PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 6, June 2010, Pages 2151–2161
S 0002-9939(10)10254-8
Article electronically published on February 12, 2010

DIAMONDS

SAHARON SHELAH

(Communicated by Julia Knight)

ABSTRACT. If $\lambda = \chi^+ = 2^\chi > \aleph_1$, then diamond on λ holds. Moreover, if $\lambda = \chi^+ = 2^\chi$ and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) \neq \mathrm{cf}(\chi)\}$ is stationary, then $\diamondsuit_S$ holds. Earlier this was known only under additional assumptions on χ and/or S.

1. INTRODUCTION

In this paper we prove several results about diamonds. Let us recall the basic definitions and sketch the (pretty long) history of related questions.

The diamond principle was formulated by Jensen, who proved that it holds in $\mathbf{L}$ for every regular uncountable cardinal κ and stationary $S \subseteq \kappa$. This is a prediction principle, which asserts the following:

Definition 1.1. $\diamondsuit_S$ (the set version).

Assume $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $S \subseteq \kappa$ is stationary; $\diamondsuit_S$ holds <u>when</u> there is a sequence $\langle A_\alpha : \alpha \in S\rangle$ such that $A_\alpha \subseteq \alpha$ for every $\alpha \in S$ and the set $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is a stationary subset of κ for every $A \subseteq \kappa$.

The diamond sequence $\langle A_\alpha : \alpha \in S\rangle$ guesses enough (i.e., stationarily many) initial segments of every $A \subseteq \kappa$. Several variants of this principle were formulated, for example:

Definition 1.2. $\diamondsuit^*_S$.

Assume $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and S is a stationary subset of κ. Now $\diamondsuit^*_S$ holds <u>when</u> there is a sequence $\langle \mathcal{A}_\alpha : \alpha \in S\rangle$ such that each $\mathcal{A}_\alpha$ is a subfamily of $\mathcal{P}(\alpha)$, $|\mathcal{A}_\alpha| \leq |\alpha|$ and for every $A \subseteq \kappa$ there exists a club $C \subseteq \kappa$ such that $A \cap \alpha \in \mathcal{A}_\alpha$ for every $\alpha \in C \cap S$.

We know that $\diamondsuit^*_S$ holds in $\mathbf{L}$ for every regular uncountable κ and stationary $S \subseteq \kappa$. Kunen proved that $\diamondsuit^*_S \Rightarrow \diamondsuit_S$. Moreover, if $S_1 \subseteq S_2$ are stationary subsets of κ, then $\diamondsuit^*_{S_2} \Rightarrow \diamondsuit^*_{S_1}$ (hence $\diamondsuit_{S_1}$). But the assumption $\mathbf{V} = \mathbf{L}$ is heavy. Trying to avoid it, we can walk in several directions. On weaker relatives see [12] and references there. We can also use other methods, aiming to prove the diamond without assuming $\mathbf{V} = \mathbf{L}$.

Received by the editors March 24, 2008, and, in revised form, July 7, 2008; February 26, 2009; and October 15, 2009.

2010 *Mathematics Subject Classification.* Primary 03E04; Secondary 03E05, 03E35.

This research was supported by the United-States-Israel Binational Science Foundation (Grant No. 2002323), Publication No. 922. The author thanks Alice Leonhardt for the beautiful typing.

There is another formulation of the diamond principle, phrased via functions (instead of sets). Since we use this version in our proof, we introduce the following:

Definition 1.3. $\diamondsuit_S$ (the functional version).

Assume $\lambda = \text{cf}(\lambda) > \aleph_0, S \subseteq \lambda, S$ is stationary. $\diamondsuit_S$ holds if there exists a diamond sequence $\langle g_\delta : \delta \in S\rangle$ which means that $g_\delta \in {}^\delta\delta$ for every $\delta \in S$, and for every $g \in {}^\lambda\lambda$ the set $\{\delta \in S : g{\restriction}\delta = g_\delta\}$ is a stationary subset of λ.

By Gregory [4] and Shelah [6] we know that assuming $\lambda = \chi^+ = 2^\chi$ and $\kappa = \text{cf}(\kappa) \neq \text{cf}(\chi)$, $\kappa < \lambda$, and that G.C.H. holds (or actually just $\chi^\kappa = \chi$ or $(\forall\alpha < \chi)(|\alpha|^\kappa < \chi) \wedge \text{cf}(\chi) < \kappa)$, then $\diamondsuit^*_{S^\lambda_\kappa}$ holds (recall that $S^\lambda_\kappa = \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$).

We also have results which show that the failures of the diamond above a strong limit cardinal are limited. For instance, if $\lambda = \chi^+ = 2^\chi > \mu$ and $\mu > \aleph_0$ is a strong limit, then (by [8]) the set $\{\kappa < \mu : \diamondsuit^*_{S^\lambda_\kappa} \text{ fails}\}$ is bounded in μ (recall that κ is regular). Note that the result here does not completely subsume the earlier results when $\lambda = 2^\chi = \chi^+$, as we get a "diamond on every stationary set $S \subseteq \lambda \setminus S^\lambda_{\text{cf}(\chi)}$" but not on $\diamondsuit^*_S$; this is inherent as noted in Observation 3.4. In [12], a similar, stronger result is proved for $\diamondsuit_{S^\lambda_\kappa}$: for every $\lambda = \chi^+ = 2^\chi > \mu$, μ a strong limit for some <u>finite</u> $\mathfrak{d} \subseteq$ Reg $\cap\, \mu$, for every regular $\kappa < \mu$ not from $\mathfrak{d}$ we have $\diamondsuit_{S^\lambda_\kappa}$, and even $\diamondsuit_S$ for "most" stationary $S \subseteq S^\lambda_\kappa$. In fact, for the relevant good stationary sets $S \subseteq S^\lambda_\kappa$ we get $\diamondsuit^*_S$. Also weaker related results are proved there for other regular λ (mainly $\lambda = \text{cf}(2^\chi)$).

The present work does not resolve:

Problem 1.4. Assume χ is singular and that $\lambda = \chi^+ = 2^\chi$. Do we have $\diamondsuit_{S^\lambda_{\text{cf}(\chi)}}$? (You may even assume G.C.H.)

However, the full analog result for Problem 1.4 consistently fails; see [7] or [10]. That is, if G.C.H., $\chi > \text{cf}(\chi) = \kappa$, then we can force a non-reflecting stationary $S \subseteq S^{\chi^+}_\kappa$ such that the diamond on S fails and cardinalities and cofinalities are preserved; also G.C.H. continue to hold. But if χ is a strong limit, $\lambda = \chi^+ = 2^\chi$, we still know something on guessing equalities for every stationary $S \subseteq S^\lambda_\kappa$; see [10].

