which guarantees that the result $\langle r_i, C_i : i < \lambda\rangle$ of the play satisfies A.3.4(5)(β). Then she has a winning strategy in $\Game^{\rm fuzzy}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{P}, \bar{F}, \bar{q})$ (i.e., one ensuring $(\alpha) + (\beta)$ of A.3.4(5)).*

(4) *If $\mathbb{P}$ is fuzzy proper over quasi D-diamonds, then it is weakly fuzzy proper*

over quasi D-diamonds. If $\mathbb{P}$ is strongly fuzzy proper over quasi D-diamonds, then it is fuzzy proper over quasi D-diamonds.

(5) *Assume that $\mathbb{P}$ is weakly fuzzy proper over quasi D-diamonds, $\mu \geq \lambda$, $Y \subseteq [\mu]^{\leq\lambda}$, $A^* \subseteq \mathcal{H}(\chi)$, $W^* \subseteq [A^*]^\lambda$ ($Y, A^*, W^* \in \mathbf{V}$). Then:*
   (a) *forcing with $\mathbb{P}$ does not collapse λ^+,*
   (b) *forcing with $\mathbb{P}$ preserves the following two properties:*
      (i) *Y is a cofinal subset of $[\mu]^{\leq\lambda}$ (under inclusion),*
      (ii) *for every $x \in \mathcal{H}(\chi)$ there is $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ such that $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$, $N \cap A^* \in W^*$ (i.e., the stationarity of W^* under the relevant filter).*

Proof:

(1) Immediate (by the $(<\lambda)$-completeness of $\mathbb{P}$; remember A.3.3(c) and that $\mathfrak{R} \in N$; note that $\mathfrak{Y}_a(\delta) \in N$).

(2) Remember that $0 \notin S$, so in the game $\Game^{\rm fuzzy}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{P}, \bar{F}, \bar{q})$ the condition r_0 is chosen by Antigeneric. So if the conclusion fails, then for some $\mathbb{P}$-name $\underset{\sim}{\alpha} \in N$ for an ordinal we have $r \nVdash$ "$\underset{\sim}{\alpha} \in N$". Thus Antigeneric can choose $r_0 \geq r$ so that $r_0 \Vdash$ "$\underset{\sim}{\alpha} = \alpha_0$" for some ordinal $\alpha_0 \notin N$, what guarantees him to win the play (remember clause (α) of A.3.4(5)).

(3) Generic modifies her original strategy as follows. During the play she builds aside a $\leq_{\mathbb{P}}$-increasing sequence of conditions $\langle p_i : i \in \lambda \setminus S\rangle \subseteq \mathbb{P} \cap N$ such that $p_i \leq r_i$ for $i \in \lambda \setminus S$. Arriving to stage $i+1$, $i \in \lambda \setminus S$, she has two sequences: $\langle r_j, C_j : j \leq i\rangle$ (of the play) and $\langle p_j : j \in i \setminus S\rangle$ such that $p_j \leq r_j$. Now Generic picks $p_i \in \mathbb{P} \cap N$ such that

$$(\forall j \in i \setminus S)(p_j \leq p_i) \quad \text{and} \quad (\forall \xi < i)(p_i \in \mathcal{I}_\xi),$$

and p_i, r_i are compatible. (Remember: the set of all $p \in \mathbb{P}$ such that $p \in \mathcal{I}_\xi$ for all $\xi < i$ and for each $j \in i \setminus S$ either $p_j \leq p$ or p_j, p are incompatible is open dense in $\mathbb{P}$ and it belongs to N. Now use the assumption that r is $(N, \mathbb{P})$-generic). Next she replaces r_i by a common upper bound of p_i and r_i, pretending that that was the condition played by her opponent, and then she plays according to her original strategy. One easily verifies that this is a winning strategy for the Generic player.

(4) Straightforward (remember that, by A.3.1(3), there is a quasi D-diamond and by A.3.8(1) there is a fuzzy candidate over it).

(5) Follows from (2) by the same arguments as used in the "standard proper forcing" version of this claim. ∎

PROPOSITION A.3.9: *($<\lambda^+$)-complete forcing notions are strongly fuzzy proper for W.*

Proof: This is essentially a variant of [16, 2.5], but since we did not give the proof there, we will present it fully here.

So suppose that a forcing notion $\mathbb{P}$ is ($<\lambda^+$)-complete. Let $\mathfrak{R}^{\rm tr} = \mathfrak{R}^{\rm tr}(\mathbb{P})$ be the trivial relation consisting of all triples $(p,\delta,0)$ such that $p \in \mathbb{P}$ and $\delta < \lambda$ and let $\bar{\mathfrak{Y}}^{\rm tr}$ be such that $\mathfrak{Y}^{\rm tr}_a(\delta) = \{0\}$ (for each $\delta < \lambda$, $a \in W$). Assume now that

- $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$, $\lambda, \mathbb{P} \in N$, and $a \stackrel{\rm def}{=} N \cap A \in W$,
- $p \in \mathbb{P} \cap N$, and $h : \lambda \longrightarrow N$ satisfies $\mathbb{P} \cap N \subseteq {\rm Rng}(h)$,
- $\bar{F} = \langle F_\delta : \delta \in S\rangle$ is a quasi D-diamond for $(N, h, \mathbb{P})$ and $\bar{q}$ is a weak fuzzy candidate over $\bar{F}$. Since $\mathfrak{Y}^{\rm tr}_a(\delta)$ has a one member only we may think of $\bar{q}$ as a sequence $\langle q_\delta : \delta \in S \text{ is limit}\rangle$.

Let $\bar{\mathcal{I}} = \langle \mathcal{I}_\xi : \xi < \lambda\rangle$ list of all open dense subsets of $\mathbb{P}$ from N.

We are going to build a condition $r \in \mathbb{P}$ stronger than p which is $(\mathfrak{R}^{\rm tr}, \bar{\mathfrak{Y}}^{\rm tr})$-fuzzy generic for $\bar{q}$. For this we inductively build a $\leq_{\mathbb{P}}$-increasing sequence $\langle r'_i : i < \lambda\rangle \subseteq \mathbb{P} \cap N$ such that

- $r'_0 = p$, $r'_{i+1} \in \bigcap_{\xi\leq i} \mathcal{I}_\xi$,
- if there is an upper bound to $\{r'_j : j < i\} \cup \{q_i\}$, then r'_i is such an upper bound.

Then we pick any upper bound r to the sequence $\langle r'_i : i < \lambda\rangle$ (remember: $\mathbb{P}$ is ($<\lambda^+$)-complete). Now we want to argue that Generic has a winning strategy in the game $\Game^{\rm fuzzy}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{P}, \bar{F}, \bar{q})$. Since r is $(N, \mathbb{P})$-generic it is enough to give a strategy for the Generic player which ensures that the result of the play satisfies A.3.4(5)(β) (by A.3.8(3)). To this end note that there is a club E_0 of λ such that

- every member of E_0 is a limit of ordinals from $\lambda \setminus S$,
- for every $\delta \in E_0$ and $i < \delta$,

$$\{q \in \mathbb{P} : q \geq r'_i \text{ or } q, r'_i \text{ are incompatible}\} \in \{\mathcal{I}_\xi : \xi < \delta\},$$

Let Generic play so that arriving to a stage $\delta \in S$ of the play she puts E_0 and the $<^*_\chi$-first upper bound to the conditions played so far. Why does this strategy work? Let $\langle r_i, C_i : i < \lambda\rangle$ be the result of the play in which Generic plays as described above and let $\delta \in S \cap \bigcap_{i<\delta} C_i$ be a limit ordinal such that $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is a $\leq_{\mathbb{P}}$-increasing $\bar{\mathcal{I}}$-exact sequence and

$$(\forall \alpha < \delta)(\exists i < \delta)(h \circ F_\delta(\alpha) \leq r_i).$$

Then no $r'_i, h \circ F_\delta(\alpha)$ (for $i, \alpha < \delta$) can be incompatible, so (since $\delta \in E_0$ and $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is $\bar{\mathcal{I}}$-exact) we have also

$$(\forall i < \delta)(\exists \alpha < \delta)(r'_i \le h \circ F_\delta(\alpha)),$$

and hence q_δ is stronger than all r'_i (for $i < \delta$). Therefore $q_\delta \le r'_\delta \le r_\delta$. ∎

THEOREM A.3.10: *Let A, W, D be as in A.3.1 and let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \zeta^*\rangle$ be a λ-support iteration of $({<}\lambda)$-complete forcing notions, and assume that $\zeta^* \subseteq A$. Suppose also that for each $\zeta < \zeta^*$ we have $\bar{\mathfrak{Y}}^\zeta$ and $\mathbb{P}_\zeta$-names $\underset{\sim}{\mathfrak{R}}_\zeta, \underset{\sim}{c}_\zeta$ such that*

$$\Vdash_{\mathbb{P}_\zeta} \text{“}\underset{\sim}{\mathbb{Q}}_\zeta \textit{ is fuzzy proper over quasi } D\textit{-diamonds for } W$$
$$\textit{with witnesses } (\underset{\sim}{\mathfrak{R}}_\zeta, \bar{\mathfrak{Y}}^\zeta) \textit{ and } \underset{\sim}{c}_\zeta\text{”}.$$

Then $\mathbb{P}_{\zeta^*} = \lim(\bar{\mathbb{Q}})$ is weakly fuzzy proper over quasi D-diamonds.

Note: in the assumptions of A.3.10, $\bar{\mathfrak{Y}}^\zeta$ are **objects** not names, i.e., $\bar{\mathfrak{Y}}^\zeta \in \mathbf{V}$.

Proof: By A.1.5, the forcing notion $\mathbb{P}_{\zeta^*}$ is $({<}\lambda)$-complete, so we have to concentrate on showing clause A.3.6(3)($(\circledast)^-$) for it. The proof, though not presented as such, is by induction on ζ^*. However, the inductive hypothesis is used only to be able to claim that A, W, D are as in A.3.1 when considered in the intermediate universes $\mathbf{V}^{\mathbb{P}_\zeta}$ (for $\zeta < \zeta^*$) — remember A.3.8(5). Thus our assumptions on $\underset{\sim}{\mathbb{Q}}_\zeta$'s are meaningful.

Let us fix a convenient D-diamond sequence $\bar{F}' = \langle F'_\delta : \delta \in S\rangle$ (so in particular, $S \in D^+$ contains all successors, $\lambda \setminus S$ is unbounded in λ and $0 \notin S$). Put

$$E_0 \overset{\text{def}}{=} \{\delta < \lambda : \delta \text{ is a limit of points from } \lambda \setminus S\}, \quad E_1 \overset{\text{def}}{=} (\lambda \setminus S) \cup E_0.$$

Plainly, E_0, E_1 are clubs of λ. Let $\langle i_\alpha : \alpha < \lambda\rangle$ be the increasing enumeration of E_1 and $E_2 = E_0 \cap \{\alpha < \lambda : i_\alpha = \alpha\}$ (So E_2 is a club of λ too).

For each $a \in W$ fix a one-to-one mapping $\pi_a \colon a \cap \zeta^* \longrightarrow \lambda$ such that $\pi_a(0) = 0$ (say, π_a is the $<^*_\chi$-first such function), and for $\alpha < \lambda$ let $w^a_\alpha = (\pi_a)^{-1}[i_\alpha]$ (so $a \cap \zeta^* = \bigcup_{\alpha<\lambda} w^a_\alpha$).

For $\zeta \le \zeta^*$ let $\mathfrak{R}^{[\zeta]}$ consist of all triples $(p, \delta, \bar{x}) \in \mathbb{P}_\zeta \times \lambda \times A$ such that for some non-empty $w \in [\zeta]^{<\lambda}$ we have

$$\bar{x} = \langle x_\varepsilon : \varepsilon \in w\rangle \quad \text{and} \quad (\forall \varepsilon \in w)(p{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}(p(\varepsilon), \delta, x_\varepsilon) \in \underset{\sim}{\mathfrak{R}}_\varepsilon\text{”}).$$

Next, for $\zeta \le \zeta^*$, $a \in W$ and $\delta < \lambda$ we put

$$\mathfrak{Y}^{[\zeta]}_a(\delta) = \prod\{\mathfrak{Y}^\varepsilon_a(\delta) : \varepsilon \in w^a_\delta \cap \zeta\},$$

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thus defining $\mathfrak{Y}^{[\zeta]}_a$ and $\bar{\mathfrak{Y}}^{[\zeta]} = \langle \mathfrak{Y}^{[\zeta]}_a : a \in W\rangle$. If $\zeta = \zeta^*$ we will omit it (so then we write $\mathfrak{R}$ and $\bar{\mathfrak{Y}}$).

Claim A.3.10.1: *For each $\zeta \leq \zeta^*$, $(\mathfrak{R}^{[\zeta]}, \bar{\mathfrak{Y}}^{[\zeta]})$ is a λ-base for $\mathbb{P}_\zeta$ over W.*

Proof of the claim: Immediate, by the definition of $\mathfrak{R}^{[\zeta]}, \bar{\mathfrak{Y}}^{[\zeta]}$ we see that clauses (a), (b) of A.3.3 hold (note: $\mathfrak{Y}^{[\zeta]}_a(\delta) \subseteq a$ by A.3.1(1)). Now, to verify A.3.3(c), suppose $q \in \mathbb{P}_\zeta$, $a \in W$ and $\delta < \lambda$ is a limit ordinal. For each $\varepsilon \in w^a_\delta \cap \zeta$ let $p'(\varepsilon), \underaccent{\tilde}{x}'_\varepsilon$ be $\mathbb{P}_\varepsilon$-names such that

$$q{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}p'(\varepsilon) \geq q(\varepsilon) \,\&\, \underaccent{\tilde}{x}'_\varepsilon \in \mathfrak{Y}^\varepsilon_a(\delta) \,\&\, (p'(\varepsilon), \delta, \underaccent{\tilde}{x}'_\varepsilon) \in \underaccent{\tilde}{\mathfrak{R}}_\varepsilon\text{”},$$

and for $\varepsilon \in \mathrm{Dom}(q) \setminus w^a_\delta$ let $p'(\varepsilon) = q(\varepsilon)$. This defines a condition $p' \in \mathbb{P}_\zeta$ stronger than q (and names $\underaccent{\tilde}{x}'_\varepsilon$). Since $\mathbb{P}_\zeta$ is $({<}\lambda)$-complete we may find a condition $p \geq p'$ and $x_\varepsilon \in \mathfrak{Y}^\varepsilon_a(\delta)$ (for $\varepsilon \in w^a_\delta \cap \zeta$) such that $p{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}\underaccent{\tilde}{x}'_\varepsilon = x_\varepsilon\text{”}$ (for $\varepsilon \in w^a_\delta \cap \zeta$). Then, by A.3.3(a), we have $p{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“ }(p(\varepsilon), \delta, x_\varepsilon) \in \underaccent{\tilde}{\mathfrak{R}}_\varepsilon\text{ ”}$ (for each $\varepsilon \in w^a_\delta \cap \zeta$), and $(p, \delta, \langle x_\varepsilon : \varepsilon \in w^a_\delta \cap \zeta\rangle) \in \mathfrak{R}^{[\zeta]}$. ∎

Our aim now is to show that $\mathbb{P}_{\zeta^*}$ is weakly fuzzy proper with witnesses $(\mathfrak{R}, \bar{\mathfrak{Y}})$ and $c = (\langle \underaccent{\tilde}{c}_\varepsilon : \varepsilon < \zeta^*\rangle, \langle \underaccent{\tilde}{\mathfrak{R}}_\varepsilon : \varepsilon < \zeta^*\rangle, \mathfrak{R}, \bar{\mathfrak{Y}}, S, D, \bar{F}', \bar{\mathbb{Q}})$. So suppose that a model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ satisfies

$$|N| = \lambda, \quad {}^{<\lambda}N \subseteq N, \quad \lambda, \mathbb{P}_{\zeta^*}, c \in N, \quad a \stackrel{\mathrm{def}}{=} N \cap A \in W,$$

and $p \in \mathbb{P}_{\zeta^*} \cap N$, and $h : \lambda \longrightarrow N$ is such that $\mathbb{P}_{\zeta^*} \cap N \subseteq \mathrm{Rng}(h)$. To simplify the notation later, let $\pi = \pi_a$, $w_\alpha = w^a_\alpha$ (for $\alpha < \lambda$).

Let us fix a list $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda\rangle$ of all open dense subsets of $\mathbb{P}_{\zeta^*}$ from N. For $\zeta \in (\zeta^* + 1) \cap N$, let $\bar{\mathcal{I}}^{[\zeta]} = \langle \mathcal{I}^{[\zeta]}_\alpha : \alpha < \lambda\rangle$, where $\mathcal{I}^{[\zeta]}_\alpha = \{p \restriction \zeta : p \in \mathcal{I}_\alpha\}$. (Note that $\bar{\mathcal{I}}^{[\zeta]}$ lists all open dense subsets of $\mathbb{P}_\zeta$ from N.) Also for $\zeta \in \zeta^* \cap N$ let $\mathcal{J}_\zeta = \{p \in \mathbb{P}_{\zeta^*} : p{\restriction}\zeta \Vdash p(\zeta) \neq \underaccent{\tilde}{\emptyset}_{\mathbb{Q}_\zeta}\}$ (so $\mathcal{J}_\zeta$ is an open dense subset of $\mathbb{P}_{\zeta^*}$ from N) and let $E_3 = \{\alpha \in E_2 : (\forall \zeta \in w_\alpha)(\exists \beta < \alpha)(\mathcal{J}_\zeta = \mathcal{I}_\beta)\}$. Clearly, E_3 is a club of λ.

Now, using the diamond $\bar{F}'$ fixed earlier, we are going to define a quasi D-diamond sequence $\bar{F}$ (and then a weak fuzzy candidate $\bar{q}$ over it) that will be as required by $(\circledast)^-$ of A.3.6(3). So, for each $\delta \in S$ we let

$$Z(\delta) = \{\zeta \in (\zeta^* + 1) \setminus \{0\} : \langle (h \circ F'_\delta(\alpha)) \restriction \zeta : \alpha < \delta\rangle \text{ is a } \leq_{\mathbb{P}_\zeta}\text{-increasing } \bar{\mathcal{I}}^{[\zeta]}\text{-exact sequence of members of } N \cap \mathbb{P}_\zeta\}$$

and if $Z(\delta) \neq \emptyset$ then we put $\gamma(\delta) = \sup(Z(\delta))$. Note that $Z(\delta) \in N$ and thus $\gamma(\delta) \in N$ (when defined). Now, the pre–diamond $\bar{F} = \langle F_\delta : \delta \in S\rangle$ is picked so that for a limit $\delta \in S$:

$(\odot)_1$ if $Z(\delta) \neq \emptyset$, then $h \circ F_\delta(\alpha) = (h \circ F'_\delta(\alpha)) \restriction \gamma(\delta)$ for all $\alpha < \delta$;

$(\odot)_2$ if $Z(\delta) = \emptyset$, then $h \circ F_\delta(\alpha) = \emptyset_{\mathbb{P}_{\zeta^*}}$ for all $\alpha < \delta$.

Then easily $\bar{F}$ is a quasi D-diamond for $(N, h, \mathbb{P}_{\zeta^*})$ and for each limit $\delta \in S$, $\langle h \circ F_\delta(\alpha) : \alpha < \delta \rangle$ is a $\leq_{\mathbb{P}_{\zeta^*}}$-increasing sequence of conditions from $\mathbb{P}_{\zeta^*} \cap N$.

Note that $\gamma(\delta)$ is chosen for F'_δ and not F_δ. It could happen that above $\gamma(\delta)$, $h \circ F'_\delta(\alpha)$ gives us something that is not a name, and this (and **not exactness**) is the reason why $\gamma(\delta)$ is not larger. Then (if our list of open dense sets is tricky) it could happen that $\gamma(\delta)$ is small but the sequence $\langle (h \circ F_\delta(\alpha)) \restriction \zeta : \alpha < \delta \rangle$ is $\bar{\mathcal{I}}^{[\zeta]}$-exact. This is exactly the reason why we will need E_3 — the exactness at $\delta \in E_3$ implies that the domains of conditions are large enough.

Just for notational simplicity, we will identify a sequence $\bar{\sigma} = \langle \sigma_0 \rangle$ with its (only) term σ_0. Thus below, when we talk about a standard $(w, 1)^{\zeta^*}$-tree $\mathcal{T}$, we think of T as a set of sequences $t = \langle (t)_\zeta : \zeta \in w \cap \mathrm{rk}(t) \rangle$ where $(t)_\zeta$'s do not have to be sequences.

