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Changing cardinal characteristics without changing ω-sequences or cofinalities

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Abstract

We show: There are pairs of universes $V_1 \subseteq V_2$ and there is a notion of forcing $P \in V_1$ such that the change mentioned in the title occurs when going from $V_1[G]$ to $V_2[G]$ for a P-generic filter G over V_2. We use forcing iterations with partial memories. Moreover, we implement highly transitive automorphism groups into the forcing orders.

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0. Introduction

In [14] it is shown that some cardinal characteristics can be changed without changing ω-sequences or cardinalities, that is we can have two models $V_1 \subseteq V_2$ of ZFC such that $({}^{\omega}V_1)^{V_2} \subseteq V_1$ and such that V_1 and V_2 have the same cardinalities and such that, e.g., $\mathfrak{d}^{V_2} < \mathfrak{d}^{V_1}$ ($\mathfrak{d}$ is the dominating number, the minimum size of a subset $\mathscr{D} \subseteq \omega^\omega$ such that every function $f \in \omega^\omega$ is eventually dominated by some member of $\mathscr{D}$). Since in such a situation the covering theorem for (V_1, V_2) fails, there is consistency strength of at least a measurable cardinal. In [14] a change of a cofinality of a regular cardinal in V_1 was the main step when changing all the entries of Cichoń's Diagram (for information on cardinal characteristics and Cichoń's Diagram see e.g. [4, 2, 6, 22])

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without changing cardinalities or the reals. In this work, we show that we do not need to change cofinalities in order to change $\mathfrak{b}$, $\mathrm{cov}(\mathscr{M})$, $\mathrm{cov}(\mathscr{N})$, $\mathrm{unif}(\mathscr{M})$ or $\mathrm{unif}(\mathscr{N})$ and both additivities without changing cardinalities or the reals. These are all entries of Cichoń's Diagram that are not norms of transitive relations. In order to cover all these cases we use two different procedures.

In Section 1, we show how to change $\mathfrak{b}$, $\mathrm{unif}(\mathscr{M})$ and $\mathrm{cov}(\mathscr{N})$ and both additivities starting from a bare set-theoretic situation. We use an iteration with partial memory.

In [14] it is shown that $\mathfrak{d}$, $\mathrm{cof}(\mathscr{M})$ and $\mathrm{cof}(\mathscr{N})$ cannot be changed if their values in V_1 are regular in V_2 and if V_1 and V_2 have the same cardinalities. At the end of Section 1, we shall show that if V_1 and V_2 have the same cofinalities, then these characteristics (and some more, whose definition exhibits a certain syntax) cannot be changed either when starting from a singular value in V_1.

In Sections 2–5, we show how to change $\mathrm{unif}(\mathscr{N})$. We work with partial random forcing as in [20, 18], however, as we need special instances of the methods presented there, we (try to) make our present work self-contained. We include some comments on the connections to Shelah (Preprint, 1998, Sh619; Fund. Math., to appear) and give references to items we use almost literally, so that the reader may also read these. In Section 6 we shall present a variation of the techniques for a case with countable cofinality.

In Section 7, we show how to obtain the set-theoretic assumptions made in Theorems 1.1 and 2.1 from Gitik's work in [8, 9].

Notation. Our notation is fairly standard, see [11, 13]. However, we adopt the Jerusalem convention that the stronger forcing condition is the larger one. We often use V^P for $V[G]$, where G is any P-generic filter over V. For two forcing notions P, Q we write $P \lessdot Q$ if P is a complete suborder of Q. A forcing notion P is called σ-linked if $P = \bigcup_{n\in\omega} P_n$ such that each P_n is linked, that is any two $p, q \in P_n$ are compatible. Martin's axiom for less than λ dense subsets of a σ-linked partial order is denoted by $\mathrm{MA}_{<\lambda}(\sigma\text{-linked})$. We speak of ω^ω, the set of all functions from ω to ω, as the reals. For $f, g \in \omega^\omega$ we write $f \leqslant^* g$ if $\exists n\, \forall k \geqslant n\; f(k) \leqslant g(k)$. The ideal of Lebesgue null sets is denoted by $\mathscr{N}$, and the ideal of meagre sets is denoted by $\mathscr{M}$. The bounding number, $\mathfrak{b}$, is the smallest size of a subset $B \subseteq \omega^\omega$ such that for any $f \in \omega^\omega$ there is some $b \in B$ such that $b \not\leqslant^* f$. Let $\mathscr{I}$ be an ideal on the reals. The uniformity of $\mathscr{I} \subseteq \mathscr{P}(\omega)$, $\mathrm{unif}(\mathscr{I})$, is the smallest size of a subset of the reals that is not a member of $\mathscr{I}$. The covering number of $\mathscr{I}$, $\mathrm{cov}(\mathscr{I})$, is the smallest size of a subfamily of $\mathscr{I}$ whose union covers the reals. The additivity of $\mathscr{I}$, $\mathrm{add}(\mathscr{I})$, is the smallest size of a subset of $\mathscr{I}$ whose union is not in $\mathscr{I}$.