Note that this S (by [7], [10]) in some circumstances has to be "small"

- $(*)$ if (χ is singular, $2^\chi = \chi^+ = \lambda, \kappa = \text{cf}(\chi)$ and) we have the square $\square_\chi$ (i.e. there exists a sequence $\langle C_\delta : \delta < \lambda, \delta \text{ is a limit ordinal}\rangle$ so that C_δ is closed and unbounded in δ, $\text{cf}(\delta) < \chi \Rightarrow |C_\delta| < \chi$, and if γ is a limit point of C_δ, then $C_\gamma = C_\delta \cap \gamma$), <u>then</u> $\diamondsuit_{S^\lambda_\kappa}$ holds. Moreover, if $S \subseteq S^\lambda_\kappa$ reflects in a stationary set of $\delta < \lambda$, then $\diamondsuit_S$ holds; see [7, §3].

Also note that our results are of the form "$\diamondsuit_S$ for every stationary $S \subseteq S^*$" for suitable $S^* \subseteq \lambda$. Usually this was deduced from the stronger statement $\diamondsuit^*_S$. However, the results on $\diamondsuit^*_S$ cannot be improved; see Observation 3.4.

Also, if χ is regular we cannot improve the result to $\diamondsuit_{S^\lambda_\chi}$ (see [7] or [9]), even assuming G.C.H. Furthermore, the question on $\diamondsuit_{\aleph_2}$ when $2^{\aleph_1} = \aleph_2 = 2^{\aleph_0}$ was raised. Concerning this we show in Claim 3.2 that $\diamondsuit_{S^{\aleph_2}_{\aleph_1}}$ may fail (this works in other cases, too).

Question 1.5. Can we deduce any ZFC result on λ strongly inaccessible?

By Džamonja-Shelah [2] we know that failure of SNR helps (SNR stands for strong non-reflection); a parallel here is Claim 2.3(2).

For $\lambda = \lambda^{<\lambda} = 2^\mu$ weakly inaccessible, we do not know if the diamond holds (in ZFC). Nevertheless, we have proved (in [11] and [5]) that the weaker version of the diamond (as formulated in Definition 3.5(2)) holds in this case. Again failure of SNR helps.

Regarding consistency results on SNR see Cummings-Džamonja-Shelah [1] and Džamonja-Shelah [3].

Notation 1.6. 1) If $\kappa = \mathrm{cf}(\kappa) < \lambda = \mathrm{cf}(\lambda)$, then we let $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

2) $\mathcal{D}_\lambda$ is the club filter on λ for λ a regular uncountable cardinal.

2. Diamond on successor cardinals

Recall (needed only for part (2) of Claim 2.3):

Definition 2.1. We say λ on S has κ-SNR or $\mathrm{SNR}(\lambda, S, \kappa)$ or λ has strong non-reflection for S in κ <u>when</u> $S \subseteq S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$, so $\lambda = \mathrm{cf}(\lambda) > \kappa = \mathrm{cf}(\kappa)$. Also, there are $h : \lambda \to \kappa$ and a club E of λ such that for every $\delta \in S \cap E$ for some club C of δ, the function $h \restriction C$ is one-to-one and even increasing (note that without loss of generality $\alpha \in \mathrm{nacc}(E) \Rightarrow \alpha$ is a successor and without loss of generality $E = \lambda$, so $\mu \in \mathrm{Reg} \cap \lambda \backslash \kappa^+ \Rightarrow \mathrm{SNR}(\mu, S \cap \mu, \kappa)$). If $S = S^\lambda_\kappa$ we may omit it.

Remark 2.2. Note by Fodor's lemma that if $\mathrm{cf}(\delta) = \kappa > \aleph_0$ and h is a function from some set $\supseteq \delta$ and the range of h is $\subseteq \kappa$, <u>then</u> the following conditions are equivalent:

- (a) h is one-to-one on some club of δ,
- (b) h is increasing on some club of δ,
- (c) $\mathrm{Rang}(h \restriction S)$ is unbounded in κ for every stationary subset S of δ.

Our main theorem is:

Claim 2.3. Assume $\lambda = 2^\chi = \chi^+$.

1) If $S \subseteq \lambda$ is stationary and $\delta \in S \Rightarrow \mathrm{cf}(\delta) \neq \mathrm{cf}(\chi)$, <u>then</u> $\diamondsuit_S$ holds.

2) If $\aleph_0 < \kappa = \mathrm{cf}(\chi) < \chi$ and $\diamondsuit_S$ fails where $S = S^\lambda_\kappa$ (or just $S \subseteq S^\lambda_\kappa$ is a stationary subset of λ), <u>then</u> $\mathrm{SNR}(\lambda, \kappa)$ or just λ has a strong non-reflection for $S \subseteq S^\lambda_\kappa$ in κ.

Definition 2.4. 1) For a filter D on a set I let $\mathrm{Dom}(D) := I$, and S is called D-positive when $S \subseteq I \wedge (I \backslash S) \notin D$ and $D^+ = \{S \subseteq \mathrm{Dom}(D) : S \text{ is } D\text{-positive}\}$. Also, we let $D + A = \{B \subseteq I : B \cup (I \backslash A) \in D\}$ (so if $D = \mathcal{D}_\lambda$, the club filter on the regular uncountable λ, then D^+ is the family of stationary subsets of X).

2) For D a filter on a regular uncountable cardinal λ which extends the club filter, let $\diamondsuit_D$ mean: there is $\bar{f} = \langle f_\alpha : \alpha \in S \rangle$ which is a diamond sequence for D (or a D-diamond sequence), which means that $S \in D^+$ and for every $g \in {}^\lambda\lambda$ the set $\{\alpha < \lambda : g \restriction \alpha = f_\alpha\}$ belongs to D^+, so $\bar{f}$ is also a diamond sequence for the filter $D + S$ (clearly $\diamondsuit_S$ means $\diamondsuit_{\mathcal{D}_\lambda + S}$ for S a stationary subset of the regular uncountable λ).

A somewhat more general version of the theorem is

Claim 2.5. 1) Assume $\lambda = \chi^+ = 2^\chi$ and D is a λ-complete filter on λ which extends the club filter. If $S \in D^+$ and $\delta \in S \Rightarrow \mathrm{cf}(\delta) \neq \mathrm{cf}(\chi)$, <u>then</u> we have $\diamondsuit_{D+S}$.

2) We have $\diamondsuit_D$ <u>when</u>:

- (a) $\lambda = \lambda^{<\lambda}$,

(b) $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ lists $\cup\{{}^\alpha\lambda : \alpha < \lambda\}$,

(c) $S \in D^+$,

(d) $\bar{u} = \langle u_\alpha : \alpha \in S\rangle$ and $u_\alpha \subseteq \alpha$ for every $\alpha \in S$,

(e) $\chi = \sup\{|u_\alpha|^+ : \alpha < \lambda\} < \lambda$,

(f) D is a χ^+-complete filter on λ extending the club filter,

(g) $(\forall g \in {}^\lambda\lambda)(\exists^{D^+}\delta \in S)[\delta = \sup\{\alpha \in u_\delta : g{\restriction}\alpha \in \{f_\beta : \beta \in u_\delta\}\}]$.