Now we are going to define sequences $\bar{p} = \langle p_i : i < \lambda \rangle \subseteq \mathbb{P}_{\zeta^*} \cap N$, $\langle \mathcal{T}_\delta : \delta \in S$ is limit $\rangle$, and $\langle q_{\delta,t} : \delta \in S$ is limit $\ \&\ t \in T_\delta \rangle \subseteq \mathbb{P}_{\zeta^*} \cap N$ such that for a limit ordinal $\delta \in S$:

(i) $\mathcal{T}_\delta = (T_\delta, \mathrm{rk}_\delta)$ is a standard $(w_\delta, 1)^{\zeta^*}$-tree, and (under the identification mentioned earlier) $\{t \in T_\delta : \mathrm{rk}_\delta(t) = \zeta\} \subseteq \mathfrak{Y}_a^{[\zeta]}(\delta)$ for $\zeta \in w_\delta \cup \{\zeta^*\}$, so $(t)_\varepsilon \in \mathfrak{Y}_a^\varepsilon(\delta)$ for $t \in T_\delta$, $\varepsilon \in w_\delta \cap \mathrm{rk}_\delta(t)$,

(ii) $\langle q_{\delta,t} : t \in T_\delta \rangle$ is a standard tree of conditions in $\bar{\mathbb{Q}}$,

(iii) $p \leq p_i \leq p_j$ for $i < j < \lambda$,

(iv) if $j < \lambda$, then $w_j \subseteq \mathrm{Dom}(p_j)$ and $(\forall \varepsilon \in w_j)(\forall j' > j)(p_j(\varepsilon) = p_{j'}(\varepsilon))$,

(v) if $t \in T_\delta$, $\mathrm{rk}_\delta(t) = \zeta$, then $q_{\delta,t} \in \mathbb{P}_\zeta \cap N$ is such that

(a) $\left(\bigcup_{\alpha<\delta} \mathrm{Dom}(h \circ F_\delta(\alpha)) \cup \bigcup_{i<\delta} \mathrm{Dom}(p_i) \right) \cap \zeta \subseteq \mathrm{Dom}(q_{\delta,t})$,

(b) $(\forall \alpha < \delta)((h \circ F_\delta)(\alpha) \restriction \zeta \leq q_{\delta,t})$, and

(c) if $\varepsilon \in \mathrm{Dom}(q_{\delta,t}) \setminus w_\delta$, then

$$q_{\delta,t} \restriction \varepsilon \Vdash \text{“if the set } \{p_i(\varepsilon) : i < \delta\} \cup \{(h \circ F_\delta(\alpha))(\varepsilon) : \alpha < \delta\}$$
$$\text{has an upper bound in } \mathop{\mathbb{Q}}_{\sim}{}_\varepsilon,$$
$$\text{then } q_{\delta,t}(\varepsilon) \text{ is such an upper bound”},$$

(d) if $\varepsilon \in \mathrm{Dom}(q_{\delta,t}) \cap w_\delta$, then

$$q_{\delta,t} \restriction \varepsilon \Vdash \text{"if the set } \{p_i(\varepsilon) : i < \delta\} \cup \{(h \circ F_\delta(\alpha))(\varepsilon) : \alpha < \delta\} \text{ has an upper bound which obeys } (t)_\varepsilon \text{ at } \delta, \text{ then } q_{\delta,t}(\varepsilon) \text{ is such an upper bound, else } q_{\delta,t}(\varepsilon) \text{ is an upper bound of } \{(h \circ F_\delta(\alpha))(\varepsilon) : \alpha < \delta\} \text{which obeys } (t)_\varepsilon \text{ at } \delta\text{''},$$

(e) $q_{\delta,t} \in \bigcap_{\xi<\delta} \mathcal{I}_\xi^{[\zeta]}$,

(vi) if $t \in T_\delta$, $\zeta = \mathrm{rk}_\delta(t) < \zeta^*$, $\zeta' \in w_\delta \cup \{\zeta^*\}$ is the successor of ζ in $w_\delta \cup \{\zeta^*\}$ and $t', t'' \in T_\delta$ are such that $\mathrm{rk}_\delta(t') = \mathrm{rk}_\delta(t'') = \zeta'$, $t \vartriangleleft t'$, $t \vartriangleleft t''$ and $t' \neq t''$, then

$$q_{\delta,t} \Vdash_{\mathbb{P}_\zeta} \text{"the conditions } q_{\delta,t'}(\zeta) \text{ and } q_{\delta,t''}(\zeta) \text{ are incompatible''}$$

(and so also the conditions $q_{\delta,t'}, q_{\delta,t''}$ are incompatible),

(vii) if $t \in T_\delta$ and $\varepsilon \in \mathrm{Dom}(q_{\delta,t}) \setminus w_\delta$, then $\varepsilon \in \mathrm{Dom}(p_\delta)$ and

$$p_\delta \restriction \varepsilon \Vdash \text{"if } q_{\delta,t} \restriction \varepsilon \in \Gamma_{\mathbb{P}_\varepsilon} \text{ and } \{p_i(\varepsilon) : i < \delta\} \cup \{q_{\delta,t}(\varepsilon)\} \text{ has an upper bound in } \underset{\sim}{\mathbb{Q}}_\varepsilon, \text{ then } p_\delta(\varepsilon) \text{ is such an upper bound''},$$

(viii) if $t \in T_\delta$, $\mathrm{rk}_\delta(t) = \zeta < \zeta^*$, $x \in \mathfrak{Y}_a^\zeta(\delta)$ and

$$q_{\delta,t} \nVdash_{\mathbb{P}_\zeta} \text{"there is no condition stronger than all } (h \circ F_\delta(\alpha))(\zeta) \text{ for } \alpha < \delta \text{ which obeys } x \text{ at } \delta\text{''},$$

then there is $t' \in T_\delta$ such that $t \vartriangleleft t'$ and $(t')_\zeta = x$.

Assume that $\delta < \lambda$ and we have defined $p_i, \mathcal{T}_i, q_{i,t}$ for relevant $i < \delta$ and t. If δ is not a limit ordinal from S, then only $p_\delta \in \mathbb{P}_{\zeta^*} \cap N$ needs to be defined, and clauses (iii), (iv) can easily be taken care of. So suppose that $\delta \in S$ is limit.

First we let $\mathcal{T}_\delta'$ be a standard $(w_\delta, 1)^{\zeta^*}$-tree such that $\{t \in T_\delta' : \mathrm{rk}_\delta(t) = \zeta\} = \mathfrak{Y}_a^{[\zeta]}(\delta)$ (for $\zeta \in w_\delta \cup \{\zeta^*\}$). For $t \in T_\delta'$ we define a condition $r_t \in \mathbb{P}_{\mathrm{rk}_\delta'(t)}$ so that

$$\mathrm{Dom}(r_t) = \Big(\bigcup_{\alpha<\delta} \mathrm{Dom}(h \circ F_\delta(\alpha)) \cup \bigcup_{i<\delta} \mathrm{Dom}(p_i) \cup w_\delta\Big) \cap \mathrm{rk}_\delta'(t),$$

and for each $\zeta \in \mathrm{Dom}(r_t)$, $r_t(\zeta)$ is the $<_\chi^*$-first $\mathbb{P}_\zeta$-name for a condition in $\underset{\sim}{\mathbb{Q}}_\zeta$ such that:

if $\zeta \in w_\delta$, then

$$r_t\restriction\zeta \Vdash_{\mathbb{P}_\zeta} \text{"if the family } \{p_i(\zeta) : i<\delta\} \cup \{(h\circ F_\delta(\alpha))(\zeta) : \alpha<\delta\}$$
$$\text{has an upper bound which obeys } (t)_\zeta \text{ at } \delta,$$
$$\text{then } r_t(\zeta) \text{ is such an upper bound,}$$
$$\text{if the previous is impossible, but there is an upper bound of}$$
$$\{(h\circ F_\delta(\alpha))(\zeta) : \alpha<\delta\} \text{ which obeys } (t)_\zeta \text{ at } \delta$$
$$\text{then } r_t(\zeta) \text{ is such an upper bound,}$$
$$\text{if neither of the previous two possibilities holds,}$$
$$\text{then } r_t(\zeta) \text{ is an upper bound of } \{(h\circ F_\delta(\alpha))(\zeta) : \alpha<\delta\}\text{"},$$

and if $\zeta \notin w_\delta$, then

$$r_t\restriction\zeta \Vdash_{\mathbb{P}_\zeta} \text{"if the family } \{p_i(\zeta) : i<\delta\} \cup \{(h\circ F_\delta(\alpha))(\zeta) : \alpha<\delta\}$$
$$\text{has an upper bound, then } r_t(\zeta) \text{ is such an upper bound,}$$
$$\text{if this is not possible, then } r_t(\zeta) \text{ is just an upper bound of}$$
$$\{(h\circ F_\delta(\alpha))(\zeta) : \alpha<\delta\}\text{"}.$$

Plainly, $|T'_\delta| < \lambda$ and $\bar{r} = \langle r_t : t \in T'_\delta\rangle$ is a standard tree of conditions, and it belongs to N (remember: ${}^{<\lambda}N \subseteq N$). So using A.1.9 in N we may pick a standard tree of conditions $\bar{r}^* = \langle r^*_t : t \in T'_\delta\rangle \in N$ such that $\bar{r} \le \bar{r}^*$ and for each $t \in T'_\delta$ and $\zeta \in w_\delta \cap \mathrm{rk}'_\delta(t)$ the condition $r^*_t\restriction\zeta$ decides the truth value of the sentence

$$\text{"}r^*_t(\zeta) \text{ obeys } (t)_\zeta \text{ at } \delta \text{ (with respect to } \mathfrak{R}_\zeta)\text{"}$$

(remember the choice of $r_t(\zeta)$ for $\zeta \in w_\delta$ and A.3.3(a)). Put

$$T_\delta = \{t \in T'_\delta : \text{ for each } \zeta \in w_\delta \cap \mathrm{rk}'_\delta(t),\ r^*_t\restriction\zeta \Vdash_{\mathbb{P}_\zeta} \text{"}r^*_t(\zeta) \text{ obeys } (t)_\zeta \text{ at } \delta\text{"}\},$$

and notice that $T_\delta \in N$ is a standard $(w_\delta, 1)^{\zeta^*}$-tree.

Let us argue that for each $t \in T_\delta$ there is $t' \in T_\delta$ such that $t \trianglelefteq t'$ and $\mathrm{rk}_\delta(t') = \zeta^*$. First note that, by our choices we have:

- $\langle (h\circ F_\delta)(\alpha) : \alpha<\delta\rangle$ is an increasing sequence of conditions in $\mathbb{P}_{\zeta^*}$,
- for each $t \in T'_\delta$ and $\alpha<\delta$, the condition r_t is stronger than $(h\circ F_\delta)(\alpha)\restriction \mathrm{rk}'_\delta(r_t)$,
- every $t \in T'_\delta$ can be extended to an element of T'_δ with rank ζ^*, as a matter of fact, if $t \in T'_\delta$, $\mathrm{rk}'_\delta(t) = \zeta < \zeta^*$ and $x \in \mathfrak{Y}^\zeta_a(\delta)$, then $t^\frown\langle x\rangle \in T'_\delta$.

Suppose now that $t \in T_\delta$, $\zeta = \mathrm{rk}_\delta(t) \in w_\delta$ (so also $t \in T'_\delta$ and $\mathrm{rk}'_\delta(t) = \zeta$). For each $x \in \mathfrak{Y}^\zeta_a(\delta)$, $t^\frown\langle x\rangle \in T'_\delta$. So $r^*_{t^\frown\langle x\rangle}$ is defined and by the choice of $\bar{r}^*$ the condition r^*_t decides the truth value of "$r^*_{t^\frown\langle x\rangle}(\zeta)$ obeys x at δ". We are going to argue that for some $z \in \mathfrak{Y}^\zeta_a(\delta)$ the decision above is positive, i.e., $r^*_t = r^*_{t^\frown\langle z\rangle}\restriction\zeta$ forces that "$r^*_{t^\frown\langle z\rangle}(\zeta)$ obeys z at δ". This will imply that $t^\frown\langle z\rangle \in T_\delta$.

The condition r^*_t is stronger than r_t, so by what we said earlier it forces that "$\langle (h\circ F_\delta)(\alpha)(\zeta) : \alpha < \delta\rangle$ is an increasing sequence of conditions in $\underset{\sim}{\mathbb{Q}}_\zeta$ and therefore it has an upper bound". Now look at A.3.3(c) and apply it to a condition q which is stronger than all $(h\circ F_\delta)(\alpha)(\zeta)$: r^*_t also forces "there are an $x \in \mathfrak{Y}^\zeta_a(\delta)$ and a condition $q' \in \underset{\sim}{\mathbb{Q}}_\zeta$ stronger than all $(h\circ F_\delta)(\alpha)(\zeta)$ for $\alpha < \delta$ and such that q' obeys x". It follows from the choice of $r_{t^\frown\langle x\rangle}(\zeta)$ for $x \in \mathfrak{Y}^\zeta_a(\delta)$ that r^*_t forces "there is an $x \in \mathfrak{Y}^\zeta_a(\delta)$ such that $r_{t^\frown\langle x\rangle}(\zeta)$ obeys x at δ", and therefore also r^*_t forces "there is an $x \in \mathfrak{Y}^\zeta_a(\delta)$ such that $r^*_{t^\frown\langle x\rangle}(\zeta)$ obeys x at δ" (remember A.3.3(a)). Therefore, it cannot be the case that for all $z \in \mathfrak{Y}^\zeta_a(\delta)$, r^*_t forces "$r^*_{t^\frown\langle z\rangle}(\zeta)$ does not obey z at δ", so we can pick z as desired.

Proceeding inductively in this manner, we may extend any sequence in T_δ to one with rank ζ^*.

Finally, using A.1.4 and A.1.9 next in N we may choose a standard tree of conditions $\langle q_{\delta,t} : t \in T_\delta\rangle \in N$ which satisfies clauses (vi) and (v)(e) and such that $r^*_t \le q_{\delta,t}$ (for $t \in T_\delta$). It should be clear that then $\mathcal{T}_\delta$ and $\langle q_{\delta,t} : t \in T_\delta\rangle$ satisfy all the relevant demands from our list ((i)–(viii)). Now finding a condition $p_\delta \in \mathbb{P}_{\zeta^*} \cap N$ which satisfies (iii)+(iv)+(vii) is straightforward. (Note that, by (vi), if $t', t'' \in T_\delta$, $\varepsilon \in \mathrm{Dom}(q_{\delta,t'}) \cap \mathrm{Dom}(q_{\delta,t''})$, $\varepsilon \notin w_\delta$ and the conditions $q_{\delta,t'}\restriction\varepsilon, q_{\delta,t''}\restriction\varepsilon$ are compatible, then $q_{\delta,t'}\restriction(\varepsilon+1) = q_{\delta,t''}\restriction(\varepsilon+1)$.)

For a limit ordinal $\delta \in S$ and $\zeta \in N \cap (\zeta^*+1)$ we let

- $\mathcal{X}^{[\zeta]}_\delta = \{t\restriction\zeta : t \in T_\delta \;\&\; \mathrm{rk}_\delta(t) = \zeta^*\}$;
- if $s \in \mathcal{X}^{[\zeta]}_\delta$, then $q^{[\zeta]}_{\delta,s} = q_{\delta,t}\restriction\zeta$ for some (equivalently: all; remember (ii)) $t \in T_\delta$ such that $s \trianglelefteq t$ and $\mathrm{rk}_\delta(t) = \zeta^*$;
- $\bar q^{[\zeta]} = \langle q^{[\zeta]}_{\delta,s} : \delta \in S$ is limit $\& s \in \mathcal{X}^{[\zeta]}_\delta\rangle$;
- $h^{[\zeta]}\colon \lambda \longrightarrow N$ is such that $h^{[\zeta]}(\gamma) = (h(\gamma)) \restriction \zeta$ provided $h(\gamma)$ is a function, and $h^{[\zeta]}(\gamma) = \emptyset_{\mathbb{P}_\zeta}$ otherwise.

Plainly, $\emptyset \ne \mathcal{X}^{[\zeta]}_\delta \subseteq \mathfrak{Y}^{[\zeta]}_a(\delta)$ (remember (i)) and $h^{[\zeta]} : \lambda \longrightarrow N$ is such that $\mathbb{P}_\zeta \cap N \subseteq \mathrm{Rng}(h^{[\zeta]})$. Moreover, one easily verifies the following

Claim A.3.10.2: *Let $\zeta \in N \cap (\zeta^*+1)$. Then $\bar F$ is a quasi D-diamond sequence for $(N, h^{[\zeta]}, \mathbb{P}_\zeta)$ and $\bar q^{[\zeta]}$ is a weak fuzzy candidate over $\bar F$ for*

$$(N, h^{[\zeta]}, \mathbb{P}_\zeta, \mathfrak{R}^{[\zeta]}, \bar{\mathfrak{Y}}^{[\zeta]}, \bar{\mathcal{I}}^{[\zeta]}).$$

We may write $\bar q, q_{\delta,t}, \mathcal{X}_\delta$ for $\bar q^{[\zeta^*]}, q^{[\zeta^*]}_{\delta,t}, \mathcal{X}^{[\zeta^*]}_\delta$, respectively. Also note that, in the context of our definitions, the functions h and $h^{[\zeta^*]}$ behave the same, so we may identify them. Of course, we are going to define an $(\mathfrak{R}, \bar{\mathfrak{Y}})$-fuzzy generic condition $r \in \mathbb{P}_{\zeta^*}$ for $\bar{q}$ over $\bar{F}$, but before that we have to introduce more notation used later and prove some important facts.

For $\zeta \in \zeta^* \cap N$ we define a function $h^{\langle\zeta\rangle}$ and $\mathbb{P}_\zeta$-names $\underset{\sim}{S}^{\langle\zeta\rangle}, \underset{\sim}{\mathcal{X}}_\delta^{\langle\zeta\rangle}, \underset{\sim}{\mathcal{I}}_\alpha^{\langle\zeta\rangle}, \underset{\sim}{\bar{F}}^{\langle\zeta\rangle}$, $\underset{\sim}{\bar{\mathcal{I}}}^{\langle\zeta\rangle}$ and $\underset{\sim}{\bar{q}}^{\langle\zeta\rangle}$ so that:

- $h^{\langle\zeta\rangle} : \lambda \longrightarrow N$ is such that if $h(\gamma)$ is a function, $\zeta \in \mathrm{Dom}(h(\gamma))$ and $(h(\gamma))(\zeta)$ is a $\mathbb{P}_\zeta$-name then $h^{\langle\zeta\rangle}(\gamma) = ((h(\gamma))(\zeta)$, otherwise $h^{\langle\zeta\rangle}(\gamma) = \underset{\sim}{\emptyset}_{\mathbb{Q}_\zeta}$;
- $\Vdash_{\mathbb{P}_\zeta}$ "$\underset{\sim}{S}^{\langle\zeta\rangle} = \{\delta \in S :$ if δ is limit then $\delta > \pi(\zeta) \,\&\, (\exists s \in \mathcal{X}_\delta^{[\zeta]})(q_{\delta,s}^{[\zeta]} \in \Gamma_{\mathbb{P}_\zeta})\}$";
- $\Vdash_{\mathbb{P}_\zeta}$ "if $\delta \in \underset{\sim}{S}^{\langle\zeta\rangle}$ is limit, then $\underset{\sim}{\mathcal{X}}_\delta^{\langle\zeta\rangle} = \{x \in \mathfrak{Y}_a^\zeta(\delta) : (\exists t \in \mathcal{X}_\delta)(q_{\delta,t}{\restriction}\zeta \in \Gamma_{\mathbb{P}_\zeta} \,\&\, (t)_\zeta = x)\}$";
- $\Vdash_{\mathbb{P}_\zeta}$ "$\underset{\sim}{\bar{q}}^{\langle\zeta\rangle} = \langle \underset{\sim}{q}_{\delta,x}^{\langle\zeta\rangle} : \delta \in \underset{\sim}{S}^{\langle\zeta\rangle}$ is limit $\& x \in \underset{\sim}{\mathcal{X}}_\delta^{\langle\zeta\rangle}\rangle$", where:
- $\Vdash_{\mathbb{P}_\zeta}$ "if $\delta \in \underset{\sim}{S}^{\langle\zeta\rangle}$ is limit and $x \in \underset{\sim}{\mathcal{X}}_\delta^{\langle\zeta\rangle}$, then $\underset{\sim}{q}_{\delta,x}^{\langle\zeta\rangle} = q_{\delta,t}(\zeta)$ for some (equivalently: all) $t \in \mathcal{X}_\delta$ such that $q_{\delta,t}{\restriction}\zeta \in \Gamma_{\mathbb{P}_\zeta}$ and $(t)_\zeta = x$"
(again, remember (ii));
- $\Vdash_{\mathbb{P}_\zeta}$ " $\underset{\sim}{\bar{F}}^{\langle\zeta\rangle} = \langle F_\delta : \delta \in \underset{\sim}{S}^{\langle\zeta\rangle}\rangle$";
- $\Vdash_{\mathbb{P}_\zeta}$ "$\underset{\sim}{\bar{\mathcal{I}}}^{\langle\zeta\rangle} = \langle \underset{\sim}{\mathcal{I}}_\alpha^{\langle\zeta\rangle} : \alpha < \lambda\rangle$", where:
- $\Vdash_{\mathbb{P}_\zeta}$ "$\underset{\sim}{\mathcal{I}}_\alpha^{\langle\zeta\rangle} = \{p(\zeta) : p \in \mathcal{I}_\alpha \,\&\, p \restriction \zeta \in \Gamma_{\mathbb{P}_\zeta}\}$".

Naturally, we treat $h^{\langle\zeta\rangle}$ as a $\mathbb{P}_\zeta$-name for a function from λ to $N[\Gamma_{\mathbb{P}_\zeta}]$. Observe that $\Vdash_{\mathbb{P}_\zeta}$ "$N[\Gamma_{\mathbb{P}_\zeta}] \cap \underset{\sim}{\mathbb{Q}}_\zeta \subseteq \mathrm{Rng}(h^{\langle\zeta\rangle})$", and
$\Vdash_{\mathbb{P}_\zeta}$ "$\underset{\sim}{\bar{\mathcal{I}}}^{\langle\zeta\rangle}$ lists all open dense subsets of $\underset{\sim}{\mathbb{Q}}_\zeta$ from $N[\Gamma_{\mathbb{P}_\zeta}]$".

Claim A.3.10.3: *Assume that $\zeta \in N \cap \zeta^*$ and $r \in \mathbb{P}_\zeta$ is a $(\mathfrak{R}^{[\zeta]}, \bar{\mathfrak{Y}}^{[\zeta]})$-fuzzy generic condition for $\bar{q}^{[\zeta]}$ over $(N, \bar{\mathcal{I}}^{[\zeta]}, h^{[\zeta]}, \mathbb{P}_\zeta, \bar{F})$. Then*

(1) *$r \Vdash_{\mathbb{P}_\zeta}$ "$\underset{\sim}{S}^{\langle\zeta\rangle} \in D^+$",*
(2) *$r \Vdash_{\mathbb{P}_\zeta}$ "$\underset{\sim}{\bar{F}}^{\langle\zeta\rangle}$ is a quasi D-diamond for $(N[\Gamma_{\mathbb{P}_\zeta}], h^{\langle\zeta\rangle}, \underset{\sim}{\mathbb{Q}}_\zeta)$", and*
(3) *$r \Vdash_{\mathbb{P}_\zeta}$ "$\underset{\sim}{\bar{q}}^{\langle\zeta\rangle}$ is a fuzzy candidate for $(N[\Gamma_{\mathbb{P}_\zeta}], h^{\langle\zeta\rangle}, \underset{\sim}{\mathbb{Q}}_\zeta, \underset{\sim}{\mathfrak{R}}_\zeta, \bar{\mathfrak{Y}}^\zeta)$ over $\underset{\sim}{\bar{F}}^{\langle\zeta\rangle}$".*

Proof: (1) Will follow from (2).

(2) Assume that this fails. Then we can find a condition $r^* \in \mathbb{P}_\zeta$, a $\mathbb{P}_\zeta$-name $\underset{\sim}{\bar{q}}' = \langle \underset{\sim}{q}'_\alpha : \alpha < \lambda\rangle$ for an increasing sequence of conditions from $\underset{\sim}{\mathbb{Q}}_\zeta \cap N[\Gamma_{\mathbb{P}_\zeta}]$, and $\mathbb{P}_\zeta$-names $\underset{\sim}{A}_\xi, \underset{\sim}{B}_\xi$ for members of $D \cap \mathbf{V}$ such that $r \le_{\mathbb{P}_\zeta} r^*$ and

$$r^* \Vdash_{\mathbb{P}_\zeta} \text{"}(\forall \delta \in \mathop{\triangle}_{\xi<\lambda} \underset{\sim}{A}_\xi)(\langle \underset{\sim}{q}'_\alpha : \alpha < \delta\rangle \text{ is } \underset{\sim}{\bar{\mathcal{I}}}^{\langle\zeta\rangle}\text{-exact}) \quad \text{and}$$
$$(\forall \delta \in \underset{\sim}{S}^{\langle\zeta\rangle} \cap \mathop{\triangle}_{\xi<\lambda} \underset{\sim}{B}_\xi)(\langle h^{\langle\zeta\rangle} \circ F_\delta(\alpha) : \alpha < \delta\rangle \neq \underset{\sim}{\bar{q}}' \restriction \delta)\text{"}.$$

Consider a play $\langle r_j, C_j : j < \lambda\rangle$ of the game $\eth_\lambda^{\mathrm{fuzzy}}(r, N, \bar{\mathcal{I}}^{[\zeta]}, h^{[\zeta]}, \mathbb{P}_\zeta, \bar{F}, \bar{q}^{[\zeta]})$ in which Generic uses her winning strategy and Antigeneric plays as follows.