1. Changing the uniformity of category

In this section, we show how to change $\mathrm{unif}(\mathscr{M})$. Since $\mathrm{add}(\mathscr{M}) \leqslant \mathfrak{b} \leqslant \mathrm{unif}(\mathscr{M})$ and $\mathrm{add}(\mathscr{N}) \leqslant \mathrm{cov}(\mathscr{N}) \leqslant \mathrm{unif}(\mathscr{M})$ (for proofs of these inequalities e.g. see [7]), and in the beginning, that is in $V_1[G]$, everything is large because of an instance of Martin's axiom, the other four mentioned characteristics drop as well.

Theorem 1.1. *Assume that we have*

(a) $V_1 \subseteq V_2$, *both models of* ZFC, $({}^{\omega}V_1)^{V_2} \subseteq V_1$,

(b) μ *is a cardinal in* V_2, $C \subseteq \mu$, $C \in V_2$, $\mathscr{I} \in V_2$ *is an* $\aleph_1$-*complete proper ideal on* $\mathscr{P}(C)$,

(c) $\exists \lambda \leqslant \mu$ *such that* $\forall B \in V_1$, *if* $V_1 \models |B| < \lambda$, *then* $B \cap C \in \mathscr{I}$,

(d) $V_1 \models \lambda > \aleph_0$ *and* λ *is regular.*

Then for some P

(α) $V_1 \models P$ *is a finite support iteration of* σ-*linked forcing notions, and the cardinality of* P *is* $\mu^{<\lambda}$,

(β) P *is c.c.c. in* V_2.

For $G \subset P$ *generic over* V_2 *we have*

(γ) $({}^{\omega}V_1[G])^{V_2[G]} \subseteq V_1[G]$,

(δ) $V_1[G]$ *and* $V_2[G]$ *have the same cardinals if* V_1 *and* V_2 *have,*

(ε) $V_1[G]$ *and* $V_2[G]$ *have the same cofinality function if* V_1 *and* V_2 *have,*

(ζ) $V_1[G] \models \mathrm{MA}_{<\lambda}(\sigma$-*linked*),

(η) *in* $V_2[G]$ *there is* $\langle r_i \mid i \in C \rangle$, $r_i \in ({}^{\omega}2)^{V_1[G]} = ({}^{\omega}2)^{V_1[G]}$, *such that* $\forall s \in ({}^{\omega}2)^{V_1[G]}$ $\exists B \subseteq \mu$, $B \in V_1$, $|B|^{V_1} < \lambda$ (*so* $C \cap B \in \mathscr{I}$) $\forall i \in C \setminus B$, r_i *is Cohen over* $V_2[s]$.

Proof. In V_1 we build a finite support iteration

$$\langle P_i, \underset{\sim}{Q}_j \mid j < \alpha^*,\ i \leqslant \alpha^* \rangle$$

of length $\alpha^* = \mu + \mu^{<\lambda}$ as follows. For $\beta < \mu$ we let $Q_\beta = ({}^{<\omega}2, \lhd)$, the Cohen forcing.

For $\beta < \mu^{<\lambda}$ we shall choose $\underset{\sim}{Q}_{\mu+\beta}$ such that it is a name built from only part of $P_{\mu+\beta}$. We first need some definitions in order to specify good parts of the past. This forcing technique has also been applied in [18–20] and their predecessors and in [21]. The part [21, 3.3–3.7] contains some lemmas showing that there are complete embeddings from specified suborders of the iteration that are not just initial segments. The organisation of our forcing will be slightly different from that in [21] in as much as we have the initial Cohen part here at once.

The support of a condition $p \in P_\beta$ is $\mathrm{supt}(p) = \{\gamma \in \beta \mid p(\gamma) \neq 1_{\underset{\sim}{Q}_\gamma}\}$, where $1_{\underset{\sim}{Q}_\gamma}$ is a name for the weakest element in $\underset{\sim}{Q}_\gamma$. In addition to having finite supports we shall require that the supports hereditarily stem only from a part of the "past" P_β. These parts of the past can be called memories.