3) Assume $\lambda = \chi^+ = 2^\chi$ and $\aleph_0 < \kappa = \mathrm{cf}(\chi) < \chi$, $S \subseteq S^\lambda_\kappa$ is stationary, and D is a λ-complete filter extending the club filter on λ to which S belongs. If $\diamondsuit_D$ fails, then $\mathrm{SNR}(\lambda, S, \kappa)$.

Proof of Claim 2.3. Part (1) follows from Claim 2.5(1) for D the filter $\mathcal{D}_\lambda + S$. Part (2) follows from Claim 2.5(3) for D the filter $\mathcal{D}_\lambda + S$. $\square$

Proof of Claim 2.5. Proof of part (1).

Clearly we can assume

$\circledast_0$ $\chi > \aleph_0$, as for $\chi = \aleph_0$ the statement is empty.

Let

$\circledast_1$ $\langle f_\alpha : \alpha < \lambda\rangle$ list the set $\{f : f$ is a function from β to λ for some $\beta < \lambda\}$.

For each $\alpha < \lambda$ clearly $|\alpha| \le \chi$, so let

$\circledast_{2,\alpha}$ $\langle u_{\alpha,\varepsilon} : \varepsilon < \chi\rangle$ be $\subseteq$-increasing continuous with union α such that $\varepsilon < \chi \Rightarrow |u_{\alpha,\varepsilon}| \le \aleph_0 + |\varepsilon| < \chi$.

For $g \in {}^\lambda\lambda$ let $h_g \in {}^\lambda\lambda$ be defined by

$\circledast_{3,g}$ $h_g(\alpha) = \mathrm{Min}\{\beta < \lambda : g \restriction \alpha = f_\beta\}$.

Let cd, $\langle \mathrm{cd}_\varepsilon : \varepsilon < \chi\rangle$ be such that

$\circledast_4$ (a) cd is a one-to-one function from ${}^\chi\lambda$ onto λ such that $\mathrm{cd}(\bar{\alpha}) \ge \sup\{\alpha_\varepsilon : \varepsilon < \chi\}$ (when $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon < \chi\rangle$),

(b) for $\varepsilon < \chi$, cd_ε is a function from λ to λ such that $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon < \chi\rangle \in {}^\chi\lambda \Rightarrow \mathrm{cd}_\varepsilon(\mathrm{cd}(\bar{\alpha})) = \alpha_\varepsilon$

(they exist as $\lambda = \lambda^\chi$; in the present case this holds as $2^\chi = \chi^+ = \lambda$).

Now we let (for $\beta < \lambda, \varepsilon < \chi$)

$\circledast_5$ $f^1_{\beta,\varepsilon}$ be the function from $\mathrm{Dom}(f_\beta)$ into λ defined by $f^1_{\beta,\varepsilon}(\alpha) = \mathrm{cd}_\varepsilon(f_\beta(\alpha))$, so $\mathrm{Dom}(f^1_{\beta,\varepsilon}) = \mathrm{Dom}(f_\beta)$.

Without loss of generality

$\circledast_6$ $\alpha \in S \Rightarrow \alpha$ is a limit ordinal.

For $g \in {}^\lambda\lambda$ and $\varepsilon < \chi$ we let

$$S^\varepsilon_g = \Big\{\delta \in S : \quad \delta = \sup\{\alpha \in u_{\delta,\varepsilon} : \text{ for some } \beta \in u_{\delta,\varepsilon} \text{ we have } g \restriction \alpha = f^1_{\beta,\varepsilon}\}\Big\}.$$

Next we shall show

$\circledast_7$ for some $\varepsilon(*) < \chi$ and for every $g \in {}^\lambda\lambda$ the set $S^{\varepsilon(*)}_g$ is a D-positive subset of λ.

Proof of $\circledast_7$. Assume this fails, so for every $\varepsilon < \chi$ there is $g_\varepsilon \in {}^\lambda\lambda$ such that $S^\varepsilon_{g_\varepsilon}$ is not D-positive and let E_ε be a member of D disjoint to $S^\varepsilon_{g_\varepsilon}$. Define $g \in {}^\lambda\lambda$ by

$g(\alpha) :=$ cd$(\langle g_\varepsilon(\alpha) : \varepsilon < \chi\rangle)$ and let $h_g \in {}^\lambda\lambda$ be as in $\circledast_{3,g}$; i.e. $h_g(\alpha) =$ Min$\{\beta : g \restriction \alpha = f_\beta\}$.

Let $E_* = \{\delta < \lambda : \delta$ is a limit ordinal such that $\alpha < \delta \Rightarrow h_g(\alpha) < \delta\}$. Clearly it is a club of λ; hence it belongs to D, and so $E = \cap\{E_\varepsilon : \varepsilon < \chi\} \cap E_*$ belongs to D as D is λ-complete and $\chi + 1 < \lambda$.

As S is a D-positive subset of λ there is $\delta_* \in E \cap S$. For each $\alpha < \delta_*$ as $\delta_* \in E \subseteq E_*$, clearly $h_g(\alpha) < \delta_*$, and α as well as $h_g(\alpha)$ belong to $\cup\{u_{\delta_*,\varepsilon} : \varepsilon < \chi\} = \delta_*$. However, $\langle u_{\delta_*,\varepsilon} : \varepsilon < \chi\rangle$ is $\subseteq$-increasing; hence $\varepsilon_{\delta_*,\alpha} =$ min$\{\varepsilon : \alpha \in u_{\delta_*,\varepsilon}$ and $h_g(\alpha) \in u_{\delta_*,\varepsilon}\}$ is not just well defined but also $\varepsilon \in [\varepsilon_{\delta_*,\alpha}, \chi) \Rightarrow \{\alpha, h_g(\alpha)\} \subseteq u_{\delta_*,\varepsilon}$. As cf$(\delta_*) \neq$ cf(χ), by an assumption on S, it follows that for some $\varepsilon(*) < \chi$ the set $B := \{\alpha < \delta_* : \varepsilon_{\delta_*,\alpha} < \varepsilon(*)\}$ is unbounded below δ_*.

So

(a) $\alpha \in B \Rightarrow \{\alpha, h_g(\alpha)\} \subseteq u_{\delta_*,\varepsilon(*)}$ and

(b) $\alpha \in B \Rightarrow g{\restriction}\alpha = f_{h_g(\alpha)} \Rightarrow \bigwedge\limits_{\varepsilon<\chi} [g_\varepsilon{\restriction}\alpha = f^1_{h_g(\alpha),\varepsilon}] \Rightarrow g_{\varepsilon(*)}{\restriction}\alpha = f^1_{h_g(\alpha),\varepsilon(*)}$.