Together with choosing r_j (for $j \in \lambda \setminus S$), Antigeneric chooses also side conditions $p^+_j \in N \cap \mathbb{P}_\zeta$, sets $A_\xi, B_\xi \in D$ and $\mathbb{P}_\zeta$-names $\underset{\sim}{q}^*_\xi \in N$ for elements of $\underset{\sim}{\mathbb{Q}}_\zeta$ (for $\xi \le j$) such that

- $r_j \ge r^*$ (so $r_0 \ge r^*$; remember Antigeneric plays at 0), $r_j \ge r_i$ (for $i < j$), and $r_j \ge p^+_j$ and
- $r_j \Vdash_{\mathbb{P}_\zeta}$ "$(\forall \xi \le j)(\underset{\sim}{A}_\xi = A_\xi \,\&\, \underset{\sim}{B}_\xi = B_\xi \,\&\, \underset{\sim}{q}'_\xi = \underset{\sim}{q}^*_\xi)$", and
- if $j' < j$ are from $\lambda \setminus S$, then $p^+_j \ge p^+_{j'}$, and $p^+_j \in \bigcap_{\xi<j} \mathcal{I}^{[\zeta]}_\xi$, and
- $p^+_j \Vdash_{\mathbb{P}_\zeta}$ "$(\forall \xi_0 < \xi_1 \le j)(\underset{\sim}{q}^*_{\xi_0} \le \underset{\sim}{q}^*_{\xi_1})$", and
- if $\delta < j$, $\delta \in \bigcap_{\xi<\delta} A_\xi$, then $p^{+\frown}_j \langle \underset{\sim}{q}^*_j \rangle \in \mathcal{I}^{[\zeta+1]}_\xi$ for all $\xi < \delta$.

(The C_j's are not that important for our argument, so we do not specify any requirements on them. Regarding the choice of the p^+_j's, remember A.3.8(2); for the last two demands remember that $\underset{\sim}{q}'_j$'s are (forced to be) increasing.) After the play, Antigeneric completes $\langle p^+_j : j \in \lambda \setminus S \rangle$ to a $\le_{\mathbb{P}_\zeta}$-increasing sequence $\langle p^+_j : j < \lambda \rangle \subseteq N \cap \mathbb{P}_\zeta$ letting $p^+_j = p^+_{\min(\lambda \setminus S \setminus (j+1))}$ for $j \in S$.

Note that if $\delta \in E_0$ is a limit of elements of $\triangle_{\xi<\lambda} A_\xi$, then the sequence $\langle p^{+\frown}_j \langle \underset{\sim}{q}^*_j \rangle : j < \delta \rangle$ is $\bar{\mathcal{I}}^{[\zeta+1]}$-exact and increasing (and $\langle p^+_j : j < \delta \rangle$ is $\bar{\mathcal{I}}^{[\zeta]}$-exact). So, as D is normal and $A_\xi, B_\xi, C_j \in D$ and $\bar{F}$ is a quasi D-diamond for $(N, h^{[\zeta+1]}, \mathbb{P}_{\zeta+1})$ (by A.3.10.2), we may find an ordinal $\delta \in S \cap E_0 \cap \triangle_{\xi<\lambda} A_\xi \cap \triangle_{\xi<\lambda} B_\xi \cap \triangle_{j<\lambda} C_j \setminus (\pi(\zeta)+1)$ which is a limit of members of $\triangle_{\xi<\lambda} A_\xi$ and such that $\langle h^{[\zeta+1]} \circ F_\delta(j) : j < \delta \rangle = \langle p^{+\frown}_j \langle \underset{\sim}{q}^*_j \rangle : j < \delta \rangle$. By the choice of $\bar{F}$ we know that $h(F_\delta(j))$ is a condition in $\mathbb{P}_{\zeta^*}$ (so a function) and hence $h^{\langle \zeta \rangle}(F_\delta(j)) = \underset{\sim}{q}^*_j$ for all $j < \delta$. Also

$$(\forall i < \delta)(\exists j \in \delta \setminus S)(h^{[\zeta]} \circ F_\delta(i) \le_{\mathbb{P}_\zeta} h^{[\zeta]} \circ F_\delta(j) = p^+_j \le_{\mathbb{P}_\zeta} r_j).$$

Since the play is won by Generic, for some $s \in \mathcal{X}^{[\zeta]}_\delta$ we have $q^{[\zeta]}_{\delta,s} \le r_\delta$. But then

$$r_\delta \Vdash \text{“}\delta \in S^{\langle \zeta \rangle} \cap \mathop{\triangle}_{\xi<\lambda} \underset{\sim}{B}_\xi \quad \& \quad \langle h^{\langle \zeta \rangle} \circ F_\delta(\alpha) : \alpha < \delta \rangle = \bar{q}' \restriction \delta\text{”},$$

a contradiction.

Comment on the proof above: for these arguments we really need more than just $(N, \mathbb{P}_\zeta)$-genericity of r, as we need to know that r_δ forces $\delta \in \underset{\sim}{S}^{\langle \zeta \rangle}$ (see the definition of $\underset{\sim}{\bar{F}}^{\langle \zeta \rangle}$; note also part (1) of A.3.10.3). Now look at the definition of $\underset{\sim}{S}^{\langle \zeta \rangle}$. Note that $\underset{\sim}{\bar{F}}^{\langle \zeta \rangle}$ is (a name for) a subsequence of $\bar{F}$; without doing something that involves $\bar{q}$ we could get into the situation where the domain of this subsequence is non-stationary. Playing the fuzzy game works well here.

(3) Straightforward (remember the choice of $q_{\delta,t}$'s, specifically clauses (v)(b,d,e) and (viii)). ∎

Now we are going to define an $(\mathfrak{R},\bar{\mathfrak{Y}})$-fuzzy generic condition $r\in\mathbb{P}$ for $\bar{q}$ over $(N,\bar{\mathcal{I}},h,\mathbb{P}_{\zeta^*},\bar{F})$ in the most natural way. Its domain is $\mathrm{Dom}(r)=\zeta^*\cap N$ and for each $\zeta\in\zeta^*\cap N$

$$r\restriction\zeta\Vdash\text{“}r(\zeta)\geq p_{\pi(\zeta)+1}(\zeta)\text{ is an }(\mathfrak{R}_\zeta,\bar{\mathfrak{Y}}^\zeta)\text{-fuzzy generic condition for }\bar{q}^{\langle\zeta\rangle}\\ \text{over }(N[\Gamma_{\mathbb{P}_\zeta}],\underset{\sim}{\bar{\mathcal{I}}}^{\langle\zeta\rangle},h^{\langle\zeta\rangle},\underset{\sim}{\mathbb{Q}}_\zeta,\underset{\sim}{\bar{F}}^{\langle\zeta\rangle})\text{”}.$$

(So $r\geq p_i$ for all $i<\lambda$.)

Claim A.3.10.4: *For every $\zeta\in(\zeta^*+1)\cap N$, the Generic player has a winning strategy in the game $\partial^{\rm fuzzy}_\lambda(r\restriction\zeta,N,\bar{\mathcal{I}}^{[\zeta]},h^{[\zeta]},\mathbb{P}_\zeta,\bar{F},\bar{q}^{[\zeta]})$.*

Proof of the Claim: We will prove the claim by induction on $\zeta\in(\zeta^*+1)\cap N$. After we are done with stage $\zeta\in(\zeta^*+1)\cap N$, we know that $r\restriction\zeta$ is $(\mathfrak{R}^{[\zeta]},\bar{\mathfrak{Y}}^{[\zeta]})$-fuzzy generic for $\bar{q}^{[\zeta]}$ over $(N,\bar{\mathcal{I}}^{[\zeta]},h^{[\zeta]},\mathbb{P}_\zeta,\bar{F})$. For $\zeta\in\zeta^*\cap N$ this implies that $r(\zeta)$ is well-defined (remember A.3.10.3). Of course for $\zeta=\zeta^*$ we finish the proof of the theorem.

Suppose that $\zeta\in(\zeta^*+1)\cap N$ and we know that $r\restriction\zeta'$ is $(\mathfrak{R}^{[\zeta']},\bar{\mathfrak{Y}}^{[\zeta']})$-fuzzy generic for $\bar{q}^{[\zeta']}$ over $(N,\bar{\mathcal{I}}^{[\zeta']},h^{[\zeta']},\mathbb{P}_{\zeta'},\bar{F})$ for all $\zeta'\in N\cap\zeta$. We are going to define a winning strategy st for Generic in the game

$$\partial^{\rm fuzzy}_\lambda(r\restriction\zeta,N,\bar{\mathcal{I}}^{[\zeta]},h^{[\zeta]},\mathbb{P}_\zeta,\bar{F},\bar{q}^{[\zeta]}).$$

First, for $\varepsilon\in\zeta\cap N$ fix a $\mathbb{P}_\varepsilon$-name $\underset{\sim}{\rm st}_\varepsilon$ such

$$r\restriction\varepsilon\Vdash\text{“}\underset{\sim}{\rm st}_\varepsilon\text{ is a winning strategy of the Generic player}\\ \text{in the game }\partial^{\rm fuzzy}_\lambda(r(\varepsilon),N[\Gamma_{\mathbb{P}_\varepsilon}],\underset{\sim}{\bar{\mathcal{I}}}^{\langle\varepsilon\rangle},h^{\langle\varepsilon\rangle},\underset{\sim}{\mathbb{Q}}_\varepsilon,\bar{F}^{\langle\varepsilon\rangle},\bar{q}^{\langle\varepsilon\rangle})\text{”}.$$

We will think of $\underset{\sim}{\rm st}_\varepsilon$ as a name for a function from $<\lambda$-sequences of members of $\underset{\sim}{\mathbb{Q}}_\varepsilon\times D$ (thought of as pairs of sequences of the same length $<\lambda$) to $\underset{\sim}{\mathbb{Q}}_\varepsilon\times D$ such that if $(\underset{\sim}{\bar{\sigma}},\underset{\sim}{\bar{C}})\in\mathrm{Dom}(\underset{\sim}{\rm st}_\varepsilon)$ and $\underset{\sim}{\bar{\sigma}}$ has an upper bound, then the first coordinate of ${\rm st}_\varepsilon(\underset{\sim}{\bar{\sigma}},\underset{\sim}{\bar{C}})$ is such an upper bound (and, of course, any play according to $\underset{\sim}{\rm st}_\varepsilon$ is won by Generic). (In a play of $\partial^{\rm fuzzy}_\lambda(r(\varepsilon),N[\Gamma_{\mathbb{P}_\varepsilon}],\underset{\sim}{\bar{\mathcal{I}}}^{\langle\varepsilon\rangle},h^{\langle\varepsilon\rangle},\underset{\sim}{\mathbb{Q}}_\varepsilon,\bar{F}^{\langle\varepsilon\rangle},\bar{q}^{\langle\varepsilon\rangle})$ only the values of $\underset{\sim}{\rm st}_\varepsilon$ at “legal partial plays according to $\underset{\sim}{\rm st}_\varepsilon$” matter, but it is notationally convenient to have $\underset{\sim}{\rm st}_\varepsilon$ giving values for all sequences of elements of $\underset{\sim}{\mathbb{Q}}_\varepsilon\times D$, even if first coordinates are not increasing, as well as for sequences after which Antigeneric should play.)

We will describe the strategy st by giving the answers of Generic on intervals $S\cap[i_\alpha,i_{\alpha+1})$ (for $\alpha<\lambda$), where, remember, $\langle i_\alpha:\alpha<\lambda\rangle$ is the increasing enumeration of E_1. Aside the Generic player will construct sequences

$\langle \underset{\sim}{r}'_{j'}(\varepsilon) : j' < \lambda, \varepsilon \in \zeta \cap N\rangle$ and $\langle \underset{\sim}{C}^{\xi}_{j'}(\varepsilon) : j', \xi < \lambda, \varepsilon \in \zeta \cap N\rangle$ so that, letting $r_j \in \mathbb{P}_\zeta$ be the conditions played in the game,

$(*)_1$ $\underset{\sim}{r}'_{j'}(\varepsilon)$ is a $\mathbb{P}_\varepsilon$-name for a member of $\underset{\sim}{\mathbb{Q}}_\varepsilon$, $\underset{\sim}{C}^{\xi}_{j'}(\varepsilon)$ is a $\mathbb{P}_\varepsilon$-name for a member of $D \cap \mathbf{V}$, and

$(*)_2$ if $\alpha < \lambda$, $\delta = \min(S \cap [i_\alpha, i_{\alpha+1}))$, and $\varepsilon \in w_{\alpha+1} \cap \zeta$, then after the δ-th move (which is a move of the Generic player) the terms $\underset{\sim}{r}'_{j'}(\varepsilon), \langle \underset{\sim}{C}^{\xi}_{j'}(\varepsilon) : \xi < \lambda\rangle$ are defined for all $j' < i_{\alpha+1}$, and

$(*)_3$ if $\alpha < \lambda$, $\varepsilon \in w_{\alpha+1} \cap \zeta$ and $p^* \in \mathbb{P}_\varepsilon$ is stronger than all $r_j{\restriction}\varepsilon$ for $j \in (i_\alpha + 1) \setminus S$, then

$$p^* \Vdash_{\mathbb{P}_\varepsilon} \text{“}(\forall j \in (i_\alpha + 1) \setminus S)(r_j(\varepsilon) \le \underset{\sim}{r}'_{i_\alpha}(\varepsilon))\text{”},$$

$(*)_4$ if $\alpha < \lambda$, $\varepsilon \in w_{\alpha+1} \cap \zeta$ and $r_{i_\alpha+1}$ is the condition played by Generic at stage $i_\alpha + 1 \in S$, then

$$r_{i_\alpha+1}{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}(\forall j' < i_{\alpha+1})(\underset{\sim}{r}'_{j'}(\varepsilon) \le r_{i_\alpha+1}(\varepsilon))\text{”},$$

$(*)_5$ for each $\varepsilon \in N \cap \zeta$,

$$r{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}\langle \underset{\sim}{r}'_j(\varepsilon), \underset{\xi<\lambda}{\triangle} \underset{\sim}{C}^{\xi}_j(\varepsilon) : j < \lambda\rangle \text{ is a legal play of the game}$$
$$\Game^{\rm fuzzy}_\lambda(r(\varepsilon), N[\Gamma_{\mathbb{P}_\varepsilon}], \bar{\mathcal{I}}^{\langle\varepsilon\rangle}, h^{\langle\varepsilon\rangle}, \underset{\sim}{\mathbb{Q}}_\varepsilon, \bar{F}^{\langle\varepsilon\rangle}, \underset{\sim}{\bar{q}}^{\langle\varepsilon\rangle})$$
$$\text{in which Generic uses } \underset{\sim}{\rm st}_\varepsilon\text{”}.$$

So suppose that $\alpha < \lambda$, $\delta = \min(S \cap [i_\alpha, i_{\alpha+1}))$ and $\langle r_j, C_j : j < \delta\rangle$ is the result of the play so far. Now Generic looks at ordinals $\varepsilon \in w_{\alpha+1} \cap \zeta$. She lets the $\mathbb{P}_\varepsilon$-names $\underset{\sim}{r}'_{j'}(\varepsilon), \underset{\sim}{C}^{\xi}_{j'}(\varepsilon)$ be such that $\langle \underset{\sim}{r}'_{j'}(\varepsilon), \triangle_{\xi<\lambda} \underset{\sim}{C}^{\xi}_{j'}(\varepsilon) : j' < i_{\alpha+1}\rangle$ is forced by $r{\restriction}\varepsilon$ to be a play of $\Game^{\rm fuzzy}_\lambda(r(\varepsilon), N[\Gamma_{\mathbb{P}_\varepsilon}], \bar{\mathcal{I}}^{\langle\varepsilon\rangle}, h^{\langle\varepsilon\rangle}, \underset{\sim}{\mathbb{Q}}_\varepsilon, \bar{F}^{\langle\varepsilon\rangle}, \underset{\sim}{\bar{q}}^{\langle\varepsilon\rangle})$ in which the moves are determined as follows. If $\varepsilon \in w_\alpha$, then we have already the play below i_α and the names $\underset{\sim}{r}'_{i_\alpha}(\varepsilon), \underset{\sim}{C}^{\xi}_{i_\alpha}(\varepsilon)$ are such that

- if $i_\alpha = \delta$ (i.e., $i_\alpha \in S$ and r_i, C_i have been chosen for $i < i_\alpha$ only), then

$$r{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}(\underset{\sim}{r}'_{i_\alpha}(\varepsilon), \underset{\xi<\lambda}{\triangle} \underset{\sim}{C}^{\xi}_{i_\alpha}(\varepsilon)) \text{ is the value of } \underset{\sim}{\rm st}_\varepsilon$$
$$\text{applied to } \langle \underset{\sim}{r}'_j(\varepsilon), \underset{\xi<\lambda}{\triangle} \underset{\sim}{C}^{\xi}_j(\varepsilon) : j < i_\alpha\rangle,\text{”}$$

- if $i_\alpha < \delta$ (i.e., $i_\alpha \notin S$ so r_i, C_i are already chosen for $i \le i_\alpha$), then

$$r{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“if } (\forall j < i_\alpha)(\underset{\sim}{r}'_j(\varepsilon) \le r_{i_\alpha}(\varepsilon)) \text{ then } \underset{\sim}{r}'_{i_\alpha}(\varepsilon) = r_{i_\alpha}(\varepsilon)$$
$$\text{otherwise } \underset{\sim}{r}'_{i_\alpha}(\varepsilon) \text{ is the first coordinate of } \underset{\sim}{\rm st}_\varepsilon \text{ applied to}$$
$$\text{the play so far, and } \underset{\sim}{C}^{\xi}_{i_\alpha}(\varepsilon) = \bigcap\nolimits_{j \le i_\alpha} C_j \text{ for all } \xi < \lambda\text{”}.$$

Then for $j \in (i_\alpha, i_{\alpha+1})$ (and $\varepsilon \in w_\alpha \cap \zeta$) the names $\underaccent{\sim}{r}'_j(\varepsilon), \underaccent{\sim}{C}^\xi_j(\varepsilon)$ are determined by applying successively $\underaccent{\sim}{\rm st}_\varepsilon$, that is

$$r{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}(\underaccent{\sim}{r}'_j(\varepsilon), \mathop{\triangle}_{\xi<\lambda} \underaccent{\sim}{C}^\xi_j(\varepsilon)) \text{ is the value of } \underaccent{\sim}{\rm st}_\varepsilon$$
$$\text{applied to } \langle \underaccent{\sim}{r}'_{j'}(\varepsilon), \mathop{\triangle}_{\xi<\lambda} \underaccent{\sim}{C}^\xi_{j'}(\varepsilon) : j' < j\rangle.\text{”}$$

If $\varepsilon \in (w_{\alpha+1} \setminus w_\alpha) \cap \zeta$, then the Generic player defines the names $\underaccent{\sim}{r}'_j(\varepsilon), \underaccent{\sim}{C}^\xi_j(\varepsilon)$ somewhat like above, but starting with subscript $j = 0$. Thus

- if $i_\alpha = \delta$, then

$$r{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}\underaccent{\sim}{r}'_0(\varepsilon) \text{ is the first coordinate of the value of } \underaccent{\sim}{\rm st}_\varepsilon$$
$$\text{at } \langle r_{j'}(\varepsilon), \bigcap\nolimits_{i<i_\alpha} C_i : j' < i_\alpha\rangle$$
$$\text{and } \underaccent{\sim}{C}^\xi_0(\varepsilon) = \bigcap\nolimits_{i<i_\alpha} C_i \text{ for all } \xi < \lambda\text{”},$$

- if $i_\alpha < \delta$, then

$$r{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“if } (\forall j < i_\alpha)(r_j(\varepsilon) \le r_{i_\alpha}(\varepsilon)) \text{ then } \underaccent{\sim}{r}'_0(\varepsilon) = r_{i_\alpha}(\varepsilon)$$
$$\text{otherwise } \underaccent{\sim}{r}'_0(\varepsilon) \text{ is the first coordinate of the value of } \underaccent{\sim}{\rm st}_\varepsilon$$
$$\text{at } \langle r_j(\varepsilon), \bigcap\nolimits_{i<i_\alpha} C_i : j < i_\alpha\rangle$$
$$\text{and } \underaccent{\sim}{C}^\xi_0(\varepsilon) = \bigcap\nolimits_{j\le i_\alpha} C_j \text{ for all } \xi < \lambda\text{”}.$$

Last, for $0 < j < i_{\alpha+1}$ (and $\varepsilon \in (w_{\alpha+1} \setminus w_\alpha) \cap \zeta$) the names $\underaccent{\sim}{r}'_j(\varepsilon), \underaccent{\sim}{C}^\xi_j(\varepsilon)$ are determined by applying successively $\underaccent{\sim}{\rm st}_\varepsilon$ (like earlier).

This defines the names $\underaccent{\sim}{r}'_j(\varepsilon), \underaccent{\sim}{C}^\xi_j(\varepsilon)$ for $j < i_{\alpha+1}, \xi < \lambda$ and $\varepsilon \in w_{\alpha+1} \cap \zeta$. It is straightforward to check that the requirements $(*)_1$–$(*)_3$ and $(*)_5$ restricted to $\varepsilon \in w_{\alpha+1} \cap \zeta$ (and with "$j < \lambda$" replaced by "$j < i_{\alpha+1}$") are satisfied.

Next, using the fact that $\mathbb{P}_\zeta$ is $({<}\lambda)$-complete and $(*)_3$ of the choice above, Generic picks a condition $r^* \in \mathbb{P}_\zeta$ such that

$(*)_6$ $r^* \ge r_j$ for every $j < \delta$,
$(*)_7$ $r^*{\restriction}\varepsilon \Vdash$ "$\underaccent{\sim}{r}'_{j'}(\varepsilon) \le r^*(\varepsilon)$" for every $j' < i_{\alpha+1}$ and $\varepsilon \in w_{\alpha+1} \cap \zeta$,
$(*)_8$ $r^* \in \bigcap_{\xi<i_{\alpha+1}} \mathcal{I}_\xi$, and
$(*)_9$ for every $j', \xi < i_{\alpha+1}$ and $\varepsilon \in w_{\alpha+1} \cap \zeta$, the condition $r^*{\restriction}\varepsilon$ decides the value of $\underaccent{\sim}{C}^\xi_{j'}(\varepsilon)$, say $r^*{\restriction}\varepsilon \Vdash$ "$\underaccent{\sim}{C}^\xi_{j'}(\varepsilon) = C^\xi_{j'}(\varepsilon)$", where $C^\xi_{j'}(\varepsilon) \in D \cap \mathbf{V}$.