First, we explain how to choose sequences $\langle a_\beta \mid \beta \in \mu^{<\lambda} \rangle$ which will allow us to define suitable memories. Given a sequence $\langle a_\beta \mid \beta \in \mu^{<\lambda} \rangle = \bar{a}$ of subsets of an ordinal, we say c is $\bar{a}$-closed, if

$$c \subseteq \alpha^* \quad \text{and} \quad \forall \beta \in c\ a_\beta \subseteq c.$$

We regard $\mu^{<\lambda}$ as an ordinal and as a set of sequences of length less than λ. The set of all subsets of a set A of size less than λ is denoted by $[A]^{<\lambda}$. For $x \in \mu^{<\lambda}$ we can also regard x as a function from some ordinal less than λ to μ and then write range(x) for its range, which is a subsets of μ. This will be used for referring to a part of the Cohen reals.

We show that there is some $\langle a_\beta \mid \beta < \mu^{<\lambda} \rangle$ such that

(1) $\forall b \in [\mu^{<\lambda}]^{<\lambda}\ \exists \beta\ b \subseteq a_\beta$,

(2) $a_\beta \subseteq \beta$,

(3) $|a_\beta| < \lambda$,

(4) $\gamma \in a_\beta \to a_\gamma \subseteq a_\beta$ (i.e. each a_β is $\bar{a}$-closed).

This can be seen as follows: Let $\langle b_\beta \mid \beta \in \mu^{<\lambda} \rangle$ enumerate $[\mu^{<\lambda}]^{<\lambda}$, where $b_\beta \subseteq \beta$. By induction on β we now choose a_β. Suppose a_γ is chosen for $\gamma < \beta$. Then we set

$$a_\beta^1 = \bigcup_{j \in b_\beta} a_j \cup b_\beta,$$

$$a_\beta^{n+1} = \bigcup_{j \in a_\beta^n} a_j \cup a_\beta^n,$$

$$a_\beta = \bigcup_{n \in \omega} a_\beta^n.$$

This is still in $[\mu^{<\lambda}]^{<\lambda}$ because λ is regular and $\mathrm{cf}(\lambda) > \aleph_0$. Now it is easy to see that $\bar{a}$ fulfils (1)–(4), and we fix such a sequence.

In order to take care of the initial Cohen part, we need shifts and write $\mu \oplus a_\beta$ for $\{\mu + \gamma \mid \gamma \in a_\beta\}$.

For each $\beta \in \mu^{<\lambda}$ we define a suborder $P^*_{\mu \oplus a_\beta}$ of $P_{\mu+\beta}$ inductively by

$$\begin{aligned} P^*_{\mu \oplus a_\beta} = \Big\{ p \in P_{\mu+\beta} \mid & \operatorname{supt}(p) \cap \mu \subseteq \bigcup \{\operatorname{range}(x) \mid x \in a_\beta\} \\ & \wedge \operatorname{supt}(p) \cap [\mu, \mu + \mu^{<\lambda}) \subseteq \mu \oplus a_\beta \\ & \wedge \forall \gamma \in \operatorname{supt}(p) \cap [\mu, \mu + \mu^{<\lambda})\ p(\gamma) \text{ is a } P^*_{\mu \oplus a_\gamma}\text{-name} \Big\}. \end{aligned}$$

If $b \subseteq \alpha \leqslant \mu^{<\lambda}$ then $p \restriction (\bigcup \{\operatorname{range}(x) \mid x \in b\} \cup \mu \oplus b))$ denotes the $\mu + \alpha$-sequence defined by

$$\Big(p \restriction \Big(\bigcup \{\operatorname{range}(x) \mid x \in b\} \cup \mu \oplus b\Big)\Big)(\gamma) = \begin{cases} p(\gamma) \text{ if } \gamma \in \Big(\bigcup \{\operatorname{range}(x) \mid x \in b\} \cup \mu \oplus b\Big), \\ 1_{\underset{\sim}{Q}_\gamma} \quad \text{else.} \end{cases}$$

Now we have for all $\alpha \in \mu^{<\lambda}$: If $b \subseteq \alpha$ is $\bar{a}$-closed, then $P^*_{\mu \oplus b} \lessdot P_{\mu+\alpha}$. If $p \in P_{\mu+\alpha}$, then $(p \restriction (\bigcup \{\operatorname{range}(x) \mid x \in b\} \cup \mu \oplus b))) \in P^*_{\mu \oplus b}$ and for $q \geqslant p \restriction (\bigcup \{\operatorname{range}(x) \mid x \in b\} \cup \mu \oplus b)$

(in the Jerusalem notation) we have that $q \cup p \upharpoonright (\alpha \setminus (\bigcup\{\text{range}(x) \mid x \in b\} \cup \mu \oplus b)) \in P_{\mu+\alpha}$ (for proofs, see [21]).