But $\delta_* \in E \subseteq E_{\varepsilon(*)}$; hence $\delta_* \notin S^{\varepsilon(*)}_{g_{\varepsilon(*)}}$ by the choice of $E_{\varepsilon(*)}$. But by $(a) + (b)$ and the definition of $S^{\varepsilon(*)}_{g_{\varepsilon(*)}}$ recalling $\delta_* \in S$, we have $\sup(B) = \delta_* \Rightarrow \delta_* \in S^{\varepsilon(*)}_{g_{\varepsilon(*)}}$ (where $h_g(\alpha)$ plays the role of β in the definition of S^ε_g above), a contradiction. So the proof of $\circledast_7$ is finished.

Let $\chi_* = (|\varepsilon(*)| + \aleph_0)$; hence $\delta \in S \Rightarrow |u_{\delta,\varepsilon(*)}| \le \chi_*$ and $\chi_*^+ < \lambda$ as $\chi_* < \chi < \lambda$ because $\aleph_0, \varepsilon(*) < \chi < \lambda$. Now we apply Claim 2.5(2), which is proved below with $\lambda, S, D, \chi_*^+, \langle f^1_{\beta,\varepsilon(*)} : \beta < \lambda\rangle, \langle u_{\delta,\varepsilon(*)} : \delta \in S\rangle$ here standing for $\lambda, S, D, \chi, \bar f, \bar u$ there. The conditions there are satisfied, hence also the conclusion which says that $\diamondsuit_D$ holds. $\square$

Proof of Claim 2.5(2). Let

$\boxtimes_1$ $\langle \mathrm{cd}_\varepsilon : \varepsilon < \chi\rangle$ and cd be as in $\circledast_4$ in the proof of part (1), possible as we are assuming $\chi < \lambda = \lambda^{<\lambda}$;

$\boxtimes_2$ for $\beta < \lambda$ and $\zeta < \chi$ let $f^2_{\beta,\zeta}$ be the function with domain Dom(f_β) such that $f^2_{\beta,\zeta}(\alpha) =$ cd$_\zeta(f_\beta(\alpha))$;

$\boxtimes_3$ for $g \in {}^\lambda\lambda$ define $h_g \in {}^\lambda\lambda$ as in $\circledast_3$ in the proof of part (1), i.e. $h_g(\alpha) =$ Min$\{\beta : g{\restriction}\alpha = f_\beta\}$.

If $2^{<\chi} < \lambda$ our life would be easier, but we do not assume this. For $\delta \in S$ let ξ^*_δ be a cardinal, and let $\langle(\alpha^1_{\delta,\xi}, \alpha^2_{\delta,\xi}) : \xi < \xi^*_\delta\rangle$ list the set $\{(\alpha_1, \alpha_2) \in u_\delta \times u_\delta :$ Dom$(f_{\alpha_2}) = \alpha_1\}$. Note that $\xi^*_\delta < \chi$, recalling $|u_\delta| < \chi$ by clause (e) of the assumption. We now try to choose $(\bar v_\varepsilon, g_\varepsilon, E_\varepsilon)$ by induction on $\varepsilon < \chi$ (note that $\bar v_\varepsilon$ is defined from $\langle g_\zeta : \zeta < \varepsilon\rangle$ (see clause (e) of $\boxtimes_4$ below), so we choose just $(g_\varepsilon, E_\varepsilon)$) such that

$\boxtimes_4$ (a) E_ε is a member of D and $\langle E_\zeta : \zeta \le \varepsilon\rangle$ is $\subseteq$-decreasing with ζ;

(b) $\bar v_\varepsilon = \langle v^\varepsilon_\delta : \delta \in S \cap E'_\varepsilon\rangle$ when $E'_\varepsilon = \cap\{E_\zeta : \zeta < \varepsilon\} \cap \lambda$, so is λ if $\varepsilon = 0$;

(c) $\langle v^\zeta_\delta : \zeta \le \varepsilon\rangle$ is $\subseteq$-decreasing with ζ for each $\delta \in S \cap E'_\varepsilon$;

(d) $g_\varepsilon \in {}^\lambda\lambda$;

(e) $v^\varepsilon_\delta = \{\xi < \xi^*_\delta$: if $\zeta < \varepsilon$, then $g_\zeta \restriction \alpha^1_{\delta,\xi} = f^2_{\alpha^2_{\delta,\xi},\zeta}\}$

(so if ε is a limit ordinal then $v^\varepsilon_\delta = \bigcap\limits_{\zeta<\varepsilon} v^\zeta_\delta$ and $\varepsilon = 0 \Rightarrow v^\varepsilon_\delta = \xi^*_\delta$);

(f) if $\delta \in E'_\varepsilon \cap S$, <u>then</u> $v^{\varepsilon+1}_\delta \subsetneqq v^\varepsilon_\delta$ or $\delta > \sup\{\alpha^1_{\delta,\xi} : \xi \in v^{\varepsilon+1}_\delta\}$.

Next

$\oplus_1$ we cannot carry the induction, that is for all $\varepsilon < \chi$.

Why? Assume by contradiction that $\langle(\bar{v}_\varepsilon, g_\varepsilon, E_\varepsilon) : \varepsilon < \chi\rangle$ is well defined. Let $E := \bigcap\{E_\varepsilon : \varepsilon < \chi\}$; it is a member of D as D is χ^+-complete. Define $g \in {}^\lambda\lambda$ by $g(\alpha) :=$ cd$(\langle g_\varepsilon(\alpha) : \varepsilon < \chi\rangle)$. Let $E_* = \{\delta < \lambda : \delta$ a limit ordinal such that $h_g(\alpha) < \delta$ and $\delta > \sup(\mathrm{Dom}(f_\alpha) \cup \mathrm{Rang}(f_\alpha))$ for every $\alpha < \delta\}$, so E_* is a club of λ; hence it belongs to D. By assumption (g) of the claim the set

$$S_g := \{\delta \in S : \delta = \sup\{\alpha \in u_\delta : (\exists\beta \in u_\delta)(f_\beta = g \restriction \alpha)\}\}$$

is D-positive, so we can choose $\delta \in E \cap E_* \cap S_g$. Hence $B := \{\alpha \in u_\delta : (\exists\beta \in u_\delta)(f_\beta = g{\restriction}\alpha)\}$ is an unbounded subset of u_δ and let $h : B \to u_\delta$ be $h(\alpha) = \min\{\beta \in u_\delta : f_\beta = g{\restriction}\alpha\}$; clearly h is a function from B into u_δ. Now $\alpha \in B \wedge \zeta < \chi \Rightarrow f_{h(\alpha)} = g{\restriction}\alpha \wedge \zeta < \chi \Rightarrow f^2_{h(\alpha),\zeta} = g_\zeta{\restriction}\alpha$, so for $\alpha \in B$ the pair $(\alpha, h(\alpha))$ belongs to $\{(\alpha^1_{\delta,\xi}, \alpha^2_{\delta,\xi}) : \xi \in v^\varepsilon_\delta\}$ for every $\varepsilon < \chi$. Hence for any $\varepsilon < \chi$ we have $B \subseteq \{\alpha^1_{\delta,\xi} : \xi \in v^\varepsilon_\delta\}$, so $\delta = \sup\{\alpha^1_{\delta,\xi} : \xi \in v^\varepsilon_\delta\}$.