If $i_\alpha \in S$ (so $\delta = i_\alpha$ is a limit ordinal), then Generic picks a condition $r^+ \in \mathbb{P}_\zeta$ stronger than r^* and such that for every $t \in \mathcal{X}^{[\zeta]}_\delta$ and $\varepsilon \in (w_\delta \cap \zeta) \cup \{\zeta\}$ we have:

$(*)_{10}$ either the conditions $r^+{\restriction}\varepsilon$ and $q^{[\zeta]}_{\delta,t}{\restriction}\varepsilon$ are incompatible, or $q^{[\zeta]}_{\delta,t}{\restriction}\varepsilon \leq_{\mathbb{P}_\varepsilon} r^+{\restriction}\varepsilon$,

$(*)_{11}$ if $\varepsilon \in w_\delta \cap \zeta$ and $q^{[\zeta]}_{\delta,t}{\restriction}\varepsilon \leq_{\mathbb{P}_\varepsilon} r^+{\restriction}\varepsilon$, and $s \in \mathcal{X}^{[\zeta]}_\delta$ is such that $t{\restriction}\varepsilon = s{\restriction}\varepsilon$, then either $r^+{\restriction}\varepsilon \Vdash$"$q^{[\zeta]}_{\delta,s}(\varepsilon), r^+(\varepsilon)$ are incompatible" or $r^+{\restriction}\varepsilon \Vdash$ "$q^{[\zeta]}_{\delta,s}(\varepsilon) \leq r^+(\varepsilon)$".

If $\delta > i_\alpha$ (i.e., $i_\alpha \notin S$) then Generic lets $r^+ = r^*$. Finally, for every $j \in [i_\alpha, i_{\alpha+1}) \cap S$ she plays

$$r_j = r^+ \quad \text{and} \quad C_j = E_3 \cap \bigcap\{C^\xi_{j'}(\varepsilon) : j', \xi < i_{\alpha+1}, \varepsilon \in w_{\alpha+1} \cap \zeta\}.$$

Plainly, $r_{i_\alpha+1} = r^+$ satisfies clause $(*)_4$.

Why does the strategy described above work?

Suppose that $\langle r_j, C_j : j < \lambda\rangle$ is a play of the game $\partial(r \restriction \zeta, N, \bar{\mathcal{I}}^{[\zeta]} h^{[\zeta]}, \mathbb{P}_\zeta, \bar{F}, \bar{q}^{[\zeta]})$ in which the Generic player used this strategy and let $\langle \underset{\sim}{r}'_{j'}(\varepsilon) : j' < \lambda, \varepsilon \in \zeta \cap N\rangle$ and $\langle \underset{\sim}{C}^\xi_{j'}(\varepsilon) : j', \xi < \lambda, \varepsilon \in \zeta \cap N\rangle$ be the sequences she constructed aside.

First let us argue that condition A.3.4(5)(β) holds. We will show slightly more than actually needed to help later with clause (α). Remember below that ordinals $\gamma(\delta)$ were defined when we picked our quasi D-diamond $\bar{F}$, and if $\varepsilon < \gamma(\delta)$ then the sequence $\langle h^{[\varepsilon+1]} \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is $\bar{\mathcal{I}}^{[\varepsilon+1]}$-exact. Now, suppose that a limit ordinal $\delta \in S \cap \bigcap_{j<\delta} C_j$ (so in particular $\delta \in E_3$) is such that

$(\boxplus)_\delta$ $w_\delta \cap \zeta \subseteq \gamma(\delta)$ and $(\forall \alpha < \delta)(\exists j < \delta)(h^{[\zeta]} \circ F_\delta(\alpha) \leq r_j)$.

(So then $(\forall \alpha < \delta)(h^{[\zeta]} \circ F_\delta(\alpha) \leq r_\delta)$. Note also that by the choice of E_3, if $\langle h^{[\zeta]} \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is $\bar{\mathcal{I}}^{[\zeta]}$-exact, then $w_\delta \cap \zeta \subseteq \gamma(\delta)$. This is the only place we need E_3; compare the discussion after the definition of $\bar{F}$.)

We are going to choose $t \in \mathcal{X}^{[\zeta]}_\delta$ and show that $q^{[\zeta]}_{\delta,t} \leq r_\delta$. We do this by induction on $\varepsilon \in (\zeta+1) \cap N$, defining $t{\restriction}\varepsilon \in T_\delta$ and showing that $q_{\delta,t}{\restriction}\varepsilon = q_{\delta,t\restriction\varepsilon}{\restriction}\varepsilon \leq r_\delta{\restriction}\varepsilon$ (and for $\varepsilon = \zeta$ we get the desired conclusion). Limit stages and the initial stage $\varepsilon = 0$ are trivial, so assume that we have defined $t{\restriction}\varepsilon$ and have shown $q_{\delta,t\restriction\varepsilon}{\restriction}\varepsilon = q_{\delta,t}{\restriction}\varepsilon \leq r_\delta{\restriction}\varepsilon$ (where $\varepsilon \in \zeta \cap N$), and let us consider the restrictions to $\varepsilon + 1$.

If $\varepsilon \notin w_\delta$ then $t{\restriction}(\varepsilon+1) = t{\restriction}\varepsilon$ (so it has been already defined). Suppose also that $\varepsilon \in \mathrm{Dom}(q_{\delta,t\restriction\varepsilon})$ (otherwise there is nothing to do). Look at the clause (v)(c) of the choice of $q_{\delta,t\restriction\varepsilon}$ at the beginning: $r_\delta \geq r \geq p_\delta$ (and $(\boxplus)_\delta$) implies that

$$r_\delta \restriction \varepsilon \Vdash \text{“}q_{\delta,t\restriction\varepsilon}(\varepsilon) \text{ is an upper bound to } \{p_i(\varepsilon) : i < \delta\}\text{”}.$$

But then also by the clause (vii) there, $r_\delta{\restriction}\varepsilon \Vdash$ "$q_{\delta,t\restriction\varepsilon}(\varepsilon) \leq p_\delta(\varepsilon) \leq r_\delta(\varepsilon)$", so we are done.

Suppose now that $\varepsilon \in w_\delta \cap \zeta$ (and thus $\varepsilon < \gamma(\delta)$). Since $\delta \in E_3 \subseteq E_2$, we know that arriving to stage δ of the game, Generic has already defined $\underset{\sim}{r}'_j(\varepsilon), \underset{\sim}{C}^\xi_j(\varepsilon)$ for $j < \delta$ and $\xi < \lambda$ (remember $(*)_2$). Moreover, the condition $r_\delta{\restriction}\varepsilon$ forces that (remember: $\langle h^{[\varepsilon+1]} \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is $\bar{\mathcal{I}}^{[\varepsilon+1]}$-exact):

- the sequence $\langle h^{\langle\varepsilon\rangle} \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is $\le_{\underset{\sim}{\mathbb{Q}}_\varepsilon}$-increasing $\underset{\sim}{\bar{\mathcal{I}}}^{\langle\varepsilon\rangle}$-exact, and
- $\langle \underset{\sim}{r}'_j(\varepsilon), \triangle_{\xi<\lambda} \underset{\sim}{C}^\xi_j(\varepsilon) : j < \delta\rangle$ is a play according to $\underset{\sim}{\rm st}_\varepsilon$ (by $(*)_5$), and
- $\delta \in \underset{\sim}{S}^{\langle\varepsilon\rangle} \cap \bigcap_{j,\xi<\delta} \underset{\sim}{C}^\xi_j(\varepsilon)$ (remember $(*)_9$ and the choice of $C_{i_\alpha+1}$ for $\alpha < \delta$), and hence also $\delta \in \underset{\sim}{S}^{\langle\varepsilon\rangle} \cap \bigcap_{j<\delta} \triangle_{\xi<\lambda} \underset{\sim}{C}^\xi_j(\varepsilon)$,
- $(\forall j < \delta)(\exists j' < \delta)(r_j(\varepsilon) \le \underset{\sim}{r}'_{j'}(\varepsilon))$ and $(\forall j < \delta)(\exists j' < \delta)(\underset{\sim}{r}'_j(\varepsilon) \le r_{j'}(\varepsilon))$ (by $(*)_3 + (*)_4$), so also

$$(\forall \alpha < \delta)(\exists j < \delta)(h^{\langle\varepsilon\rangle} \circ F_\delta(\alpha) \le_{\underset{\sim}{\mathbb{Q}}_\varepsilon} \underset{\sim}{r}'_j(\varepsilon)).$$

Since $\underset{\sim}{\rm st}_\varepsilon$ is a name for a winning strategy, we may conclude that (by $(*)_7$)

$$r_\delta{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}(\exists x \in \underset{\sim}{\mathcal{X}}^{\langle\varepsilon\rangle}_\delta)(q^{\langle\varepsilon\rangle}_{\delta,x} \le \underset{\sim}{r}'_\delta(\varepsilon) \le r_\delta(\varepsilon))\text{”}.$$

Now look at $(*)_{11}$ remembering clause (vi) of the choice of $\bar{q}$: by them there is a unique $x \in \mathfrak{Y}^\varepsilon_a(\delta)$ such that letting $(t)_\varepsilon = x$ we get $t{\restriction}(\varepsilon+1) \in T_\delta$ satisfying $q_{\delta,t\restriction(\varepsilon+1)}{\restriction}(\varepsilon+1) \le r_\delta{\restriction}(\varepsilon+1)$.

This completes the inductive proof of A.3.4(5)(β).

Why does A.3.4(5)(α) hold? To show this condition, it is enough to prove that $(\boxplus)_\delta$ holds for unboundedly many $\delta \in S \cap \triangle_{\xi<\lambda} C_j$ (remember clause (v)(e) of the choice of $q_{\delta,t}$'s and what we have already shown). We do this considering various characters of ζ.

ζ is a limit ordinal of cofinality $\mathrm{cf}(\zeta) < \lambda$.

Pick a closed set $u \subseteq \zeta$ such that $u \in N$, $0 \in u$, $\mathrm{otp}(u) = \mathrm{cf}(\zeta)$ and $\sup(u) = \zeta$. For $\alpha < \lambda$ let $\varepsilon_\alpha \in u$ be such that

$$\alpha = \mathrm{otp}(u \cap \varepsilon_\alpha) \bmod \mathrm{cf}(\zeta).$$

Now, by induction on $\alpha < \lambda$ we choose conditions $s_\alpha \in N \cap \mathbb{P}_\zeta$ such that

(a)$_\alpha$ $(\exists j < \lambda)(s_\alpha \le r_j)$,

(b)$_\alpha$ $s_\alpha \in \mathbb{P}_{\varepsilon_\alpha} \cap N$,

(c)$_\alpha$ if $\beta < \alpha < \lambda$, then $s_\beta{\restriction}(\varepsilon_\alpha \cap \varepsilon_\beta) \le s_\alpha{\restriction}(\varepsilon_\alpha \cap \varepsilon_\beta)$,

(d)$_\alpha$ $s_\alpha \in \bigcap_{\gamma<\alpha} \underset{\sim}{\mathcal{I}}^{[\varepsilon_\alpha]}_\gamma$.

So suppose that we have defined s_β's for $\beta < \alpha$. For $\beta < \alpha$ let

$$\mathcal{I}_{\alpha,\beta} = \{s \in \mathbb{P}_{\varepsilon_\alpha} : \text{either } s_\beta{\restriction}\varepsilon_\alpha \le s, \text{ or } s_\beta{\restriction}\varepsilon_\alpha, s \text{ are incompatible}\}.$$

Clearly $\mathcal{I}_{\alpha,\beta} \in N$ is an open dense subset of $\mathbb{P}_{\varepsilon_\alpha}$. Since the condition $r{\restriction}\varepsilon_\alpha$ is $(N, \mathbb{P}_{\varepsilon_\alpha})$-generic and the increasing sequence $\langle r_j{\restriction}\varepsilon_\alpha : j < \lambda\rangle$ enters all open dense subsets of $\mathbb{P}_{\varepsilon_\alpha}$ from N (by $(*)_8$), we may find $s_\alpha \in \bigcap_{\beta<\alpha} \mathcal{I}_{\alpha,\beta} \cap \bigcap_{j<\alpha} \mathcal{I}_j^{[\varepsilon_\alpha]}$ such that $s_\alpha \leq r_j{\restriction}\varepsilon_\alpha$ for all $j < \lambda$ large enough. By $(a)_\beta$ (for $\beta < \alpha$) we conclude that s_α and $s_\beta{\restriction}\varepsilon_\alpha$ cannot be incompatible, and hence clauses $(a)_\alpha$–$(d)_\alpha$ are satisfied.

Now, let conditions $s'_\alpha \in \mathbb{P}_\zeta \cap N$ (for $\alpha < \lambda$) be such that $\mathrm{Dom}(s'_\alpha) = \bigcup_{\beta\leq\alpha} \mathrm{Dom}(s_\beta)$ and $s'_\alpha(\varepsilon)$ (for $\varepsilon \in \mathrm{Dom}(s'_\alpha)$) is the $<^*_\chi$-first $\mathbb{P}_\varepsilon$-name for a condition in $\underset{\sim}{\mathbb{Q}}_\varepsilon$ satisfying

$$\begin{aligned} s'_\alpha{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}&(\forall\beta < \alpha)(s'_\beta(\varepsilon) \leq s'_\alpha(\varepsilon)) \text{ and}\\ &\text{if there is a } \gamma \in [\alpha, \lambda) \text{ such that } (\forall\beta < \alpha)(s'_\beta(\varepsilon) \leq s_\gamma(\varepsilon))\\ &\text{then } s'_\alpha(\varepsilon) = s_\gamma(\varepsilon) \text{ for the first such } \gamma\text{”}. \end{aligned}$$

Then the sequence $\langle s'_\alpha : \alpha < \lambda\rangle$ is $\leq_{\mathbb{P}_\zeta}$-increasing and

$$(\forall\alpha < \lambda)(\forall\varepsilon < \varepsilon_\alpha)(s_\alpha{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“}s_\alpha(\varepsilon) = s'_\alpha(\varepsilon)\text{”}).$$

So it follows from $(d)_\alpha$ that for each $\varepsilon \in u$ there is a club $C'_\varepsilon \subseteq \lambda$ such that $\langle s'_\alpha{\restriction}\varepsilon : \alpha < \delta\rangle$ is $\bar{\mathcal{I}}^{[\varepsilon]}$-exact for all $\delta \in C'_\varepsilon$. Also, we may pick a club C^* of λ such that

$$(\forall\delta \in C^*)(\forall\alpha < \delta)(\exists j < \delta)(s'_\alpha \leq r_j),$$

Now, as $\bar{F}'$ is a D-diamond, for unboundedly many $\delta \in S \cap \triangle_{j<\lambda} C_j \cap \bigcap_{\varepsilon\in u} C'_\varepsilon \cap C^*$ we have $\langle s'_\alpha : \alpha < \delta\rangle = \langle h \circ F'_\delta(\alpha) : \alpha < \delta\rangle$. Plainly, defining F_δ for those δ we had clause $(\odot)_1$ with $\gamma(\delta) \geq \zeta$ and hence $\langle s'_\alpha : \alpha < \delta\rangle = \langle h^{[\zeta]} \circ F_\delta(\alpha) : \alpha < \delta\rangle$. Therefore $(\boxplus)_\delta$ holds for those δ (remember the choice of C^*).

ζ is a limit ordinal of cofinality $\geq \lambda$.

Let $\langle \varepsilon_\alpha : \alpha < \lambda\rangle \subseteq \zeta \cap N$ be an increasing continuous sequence cofinal with $\zeta \cap N$, $\varepsilon_0 = 0$. By induction on $\alpha < \lambda$ choose conditions s_α such that

$(a)_\alpha$ $(\exists j < \lambda)(s_\alpha \leq r_j)$,

$(b)_\alpha$ $s_\alpha \in \mathbb{P}_{\varepsilon_\alpha} \cap N$,

$(c)_\alpha$ if $\beta < \alpha < \lambda$, then $s_\beta \leq s_\alpha$,

$(d)_\alpha$ $s_\alpha \in \bigcap_{\gamma<\alpha} \mathcal{I}_\gamma^{[\varepsilon_\alpha]}$.

(Possible as $r{\restriction}\varepsilon_\alpha$ is $(N, \mathbb{P}_{\varepsilon_\alpha})$-generic and by $(*)_8$.) For each $\alpha < \lambda$, for some club C'_α of λ we have

$$(\forall\delta \in C'_\alpha)(\langle s_\gamma{\restriction}\varepsilon_\alpha : \gamma < \delta\rangle \text{ is } \bar{\mathcal{I}}^{[\varepsilon_\alpha]}\text{-exact}).$$

Take a club C^* of λ such that for every $\delta \in C^*$ we have:

- $w_\delta \cap \zeta \subseteq \varepsilon_\delta$, and
- $(\forall \alpha < \delta)(\exists j < \delta)(s_\alpha \leq r_j)$.

Like before, as $\bar{F}'$ is a D-diamond, for unboundedly many $\delta \in S \cap \triangle_{j<\lambda} C_j \cap \triangle_{\alpha<\lambda} C'_\alpha \cap C^*$ we have $\langle s_\alpha : \alpha < \delta \rangle = \langle h \circ F'_\delta(\alpha) : \alpha < \delta \rangle$. Plainly, for those δ we have $\gamma(\delta) \geq \varepsilon_\delta$ and also $\langle s_\alpha : \alpha < \delta \rangle = \langle h^{[\zeta]} \circ F_\delta(\alpha) : \alpha < \delta \rangle$, and thus $(\boxplus)_\delta$ holds (remember the choice of C^*).

ζ **is a successor ordinal**.
Like before (remember that, letting $\zeta = \zeta' + 1$, the condition $r{\restriction}\zeta'$ is $(N, \mathbb{P}_{\zeta'})$-generic and it forces that $r(\zeta')$ is $(N[\underset{\sim}{G}_{\mathbb{P}_{\zeta'}}], \underset{\sim}{\mathbb{Q}}_{\zeta'})$-generic). ∎

This ends the proof of Theorem A.3.10. ∎

Remark A.3.11:

1. In A.3.1 we may have $\bar{S} = \langle S_a : a \in W \rangle$ and $\bar{D} = \langle D_a : a \in W \rangle$ be such that each D_a is a normal filter on λ, $S_a \in D_a^+$ satisfies the relevant demands of A.3.2(1), and require that there is a D_a diamond $\langle F^a_\delta : \delta \in S_a \rangle$. Then in all definitions and results we may replace D, S by D_a, S_a, where $a = N \cap A$. In particular, this way we get the notions of **fuzzy properness over quasi $\bar{D}$-diamonds** which behave nicely in iterations.
2. Everything in this section goes through if we skip "exact" (and deal just with increasing sequences of conditions). There would be almost no changes in the proof of the iteration theorem. The reason why we add "exact" everywhere is in examples we have in mind: we do not know how to show that (some of) the forcings built later are fuzzy-but-without-exact proper. Exactness makes fuzzy properness a weaker condition as $(\exists x \in \mathcal{X}_\delta)(q_{\delta,x} \leq r_\delta)$ of A.3.4(5)(β) has to be fulfilled for somewhat more special δ only. And with that our forcings are fuzzy proper, see §B.7.

B. Building suitably proper forcing notions

B.4. A creature–free example. In this section we show that a natural forcing notion uniformizing colourings on ladder systems is fuzzy proper. (This forcing is a relative of $\mathbb{Q}^*$ from [16, 4.6–4.8].)

Here we assume that:

Context B.4.1:

1. $\lambda^* > \lambda$ is a regular cardinal, $A = \mathcal{H}_{<\lambda}(\lambda^*)$ and $W \subseteq [A]^\lambda$ is as in A.3.1, and $\lambda \subseteq a$ for each $a \in W$,

(2) $\xi^* < \lambda$, S^* is a stationary subset of $S^{\lambda^+}_\lambda \stackrel{\rm def}{=} \{\delta < \lambda^+ : {\rm cf}(\delta) = \lambda\}$ and for $\beta \in S^*$:

(α) $B_\beta \subseteq \beta$ is a club of β of order type ${\rm otp}(B_\beta) = \lambda$,

(β) $h_\beta \colon B_\beta \longrightarrow \xi^*$.

Let $\bar{B} = \langle B_\beta : \beta \in S^* \rangle$, $\bar{h} = \langle h_\beta : \beta \in S^* \rangle$.

The forcing notion $\mathbb{Q}^* = \mathbb{Q}^*(S^*, \bar{B}, \bar{h})$ is defined as follows:

a condition in $\mathbb{Q}^*$ is a tuple $p = (u^p, v^p, \bar{e}^p, h^p)$ such that

(a) $u^p \in [\lambda^+]^{<\lambda}$, $v^p \in [S^*]^{<\lambda}$,

(b) $\bar{e}^p = \langle e^p_\beta : \beta \in v^p \rangle$, where each e^p_β is a closed bounded subset of B_β, and $e^p_\beta \subseteq u^p$, and

(c) $\sup(e^p_\beta) = \sup(u^p \cap \beta)$ (for $\beta \in v^p$), and if $\beta_1 < \beta_2$ are from v^p, then

$$\sup(e^p_{\beta_2}) > \beta_1 \quad \text{and} \quad \sup(e^p_{\beta_1}) > \sup(B_{\beta_2} \cap \beta_1),$$

(d) $h^p \colon u^p \longrightarrow \xi^*$ is such that

$$(\forall \beta \in v^p)(\forall \alpha \in e^p_\beta)(h^p(\alpha) = h_\beta(\alpha));$$

the order $\leq$ of $\mathbb{Q}^*$ is such that $p \leq q$ if and only if $u^p \subseteq u^q$, $h^p \subseteq h^q$, $v^p \subseteq v^q$, and for each $\beta \in v^p$ the set e^q_β is an end-extension of e^p_β.

A tuple $p = (u^p, v^p, \bar{e}^p, h^p)$ satisfying clauses (a), (b) and (d) above will be called **a pre-condition**. Note that every pre-condition can be extended to a condition in $\mathbb{Q}^*$.