We choose $\underset{\sim}{Q}_{\mu+\beta}$ such that $|\text{dom}(\underset{\sim}{Q}_{\mu+\beta})| < \lambda$, $\underset{\sim}{Q}_{\mu+\beta}$ is a $P^*_{\mu\oplus a_\beta}$-name, $1 \Vdash_{P^*_{\mu\oplus a_\beta}}$ "$\underset{\sim}{Q}_{\mu+\beta}$ is σ-linked", and with some bookkeeping such that $\underset{\sim}{Q}_{\mu+\beta}$ ranges cofinally often over all $P^*_{\mu\oplus a_\gamma}$-names for σ-linked forcings for every $\gamma \in \mu^{<\lambda}$. In order to allow such a bookkeeping, we assume that $\forall b \in [\mu^{<\lambda}]^{<\lambda}\ \exists^{\mu^{<\lambda}} \beta\ b \subseteq a_\beta$, which can easily be reached by starting with suitable $\langle b_\beta \mid \beta \in \mu^{<\lambda} \rangle$.

Now we are in a position to check all the items of the theorem:

(α) It follows immediately from our definition of P.

(β) If $P = \bigcup_{n\in\omega} P_n$ witnesses σ-linkedness in V_1 then it does so in V_2 as well. Thus in V_2, P is a finite support iteration of σ-linked forcing notions and hence c.c.c.

(γ) $({}^\omega V_1[G])^{V_2[G]} \subseteq V_1[G]$ follows from $({}^\omega V_1)^{V_2} \subseteq V_1$ and the countable chain condition of P in V_2. (There are also proofs in [11, Section 37] and more explicit in [5].)

(δ) and (ε) V_i and $V_i[G]$ have the same cofinalities.

(ζ) Let Q be in $V_1[G]$ be a σ-linked notion of forcing such that $Q \subseteq \lambda' < \lambda$. Let $\mathscr{D} = \{D_\alpha \mid \alpha < \lambda'\}$ be a set of dense sets in Q. Since the supports are finite and since we have c.c.c., there is some $A \subseteq \mu + \mu^{<\lambda}$ of size less than λ such that there is a name for $(Q, \mathscr{D})$ that contains only conditions whose support is in A. Then we take $\alpha \in \mu^{<\lambda}$ such that

$$x = \bigcup\{\text{range}(x) \mid x \in a_\alpha\} \supseteq A \cap \mu \quad \text{and} \quad y = \mu \oplus a_\alpha \supseteq A \cap [\mu, \mu + \mu^{<\lambda})$$

and have that $\mathscr{D}, Q \in V_1^{P^*_{\mu\oplus a_\alpha}}$. Hence a Q-generic $G \subseteq Q$ is added at some stage in our iteration.

(η) Let $\langle r_i \mid i \in \mu \rangle$ be the Cohen reals added by P_μ. We show that $\{r_i \mid i \in C\}$ is as claimed. Let $s \in (2^\omega)^{V_1[G]}$. Say s was added by forcing with $\underset{\sim}{Q}_{\mu+\beta}$ (the case when s was added before stage μ is similar), a $P_{\mu\oplus a_\beta}$-name. We take $B = a_\beta$. Then $B \in V_1$, $B \subseteq \mu$, and $|B|^{V_1} < \lambda$. As $C \cap B \in \mathscr{I}$, we have $C \setminus B \neq \emptyset$. For $i \in C \setminus B$ r_i is Cohen over $V_1[s]$. Proof: For $Q_i = ({}^{<\omega}2, \lhd)$ we have

$$Q_i * P^*_{\mu\oplus a_\beta} = Q_i \times P^*_{\mu\oplus a_\beta}.$$

Remark. This equation is very crucial: Note that there is "no time-dependence", i.e. the location of i in $\mu + \mu^{<\lambda}$ as compared to the location of $x \cup y$ does not have any influence. Neither Q_i nor $P^*_{\mu\oplus a_\beta}$ is the "later" forcing, because neither of them is influenced by the extension performed by the other. All the work with the partial memory was done in order to get this equation. Counting cardinalities of unions of supports of conditions appearing in nice names seems not to suffice for it.