So for the present δ, in clause (f) of $\boxtimes_4$ the second possibility never occurs.

So clearly $\langle v^\varepsilon_{\delta_*} : \varepsilon < \chi\rangle$ is strictly $\subseteq$-decreasing, i.e. is $\subset$-decreasing, which is impossible as $|v^0_{\delta_*}| = \xi^*_{\delta_*} < \chi$. So we have proved $\oplus_1$; hence we can assume

- $\oplus_2$ there is $\varepsilon < \chi$ such that we have defined our triple for every $\zeta < \varepsilon$, but we cannot define it for ε. So we have $\langle(\bar{v}_\zeta, g_\zeta, E_\zeta) : \zeta < \varepsilon\rangle$.

As in $\boxplus_4(e)$, let

- $\odot_1$ E'_ε be λ if $\varepsilon = 0$ and $\bigcap\{E_\zeta : \zeta < \varepsilon\}$ if $\varepsilon > 0$, and let $S_* := S \cap E'_\varepsilon$.

Clearly $\bar{v}_\varepsilon$ is well defined (see clauses (b) and (e) of $\boxtimes_4$), and for $\delta \in S_*$ let $\mathcal{F}_\delta = \{f^2_{\alpha^2_{\delta,\xi},\varepsilon} : \xi \in v^\varepsilon_\delta\}$, so each member is a function from some $\alpha \in u_\delta \subseteq \delta$ into some ordinal $< \delta$.

Let

- $\odot_2$ $S^*_1 := \{\delta \in S_* :$ there are $f', f'' \in \mathcal{F}_\delta$ which are incompatible as functions$\}$,
- $\odot_3$ $S^*_2 := \{\delta \in S_* : \delta \notin S^*_1$, but the function $\bigcup\{f : f \in \mathcal{F}_\delta\}$, has domain $\neq \delta\}$,
- $\odot_4$ $S^*_3 = S_* \backslash (S^*_1 \cup S^*_2)$.

For $\delta \in S^*_3$ let $g^*_\delta = \bigcup\{f : f \in \mathcal{F}_\delta\}$, so by the definition of $\langle S^*_\ell : \ell = 1, 2, 3\rangle$, clearly $g^*_\delta \in {}^\delta\delta$. Now if $\langle g^*_\delta : \delta \in S^*_3\rangle$ is a diamond sequence for D, we are done.

Assume that this fails. So for some $g \in {}^\lambda\lambda$ and member E of D we have $\delta \in S^*_3 \cap E \Rightarrow g^*_\delta \neq g \restriction \delta$. Without loss of generality E is included in E'_ε. But then we could have chosen (g, E) as $(g_\varepsilon, E_\varepsilon)$, recalling that $\bar{v}_\varepsilon$ was already chosen. Easily the triple $(g_\varepsilon, E_\varepsilon, \bar{v}_\varepsilon)$ is as required in $\oplus_1$, contradicting the choice of ε in $\oplus_2$, so we are done proving part (2) of Claim 2.3 and hence also part (1). $\square$

Proof of Claim 2.5(3). We use cd, cd$_\varepsilon$ (for $\varepsilon < \chi$), $\langle\langle u_{\alpha,\varepsilon} : \varepsilon < \chi\rangle : \alpha < \lambda\rangle, \langle f_\alpha : \alpha < \lambda\rangle, \langle f^1_{\alpha,\varepsilon} : \alpha < \lambda, \varepsilon < \chi\rangle$ and S^ε_g for $\varepsilon < \kappa$ as in the proof of part (1).

Recall that κ, a regular uncountable cardinal, is the cofinality of the singular cardinal χ and let $\langle\chi_\gamma : \gamma < \kappa\rangle$ be increasing with limit χ. For every $\gamma < \kappa$ we ask:

The γ-Question. For every $g \in {}^\lambda\lambda$, do we have that the following is a D-positive subset of λ?

$\{\delta \in S : S_\gamma[g] \cap \delta$ is a stationary subset of $\delta\}$, where $S_\gamma[g] := \{\zeta < \lambda$: cf$(\zeta) \in [\aleph_0, \kappa)$, $\sup(u_{\zeta,\chi_\gamma}) = \zeta$, and for arbitrarily large $\alpha \in u_{\zeta,\chi_\gamma}$ for some $\beta \in u_{\zeta,\chi_\gamma}$ and $\varepsilon < \chi_\gamma$, we have $\mathrm{Dom}(f_\beta) = \alpha$ and $g \restriction \alpha = f^1_{\beta,\varepsilon}\}$.

Case 1. For some $\gamma < \kappa$, the answer is yes.

Choose $\langle C_\delta : \delta \in S\rangle$ such that C_δ is a club of δ of order type $\mathrm{cf}(\delta) = \kappa$.

For $\delta \in S \subseteq S^\lambda_\kappa$ let $u_\delta := \bigcup\{u_{\alpha,\chi_\gamma} : \alpha \in C_\delta\}$.

Clearly,

$\boxplus_2$ $|u_\delta| \le \kappa + \chi_\gamma < \chi$;

$\boxplus_3$ for every $g \in {}^\lambda\lambda$ for D-positively many $\delta \in S$, we have $\delta = \sup\{\alpha \in u_\delta : g{\restriction}\alpha \in \{f^1_{\beta,\varepsilon} : \varepsilon < \chi_\gamma \text{ and } \beta \in u_\delta\}\}$.

Why does $\boxplus_3$ hold? Given $g \in {}^\lambda\lambda$, let $h_g \in {}^\lambda\lambda$ be defined by $h_g(\alpha) = \min\{\beta < \lambda : g{\restriction}\alpha = f_\beta\}$, so $h_g(\alpha) \ge \alpha$ (but is less than λ). Let $E_g = \{\delta < \lambda : \delta$ is a limit ordinal such that $(\forall\alpha < \delta) h_g(\alpha) < \delta\}$, so E_g is a club of λ, and let E'_g be the set of accumulation points of E_g, so that E'_g, too, is a club of λ. By the assumption of this case, the set $S' := \{\delta \in S : \delta \cap S_\gamma[g]$ is a stationary subset of $\lambda\}$ is D-positive; hence $S'' := S' \cap E'_g$ is a D-positive subset of λ. Let $\delta \in S''$. By E'_g's definition, we can find that $B^0_\delta \subseteq E_g \cap \delta$ unbounded in δ, so without loss of generality B^0_δ is closed. But $S_\gamma[g] \cap \delta$ is a stationary subset of δ, recalling $\delta \in S''$, so $B^1_\delta = B^0_\delta \cap S_\gamma[g] \cap C_\delta$ is a stationary subset of δ as B^0_δ, C_δ are closed unbounded subsets of δ.