PROPOSITION B.4.2:

(1) *The forcing notion $\mathbb{Q}^*$ is $(<\lambda)$-complete, it satisfies the λ^+-chain condition and $|\mathbb{Q}^*| = \lambda^+$.*

(2) *If $p \in \mathbb{Q}^*$, $\alpha < \lambda^+$, $\beta \in S^*$ and $\delta < \lambda$, then there is a condition $q \geq p$ such that*

$$\alpha \in u^q, \quad \beta \in v^q \quad \text{and} \quad (\forall \beta' \in v^q)({\rm otp}(e^q_{\beta'}) > \delta).$$

Proof:

(1) Verification of the chain condition is a straightforward application of the Δ-lemma. To check that $\mathbb{Q}^*$ is $(<\lambda)$-complete suppose that $\langle p_i : i < j \rangle$ is a $\leq_{\mathbb{Q}^*}$-increasing sequence of conditions from $\mathbb{Q}^*$, $j < \lambda$. Let $r = (u^r, v^r, \bar{e}^r, h^r)$ be such that

$$v^r = \bigcup_{i<j} v^{p_i}, \text{ and for } \beta \in v^r$$

$$e^r_\beta = \bigcup\{e^{p_i}_\beta : \beta \in v^{p_i} \& i < j\} \cup \{\sup(\bigcup\{e^{p_i}_\beta : \beta \in v^{p_i} \& i < j\})\}$$

$$u^r = \bigcup_{i<j} u^{p_i} \cup \bigcup\{e^r_\beta : \beta \in v^r\}$$

$$h^r \supseteq \bigcup_{i<j} h^{p_i},$$

and if $\alpha \in e^r_\beta \setminus \bigcup\{e^{p_i}_\beta : \beta \in v^{p_i}, i < j\}$, then $h^r(\alpha) = h_\beta(\alpha)$. Using clause (c) for p_i's one easily sees that r is a pre-condition. Extend it to a condition $q \in \mathbb{Q}^*$.

(2) Should be clear. ∎

Proposition B.4.3: *$\mathbb{Q}^*$ is fuzzy proper for W.*

Proof: Suppose that D is a normal filter on λ such that there is a D-diamond. We will show that $\mathbb{Q}^*$ is fuzzy proper over quasi D-diamonds. First we define a λ-base $(\mathfrak{R}^*, \bar{\mathfrak{Y}}^*)$ for $\mathbb{Q}^*$ over W. We let $\mathfrak{R}^*$ be the set of all triples (p, δ, x) such that $p \in \mathbb{Q}^*$, $\delta \in \lambda$ and x is a function with $\mathrm{Dom}(x) \subseteq u^p$ and $(\forall \alpha \in \mathrm{Dom}(x))(h^p(\alpha) = x(\alpha))$.

Now suppose that $a \in W$ and let π_a be the $<^*_\chi$-first one-to-one mapping from $a \cap \lambda^+$ to λ. For a limit ordinal $\delta < \lambda$ we put

$$x^\delta_0 = (\pi_a)^{-1}[\delta] \cup \{\alpha < \lambda^+ : \alpha = \sup(\alpha \cap (\pi_a)^{-1}[\delta])\},$$

and then

$$\mathfrak{Y}^*_a(\delta) = \{x : x \text{ is a function from } x^\delta_0 \cap a \text{ to } \xi^*\}.$$

For non-limit $\alpha < \lambda$ we put $\mathfrak{Y}^*_a(\alpha) = \{0\}$. This defines $\mathfrak{Y}^*_a$ and $\bar{\mathfrak{Y}}^* = \langle \mathfrak{Y}^*_a : a \in W\rangle$. It is easy to check that $(\mathfrak{R}^*, \bar{\mathfrak{Y}}^*)$ is a λ-base for $\mathbb{Q}^*$ (for A.3.3(c) use repeatedly B.4.2). Assume now that

- $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$, $\lambda, \mathbb{Q}^*, \bar{B}, \bar{h}, S^*, \mathfrak{R}^* \in N$, and $a \stackrel{\text{def}}{=} N \cap A \in W$, $p \in \mathbb{Q}^* \cap N$,
- $\bar{\mathcal{I}} = \langle \mathcal{I}_\xi : \xi < \lambda\rangle$ lists all open dense subsets of $\mathbb{Q}^*$ from N,
- $h\colon \lambda \longrightarrow N$ satisfies $\mathbb{Q}^* \cap N \subseteq \mathrm{Rng}(h)$, and
- $\bar{F} = \langle F_\delta : \delta \in S\rangle$ is a quasi D-diamond for $(N, h, \mathbb{Q}^*)$ and $\bar{q}$ is a fuzzy candidate over $\bar{F}$.

For limit $\delta \in S$ let $\mathcal{Y}(\delta) = \mathcal{Y}(N, \mathbb{Q}^*, h, \bar{F}, \mathfrak{R}^*, \bar{\mathfrak{Y}}^*, \delta)$ be as defined in A.3.4(3) (and thus $\bar{q} = \langle q_{\delta,x} : \delta \in S \text{ is limit } \& x \in \mathcal{Y}(\delta)\rangle$). Also let E_0 be the set of all $\delta < \lambda$ which are limits of members of $\lambda \setminus S$ (so it is a club of λ).

We are going to show that the condition p is $(\mathfrak{R}^*, \bar{\mathfrak{Y}}^*)$-fuzzy generic for $\bar{q}$. Note that, as $\mathbb{Q}^*$ satisfies the λ^+-cc, the condition p is $(N, \mathbb{Q}^*)$-generic (in the standard sense). So, by A.3.8(3), it is enough to give a strategy of the Generic player in the game $\eth^{\rm fuzzy}_\lambda(p, N, \bar{\mathcal{I}}, h, \mathbb{Q}^*, \bar{F}, \bar{q})$ which guarantees that the result $\langle r_i, C_i : i < \lambda\rangle$ of the play satisfies A.3.4(5)(β).

Suppose that we arrive to a stage $\delta \in S$ and $\langle r_i, C_i : i < \delta\rangle$ is the sequence played so far. First, Generic picks the $<^*_\chi$-first condition r'_δ stronger than all r_i's played so far and such that

if δ is limit and $(\exists x \in \mathcal{Y}(\delta))(\exists r \in \mathbb{Q}^*)(q_{\delta,x} \le r \& (\forall i < \delta)(r_i \le r))$,
then $q_{\delta,x} \le r'_\delta$ for some $x \in \mathcal{Y}(\delta)$.

Then she plays the $<^*_\chi$-first condition r_δ above r'_δ such that

$(*)_1$ if $\beta \in v^{r_\delta}$, then $\mathrm{otp}(e^{r_\delta}_\beta) > \delta$, and
$(*)_2$ $(\pi_a)^{-1}[\delta] \subseteq u^{r_\delta}$ and $(\pi_a)^{-1}[\delta] \cap S^* \subseteq v^{r_\delta}$.

The set C_δ played at this stage is $(\alpha, \lambda) \cap E_0$, where α is the first ordinal such that

$(*)_3$ $\pi_a[u^{r_\delta} \cap N] \subseteq \alpha$, and the set

$$\{q \in \mathbb{Q}^* : (\pi_a)^{-1}[\delta] \subseteq u^q \& (\pi_a)^{-1}[\delta] \cap S^* \subseteq v^q \& (\forall \beta \in v^q)(\mathrm{otp}(e^q_\beta) > \delta)\}$$

(which is an open dense subset of $\mathbb{Q}^*$ from N; remember B.4.2) is in $\{\mathcal{I}_\xi : \xi < \alpha\}$,

$(*)_4$ $\mathrm{otp}(B_\beta \cap (\sup(e^{r_\delta}_\beta) + 1)) < \alpha$ for all $\beta \in v^{r_\delta}$,
$(*)_5$ if $\beta \in v^{r_\delta}$ and $a \cap \beta \setminus (\sup(e^{r_\delta}_\beta)+1) \neq \emptyset$, then there is $\gamma \in a \cap \beta \setminus (\sup(e^{r_\delta}_\beta)+1)$ with $\pi_a(\gamma) < \alpha$.

Why does this strategy work (i.e., why does it ensure A.3.4(5)(β))?
Let $\langle r_i, C_i : i < \lambda\rangle$ be a play according to this strategy, and suppose that $\delta \in S \cap \triangle_{i<\lambda} C_i$ is a limit ordinal such that $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is a $\le_{\mathbb{Q}^*}$-increasing $\bar{\mathcal{I}}$-exact sequence of conditions from $\mathbb{Q}^* \cap N$ such that $(\forall \alpha < \delta)(\exists i < \delta)(h \circ F_\delta(\alpha) \le r_i)$. Note that then

$(*)_6$ if $\beta \in \bigcup_{i<\delta} v^{r_i}$, then $\mathrm{otp}(\bigcup_{i<\delta} e^{r_i}_\beta) = \delta$ and $\bigcup_{i<\delta} e^{r_i}_\beta$ is an unbounded subset of $\{\varepsilon \in B_\beta : \mathrm{otp}(\varepsilon \cap B_\beta) < \delta\}$, and
$(*)_7$ $\bigcup_{i<\delta} u^{r_i} \cap N = (\pi_a)^{-1}[\delta] = \bigcup_{\alpha<\delta} u^{h\circ F_\delta(\alpha)}$ and

$$\bigcup_{i<\delta} v^{r_i} \cap N = (\pi_a)^{-1}[\delta] \cap S^* = \bigcup_{\alpha<\delta} v^{h\circ F_\delta(\alpha)},$$

$(*)_8$ if $\beta \in (\pi_a)^{-1}[\delta] \cap S^*$, then

$$\bigcup\{e^{h\circ F_\delta(\alpha)}_\beta : \alpha < \delta \& \beta \in v^{h\circ F_\delta(\alpha)}\} = \bigcup\{e^{r_i}_\beta : i < \delta \& \beta \in v^{r_i}\}.$$

We want to show that

($\boxdot$) for some $x \in \mathcal{Y}(\delta)$, there is a common upper bound to $\{r_i : i < \delta\} \cup \{q_{\delta,x}\}$

(which, by the definition of our strategy, will finish the proof). For $\beta \in S^*$ let $\gamma_\beta \in B_\beta$ be such that $\mathrm{otp}(B_\beta \cap \gamma_\beta) = \delta$. Now, let a pre-condition $r' = (u^{r'}, v^{r'}, \bar{e}^{r'}, h^{r'})$ be such that

- $v^{r'} = \bigcup_{i<\delta} v^{r_i}$, $u^{r'} = \bigcup_{i<\delta} u^{r_i} \cup \{\gamma_\beta : \beta \in v^{r'}\}$,
- $e^{r'}_\beta = \bigcup\{e^{r_i}_\beta : i < \delta \& \beta \in v^{r_i}\} \cup \{\gamma_\beta\}$ (for $\beta \in v^{r'}$), and
- $h^{r'} : u^{r'} \longrightarrow \xi^*$ is such that $\bigcup_{i<\delta} h^{r_i} \subseteq h^{r'}$ and $h^{r'}(\gamma_\beta) = h_\beta(\gamma_\beta)$.

One easily verifies that the above conditions indeed define a pre-condition (remember $(*)_6$). Also, note that if $\beta \in v^{r'}$, then $\gamma_\beta \in e^{r'}_\beta \setminus \bigcup\{e^{r_i}_\beta : i < \delta \& \beta \in v^{r_i}\}$ and each $e^{r_i}_\beta$ is a proper subset of $\bigcup\{e^{r_i}_\beta : i < \delta \& \beta \in v^{r_i}\}$. Moreover, if $\beta \in v^{r'}$ and $\gamma_\beta \in N$, then $\gamma_\beta = \sup(u^{r'} \cap N \cap \gamma_\beta) = \sup((\pi_a)^{-1}[\delta] \cap \gamma_\beta)$ (by $(*)_5 + (*)_2$; remember also $(*)_7$). Now, extend r' to a pre-condition r'' such that $u^{r''} = u^{r'} \cup x^\delta_0$, $v^{r''} = v^{r'}$ and $e^{r''}_\beta = e^{r'}_\beta$ for $\beta \in v^{r''}$ (clearly possible). Let $x = h^{r''} \restriction x^\delta_0$ (note that $x^\delta_0 \subseteq a$). Since r'' is stronger than all $h \circ F_\delta(\alpha)$ (for $\alpha < \delta$), any condition stronger than r'' witnesses that $x \in \mathcal{Y}(\delta)$. Now we put

- $u^* = u^{q_{\delta,x}} \cup u^{r''}$, $v^* = v^{q_{\delta,x}} \cup v^{r''}$, $h^* = h^{q_{\delta,x}} \cup h^{r''}$,
- if $\beta \in v^{q_{\delta,x}}$, then $e^*_\beta = e^{q_{\delta,x}}_\beta$, and if $\beta \in v^{r''} \setminus N$, then $e^*_\beta = e^{r''}_\beta$.

Note that h^* is a function from u^* to ξ^* by $(*)_7$ (remember the choice of x and that $q_{\delta,x} \in N$ is stronger than all $h \circ F_\delta(\alpha)$'s). Also, if $\beta \in v^{r''} \cap N$ then ($\beta \in v^{q_{\delta,x}}$ and) $e^{q_{\delta,x}}_\beta$ is an end-extension of $e^{r''}_\beta$ (remember $(*)_7 + (*)_8$). Hence $(u^*, v^*, \bar{e}^*, h^*)$ is a pre-condition stronger than both $q_{\delta,x}$ and r''. Extending it to a condition in $\mathbb{Q}^*$ we conclude ($\boxdot$), thus completing the proof of B.4.3. ∎

Corollary B.4.4: *Assume that λ is a strongly inaccessible cardinal, $2^\lambda = \lambda^+$, $2^{\lambda^+} = \lambda^{++}$ and D is a normal filter on λ such that there is a D-diamond. Then there is a forcing notion $\mathbb{P}$ such that:*

- *$\mathbb{P}$ is $({<}\lambda)$-complete weakly fuzzy proper over quasi D-diamonds for W and it satisfies the λ^{++}-cc,*
- *in $\mathbf{V}^{\mathbb{P}}$, $2^\lambda = 2^{\lambda^+} = \lambda^{++}$ and for every $\xi^*, S^*, \bar{B}, \bar{h}$ as in B.4.1(2) there is $h \colon \lambda^{++} \longrightarrow \xi^*$ such that for every $\beta \in S^*$ the set*

$$\{\alpha \in B_\beta : h_\beta(\alpha) = h(\alpha)\}$$

contains a club.

Proof: The forcing notion $\mathbb{P}$ is the limit $\mathbb{P}_{\lambda^{++}}$ of a λ-support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda^{++}\rangle$, where each $\underset{\sim}{\mathbb{Q}}_\alpha$ is forced to be $\mathbb{Q}^*(S^{\lambda^+}_\lambda, \underset{\sim}{\bar{B}}_\alpha, \underset{\sim}{\bar{h}}_\alpha)$ for some

$\underset{\sim}{\bar{B}}_\alpha, \underset{\sim}{\bar{h}}_\alpha$. Then, by A.1.10, A.3.10 and B.4.3 we are sure that $\mathbb{P}$ satisfies the λ^{++}-cc, it is weakly fuzzy proper over quasi D-diamonds for W and it has a dense subset of size λ^{++}. Consequently we may arrange suitable bookkeeping to take care of all $\mathbb{P}_{\lambda^{++}}$-names $\underset{\sim}{\bar{B}}, \underset{\sim}{\bar{h}}$ for objects as in B.4.1(2) — the details and the rest should be clear. ∎

B.5. Trees and creatures. Let us introduce the notation used in the forcing notions we want to build. The terminology here is somewhat parallel to that of [15, §1.2, §1.3], but there are some differences as the context is different. We start with the tree case.

Definition B.5.1: Let $\mathbf{H}\colon \lambda \longrightarrow \mathcal{H}(\lambda^+)$.

(1) A **λ-tree creature for H** is a tuple

$$t = (\eta, \mathbf{dis}, \mathbf{pos}, \mathbf{nor}) = (\eta[t], \mathbf{dis}[t], \mathbf{pos}[t], \mathbf{nor}[t])$$

such that $\mathbf{dis} \in \mathcal{H}(\lambda^+)$, $\mathbf{nor} \in \lambda + 1$,

$$\eta \in \bigcup_{\alpha<\lambda} \prod_{\beta<\alpha} \mathbf{H}(\beta), \quad \text{and} \quad \emptyset \neq \mathbf{pos} \subseteq \Big\{\nu \in \bigcup_{\alpha<\lambda} \prod_{\beta<\alpha} \mathbf{H}(\beta) : \eta \lhd \nu\Big\}.$$

$\mathrm{TCR}^\lambda[\mathbf{H}]$ is the family of all λ-tree creatures for $\mathbf{H}$.

For $\eta \in \bigcup_{\alpha<\lambda} \prod_{\beta<\alpha} \mathbf{H}(\beta)$ we let $\mathrm{TCR}^\lambda_\eta[\mathbf{H}] = \{t \in \mathrm{TCR}^\lambda[\mathbf{H}] : \eta[t] = \eta\}$.

(2) Let $K \subseteq \mathrm{TCR}^\lambda[\mathbf{H}]$. A **tree-composition operation on** K is a mapping Σ with values in $\mathcal{P}(K)$ and the domain consisting of systems $\langle t_\nu : \nu \in \hat{T}\rangle$ such that

- T is a complete λ-quasi tree of height $\mathrm{ht}(T) < \lambda$, $\hat{T} = T \setminus \max(T)$,
- for each $\nu \in \hat{T}$, $t_\nu \in K$ satisfies $\nu = \eta[t_\nu]$ and $\mathrm{succ}_T(\nu) = \mathbf{pos}[t_\nu]$,

and

- if $t \in \Sigma(t_\nu : \nu \in \hat{T})$, then $\eta[t] = \mathrm{root}(T)$ and $\mathbf{pos}[t] \subseteq \max(T)$,
- if $t \in \Sigma(t_\nu : \nu \in \hat{T})$ and $t_\nu \in \Sigma(s^\nu_\rho : \rho \in \hat{T}_\nu)$ (for $\nu \in \hat{T}$), then $t \in \Sigma(s^\nu_\rho : \rho \in \bigcup_{\nu\in\hat{T}} \hat{T}_\nu)$, and
- for each $t \in K$ we have $\langle t\rangle \in \mathrm{Dom}(\Sigma)$ and $t \in \Sigma(t)$.

Then (K, Σ) is called **a λ-tree creating pair** (for $\mathbf{H}$).

(3) A λ-tree creating pair (K, Σ) is **local** if

- $(t_\nu : \nu \in T) \in \mathrm{Dom}(\Sigma)$ implies $\mathrm{ht}(T) = \mathrm{lh}(\mathrm{root}(T)) + 1$ (and so $T = \{\mathrm{root}(T)\} \cup \mathbf{pos}[t_{\mathrm{root}(T)}]$), and
- $t' \in \Sigma(t)$ implies $\mathbf{nor}[t'] \leq \mathbf{nor}[t]$.

We say that (K, Σ) is **very local** if, additionally, for every

$$\nu \in \bigcup_{\alpha<\lambda} \prod_{\beta<\alpha} \mathbf{H}(\beta)$$

such that $K \cap \mathrm{TCR}^{\lambda}_{\nu}[\mathbf{H}] \neq \emptyset$ there is $t^*_\nu \in K \cap \mathrm{TCR}^{\lambda}_{\nu}[\mathbf{H}]$ satisfying $(\forall t \in K \cap \mathrm{TCR}^{\lambda}_{\nu}[\mathbf{H}])(t \in \Sigma(t^*_\nu))$. The tree creature t^*_ν may be called **the minimal creature at** ν.

(4) If (K,Σ) is a very local λ-tree creating pair, then **the minimal tree** T^* **for** (K,Σ) and **the minimal condition** p^* **for** (K,Σ) are defined by

$$T^* = T^*(K,\Sigma) = \{\eta \in \bigcup_{\alpha<\lambda} \prod_{\beta<\lambda} \mathbf{H}(\beta) : (\forall \alpha < \mathrm{lh}(\eta))(\eta{\restriction}(\alpha+1) \in \mathbf{pos}[t^*_{\eta\restriction\alpha}])\}$$

$$p^* = p^*(K,\Sigma) = \langle t^*_\nu : \nu \in T^*\rangle.$$

(Note that, in the general case, T^* could be of small height, but in real applications this does not happen.)

Definition B.5.2: Let (K,Σ) be a λ-tree creating pair for $\mathbf{H}$.

(1) We define the forcing notion $\mathbb{Q}^{\mathrm{tree}}_1(K,\Sigma)$ by:

conditions are systems $p = \langle t_\eta : \eta \in T\rangle$ such that

(a) $\emptyset \neq T \subseteq \bigcup_{\alpha<\lambda} \prod_{\beta<\alpha} \mathbf{H}(\beta)$ is a complete λ-quasi tree with $\max(T) = \emptyset$,

(b) $t_\eta \in \mathrm{TCR}^{\lambda}_{\eta}[\mathbf{H}] \cap K$ and $\mathbf{pos}[t_\eta] = \mathrm{succ}_T(\eta)$,

(c)$_1$ for every $\eta \in \lim_\lambda(T)$, $\lim(\mathbf{nor}[t_{\eta\restriction\alpha}] : \alpha < \lambda,\ \eta{\restriction}\alpha \in T) = \lambda$;

the order is given by:

$\langle t^1_\eta : \eta \in T^1\rangle \leq \langle t^2_\eta : \eta \in T^2\rangle$ if and only if $T^2 \subseteq T^1$ and for each $\eta \in T^2$ there is a complete λ-quasi tree $T_{0,\eta} \subseteq (T^1)^{[\eta]}$ of height $\mathrm{ht}(T_{0,\eta}) < \lambda$ such that $t^2_\eta \in \Sigma(t^1_\nu : \nu \in \hat{T}_{0,\eta})$.

If $p = \langle t_\eta : \eta \in T\rangle$ then we write $\mathrm{root}(p) = \mathrm{root}(T)$, $T^p = T$, $t^p_\eta = t_\eta$ etc.

(2) Let D^* be a filter on λ. The forcing notion $\mathbb{Q}^{\mathrm{tree}}_{D^*}(K,\Sigma)$ is defined similarly, replacing the condition (c)$_1$ by

(c)$_{D^*}$ for some set $Y = Y^p \in D^*$ we have

$$(\forall \delta \in Y)(\forall \eta \in (T)_\delta)(\mathbf{nor}[t_\eta] \geq |\delta|).$$

(The set Y^p above may be called **a witness for** $p \in \mathbb{Q}^{\mathrm{tree}}_{D^*}(K,\Sigma)$.)

(3) The forcing notion $\mathbb{Q}^{\mathrm{tree}}_{\mathrm{cl}}(K,\Sigma)$ is defined by replacing the condition (c)$_1$ by

(c)$_{\mathrm{cl}}$ (α) $(\forall \eta \in T)(\exists \nu \in T)(\eta \vartriangleleft \nu \,\&\, \mathbf{nor}[t_\nu] \geq |\,\mathrm{lh}(\nu)|)$, and

(β) $(\forall \eta \in T)(\mathbf{nor}[t_\eta] = 0$ or $\mathbf{nor}[t_\eta] \geq |\,\mathrm{lh}(\eta)|)$, and

(γ) $\mathbf{nor}[t_{\mathrm{root}(p)}] \geq |\,\mathrm{lh}(\mathrm{root}(p))|$, and

(δ) if $\delta < \lambda$ is a limit ordinal and $\langle \eta_i : i < \delta \rangle \subseteq T$ is a $\vartriangleleft$-increasing sequence such that $\mathbf{nor}[t_{\eta_i}] \geq |\operatorname{lh}(\eta_i)|$ for each $i < \delta$ and $\eta = \bigcup_{i<\delta} \eta_i$, then ($\eta \in T$ and) $\mathbf{nor}[t_\eta] \geq |\operatorname{lh}(\eta)|$.