The analogue of the crucial equation is true for the subforcing of $P^*_{\mu\oplus a_\beta}$ that has s as a generic. Now in product forcing, the factors commute, hence we have $V_1[r_i][s] = V_1[s][r_i]$. $\square$

Putting things together we get

Corollary 1.2. (1) *The following are equiconsistent (even* (B)$\Rightarrow$(A), (A)$\Rightarrow$(B) *in some c.c.c. forcing extension*):

(A)(α) *there are* $V_1, V_2, \mu, \theta, \lambda, \sigma, C$, *such that*

$V_1 \subseteq V_2$,
$V_1 \models \lambda$ *regular* $> \aleph_0$,
$({}^{\omega}V_1)^{V_2} \subseteq V_1$,
$\mu \geqslant \theta$, $\mu \geqslant \lambda > \sigma \geqslant \aleph_1$,
$C \subseteq \mu$,
$|C|^{V_2} = \theta$,
$\forall B \in V_1\ (|B|^{V_1} < \lambda \to |B \cap C|^{V_2} < \sigma)$.

(β) V_1 *and* V_2 *have the same cardinals.*

(γ) V_1 *and* V_2 *have the same cofinality function on ordinals.*

(B)(α) *like* (A)(α) *but in addition*

($*_1$) $V_1 \models \mathrm{MA}_{<\lambda}(\sigma$-*linked*)

($*_2$) *in* V_2 *there are* $\langle r_i \mid i \in C\rangle$, $r_i \in 2^{\omega}$ *and a submodel* V *such that* $\forall s \in 2^{\omega}\ \exists B \in [C]^{<\sigma}$ *such that* $\langle r_i \mid i \in C \setminus B\rangle$ *is Cohen over* $V[s]$.

(β) *as* (β) *above.*

(γ) *as* (γ) *above.*

(2) *We can leave out* (β) *or* ((β) *and* (γ)) *in both* (A) *and* (B).

(3) *If we strengthen* (A)(α) *by adding*
$({}^{\omega_1}V_1)^{V_2} \subseteq V_1$, *then we can get* $\mathrm{MA}_{<\lambda}$ (*ccc*) *in* (B).

Proof. (A) is as the premise of Theorem 1.1 with $\mathscr{I} = \{C' \subset C \mid C' \in V_2, |C'|^{V_2} < \sigma\}$. Note that σ as in (A)(α) is uncountable because we have the condition $({}^{\omega}V_1)^{V_2} \subseteq V_1$. For (3), take all names for c.c.c forcing notions, not only for the σ-linked ones. The additional premise ensures that (the new) P has the c.c.c. in V_2 as well. $\square$

We get the following conclusion for cardinal characteristics in (B) of Corollary 1.2.

Theorem 1.3. *Suppose that we have*

(α) *There are* V_1, V_2, μ, θ, λ, σ, C, *such that*

$V_1 \subseteq V_2$,
$V_1 \models \lambda$ *regular* $> \aleph_0$,
$({}^{\omega}V_1)^{V_2} \subseteq V_1$,
$\mu \geqslant \theta$, $\mu \geqslant \lambda > \sigma \geqslant \aleph_1$,
$C \subseteq \mu$,
$|C|^{V_2} = \theta$,
$\forall B \in V_1(|B|^{V_1} < \lambda \to |B \cap C|^{V_2} < \sigma)$,
$V_1 \models \mathrm{MA}_{<\lambda}(\sigma$-*linked*),
in V_2 *there are* $\langle r_i \mid i \in C\rangle$, $r_i \in 2^{\omega}$ *and a submodel* V *such that* $\forall s \in 2^{\omega} \exists B \in [C]^{<\sigma}$ *such that* $\langle r_i \mid i \in C \setminus B\rangle$ *is Cohen over* $V[s]$.

(β) *V_1 and V_2 have the same cardinals.*
(γ) *V_1 and V_2 have the same cofinality function on ordinals.*

Then (a) $\mathfrak{b}^{V_1} \geqslant \lambda$, $\mathfrak{b}^{V_2} \leqslant \sigma$ *(and in the construction from the proof of Theorem* 1.1, *we have* $\mathfrak{b}^{V_1} = \lambda$. *Moreover, if* $\forall B \in ([[\mu]^{<\lambda}]^{<\sigma})^{V_2}\ \exists B' \in ([[\mu]^{<\lambda}]^{<\lambda})^{V_1} B \subseteq B'$, *then the construction from Theorem* 1.1 *gives* $\mathfrak{b}^{V_2} = \sigma$).