Clearly $\zeta \in B^1_\delta \Rightarrow \zeta \in C_\delta \Rightarrow u_{\zeta,\chi_\gamma} \subseteq u_\delta$ by the definitions of B^1_δ and u_δ. Also, $\zeta \in B^1_\delta \Rightarrow \zeta \in S_\gamma[g] \Rightarrow$ (ζ is a limit ordinal) $\wedge \zeta = \sup(u_{\zeta,\chi_\gamma}) = \sup\{\alpha \in u_{\zeta,\chi_\gamma} : (\exists\beta \in u_{\zeta,\chi_\gamma})(\exists\varepsilon < \chi_\gamma)(g{\restriction}\alpha = f^1_{\beta,\varepsilon})\} \Rightarrow$ ((ζ is a limit ordinal) $\wedge \zeta = \sup\{\alpha \in u_\delta \cap \zeta : (\exists\beta \in u_\delta\backslash\alpha)(\exists\varepsilon < \chi_\gamma)(g{\restriction}\alpha = f^1_{\beta,\varepsilon})\}$).

As B^1_δ is unbounded in δ being stationary, we are done proving $\boxplus_3$.

Now without loss of generality every $\delta \in S$ is divisible by χ; hence $\delta = \chi_\gamma\delta$ and let $u'_\delta = u_\delta \cup \{\chi_\gamma\alpha + \varepsilon : \alpha \in u_\delta, \varepsilon < \chi_\gamma\}$, so u_δ is an unbounded subset of δ, and let $f'_\beta = f^1_{\alpha,\varepsilon}$ when $\beta = \chi_\gamma\alpha + \varepsilon, \varepsilon < \chi_\gamma$. So translating what we have:

$\boxplus_4$ (a) $\langle f'_\alpha : \alpha < \lambda\rangle$ is a sequence of members of $\bigcup\{{}^\beta\lambda : \beta < \lambda\}$,

(b) for $\delta \in S, u'_\delta$ is an unbounded subset of δ of cardinality $\le \chi_\gamma \times \chi_\gamma = \chi_\gamma (< \chi)$,

(c) for every $g \in {}^\lambda\lambda$ for D-positively many $\delta \in S$, we have $\delta = \sup\{\alpha \in u'_\delta : (\exists\beta \in u'_\delta)(g{\restriction}\alpha = f'_\beta)\}$.

Now we can apply part (2) with $\langle f'_\alpha : \alpha < \lambda\rangle, \langle u'_\delta : \delta \in S\rangle$ replacing $\bar f, \langle u_\delta : \delta \in S\rangle$.

Therefore we can prove $\diamondsuit_S$ and we are done.

Case 2. For every $\gamma < \kappa$ the answer is no.

Let (g_γ, E_γ) exemplify that the answer for γ is no, so $g_\gamma \in {}^\lambda\lambda$ and $E_\gamma \in D$. Let $E = \bigcap\limits_{\gamma<\kappa} E_\gamma$, so E is a member of D. Let $g \in {}^\lambda\lambda$ be defined by $g(\alpha) = \mathrm{cd}(\langle g_\gamma(\alpha) : \gamma < \kappa\rangle{}^\frown(0)_\chi)$; i.e. $\mathrm{cd}_\varepsilon(g(\alpha))$ is $g_\gamma(\alpha)$ if $\gamma < \kappa$ and is 0 if $\varepsilon \in [\kappa, \chi)$.

Let

$$E_g := \{\delta < \lambda : \delta \text{ a limit ordinal such that if } \alpha < \lambda, \text{ then } h_g(\alpha) < \delta \text{ and } \delta > \sup(\mathrm{Dom}(f_\alpha) \cup \mathrm{Rang}(f_\alpha))\}.$$

We now define $h : \lambda \to \kappa$ as follows:

$\boxplus_5$ For $\beta < \lambda$

(a) if $\mathrm{cf}(\beta) \notin [\aleph_0, \kappa)$ or $\beta \notin E_g$, then $h(\beta) = 0$;

(b) otherwise

$$h(\beta) = \min\{\gamma < \kappa : \beta = \sup\{\alpha_1 \in u_{\beta,\chi_\gamma} : \text{ for some } \alpha_2 \in u_{\beta,\chi_\gamma} \text{ and } \varepsilon < \chi_\gamma \text{ we have } g \restriction \alpha_1 = f^1_{\alpha_2,\varepsilon}\}\}.$$

Now

$\boxplus_6$ $h : \lambda \to \kappa$ is well defined.

Why does $\boxplus_6$ hold? Let $\beta < \lambda$. If $\mathrm{cf}(\beta) \notin [\aleph_0, \kappa)$ or $\beta \notin E_g$, then $h(\alpha) = 0 < \kappa$ by clause (a) of $\boxplus_5$. So assume $\mathrm{cf}(\beta) \in [\aleph_0, \kappa)$ and $\beta \in E_g$. Let $\langle \gamma^1_{\beta,\varepsilon} : \varepsilon < \mathrm{cf}(\beta)\rangle$ be increasing with limit β and let $\gamma^2_{\beta,\varepsilon} = \min\{\gamma : g{\restriction}\gamma^1_{\beta,\varepsilon} = f_\gamma\}$, so $\varepsilon < \mathrm{cf}(\beta) \Rightarrow \gamma^2_{\beta,\varepsilon} < \beta$ as $\beta \in E_g$. But $\langle u_{\beta,\chi_\zeta} : \zeta < \mathrm{cf}(\chi)\rangle$ is $\subseteq$-increasing with union β, so for each $\varepsilon < \mathrm{cf}(\beta)$ there is $\zeta = \zeta_{\beta,\varepsilon} < \mathrm{cf}(\chi)$ such that $\{\gamma^1_{\beta,\varepsilon}, \gamma^2_{\beta,\varepsilon}\} \subseteq u_{\beta,\chi_\zeta}$. As $\mathrm{cf}(\beta) < \kappa = \mathrm{cf}(\chi)$ for some $\zeta < \kappa$, the set $\{\varepsilon < \mathrm{cf}(\beta) : \zeta_{\beta,\varepsilon} < \zeta\}$ is unbounded in $\mathrm{cf}(\beta)$. So ζ can serve as γ in clause (b) of $\boxplus_5$, so $h(\beta)$ is well defined. In particular, it is less than κ, so we have proved $\boxplus_6$.

$\boxplus_7$ If $\delta \in S \cap E_\gamma$, then for some club C of δ the function $h \restriction C$ is increasing.