(4) If $e \in \{1, D^*, \mathrm{cl}\}$, $p \in \mathbb{Q}^{\mathrm{tree}}_e(K, \Sigma)$ and $\eta \in T^p$, then we let

$$p^{[\eta]} = \langle t^p_\nu : \nu \in (T^p)^{[\eta]} \rangle.$$

(5) For the sake of notational convenience we define partial order $\mathbb{Q}^{\mathrm{tree}}_\emptyset(K, \Sigma)$ in the same manner as $\mathbb{Q}^{\mathrm{tree}}_e(K, \Sigma)$ above but we omit the requirement $(\mathrm{c})_e$.

Definition B.5.3: Let (K, Σ) be a λ-tree creating pair for $\mathbf{H}$, $t \in K$. We define a relation $\preceq^t_\Sigma$ on $\Sigma(t)$ by

$t' \preceq^t_\Sigma t''$ if and only if ($t', t'' \in \Sigma(t)$ and) $t'' \in \Sigma(t')$.

If (K, Σ) is very local, t^*_ν is the minimal creature at ν, then $\preceq^{t^*_\nu}_\Sigma$ is also denoted by $\preceq^\nu_\Sigma$.

Remark B.5.4:

(1) Note that the relation $\preceq^t_\Sigma$ is transitive and reflexive.
(2) If (K, Σ) is local and $p \in \mathbb{Q}^{\mathrm{tree}}_\emptyset(K, \Sigma)$, then T^p is a complete λ-tree.

Now we are going to describe the non-tree case of forcing with creatures. For sake of simplicity we restrict ourselves to what corresponds to forgetful creatures of [15, 1.2.5].

Definition B.5.5: Let $\mathbf{H}$: $\lambda \longrightarrow \mathcal{H}(\lambda^+)$.

(1) A **forgetful λ-creature for H** is a tuple

$$t = (\alpha_{\mathrm{dn}}, \alpha_{\mathrm{up}}, \mathbf{dis}, \mathbf{val}, \mathbf{nor}) = (\alpha_{\mathrm{dn}}[t], \alpha_{\mathrm{up}}[t], \mathbf{dis}[t], \mathbf{val}[t], \mathbf{nor}[t])$$

such that $\mathbf{dis} \in \mathcal{H}(\lambda^+)$, $\mathbf{nor} \in \lambda + 1$, $\alpha_{\mathrm{dn}} < \alpha_{\mathrm{up}} < \lambda$ and $\emptyset \neq \mathbf{val} \subseteq \prod_{\alpha_{\mathrm{dn}} \leq \beta < \alpha_{\mathrm{up}}} \mathbf{H}(\beta)$.

$\mathrm{CR}^\lambda[\mathbf{H}]$ is the family of all forgetful λ-creatures for $\mathbf{H}$.

Since we will consider only forgetful λ-creatures, from now on we will omit the adjective “forgetful”.

(2) Let $K \subseteq \mathrm{CR}^\lambda[\mathbf{H}]$. A **composition operation on** K is a mapping Σ with values in $\mathcal{P}(K)$ and the domain consisting of systems $\langle t_i : i < j \rangle \subseteq K$ such that $j < \lambda$ and

$$\alpha_{\mathrm{up}}[t_i] = \alpha_{\mathrm{dn}}[t_{i+1}] \quad \text{for } i < i+1 < j, \text{ and}$$
$$\sup\{\alpha_{\mathrm{up}}[t_{i'}] : i' < i\} = \alpha_{\mathrm{dn}}[t_i] \quad \text{for limit } i < j,$$

and if $t \in \Sigma(t_i : i < j)$, then

- $\alpha^- = \alpha_{\rm dn}[t] = \alpha_{\rm dn}[t_0]$, $\alpha^+ = \alpha_{\rm up}[t] = \sup\{\alpha_{\rm up}[t_i] : i < j\}$, and
- $\mathbf{val}[t] \subseteq \{\nu \in \prod\limits_{\alpha^- \leq \beta < \alpha^+} \mathbf{H}(\beta) : (\forall i < j)(\nu{\restriction}[\alpha_{\rm dn}[t_i], \alpha_{\rm up}[t_i]) \in \mathbf{val}[t_i])\}$,

and

- if $t_i \in \Sigma(s^i_\zeta : \zeta < \zeta_i)$ (for $i < j$) and $t \in \Sigma(t_i : i < j)$, then $t \in \Sigma(s^i_\zeta : \zeta < \zeta_i,\ i < j)$, and
- for each $t \in K$ we have $\langle t \rangle \in \mathrm{Dom}(\Sigma)$ and $t \in \Sigma(t)$.

Then (K, Σ) is called **a λ-creating pair** (for $\mathbf{H}$).

(3) We say that (K, Σ) is **local** if for each $t \in K$

- $\alpha_{\rm up}[t] = \alpha_{\rm dn}[t] + 1$, and
- $t' \in \Sigma(t)$ implies $\mathbf{nor}[t'] \leq \mathbf{nor}[t]$.

It is **very local** if, additionally, for each $\alpha < \lambda$ there is $t^*_\alpha \in K$ such that $\alpha_{\rm dn}[t^*_\alpha] = \alpha$ and for every $t \in K$ with $\alpha_{\rm dn}[t] = \alpha$ we have $t \in \Sigma(t^*_\alpha)$. The creature t^*_α will be called **the minimal creature t^*_α at α**.

(4) For $j < \lambda$, **a j-approximation for** (K, Σ) is a pair $(w, \langle t_i : i < j \rangle)$ such that $t_i \in K$,

$$\begin{aligned} &\alpha_{\rm up}[t_i] = \alpha_{\rm dn}[t_{i+1}] \quad \text{for } i < i+1 < j, \text{ and} \\ &\sup\{\alpha_{\rm up}[t_{i'}] : i' < i\} = \alpha_{\rm dn}[t_i] \quad \text{for limit } i < j, \end{aligned}$$

and $w \in \prod_{\alpha < \alpha_{\rm dn}[t_0]} \mathbf{H}(\alpha)$.

(5) For a j-approximation $(w, \langle t_i : i < j \rangle)$ for (K, Σ) we let

$$\mathbf{pos}(w, \langle t_i : i < j \rangle) = \Big\{ v \in \prod_{\alpha < \alpha^*} \mathbf{H}(\alpha) : w \vartriangleleft v \text{ and for all } i < j \\ v{\restriction}[\alpha_{\rm dn}[t_i], \alpha_{\rm up}[t_i]) \in \mathbf{val}[t_i] \Big\},$$

where $\alpha^* = \sup\{\alpha_{\rm up}[t_i] : i < j\}$.

Definition B.5.6: Let (K, Σ) be a λ-creating pair for $\mathbf{H}$.

(1) We define the forcing notion $\mathbb{Q}^*_1(K, \Sigma)$:
conditions are pairs $p = (w, \bar{t})$ such that
(a) $\bar{t} = \langle t_i : i < \lambda \rangle$ is a sequence of λ-creatures from K satisfying

$$\begin{aligned} &\alpha_{\rm up}[t_i] = \alpha_{\rm dn}[t_{i+1}] && \text{for } i < i+1 < \lambda, \text{ and} \\ &\sup\{\alpha_{\rm up}[t_{i'}] : i' < i\} = \alpha_{\rm dn}[t_i] && \text{for limit } i < \lambda, \end{aligned}$$

(b) $w \in \prod_{\alpha < \alpha_{\rm dn}[t_0]} \mathbf{H}(\alpha)$
(c)$_1$ $\lim(\mathbf{nor}[t_i] : i < \lambda) = \lambda$

the order is given by:
$(w^1, \langle t^1_i : i < \lambda\rangle) \leq (w^2, \langle t^2_i : i < \lambda\rangle$ if and only if
for some continuous strictly increasing sequence $\langle i_\zeta : \zeta < \lambda\rangle$ we have

$$w^2 \in \mathbf{pos}(w^1, \langle t^1_i : i < i_0\rangle) \quad \text{and} \quad (\forall \zeta < \lambda)(t^2_\zeta \in \Sigma(t^1_i : i_\zeta \leq i < i_{\zeta+1})).$$

If $p = (w, \langle t_i : i < \lambda\rangle)$, then we write $w^p = w$, $t^p_i = t_i$ (for $i < \lambda$).

(2) Let D^* be a filter on λ. The forcing notion $\mathbb{Q}^*_{D^*}(K, \Sigma)$ is defined similarly, replacing the condition $(\mathrm{c})_1$ by

$(\mathrm{c})_{D^*}$ for some set $Y = Y^p \in D^*$ we have

$$(\forall i \in Y)(\mathbf{nor}[t_i] \geq |\alpha_{\mathrm{dn}}[t_i]|).$$

(The set Y^p above may be called **a witness for** $p \in \mathbb{Q}^*_{D^*}(K, \Sigma)$.)

(3) For the sake of notational convenience we define partial order $\mathbb{Q}^*_\emptyset(K, \Sigma)$ in the same manner as $\mathbb{Q}^*_e(K, \Sigma)$ above but we omit the requirement $(\mathrm{c})_e$. If (K, Σ) is very local, then **the minimal condition** p^* **for** (K, Σ) is

$$p^* = p^*(K, \Sigma) = (\langle\rangle, \langle t^*_\alpha : \alpha < \lambda\rangle) \in \mathbb{Q}^*_\emptyset(K, \Sigma),$$

where t^*_α is the minimal creature at α.

(4) The relations $\preceq^t_\Sigma$ and $\preceq^{t^*_\alpha}_\Sigma = \preceq^\alpha_\Sigma$ are defined in a way parallel to B.5.3.

B.6. Getting completeness and bounding properties. In this section we introduce properties of λ-tree creating pairs ensuring that the resulting forcing notions are complete or strategically complete. Next we show that adding bounds on the size of $\mathbf{H}(\alpha)$ guarantees strong bounding properties from Section A.2. Finally we will introduce parallel completeness conditions for the case of λ-creating pairs.

Definition B.6.1: Let (K, Σ) be a λ-tree creating pair for $\mathbf{H}$, κ be a cardinal (and $\lambda, \bar{\lambda}$ be as in 0.3).

(1) We say that a λ-tree creature $t \in K$ is $(<\kappa)$**-complete** (for (K, Σ)) if
  - (α) for every $\preceq^t_\Sigma$-increasing chain $\langle t_\alpha : \alpha < \delta\rangle \subseteq \Sigma(t)$ with $\delta < \kappa$ and $\mathbf{nor}[t_\alpha] > 0$, there is $t_\delta \in \Sigma(t)$ such that $(\forall \alpha < \delta)(t_\alpha \preceq^t_\Sigma t_\delta)$ and $\mathbf{nor}[t_\delta] \geq \min\{\mathbf{nor}[t_\alpha] : \alpha < \delta\}$,
  - (β) if $t' \in \Sigma(t)$, $\mathbf{nor}[t'] = 0$, then $|\mathbf{pos}[t']| = 1$ and $\Sigma(t') = \{t'\}$,
  - (γ) if $\nu \in \mathbf{pos}[t]$, then there is $t' \in \Sigma(t)$ such that $\mathbf{pos}[t'] = \{\nu\}$ and $\mathbf{nor}[t'] = 0$.

(2) $t \in K$ is said to be **exactly** $(<\kappa)$**-complete** if it is $(<\kappa)$-complete and

($\otimes$) if $\bar{t} = \langle t_\alpha : \alpha < \kappa \rangle \subseteq \Sigma(t)$ is a strictly $\preceq^t_\Sigma$-increasing chain, then $\bar{t}$ has no $\preceq^t_\Sigma$-upper bound in $\Sigma(t)$, but $\bigcap_{\alpha<\kappa} \mathbf{pos}[t_\alpha] \neq \emptyset$.

(3) We say that (K,Σ) is **$\bar{\lambda}$-complete** (*exactly $\bar{\lambda}$-complete*, respectively) if

(a) (K,Σ) is very local, and

$(\mathrm{b})_{\bar{\lambda}}$ each minimal creature t^*_ν is $(<\lambda^+_{\mathrm{lh}(\nu)})$-complete (exactly $(<\lambda_{\mathrm{lh}(\nu)})$-complete, respectively).

We say that (K,Σ) is **just $(<\lambda)$-complete** if it satisfies (a) above and

$(\mathrm{b})^\lambda$ each minimal creature t^*_ν is $(<\lambda)$-complete.

Proposition B.6.2: *Assume that (K,Σ) is a very local λ-tree creating pair for $\mathbf{H}$, D^* is a $<\lambda$-complete uniform filter on λ. Let $\mathbb{P}$ be one of the forcing notions $\mathbb{Q}^{\rm tree}_1(K,\Sigma)$, $\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$, or $\mathbb{Q}^{\rm tree}_{\rm cl}(K,\Sigma)$.*

1. *If (K,Σ) is $\bar{\lambda}$-complete, then $\mathbb{P}$ is strategically $(<\lambda)$-complete.*
2. *If (K,Σ) is just $(<\lambda)$-complete, then $\mathbb{P}$ is $(<\lambda)$-complete.*
3. *If (K,Σ) is exactly $\bar{\lambda}$-complete, then $\mathbb{P}$ is $(<\lambda)$-complete.*

Proof: (1) Let $\mathbb{P} \in \{\mathbb{Q}^{\rm tree}_1(K,\Sigma), \mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma), \mathbb{Q}^{\rm tree}_{\rm cl}(K,\Sigma)\}$ and let $r \in \mathbb{P}$. Consider the following strategy st of Complete in the game $\Game^\lambda_0(\mathbb{P},\emptyset,r)$. At stage $j<\lambda$ of the game, after a sequence $\langle (p_i,q_i) : i<j\rangle^\frown\langle p_j\rangle \subseteq \mathbb{P}$ has been constructed (so p_j is the jth inning of Incomplete), she plays the $<^*_\chi$-first condition $q_j \in \mathbb{P}$ stronger than p_j and such that $\mathrm{lh}(\mathrm{root}(q_j)) > j+\omega$.

Why is this a winning strategy? Suppose that the players have arrived at a limit stage $\delta<\lambda$ of the game, Complete has used st and $\langle (p_i,q_i) : i<\delta\rangle$ is the result of the game so far. Our aim is to show that the (increasing) sequence $\langle q_i : i<\delta\rangle$ has an upper bound in $\mathbb{P}$. To this end we are going to define a condition $q = \langle t_\eta : \eta \in T\rangle \in \mathbb{P}$ inductively defining $(T)_\alpha$ and t_η for $\alpha<\lambda$, $\eta \in (T)_\alpha$. First we declare $\mathrm{root}(T) = \bigcup_{i<\delta} \mathrm{root}(q_i)$ and we note that

$$\mathrm{root}(T) \in \bigcap_{i<\delta} T^{q_i} \quad \text{and} \quad \delta \le \mathrm{lh}(\mathrm{root}(T)) < \lambda^+_{\mathrm{lh}(\mathrm{root}(T))}.$$

Now we may choose $t_{\mathrm{root}(T)} \in \mathrm{TCR}^\lambda_{\mathrm{root}(T)}[\mathbf{H}]$ so that $t_{\mathrm{root}(T)} \in \Sigma(t^{q_i}_{\mathrm{root}(T)})$ (for all $i<\delta$) and $\mathbf{nor}[t_{\mathrm{root}(T)}] \ge \min\{\mathbf{nor}[t^{q_i}_{\mathrm{root}(T)}] : i<\delta\}$, and we declare $\mathbf{pos}[t_{\mathrm{root}(T)}] \subseteq T$ (thus defining $(T)_{\mathrm{lh}(\mathrm{root}(T))+1}$). Next we proceed inductively in a similar manner: suppose that $(T)_\alpha$ has been already defined and it is included in $\bigcap_{i<\delta} T^{q_i}$. For each $\eta \in (T)_\alpha$ choose t_η such that

$$(\forall i<\delta)(t_\eta \in \Sigma(t^{q_i}_\eta)) \quad \text{and} \quad \mathbf{nor}[t_\eta] \ge \min\{\mathbf{nor}[t^{q_i}_\eta] : i<\delta\},$$

and declare $\mathbf{pos}[t_\eta] \subseteq T$. (So after this step $(T)_{\alpha+1}$ is defined.) If $\alpha<\lambda$ is limit and $(T)_\beta$ has been defined for $\beta<\alpha$, then we let $(T)_\alpha$ consist of all sequences

$\eta \in \prod_{\beta<\alpha} \mathbf{H}(\beta)$ such that $\eta{\restriction}\beta \in (T)_\beta$ whenever $\mathrm{lh}(\mathrm{root}(T)) \le \beta < \alpha$, and then we choose t_η (for $\eta \in (T)_\alpha$) like above.

This way we build a condition $\langle t_\eta : \eta \in T\rangle \in \mathbb{Q}^{\rm tree}_\emptyset(K,\Sigma)$, and it is very straightforward to verify that this condition is actually in $\mathbb{P}$ and is stronger than all q_i (for $i < \delta$).

(2), (3) Similar. ∎

The exact $\bar\lambda$-completeness may seem to be very strange and/or strong. But as a matter of fact it is easy to modify any $\bar\lambda$-complete λ-tree creating pair to one that is exactly complete (and the respective forcing notions are very close).

Definition B.6.3: Let (K,Σ) be a very local $\bar\lambda$-complete λ-tree creating pair for $\mathbf{H}$. We define **the $\bar\lambda$-exactivity** $(K^{{\rm ex}(\bar\lambda)},\Sigma^{{\rm ex}(\bar\lambda)})$ **of** (K,Σ) as follows.

Let $\eta \in \bigcup_{\alpha<\lambda}\prod_{\beta<\alpha}\mathbf{H}(\beta)$. We let $K^{{\rm ex}(\bar\lambda)} \cap \mathrm{TCR}^\lambda_\eta[\mathbf{H}]$ consist of all λ-tree creatures t such that

- $\eta[t] = \eta$,
- $\mathbf{dis}[t] = \langle t_\xi : \xi \le \delta\rangle$, where $t_0 = t^*_\eta$ is the minimal creature at η for (K,Σ), $\delta < \lambda_{\mathrm{lh}(\eta)}$, and $\xi < \zeta \le \delta \quad \Rightarrow \quad t_\xi \preceq^\eta_\Sigma t_\zeta \& t_\xi \ne t_\zeta$,
- $\mathbf{pos}[t] = \mathbf{pos}[t_\delta]$,
- $\mathbf{nor}[t] = \min\{\mathbf{nor}[t_\xi] : \xi \le \delta\}$.

Then, for $t', t \in K^{{\rm ex}(\bar\lambda)} \cap \mathrm{TCR}^\lambda_\eta[\mathbf{H}]$ we let $t' \in \Sigma^{{\rm ex}(\bar\lambda)}(t)$ if and only if $\mathbf{dis}[t] \trianglelefteq \mathbf{dis}[t']$.

Proposition B.6.4: *Assume (K,Σ) is a very local $\bar\lambda$-complete λ-tree creating pair. Then $(K^{{\rm ex}(\bar\lambda)},\Sigma^{{\rm ex}(\bar\lambda)})$ is a very local exactly $\bar\lambda$-complete λ-tree creating pair. The minimal creature for it at η is t^{**}_η such that $\mathbf{dis}[t^{**}_\eta] = \langle t^*_\eta\rangle$.*

Proof: Easy. ∎

Theorem B.6.5: *Suppose that*

(a) *$(\forall\alpha<\lambda)(|\mathbf{H}(\alpha)| < \lambda_\alpha)$, and*

(b) *(K,Σ) is a $\bar\mu$-complete very local λ-tree creating pair for $\mathbf{H}$ for some strictly increasing sequence $\bar\mu = \langle\mu_\alpha : \alpha<\lambda\rangle$ of regular cardinals such that $\mu_\alpha < \lambda$ (for $\alpha<\lambda$), and*

(c) *D^* is a normal filter on λ.*

Then the forcing notion $\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$ has the strong $\bar\lambda$-Sacks property

Proof: Let $i_0 < \lambda$ and $p \in \mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$. Just for notational simplicity we assume that $\mathbf{H}(\alpha) \in \lambda_\alpha$ for all $\alpha < \lambda$ and $\mathrm{lh}(\mathrm{root}(p)) \le i_0$. We are going to describe a strategy for Generic in the game $\Game^{\rm Sacks}_{\lambda}(i_0,p,\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma))$. In the course of the play she will also choose sets $Y_{i+1}\in D^*$ and λ-tree creatures t_ν.

First Generic picks $\eta\in T^p$ such that ${\rm lh}(\eta)>i_0$ and she starts the game with playing $s_{i_0}=\{\eta{\restriction}(i_0+1)\}$ and $q^{i_0}_{\eta\restriction(i_0+1)}=p^{[\eta]}$. She also picks $t_{\eta\restriction i_0}\in\Sigma(t^p_{\eta\restriction(i_0+1)})$ such that $\mathbf{pos}[t_{\eta\restriction(i_0+1)}]=\{\eta{\restriction}(i_0+1)\}$ (remember B.6.1(1)(γ)).

Arriving at a successor stage $j=i+1$ of the play the players have determined $s_i,\bar{q}^i,\bar{p}^i$ and Y_i so that, in addition to the demands of the game, for each $\eta\in s_i\cap{}^{i+1}\lambda$ we have $\eta\trianglelefteq{\rm root}(q^i_\eta)$. Now for each $\eta\in s_i\cap{}^{i+1}\lambda$ Generic picks $\nu_\eta\in T^{p^i_\eta}$ strictly extending η and she plays

$$s_{i+1}=s_i\cup\{\nu_\eta{\restriction}(i+2):\eta\in s_i\},\quad q^{i+1}_{\nu_\eta\restriction(i+2)}=(p^i_\eta)^{[\nu_\eta]}\quad\text{for }\eta\in s_i\cap{}^{i+1}\lambda.$$

She also fixes a set $Y_{i+1}\in D^*$ of limit ordinals included in $\bigcap_{\eta\in s_i}Y^{q^{i+1}_{\nu_\eta\restriction(i+2)}}$ (recall B.5.2(2)) and for $\eta\in s_i\cap{}^{i+1}\lambda$ she lets $t_\eta\in\Sigma(t^p_\eta)$ be such that $\mathbf{pos}[t_\eta]=\{\nu_\eta\}$.

Now suppose that the players have arrived to a limit stage $\delta<\lambda$ of the game, and assume that $\delta\notin\bigcap_{i<\delta}Y_{i+1}$. Generic lets s^*_δ consist of all sequences η of length δ such that $\eta{\restriction}(i+1)\in s_i$ whenever $i_0\leq i<\delta$. For each $\eta\in s^*_\delta$ she first picks a condition r_η stronger than all $p^i_{\eta\restriction(i+1)}$ for $i_0\leq i<\delta$ (there is one by arguments as in the proof of B.6.2(1)) and then she chooses $\nu_\eta\in T^{r_\eta}$ strictly extending η. Then she plays

$$s_\delta=s^*_\delta\cup\{\nu_\eta{\restriction}(\delta+1):\eta\in s^*_\delta\},\quad q^\delta_{\nu_\eta\restriction(\delta+1)}=(r_\eta)^{[\nu_\eta]}\quad\text{for }\eta\in s^*_\delta.$$

The λ-tree creatures t_η (for $\eta\in s^*_\delta$) are chosen as above: $t_\eta\in\Sigma(t^p_\eta)$, $\mathbf{pos}[t_\eta]=\{\nu_\eta\}$.

Finally suppose that we are at a limit stage $\delta<\lambda$ of the game and $\delta\in\bigcap_{i<\delta}Y_i$. Let s^*_δ be defined as above and for each $\eta\in s^*_\delta$ let r_η be a condition stronger than all $p^i_{\eta\restriction(i+1)}$ for $i_0\leq i<\delta$ and such that ${\rm root}(r_\eta)=\eta$ and $\mathbf{nor}[t^{r_\eta}_\eta]\geq|\delta|$ (there is one by arguments as in the proof of B.6.2(1) and the choice of the Y_i's). Then she plays

$$s_\delta=s^*_\delta\cup\bigcup\{\mathbf{pos}[t^{r_\eta}_\eta]:\eta\in s^*_\delta\},\quad q^\delta_\nu=(r_\eta)^{[\nu]}\quad\text{for }\nu\in\mathbf{pos}[t^{r_\eta}_\eta],\eta\in s^*_\delta.$$

She also lets $t_\eta=t^{r_\eta}_\eta\in\Sigma(t^p_\eta)$ (for $\eta\in s^*_\delta$).

It should be clear that the strategy described above always tells Generic to play legal moves (remember 0.3(c)). It should also be clear that if $\langle(s_i,\bar{q}^i,\bar{p}^i):i_0\leq i<\lambda\rangle$ is the result of a play of $\Game^{\rm Sacks}_{\lambda}(i_0,p,\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma))$ in which Generic uses that strategy, then letting $T=\bigcup\{s_i:i_0\leq i<\lambda\}$ and $q=\langle t_\eta:\eta\in T\rangle$ we get a condition in $\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$ (as witnessed by $\triangle_{i<\lambda}Y_{i+1}$) stronger than p and forcing that

$$\text{“}(\exists\rho\in{}^{\lambda}\lambda)(\forall i\in[i_0,\lambda))(\rho\restriction(i+1)\in s_i\&q^i_{\rho\restriction(i+1)}\in\Gamma_{\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)})\text{”}. \qquad \blacksquare$$

Theorem B.6.6: *Assume that* $(\forall\alpha<\lambda)(|\mathbf{H}(\alpha)|<\lambda)$, *and* $(K,\Sigma), D^*$ *satisfy* (b), (c) *of B.6.5. Then the forcing notion* $\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$ *has the strong* λ*-bounding property.*

Proof: Similar to B.6.5. ∎

The above two theorems are applicable to forcing notions of the type $\mathbb{Q}^{\rm tree}_{\rm cl}(K,\Sigma)$ as they may be treated as a special case (under the assumptions as there):

Proposition B.6.7: *Assume that* $(\forall\alpha<\lambda)(|\mathbf{H}(\alpha)|<\lambda)$ *and* (K,Σ) *is a* $\bar\mu$*-complete very local* λ*-tree creating pair for* $\mathbf{H}$ *(for some strictly increasing* $\bar\mu$*). Then the forcing notions* $\mathbb{Q}^{\rm tree}_{\rm cl}(K,\Sigma)$ *and* $\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K,\Sigma)$ *are equivalent.*

Turning to the case of λ-creating pairs (and forcing notions of the form $\mathbb{Q}^*_e(K,\Sigma)$), we have easy ways to ensure they are suitably complete (parallel to B.6.1, B.6.2).

Definition B.6.8:

(1) For a λ-creating pair (K,Σ) and $t\in K$ we define when t is $(<\kappa)$**-complete** and **exactly** $(<\kappa)$**-complete** like in B.6.1(1,2) (but with **val** replacing **pos**).

(2) If (K,Σ) is very local, then we say that it is $\bar\lambda$**-complete** (**exactly** $\bar\lambda$**-complete**, respectively) if each minimal creature t^*_α is $(<\lambda^+_\alpha)$-complete (exactly $(<\lambda_\alpha)$-complete, respectively).

Proposition B.6.9: *Assume that* (K,Σ) *is a very local* λ*-creating pair for* $\mathbf{H}$, D^* *is a normal filter on* λ*. Let* $\mathbb{P}$ *be either the forcing notion* $\mathbb{Q}^*_1(K,\Sigma)$ *or* $\mathbb{Q}^*_{D^*}(K,\Sigma)$.

(1) *If* (K,Σ) *is* $\bar\lambda$*-complete, then* $\mathbb{P}$ *is strategically* $(<\lambda)$*-complete.*

(2) *If* (K,Σ) *is exactly* $\bar\lambda$*-complete, then* $\mathbb{P}$ *is* $(<\lambda)$*-complete.*

More results on strong bounding properties for forcing notions determined by λ-creating pairs will be presented in [14].

B.7. Getting fuzzy properness. In this section we show that the forcing notions with trees and creatures may fit the fuzzy proper framework. Note that even though the forcing notions covered by Theorems B.7.2 and B.7.3 below are also covered by Theorem B.6.6, the results here still have value if we want to iterate that forcing notions with ones which do not have the strong λ-bounding property. Here we assume the following:

CONTEXT B.7.1:

(1) $\lambda, \bar{\lambda}$ are as in 0.3,
(2) $\lambda^*, A = \mathcal{H}_{<\lambda}(\lambda^*), W, D$ are as in A.3.1,
(3) $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$ is an incresing sequence of cardinals cofinal in λ and such that $\alpha < \mu_\alpha$ for $\alpha < \lambda$.

THEOREM B.7.2: *Let D^* be a normal filter on λ such that for some $S_0 \in D^*$ we have $\lambda \setminus S_0 \in D$. Assume that (K, Σ) is an exactly $\bar{\mu}$-complete very local λ-tree creating pair for $\mathbf{H}$, and $|\mathbf{H}(\alpha)| < \lambda$ for each $\alpha < \lambda$. Then the forcing notion $\mathbb{Q}^{\rm tree}_{D^*}(K, \Sigma)$ is strongly fuzzy proper over quasi D-diamonds for W.*

Proof: By B.6.2 we know that $\mathbb{Q}^{\rm tree}_{D^*}(K, \Sigma)$ is $({<}\lambda)$-complete.

Let $\mathfrak{R}^{\rm tr}, \bar{\mathfrak{Y}}^{\rm tr}$ be the trivial λ-base defined as in the proof A.3.9 (but for $\mathbb{P} = \mathbb{Q}^{\rm tree}_{D^*}(K, \Sigma)$). We are going to show that for this λ-base and for $c = (\bar{\lambda}, K, \Sigma)$ the condition A.3.6(2)($(\circledast)^+$) holds. So assume that $N, h, \bar{F} = \langle F_\delta : \delta \in S \rangle$ and $\bar{q} = \langle q_{\delta,x} : \delta \in S$ limit $\& x \in \mathcal{X}_\delta \rangle$ are as there and $p \in \mathbb{Q}^{\rm tree}_{D^*}(K, \Sigma) \cap N$. Note that $\mathcal{X}_\delta = \{0\}$ (for all relevant δ) and thus we may think that $\bar{q} = \langle q_\delta : \delta \in S$ limit$\rangle$.

Let $\bar{\mathcal{I}} = \langle \mathcal{I}_\xi : \xi < \lambda \rangle$ list all open dense subsets of $\mathbb{Q}^{\rm tree}_{D^*}(K, \Sigma)$ from N. For $i < \lambda$ let ξ_i be such that $\mathcal{I}_{\xi_i}$ consist of conditions $p \in \mathbb{Q}^{\rm tree}_{D^*}(K, \Sigma)$ with $\mathrm{lh}(\mathrm{root}(p)) > i$, and let E be a club of λ such that

$$(\forall \delta \in E)(\forall i < \delta)(\delta \text{ is limit and } \xi_i < \delta).$$

By induction on $\alpha < \lambda$ choose conditions $p_\alpha \in \mathbb{Q}^{\rm tree}_{D^*}(K, \Sigma) \cap N$ and sets $Y_\alpha \in D^*$ such that

(i) $p_0 = p$, $\mathrm{root}(p_\alpha) = \mathrm{root}(p)$, and $p_\alpha \leq p_\beta$ and $Y_\beta \subseteq Y_\alpha \subseteq S_0$ for $\alpha < \beta < \lambda$,
(ii) Y_α witnesses $p_\alpha \in \mathbb{Q}^{\rm tree}_{D^*}(K, \Sigma)$ (see B.5.2(2)),
(iii) for every $\alpha < \beta < \lambda$ and $\nu \in (T^{p_\alpha})_\alpha$ we have $\nu \in T^{p_\beta}$ and $t^{p_\alpha}_\nu = t^{p_\beta}_\nu$,
(iv) if $\alpha < \lambda$ is a successor, $\xi < \alpha$ and $\eta \in (T^{p_\alpha})_\alpha$, then for some $\nu \in (T^{p_\alpha})^{[\eta]}$ we have: $(p_\alpha)^{[\nu]} \in \mathcal{I}_\xi$ and $(\forall \rho \in T^{p_\alpha})(\eta \trianglelefteq \rho \vartriangleleft \nu \Rightarrow \mathbf{nor}[t^{p_\alpha}_\rho] = 0)$,
(v) if $\delta \in \bigcap_{\alpha<\delta} Y_\alpha$ is a limit ordinal, then $\delta \in Y_\beta$ for every $\beta \geq \delta$,
(vi) if $\delta \in S \cap E \setminus S_0$ and $\langle h \circ F_\delta(i) : i < \delta \rangle$ is an increasing $\bar{\mathcal{I}}$-exact sequence of members of $N \cap \mathbb{Q}^{\rm tree}_{D^*}(K, \Sigma)$ such that

$$(\forall \alpha < \delta)(\exists i < \delta)(p_\alpha \leq h \circ F_\delta(i)),$$

and $\eta \in (T^{p_\delta})_\delta$ is such that every $h \circ F_\delta(i)$ is compatible with $(p_\delta)^{[\eta]}$, then $(p_\delta)^{[\eta]} \le q_\delta = (p_\delta)^{[\mathrm{root}(q_\delta)]}$ and

$$(\forall \rho \in T^{p_\delta})(\eta \trianglelefteq \rho \triangleleft \mathrm{root}(q_\delta) \Rightarrow \mathbf{nor}[t^{p_\delta}_\rho] = 0).$$

(Note that there is at most one η as above; remember the choice of E.) It should be clear that the inductive construction of the p_α's and Y_α's is possible (for (v) remember $\delta < \mu_\delta$; note also that there is no collision between (v) and (vi) because $Y_\alpha \subseteq S_0$). Now letting $\mathrm{root}(r) = \mathrm{root}(p)$, $T^r = \bigcup_{\alpha<\lambda}(T^{p_\alpha})_\alpha$, $t^r_\nu = t^{p_{\mathrm{lh}(\nu)+1}}_\nu$ we get a condition $r \in \mathbb{Q}^{\mathrm{tree}}_{D^*}(K,\Sigma)$ (as witnessed by $\triangle_{\alpha<\lambda} Y_\alpha$). Also note that r is stronger than all p_α's.

Claim B.7.2.1: *The condition r is $(\mathfrak{R}^{\mathrm{tr}}, \mathfrak{Y}^{\mathrm{tr}})$-fuzzy generic for $\bar q$.*

Proof of the Claim: First note that the condition r is $(N, \mathbb{Q}^{\mathrm{tree}}_{D^*}(K,\Sigma))$-generic by clause (iv) above. Therefore we may use A.3.8(3), and it is enough that we show that Generic has a strategy in the game $\eth^{\mathrm{fuzzy}}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{Q}^{\mathrm{tree}}_{D^*}(K,\Sigma), \bar F, \bar q)$ which guarantees that the result $\langle r_i, C_i : i < \lambda\rangle$ of the play satisfies A.3.4(5)(β). Let us describe such a strategy.

First, for $\alpha < \lambda$ let $\zeta_\alpha < \lambda$ be such that

$$(\forall q \in \mathcal{I}_{\zeta_\alpha})(\text{ either } p_\alpha \le q \text{ or } p_\alpha, q \text{ are incompatible}),$$

and let $E' = \{\delta \in E : (\forall \alpha < \delta)(\zeta_\alpha < \delta)\}$ (it is a club of λ).

Now, suppose that during a play of $\eth^{\mathrm{fuzzy}}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{Q}^{\mathrm{tree}}_{D^*}(K,\Sigma), \bar F, \bar q)$ the players have arrived at stage $i \in S$ having constructed a sequence $\langle r_j, C_j : j < i\rangle$.

If either i is a successor ordinal or $i \notin \bigcap_{j<i} C_j$, then the Generic player plays the $<^*_\chi$-first condition $r_i \in \mathbb{Q}^{\mathrm{tree}}_{D^*}(K,\Sigma)$ such that $(\forall j < i)(r_j \le r_i)$ and $\mathrm{lh}(\mathrm{root}(r_i)) > i$, and the set $C_i = E' \setminus (S_0 \cup \mathrm{lh}(\mathrm{root}(r_i)))$.

If $i \in \bigcap_{j<i} C_j$ is a limit ordinal (so also $i \in E' \setminus S_0$), then Generic asks

($*$) is $\langle h \circ F_i(\alpha) : \alpha < i\rangle$ an increasing $\bar{\mathcal{I}}$-exact sequence such that

$$(\forall j < i)(\exists \alpha < i)(p_j \le h \circ F_i(\alpha))?$$

If the answer to ($*$) is "no", then she plays like at the successor stage. (Note that if the answer to ($*$) is "no" and $\langle h \circ F_i(\alpha) : \alpha < i\rangle$ is increasing $\bar{\mathcal{I}}$-exact, then for some $j < i$ and $\alpha < i$ the conditions p_j and $h \circ F_i(\alpha)$ are incompatible, and hence r_i and $h \circ F_i(\alpha)$ are incompatible.)
If the answer to ($*$) is "yes", then Generic looks at clause (vi) (of the choice of p_α's) and $\eta = \bigcup_{j<i} \mathrm{root}(r_j)$ (note that $\mathrm{lh}(\eta) = i$). If $(p_i)^{[\eta]}$ is incompatible with some $h \circ F_i(\alpha)$, $\alpha < i$, then she plays C_i, r_i as in the successor case.

(Note that then $r_i, h\circ F_i(\alpha)$ are incompatible.)
Otherwise $\eta\trianglelefteq\mathrm{root}(q_i)\in T^{p_i}$, $q_i=(p_i)^{[\mathrm{root}(q_i)]}$ and $(\forall\rho\in T^{p_i})(\eta\trianglelefteq\rho\vartriangleleft\mathrm{root}(q_i)\Rightarrow|\mathbf{pos}[t^{p_i}_\rho]|=1)$. Therefore, $\mathrm{root}(q_i)\in T^{r_j}$ and $q_i\leq(r_j)^{[\mathrm{root}(q_i)]}$ (for each $j<i$). So Generic can put $C_i=E'\setminus i$ and the $<^*_\chi$-first condition r_i stronger than all r_j (for $j<i$) and q_i.

It follows immediately from the comments stated during the description of the strategy that every play according to it satisfies A.3.4(5)(β), finishing the proof of the claim. ∎

This finish the proof of Theorem B.7.2. ∎

THEOREM B.7.3: *Let D^* be a normal filter on λ such that for some $S_0\in D^*$ we have $\lambda\setminus S_0\in D$. Assume that (K,Σ) is an exactly $\bar{\lambda}$-complete very local λ-creating pair for $\mathbf{H}$, $|\mathbf{H}(\alpha)|<\lambda$ for each $\alpha<\lambda$. Then the forcing notion $\mathbb{Q}^*_{D^*}(K,\Sigma)$ is strongly fuzzy proper over quasi D-diamonds for W.*

Proof: The proof is the same as the proof of Theorem B.7.2. ∎

THEOREM B.7.4: *Suppose that (K,Σ) is an exactly $\bar{\mu}$-complete very local λ-tree creating pair for $\mathbf{H}$, $(\forall\alpha<\lambda)(|\mathbf{H}(\alpha)|<\lambda)$, and D^* is a normal filter on λ. Then the forcing notion $\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$ is fuzzy proper for W.*

Proof: The proof closely follows the lines of that of B.7.2. Let D be a normal filter on λ such that there is a D-diamond.

Just only to simplify somewhat the definition of a λ-base which we will use, let us assume that $\bigcup_{\delta<\lambda}\prod_{\alpha<\delta}\mathbf{H}(\alpha)\subseteq a$ for every $a\in W$. Now we let $\mathfrak{R}=\mathfrak{R}(K,\Sigma)$ consist of all triples (p,δ,η) such that $\delta<\lambda$, $\eta\in\prod_{\alpha<\delta}\mathbf{H}(\alpha)$ and $p\in\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$ satisfies $\eta\vartriangleleft\mathrm{root}(p)$. Next, for $a\in W$ let $\mathfrak{Y}_a=\mathfrak{Y}_a(K,\Sigma)\colon\lambda\longrightarrow[a]^{<\lambda}$ be given by $\mathfrak{Y}_a(\delta)=\prod_{\alpha\leq\delta}\mathbf{H}(\alpha)\subseteq a$ (for $\delta<\lambda$). It should be clear that $(\mathfrak{R},\bar{\mathfrak{Y}})$ is a λ-base for $\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$ over W.

We claim that $(\mathfrak{R},\bar{\mathfrak{Y}})$ and $c=(\bar{\lambda},\mathbf{H},K,\Sigma)$ witness the condition ($\circledast$) of A.3.6(1). To this end, let $N,h,\bar{F}=\langle F_\delta:\delta\in S\rangle$ and $\bar{q}=\langle q_{\delta,x}:\delta\in S$ limit $\&x\in\mathcal{X}_\delta\rangle$ be as in A.3.6(1)($\circledast$), $p\in\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)\cap N$. Let $\bar{\mathcal{I}}=\langle\mathcal{I}_\xi:\xi<\lambda\rangle$ list all open dense subsets of $\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$ from N. For $i<\lambda$ let ξ_i be such that $\mathcal{I}_{\xi_i}$ consist of conditions $p\in\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$ with $\mathrm{lh}(\mathrm{root}(p))>i$, and let E be a club of λ such that

$$(\forall\delta\in E)(\forall i<\delta)(\delta\text{ is limit and }\xi_i<\delta).$$

By induction on $\alpha<\lambda$, like in B.7.2 (but note the change in (vi) below!), we choose conditions $p_\alpha\in\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)\cap N$ and sets $Y_\alpha\in D^*$ such that

(i) $p_0 = p$, $\text{root}(p_\alpha) = \text{root}(p)$, and $p_\alpha \le p_\beta$ and $Y_\beta \subseteq Y_\alpha$ for $\alpha < \beta < \lambda$,

(ii) Y_α witnesses $p_\alpha \in \mathbb{Q}^{\text{tree}}_{D^*}(K,\Sigma)$,

(iii) for every $\alpha < \beta < \lambda$ and $\nu \in (T^{p_\alpha})_\alpha$ we have $\nu \in T^{p_\beta}$ and $t^{p_\alpha}_\nu = t^{p_\beta}_\nu$,

(iv) if $\alpha < \lambda$ is a successor ordinal and $\eta \in (T^{p_\alpha})_\alpha$, then for some $\nu \in (T^{p_\alpha})^{[\eta]}$ we have: $(p_\alpha)^{[\nu]} \in \bigcap_{\xi<\alpha} \mathcal{I}_\xi$ and $(\forall \rho \in T^{p_\alpha})(\eta \trianglelefteq \rho \triangleleft \nu \Rightarrow \mathbf{nor}[t^{p_\alpha}_\rho] = 0)$,

(v) if $\delta \in \bigcap_{\alpha<\delta} Y_\alpha$ is a limit ordinal, then $\delta \in Y_\beta$ for every $\beta \ge \delta$,

(vi) if $\delta \in S \cap E$, $\langle h \circ F_\delta(i) : i < \delta\rangle$ is increasing $\bar{\mathcal{I}}$-exact, $\eta = \bigcup_{i<\delta} \text{root}(h \circ F_\delta(i))$ and $\text{lh}(\eta) = \delta$, and $(\forall \alpha < \delta)(\exists i < \delta)(p_\alpha \le h \circ F_\delta(i))$, then ($\eta \in T^{p_\delta}$ and) for every $\nu \in \mathbf{pos}[t^{p_\delta}_\eta] \cap \bigcap_{i<\delta} \mathbf{pos}[t^{h\circ F_\delta(i)}_\eta]$ we have

$$(p_\delta)^{[\nu]} \le q_{\delta,\nu} = (p_\delta)^{[\text{root}(q_{\delta,\nu})]} \text{ and}$$
$$(\forall \rho \in T^{p_\delta})(\nu \trianglelefteq \rho \triangleleft \text{root}(q_{\delta,\nu}) \Rightarrow \mathbf{nor}[t^{p_\delta}_\rho] = 0).$$

(Note that, in the situation as in (vi), $\mathcal{X}_\delta = \bigcap_{i<\delta} \mathbf{pos}[t^{h\circ F_\delta(i)}_\eta]$.)

Plainly, the inductive construction of the p_α's and Y_α's is possible (for (v) remember $\delta < \mu_\delta$). Now letting $\text{root}(r) = \text{root}(p)$, $T^r = \bigcup_{\alpha<\lambda} (T^{p_\alpha})_\alpha$, $t^r_\nu = t^{p_{\text{lh}(\nu)+1}}_\nu$ we get a condition $r \in \mathbb{Q}^{\text{tree}}_{D^*}(K,\Sigma)$ stronger than all p_α's.

Claim B.7.4.1: *The condition r is $(\mathfrak{R}, \bar{\mathfrak{Y}})$-fuzzy generic for $\bar{q}$.*

Proof of the Claim: It is very much like the proof of claim B.7.2.1. We note that r is $(N, \mathbb{Q}^{\text{tree}}_{D^*}(K,\Sigma))$-generic (by clause (iv)), and therefore it is enough to show that Generic has a strategy in the game $\Game^{\text{fuzzy}}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{Q}^{\text{tree}}_{D^*}(K,\Sigma), \bar{F}, \bar{q})$ which guarantees that the result $\langle r_i, C_i : i < \lambda\rangle$ of the play satisfies A.3.4(5)(β) (remember A.3.8(3)). Let us describe such a strategy. First, for $\alpha < \lambda$ let $\zeta_\alpha < \lambda$ be such that

$$(\forall q \in \mathcal{I}_{\zeta_\alpha})(\text{ either } p_\alpha \le q \text{ or } p_\alpha, q \text{ are incompatible}),$$

and let $E' = \{\delta \in E : (\forall \alpha < \delta)(\zeta_\alpha < \delta)\}$ (it is a club of λ).