Why does $\boxplus_7$ hold? If not, then by Fodor's lemma for some $\gamma < \kappa$ the set $\{\delta' \in \delta \cap S : h(\delta') \leq \gamma\}$ is a stationary subset of δ, and we get a contradiction to the choice of E_γ so $\boxplus_7$ holds indeed.

So h is as promised in the claim. $\square$

Note that

Observation 2.6. If $\kappa_* < \lambda$ are regular, $S^\lambda_{\kappa_*}$ strongly does not reflect in λ for every $\kappa \in \mathrm{Reg} \cap \kappa_*$ and $\Pi(\mathrm{Reg} \cap \kappa_*) < \lambda$, <u>then</u>:

- (a) $S^\lambda_{<\kappa_*}$ can be divided to $\leq \Pi(\mathrm{Reg} \cap \kappa_*)$ sets, each not reflecting any $\delta \in S^\lambda_{<\kappa_*}$; in particular,
- (b) $S^\lambda_{\aleph_0}$ can be divided to $\leq \Pi(\mathrm{Reg} \cap \kappa_*)$ sets each not reflecting any $\delta \in S^\lambda_{<\kappa_*}$.

Remark 2.7. 1) Of course if λ has κ-SNR, then this holds for every regular $\lambda' \in (\kappa, \lambda)$.

2) We may state the results using λ^*_κ (see below).

Definition 2.8. For each regular κ let $\lambda^*_\kappa = \mathrm{Min}\{\lambda : \lambda$ regular fails to have κ-SNR$\}$, and let λ^*_κ be ∞ (or not defined) if there is no such λ.

3. Consistent failure on S^2_1

A known question was:

Question 3.1. For $\theta \in \{\aleph_0, \aleph_1\}$ do we have $(2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 \Rightarrow \diamondsuit_{S^{\aleph_2}_\theta})$?

So for $\theta = \aleph_0$ the answer is yes (by Claim 2.3(1)), but what about $\theta = \aleph_1$? We noted some years ago the following:

Claim 3.2. Assume $\mathbf{V} \models \mathrm{GCH}$ or even just $2^{\aleph_\ell} = \aleph_{\ell+1}$ for $\ell = 0, 1, 2$. <u>Then</u> some forcing notion $\mathbb{P}$ satisfies

- (a) $\mathbb{P}$ is of cardinality $\aleph_3$;
- (b) forcing with $\mathbb{P}$ preserves cardinals and cofinalities;
- (c) in $\mathbf{V}^{\mathbb{P}}, 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2, 2^{\aleph_2} = \aleph_3$;
- (d) in $\mathbf{V}^{\mathbb{P}}, \diamondsuit_S$ fails where $S = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\}$. Moreover,
  - $(*)$ there is a sequence $\bar{A} = \langle A_\delta : \delta \in S\rangle$ where A_δ an unbounded subset of δ of order type ω_1 satisfying
  - $(**)$ if $\bar{f} = \langle f_\delta : \delta \in S\rangle, f_\delta \in {}^{(A_\delta)}(\omega_1)$, <u>then</u> there is $f \in {}^{(\omega_2)}(\omega_1)$ such that $\delta \in S \Rightarrow \delta > \sup(\{\alpha \in A_\delta : f(\alpha) \leq f_\delta(\alpha)\})$.

Remark 3.3. Similarly for other cardinals.

Proof. There is an $\aleph_1$-complete $\aleph_3$-c.c. forcing notion $\mathbb{P}$ not collapsing cardinals and not changing cofinalities, preserving $2^{\aleph_\ell} = \aleph_\ell$ for $\ell = 0, 1, 2$ and $|\mathbb{P}| = \aleph_3$ such that in $\mathbf{V}^{\mathbb{P}}$, we have $(*)$. In fact more[1] than $(*)$ holds; see [9]. Let $\mathbb{Q}$ be the forcing of adding an $\aleph_2$ Cohen or just any c.c.c. forcing notion of cardinality $\aleph_2$ adding $\aleph_2$ reals (this can be $\underset{\sim}{\mathbb{Q}}$, a $\mathbb{P}$-name). Now we shall show that $\mathbb{P} * \mathbb{Q}$, equivalently $\mathbb{P} \times \mathbb{Q}$, is as required:

Clause (a):

$|\mathbb{P} * \mathbb{Q}| = \aleph_3$, trivial.

Clause (b):

Preserving cardinals and cofinalities; obvious as both $\mathbb{P}$ and $\mathbb{Q}$ do this.

Clause (c): Easy.

Clause (d): In $\mathbf{V}^{\mathbb{P}}$ we have $(*)$ as exemplified by, say, $\bar{A} = \langle A_\delta : \delta \in S \rangle$. We shall show that $\mathbf{V}^{\mathbb{P}*\mathbb{Q}} \models$ "$\bar{A}$ satisfies $(**)$". Otherwise in $\mathbf{V}^{\mathbb{P}*\mathbb{Q}}$ we have $\bar{f} = \langle f_\delta : \delta \in S \rangle$, say in $\mathbf{V}[G_{\mathbb{P}}, G_{\mathbb{Q}}]$, a counterexample. Then in $\mathbf{V}[G_{\mathbb{P}}]$ for some $q \in \mathbb{Q}$ and $\underset{\sim}{\bar{f}}$ we have

$$\mathbf{V}[G_{\mathbb{P}}] \models (q \Vdash_{\mathbb{Q}} \text{"}\underset{\sim}{\bar{f}} = \begin{array}[t]{l} \langle \underset{\sim}{f}_\delta : \delta \in S \rangle, \text{ where } \underset{\sim}{f}_\delta : A_\delta \to \omega_1 \text{ for each } \delta \in S \\ \text{form a counterexample to } (*)\text{"}). \end{array}$$

Now in $\mathbf{V}[G_{\mathbb{P}}]$ we can define $\bar{g} = \langle g^1_\delta : \delta \in S \rangle \in \mathbf{V}[G_{\mathbb{P}}]$, where g^1_δ is a function with domain A_δ, by

$$g^1_\delta(\alpha) = \{ i : q \nVdash \underset{\sim}{f}_\delta(\alpha) \neq i \}.$$

So in $\mathbf{V}[G_{\mathbb{P}}]$ we have $q \Vdash_{\mathbb{Q}}$ "$\bigwedge\limits_{\delta \in S} (\forall \alpha \in A_\delta) \underset{\sim}{f}_\delta(\alpha) \in g^1_\delta(\alpha)\}$". Also $g^1_\delta(\alpha)$ is a countable subset of ω_1 as $\mathbb{Q}$ satisfies the c.c.c.

For $\delta \in S$ we define a function $g_\delta : A_\delta \to \omega_1$ by letting $g_\delta(\alpha) = \sup(g^1_\delta(\alpha)) + 1$; hence $g_\delta(\alpha) < \omega_1$, so $\langle g_\delta : \delta \in S \rangle$ is as required on $\bar{f}$ in $(**)$ in $\mathbf{V}[G_{\mathbb{P}}]$, of course. Apply clause $(**)$ in $\mathbf{V}[G_{\mathbb{P}}]$ to $\langle g_\delta : \delta \in S \rangle$ so we can find $g : \omega_2 \to \omega_1$ such that $\bigwedge\limits_{\delta \in S} \delta > \sup\{\alpha \in A_\delta, g_\delta(\alpha) > g(\alpha)\}$. Now g is as also required in $\mathbf{V}[G_{\mathbb{P}}][G_{\mathbb{Q}}]$. $\square$

We may wonder whether we can strengthen the conclusion of Claim 2.3 to $\diamondsuit^*_S$ (of course the demand in clauses (e) and (f) in Observation 3.4 below are necessary; i.e. otherwise $\diamondsuit^*_S$ holds). The answer is no as: (the restriction in (e) and in (f) are best possible).

Observation 3.4. Assume $\lambda = \lambda^{<\lambda}, S \subseteq S^\lambda_\kappa$.

Then for some $\mathbb{P}$

- (a) $\mathbb{P}$ is a forcing notion,
- (b) $\mathbb{P}$ is of cardinality λ^+ satisfying the λ^+-c.c.,
- (c) forcing with $\mathbb{P}$ does not collapse cardinals and does not change cofinality,
- (d) forcing with $\mathbb{P}$ adds no new $\eta \in {}^{\lambda>}\mathrm{Ord}$,
- (e) $\diamondsuit^*_S$ fails for every stationary subset S of λ such that

[1] I.e. there is $\bar{A} = \langle A_\delta : \delta \in S \rangle$, where A_δ is an unbounded subset of δ of order type ω_1 satisfying:

$\oplus$ if $\bar{f} = \langle f_\delta : \delta \in S \rangle, f_\delta \in {}^{(A_\delta)}\omega_1$, then there is $f \in {}^{(\omega_2)}\omega_1$ such that for every $\delta \in S^{\aleph_2}_{\aleph_1}$ for every $\alpha \in A_\delta$ large enough, we have $f(\alpha) = f_\delta(\alpha)$.

(α) $S \subseteq S^\lambda_\kappa$ when $(\exists \mu < \lambda)[\mu^{<\kappa>_{\rm tr}} = \lambda]$
or just

(β) $\alpha \in S \Rightarrow |\alpha|^{\langle {\rm cf}(\alpha)\rangle_{\rm tr}} > |\alpha|$,

(f) $(D\ell)_S$ (see below) fails for every $S \subseteq S^\lambda_\kappa$ when $\alpha \in S \Rightarrow |\alpha|^{\langle {\rm cf}(\alpha)\rangle_{\rm tr}} = \lambda$.

Recalling

Definition 3.5. 1) For $\mu \geq \kappa = {\rm cf}(\kappa)$ let $\mu^{\langle\kappa\rangle_{\rm tr}} = \{|\mathcal{T}| : \mathcal{T} \subseteq {}^{\kappa \geq}\mu$ is closed under initial segments (i.e. a subtree) such that $|\mathcal{T} \cap {}^{\kappa >}\mu| \leq \mu\}$.

2) For λ regular uncountable and stationary $S \subseteq \lambda$ let $(D\ell)_S$ mean that there is a sequence $\bar{\mathcal{P}} = \langle \mathcal{P}_\delta : \delta \in S\rangle$ witnessing it, which means:

$(*)_{\bar{\mathcal{P}}}$ (a) $\mathcal{P}_\delta \subseteq {}^\delta\delta$ has cardinality $< \lambda$,

(b) for every $f \in {}^\lambda\lambda$ the set $\{\delta \in S : f{\restriction}\delta \in \mathcal{P}_\delta\}$ is stationary

(for λ a successor it is equivalent to $\diamondsuit_S$; for λ strong inaccessible it is trivial).

Proof of Observation 3.4. Use $\mathbb{P} =$ adding λ^+, λ-Cohen subsets.
The proof is straightforward. $\square$

Remark 3.6. The consistency results in Observation 3.4 are best possible; see [12].

Acknowledgments

We thank the audience at the Jerusalem Logic Seminar for their comments on the lecture of this work in Fall 2006. We thank the referee for many helpful remarks. In particular, he urged more detail in the proof of Claim 2.5(1),(2), which said only "like 2.3." Instead we make the old proof of Claim 2.3(1) prove Claim 2.5(1),(2) by minor changes and make explicit Claim 2.5(3), which earlier was proved by "repeat of the second half of the proof of 2.3(1)." We thank Shimoni Garti for his help in the proofreading.

References

[1] James Cummings, Mirna Džamonja, and Saharon Shelah, *A consistency result on weak reflection*, Fundamenta Mathematicae **148** (1995), 91–100. MR1354940 (96k:03118)

[2] Mirna Džamonja and Saharon Shelah, *Saturated filters at successors of singular, weak reflection and yet another weak club principle*, Annals of Pure and Applied Logic **79** (1996), 289–316. MR1395679 (97d:03062)

[3] ———, *Weak reflection at the successor of a singular cardinal*, Journal of the London Mathematical Society (2) **67** (2003), 1–15. MR1942407 (2004a:03051)

[4] John Gregory, *Higher Souslin trees and the generalized continuum hypothesis*, Journal of Symbolic Logic **41** (1976), no. 3, 663–671. MR0485361 (58:5208)

[5] Saharon Shelah, *pcf and abelian groups*, Forum Mathematicum, preprint, arXiv 0710.0157.

[6] ———, *On successors of singular cardinals*, Logic Colloquium '78 (Mons, 1978), Stud. Logic Foundations Math, vol. 97, North-Holland, Amsterdam-New York, 1979, pp. 357–380. MR567680 (82d:03079)

[7] ———, *Diamonds, uniformization*, The Journal of Symbolic Logic **49** (1984), 1022–1033. MR771774 (86g:03083)

[8] ———, *The Generalized Continuum Hypothesis revisited*, Israel Journal of Mathematics **116** (2000), 285–321. MR1759410 (2001g:03095)

[9] ———, *Not collapsing cardinals $\leq \kappa$ in $(< \kappa)$–support iterations*, Israel Journal of Mathematics **136** (2003), 29–115. MR1998104 (2004m:03182)

[10] ———, *Successor of singulars: combinatorics and not collapsing cardinals $\leq \kappa$ in $(< \kappa)$-support iterations*, Israel Journal of Mathematics **134** (2003), 127–155. MR1972177 (2004d:03109)

[11] ——, *Middle diamond*, Archive for Mathematical Logic **44** (2005), 527–560. MR2210145 (2006k:03087)

[12] ——, *More on the revised GCH and the black box*, Annals of Pure and Applied Logic **140** (2006), 133–160. MR2224056 (2007m:03105)

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel – and – Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019

E-mail address: shelah@math.huji.ac.il

URL: http://shelah.logic.at