Now, suppose that during a play of $\Game^{\text{fuzzy}}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{Q}^{\text{tree}}_{D^*}(K,\Sigma), \bar{F}, \bar{q})$ the players have arrived at stage $i \in S$ having constructed a sequence $\langle r_j, C_j : j < i\rangle$.

If either i is a successor ordinal or $i \notin \bigcap_{j<i} C_j$, then Generic plays the $<^*_\chi$-first condition $r_i \in \mathbb{Q}^{\text{tree}}_{D^*}(K,\Sigma)$ such that $(\forall j < i)(r_j \le r_i)$ and $\text{lh}(\text{root}(r_i)) > i$ and $C_i = E' \setminus \text{lh}(\text{root}(r_i))$.

If $i \in \bigcap_{j<i} C_j \subseteq E'$ is a limit ordinal, then Generic asks

($*$) is $\langle h \circ F_i(\alpha) : \alpha < i\rangle$ an increasing $\bar{\mathcal{I}}$-exact sequence such that

$$(\forall \alpha < i)(\exists j < i)(h \circ F_i(\alpha) \le r_j)?$$

If the answer to $(*)$ is "no", then she plays like at the successor stage. If the answer to $(*)$ is "yes", then Generic takes $\eta = \bigcup_{j<i} \mathrm{root}(r_j)$ and she notes that $\mathrm{lh}(\eta) = \delta$ (by the choice of C_j's at successor stages) and $\eta = \bigcup_{i<\delta} \mathrm{root}(h \circ F_\delta(i))$. Also, by the exactness and the choice of E', we have

$$(\forall j < i)(\exists \alpha < i)(p_j \le h \circ F_i(\alpha)).$$

So now Generic looks at clause (vi) of the choice of p_α's. She picks (say, the $<^*_\chi$-first) $\nu \in \bigcap_{j<i} \mathbf{pos}[t^{r_j}_\eta] \subseteq \bigcap_{j<i} \mathbf{pos}[t^{h\circ F_i(\alpha)}_\eta]$ and notices that (by (vi)) $\nu \trianglelefteq \mathrm{root}(q_{i,\nu}) \in T^{p_i}$, $q_{i,\nu} = (p_i)^{[\mathrm{root}(q_{i,\nu})]}$ and $(\forall \rho \in T^{p_i})(\nu \trianglelefteq \rho \vartriangleleft \mathrm{root}(q_{i,\nu}) \Rightarrow |\mathbf{pos}[t^{p_i}_\rho]| = 1)$. Therefore, $\mathrm{root}(q_{i,\nu}) \in T^{r_j}$ and $q_{i,\nu} \le (T^{r_j})^{[\mathrm{root}(q_{i,\nu})]}$ (for each $j < i$). So Generic can play $C_i = E' \setminus i$ and the $<^*_\chi$-first condition r_i stronger than all r_j (for $j < i$) and $q_{i,\nu}$.

Easily, the strategy described above has the required property, and the proof is completed. ∎

This ends the proof of theorem B.7.4. ∎

Problem B.7.5: *Unlike the case of B.7.2, it is not clear how the proof of B.7.4 can be modified to get the parallel result for non-tree case. So, assuming that $A, W, \mathbf{H}$ and D^* are as in B.7.4 and (K, Σ) is an exactly $\bar{\lambda}$-complete very local λ-creating pair for $\mathbf{H}$, is the forcing notion $\mathbb{Q}^*_{D^*}(K, \Sigma)$ fuzzy proper for W?*

B.8. More examples and applications. Here we are going to present some direct applications of the methods developed in this paper. Though we do keep our basic assumptions from 0.3, we are going to introduce more parameters, so let us fully state the context we are working in now.

Context B.8.1:

(a) λ is a strongly inaccessible cardinal, $2^\lambda = \lambda^+$, and $2^{\lambda^+} = \lambda^{++}$, and
(b) $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$, $\bar{\lambda} = \langle \lambda_\alpha : \alpha < \lambda \rangle$ and $\bar{\kappa} = \langle \kappa_\alpha : \alpha < \lambda \rangle$ are strictly increasing sequences of uncountable regular cardinals, each cofinal in λ,
(c) for each $\alpha < \lambda$,
  - $\alpha < \mu_\alpha < \mu^+_\alpha < \lambda_\alpha < \kappa_\alpha$,
  - $\prod_{\beta<\alpha} \lambda_\beta < \lambda_\alpha$ and $(\forall \xi < \lambda_\alpha)(|\xi|^\alpha < \lambda_\alpha)$,
(d) $A = \mathcal{H}_{<\lambda}(\lambda^*)$, $\lambda^* > \lambda$ and $W \subseteq [A]^\lambda$ are as in A.3.1,
(e) D is a normal filter on λ such that there is a D-diamond.

Let us recall some notions related to cardinal characteristics of λ-reals.

Definition B.8.2:

(1) Let $\mathcal{S}_{\bar\mu}$ be the family of all sequences $\bar a = \langle a_\alpha : \alpha < \lambda\rangle$ such that $a_\alpha \in [\lambda]^{<\mu_\alpha}$ (for all $\alpha < \lambda$). We define

$$c(\bar\mu) = \min\{|\mathcal{Y}| : \mathcal{Y} \subseteq \mathcal{S}_{\bar\mu} \& (\forall f \in {}^\lambda\lambda)(\exists \bar a \in \mathcal{Y})(\forall \alpha < \lambda)(f(\alpha) \in a_\alpha)\},$$

$$c^-_{\rm cl}(\bar\mu) = \min\{|\mathcal{Y}| : \mathcal{Y} \subseteq \mathcal{S}_{\bar\mu} \& (\forall f \in {}^\lambda\lambda)(\exists \bar a \in \mathcal{Y})(\{\alpha < \lambda : f(\alpha) \in a_\alpha\} \in (\mathcal{D}_\lambda)^+)\},$$

and also

$$e_{\rm cl}(\bar\mu) = \min\{|\mathcal{G}| : \mathcal{G} \subseteq \prod_{\alpha<\lambda} \mu_\alpha \text{ and } (\forall f \in \prod_{\alpha<\lambda} \mu_\alpha)(\exists g \in \mathcal{G})(\{\alpha < \lambda : f(\alpha) \neq g(\alpha)\} \in \mathcal{D}_\lambda)\},$$

(2) For an ideal $\mathcal{J}$ of subsets of a set $\mathcal{X}$, the **covering number** $\operatorname{cov}(\mathcal{J})$ of $\mathcal{J}$ is

$$\operatorname{cov}(\mathcal{J}) = \min\{|\mathcal{Y}| : \mathcal{Y} \subseteq \mathcal{J} \& \cup \mathcal{Y} = \mathcal{X}\}.$$

PROPOSITION B.8.3: *It is consistent that $c(\bar\lambda) < e_{\rm cl}(\bar\mu)$.*

Proof: Let $\mathbf{H}_0(\alpha) = \mu_\alpha$ (for $\alpha < \lambda$) and let K_0 consist of all λ-tree creatures $t \in \mathrm{TCR}^\lambda[\mathbf{H}_0]$ such that:

- $\mathbf{dis}[t] \in \mu_{\mathrm{lh}(\eta[t])} + 1$,
- if $\mathbf{dis}[t] = \mu_{\mathrm{lh}(\eta[t])}$, then $\mathbf{pos}[t] = \{\eta[t]^\frown\langle\xi\rangle : \xi < \mu_{\mathrm{lh}(\eta[t])}\}$ and $\mathbf{nor}[t] = \mu_{\mathrm{lh}(\eta[t])}$,
- if $\mathbf{dis}[t] < \mu_{\mathrm{lh}(\eta[t])}$, then $\mathbf{pos}[t] = \{\eta[t]^\frown\langle\mathbf{dis}[t]\rangle\}$ and $\mathbf{nor}[t] = 0$.

Let Σ_0 be a local tree-composition operation on K_0 (so its domain consists of singletons only) such that

- if $\mathbf{dis}[t] < \mu_{\mathrm{lh}(\eta[t])}$, then $\Sigma_0(t) = \{t\}$,
- if $\mathbf{dis}[t] = \mu_{\mathrm{lh}(\eta[t])}$, then $\Sigma_0(t) = \{t' \in K_0 : \eta[t'] = \eta[t]\}$.

It should be clear that (K_0, Σ_0) is a very local exactly $\bar\lambda$-complete tree creating pair. The forcing notion $\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_0, \Sigma_0)$ has the strong $\bar\lambda$-Sacks property (by B.6.5). Let $\underset{\sim}{W}$ be the canonical $\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_0, \Sigma_0)$-name for the generic function in $\prod_{\alpha<\lambda} \mu_\alpha$, so

$$p \Vdash_{\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_0,\Sigma_0)} \text{“}\mathrm{root}(p) \lhd \underset{\sim}{W}\text{”}.$$

Then we have

$$\Vdash_{\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_0,\Sigma_0)} \text{“}(\forall f \in \prod_{\alpha<\lambda} \mu_\alpha \cap \mathbf{V})(\{\alpha < \lambda : \underset{\sim}{W}(\alpha) = f(\alpha)\} \in (\mathcal{D}_\lambda)^+)\text{”}.$$

Now let $\mathbb{P}$ be the limit of a λ-support iteration, λ^{++} in length, of the forcing notions $\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_0,\Sigma_0)$. Then, by A.2.4 + A.1.10 + A.1.5 + A.3.10,

- $\mathbb{P}$ is $(<\lambda)$-complete, λ-proper and satisfies the λ^{++}-cc, and it has a dense subset of size λ^{++}, thus forcing with $\mathbb{P}$ does not collapse cardinals,
- $\mathbb{P}$ has the $\bar{\lambda}$-Sacks property, it is weakly fuzzy proper for W,
- $\Vdash_{\mathbb{P}}$"$2^\lambda = 2^{\lambda^+} = \lambda^{++} = \mathfrak{e}_{\rm cl}(\bar{\mu})$ and $\mathfrak{c}(\bar{\lambda}) = \lambda^+$" ∎

Remark B.8.4: The forcing $\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_0,\Sigma_0)$ is a "bounded relative" of $\mathbb{D}_\lambda$ from [16, 4.10] (remember B.6.7). It is also a generalization of the forcing notions $\mathbb{D}_{\mathcal{X}}$ from [13].

PROPOSITION B.8.5: *It is consistent that $\mathfrak{c}(\bar{\lambda}) < \mathfrak{c}^-_{\rm cl}(\bar{\mu}^+) = \mathfrak{c}(\bar{\mu}^+)$, where $\bar{\mu}^+ = \langle \mu^+_\alpha : \alpha < \lambda\rangle$.*

Proof: Let $\mathbf{H}_1(\alpha) = \mu^+_\alpha$ (for $\alpha < \lambda$) and let K'_1 consist of all λ-tree creatures $t \in {\rm TCR}^\lambda[\mathbf{H}_1]$ such that:

- $\mathbf{dis}[t] \subseteq \mu^+_{{\rm lh}(\eta[t])}$, either $|\mathbf{dis}[t]| = 1$ or $\mathbf{dis}[t]$ is a club of $\mu^+_{{\rm lh}(\eta[t])}$,
- $\mathbf{pos}[t] = \{\eta[t]^\frown\langle\xi\rangle : \xi \in \mathbf{dis}[t]\}$,
- if $|\mathbf{dis}[t]| = 1$ then $\mathbf{nor}[t] = 0$, if $|\mathbf{dis}[t]| > 1$ then $\mathbf{nor}[t] = \mu_{{\rm lh}(\eta[t])}$.

Let Σ'_1 be a local tree-composition operation on K'_1 such that

$$\Sigma'_1(t) = \{t' \in K'_1 : \eta[t'] = \eta[t] \& \mathbf{dis}[t'] \subseteq \mathbf{dis}[t]\}.$$

Then (K'_1,Σ'_1) is a very local $\bar{\mu}$-complete λ-tree creating pair. Let (K_1,Σ_1) be the $\bar{\mu}$-exactivity of (K'_1,Σ'_1) (see B.6.3); thus (K_1,Σ_1) is a very local exactly $\bar{\mu}$-complete λ-tree creating pair. The forcing notion $\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_1,\Sigma_1)$ is λ-complete fuzzy proper for W and it has the strong $\bar{\lambda}$-Sacks property. Also, letting $\underset{\sim}{W}$ be the canonical name for the generic function in $\prod_{\alpha<\lambda}\mu^+_\alpha$ (i.e., $p \Vdash_{\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_1,\Sigma_1)}$ "${\rm root}(p) \vartriangleleft \underset{\sim}{W}$"), we have

$$\Vdash_{\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_1,\Sigma_1)} \text{“}(\forall \bar{a} \in \mathcal{S}_{\bar{\mu}^+} \cap \mathbf{V})(\{\alpha < \lambda : \underset{\sim}{W}(\alpha) \notin a_\alpha)\} \in \mathcal{D}_\lambda)\text{”}.$$

Let $\mathbb{P}$ be the limit of a λ-support iteration, λ^{++} in length, of the forcing notions $\mathbb{Q}^{\rm tree}_{\mathcal{D}_\lambda}(K_1,\Sigma_1)$. Then (by A.2.4 + A.1.10 + A.1.5 + A.3.10) we have:

- $\mathbb{P}$ is $(<\lambda)$-complete, λ-proper and satisfies the λ^{++}-cc, and it has a dense subset of size λ^{++}, thus forcing with $\mathbb{P}$ does not collapse cardinals,
- $\mathbb{P}$ has the $\bar{\lambda}$-Sacks property, it is weakly fuzzy proper for W,
- $\Vdash_{\mathbb{P}}$"$2^\lambda = 2^{\lambda^+} = \lambda^{++} = \mathfrak{c}^-_{\rm cl}(\bar{\mu}^+)$ and $\mathfrak{c}(\bar{\lambda}) = \lambda^+$" ∎

Remark B.8.6: The result in B.8.5 is of interest as it shows that the λ-versions of cardinal characteristics of the reals may behave totally differently from their "ancestors". Recall that if for an increasing function $f \in {}^\omega\omega$ we let $\mathcal{S}^f$ consist of all sequences $\bar{a} = \langle a_n : n < \omega\rangle$ with $a_n \in [\omega]^{\leq f(n)+1}$ (for $n < \omega$), then

$$\min\{|\mathcal{Y}| : \mathcal{Y} \subseteq \mathcal{S}^f \& (\forall h \in {}^\omega\omega)(\exists \bar{a} \in \mathcal{Y})(\forall n < \omega)(h(n) \in a_n)\}$$
$$= \min\{|\mathcal{Y}| : \mathcal{Y} \subseteq \mathcal{S}^g \& (\forall h \in {}^\omega\omega)(\exists \bar{a} \in \mathcal{Y})(\forall n < \omega)(h(n) \in a_n)\}$$

for any increasing $f, g \in {}^\omega\omega$

The λ-tree creating pair (K_1, Σ_1) may be treated (in some sense) as a special case of the λ-tree creating pairs $(K(\bar{\mathcal{A}}), \Sigma(\bar{\mathcal{A}}))$ from B.8.10 below.

Definition B.8.7: Let $\mathcal{A}$ be a family of subsets of κ such that $\kappa \in \mathcal{A}$.

(1) A game $\Game^*(\mathcal{A}, \mu)$ of two players, I and II, is defined as follows. A play lasts μ moves, in the α^{th} move a set $A_\alpha \in \mathcal{A}$ is chosen, and player I chooses A_α for even α's. In the end player II wins if $\bigcap_{\alpha<\mu} A_\alpha \neq \emptyset$.
(2) The family $\mathcal{A}$ is a **μ-category prebase on κ** if player II has a winning strategy in the game $\Game^*(\mathcal{A}, \mu)$ and $(\forall A \in \mathcal{A})(\forall \xi < \kappa)(\exists B \in \mathcal{A})(B \subseteq A \setminus \{\xi\})$.
(3) A set $X \subseteq \kappa$ is **$\mathcal{A}$-presmall** if

$$(\forall A \in \mathcal{A})(\exists B \in \mathcal{A})(B \subseteq A \setminus X).$$

Of course, every μ^+-complete uniform filter D^* on κ is a μ-category base on κ and then a set is D-presmall if and only if its complement is in D^*.

Definition B.8.8:

(1) **A $\bar{\lambda}$-smallness base on** $\bar{\kappa}$ is a sequence $\bar{\mathcal{A}} = \langle \mathcal{A}_\alpha : \alpha < \lambda\rangle$ such that each $\mathcal{A}_\alpha$ is a λ_α-category prebase on κ_α.
Let $\bar{\mathcal{A}}$ be a $\bar{\lambda}$-smallness base on $\bar{\kappa}$.
(2) Let $T \subseteq \bigcup_{\alpha<\lambda} \prod_{\beta<\lambda} \kappa_\alpha$ be a complete λ-tree with $\max(T) = \emptyset$ and D^* be a filter on λ. We say that
   - T **is $\bar{\mathcal{A}}$-small** if for every $\eta \in (T)_\alpha$, $\alpha < \lambda$, the set $\{\xi < \kappa_\alpha : \eta^\frown\langle\xi\rangle \in T\}$ is $\mathcal{A}_\alpha$-presmall;
   - T **is $(D^*, \bar{\mathcal{A}})$-small** if

$$\{\alpha < \lambda : \text{ for every } \eta \in (T)_\alpha \text{ the set } \{\xi < \kappa_\alpha : \eta^\frown\langle\xi\rangle \in T\} \text{ is } \mathcal{A}_\alpha\text{-presmall}\} \in D^*.$$

(3) Let $\mathcal{J}_{\bar{\kappa}}(\bar{\mathcal{A}})$ consist of all subsets X of $\prod_{\alpha<\lambda} \kappa_\alpha$ such that $X \subseteq \bigcup_{\varepsilon<\lambda} \lim_\lambda(T_\varepsilon)$ for some $\bar{\mathcal{A}}$-small trees $T_\varepsilon \subseteq \bigcup_{\alpha<\lambda} \prod_{\beta<\lambda} \kappa_\alpha$ (for $\varepsilon < \lambda$).

$\mathcal{J}_{\bar\kappa}(D^*,\bar{\mathcal{A}})$ is defined similarly, replacing "$\bar{\mathcal{A}}$-small" by "$(D^*,\bar{\mathcal{A}})$-small".

Proposition B.8.9: *Let $\bar{\mathcal{A}}$ be a $\bar\lambda$-smallness base on $\bar\kappa$. Then both $\mathcal{J}_{\bar\kappa}(\bar{\mathcal{A}})$ and $\mathcal{J}_{\bar\kappa}(D^*,\bar{\mathcal{A}})$ are proper λ^+-complete ideals of subsets of $\prod_{\alpha<\lambda}\kappa_\alpha$, $\mathcal{J}_{\bar\kappa}(\bar{\mathcal{A}})\subseteq\mathcal{J}_{\bar\kappa}(D^*,\bar{\mathcal{A}})$. They contain singletons and $\lambda<\mathrm{cov}(\mathcal{J}_{\bar\kappa}(D^*,\bar{\mathcal{A}}))\le\mathrm{cov}(\mathcal{J}_{\bar\kappa}(\bar{\mathcal{A}}))$.*

Proposition B.8.10: *Let $\bar{\mathcal{A}}$ be a $\bar\lambda$-smallness base on $\bar\kappa$ and D^* be a normal filter on λ. It is consistent that $\mathrm{cov}(\mathcal{J}_{\bar\kappa}(D^*,\bar{\mathcal{A}}))>\lambda^+$.*

Proof: First we define a λ-tree creating pair $(K(\bar{\mathcal{A}}),\Sigma(\bar{\mathcal{A}}))=(K,\Sigma)$. For $\alpha<\lambda$ let $\mathbf{H}(\alpha)=\kappa_\alpha$ and let st_α be a winning strategy of player II in the game $\eth^*(\mathcal{A}_\alpha,\lambda_\alpha)$.

K consists of all λ-tree creatures $t\in\mathrm{TCR}^\lambda[\mathbf{H}]$ such that letting $\alpha=\mathrm{lh}(\eta[t])$:

- **either dis**$[t]=(\delta,\langle A^t_i:i<\delta\rangle)$, where $\delta<\lambda_\alpha$ and $\langle A^t_i:i<\delta\rangle$ is (an initial segment of) a play of $\eth^*(\mathcal{A}_\alpha,\lambda_\alpha)$ in which player II uses strategy st_α,
  or dis$[t]=\langle\xi\rangle$ for some $\xi<\kappa_\alpha$;
- if **dis**$[t]=\langle\xi\rangle$, then **pos**$[t]=\{\eta[t]^\frown\langle\xi\rangle\}$ and **nor**$[t]=0$;
- if **dis**$[t]=(\delta,\langle A^t_i:i<\delta\rangle)$, then **pos**$[t]=\{\eta[t]^\frown\langle\xi\rangle:\xi\in\bigcap_{i<\delta}A_i\}$ and **nor**$[t]=\alpha+1$. (If $\delta=0$ then we stipulate **pos**$[t]=\{\eta[t]^\frown\xi:\xi<\kappa_\alpha\}$.)

The domain of the tree composition operation Σ consists of singletons only, and
if **nor**$[t]=0$ then $\Sigma(t)=\{t\}$,
if **nor**$[t]>0$, $\alpha=\mathrm{lh}(\eta[t])$ and **dis**$[t]=(\delta,\langle A^t_i:i<\delta\rangle)$, then $\Sigma(t)$ consists of those $t'\in K\cap\mathrm{TCR}^\lambda_{\eta[t]}[\mathbf{H}]$ for which:

- either **nor**$[t']=0$ and **pos**$[t']\subseteq$ **pos**$[t]$,
- or **nor**$[t']>0$, **dis**$[t']=(\delta',\langle A^{t'}_i:i<\delta'\rangle)$ and $\langle A^t_i:i<\delta\rangle\trianglelefteq\langle A^{t'}_i:i<\delta'\rangle$.

Claim B.8.10.1: *(K,Σ) is an exactly $\bar\lambda$-complete very local tree creating pair for $\mathbf{H}$. Hence the forcing notion $\mathbb{Q}^{\rm tree}_{D^*}(K,\Sigma)$ is fuzzy proper for W.*

Proof of the Claim: The proof is straightforward. ∎

We finish the proof of the proposition in a standard way: we force with λ-support iteration, λ^{++} in length, of the forcing notion $\mathbb{Q}^{\rm tree}_{D^*}(K(\bar{\mathcal{A}}),\Sigma(\bar{\mathcal{A}}))$. ∎

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