(b) $\mathrm{unif}(\mathscr{M})^{\mathscr{V}} \geqslant \lambda$, $\mathrm{unif}(\mathscr{M})^{\mathscr{V}} \leqslant \sigma$,
(c) $\mathrm{cov}(\mathscr{N})^{\mathscr{V}} \geqslant \lambda$, $\mathrm{cov}(\mathscr{N})^{\mathscr{V}} \leqslant \sigma$.

Proof. The V_1-part of (a)–(c): $\mathrm{MA}_{<\lambda}$(σ-linked) implies that the three cardinal characteristics (and add($\mathscr{M}$), add($\mathscr{N}$)) are $\geqslant \lambda$, because all of them can be increased by σ-linked notions of forcing (see e.g. [2]).

In order to show unif($\mathscr{M}$), $\mathfrak{b} \leqslant \sigma$, we take $\{r_i \mid i \in C'\}$, $C' \subset C$, $|C'| = \sigma$. This set is unbounded and not meagre in V_2, because for any $s \in V_2$ (either in ω^ω or as a name for a meagre (F_σ-)set) there is some $B_s \in [C]^{<\sigma}$ such that for $i \in C' \setminus B_s \neq \emptyset$ we have the r_i is Cohen over $V_2[s]$, hence it is not bounded by s nor in a meagre set coded by s.

Proof of $\mathrm{cov}(\mathscr{N}) \leqslant \sigma$: This follows from Rothberger's inequality $\mathrm{cov}(\mathscr{N}) \leqslant \mathrm{unif}(\mathscr{M})$ (see [16, 7]). In order to give a proof not using this inequality, we can take $\{r_i \mid i \in C'\}$ as above. We set $M(r_i) = \{m \mid r_i$ is Cohen over $V[m]\}$. Then (by Fubini) we have that $M(r_i)$ is a Lebesgue null set and for $s \in (2^\omega)^{V_2}$ we have there is some $B_s \in [C']^{<\sigma}$ such that for $i \in C' \setminus B_s$ the real r_i is Cohen over $V[s]$, hence $s \in M(r_i)$, so $\{M(r_i) \mid i \in C'\}$ covers $(2^\omega)^{V_2}$.

Regarding the part of (a) in parentheses: Any λ of the Cohen reals added in the beginning are unbounded and show that $\mathfrak{b}^{V_1} \leqslant \lambda$. Under the additional premises, we have that $\mathfrak{b}^{V_2} \geqslant \sigma$: Suppose that $M \subset ({}^\omega 2)^{V_2}$ and $|M|^{V_2} < \sigma$. We take $M_1 \subseteq \mu$ and $M_2 \subseteq \mu^{<\lambda}$ such that each member of M has a name containing only conditions from $\{C_i \mid i \in M_1\} \cup \{P^*_{\mu \oplus a_\beta} \mid \beta \in M_2\}$. Then $B = \{\{i\} \mid i \in M_1\} \cup \{a_\beta \mid \beta \in M_2\} \in ([\mu^{<\lambda}]^{<\sigma})^{V_2}$. Hence, there is some $B' \in ([\mu^{<\lambda}]^{<\lambda})^{V_1}$ such that $B' \supseteq B$. We take β such that $a_\beta \supseteq \bigcup B'$. Hence at some later stage Hechler forcing over $V^{P^*_{\mu \oplus a_\beta}}$ will be done in the iteration and add a real that dominates all reals in M. $\square$

Remark on the violation of covering: Assume that for some first-order sentence $\phi = \phi(P, \in)$, where $\in$ is a two place predicate and P is a unary predicate, we have that

$$\vdash \quad \forall x P x \rightarrow \phi,$$

$$\phi \quad \text{is preserved by increasing } P.$$

Then we define

$$\mathrm{inv}^\phi = \min\{|A| \mid (H(\aleph_1), \in, A) \models \phi\}.$$

$H(\mu)$ is the set of all sets that are hereditarily of cardinality less that μ. Now, if we have two models V_1, V_2 of set theory such that

$$V_1 \subseteq V_2, \text{ and}$$
$$V_1 \text{ and } V_2 \text{ have the same cardinals and the same } H(\aleph_1)$$
$$\text{(which is the same as having the same reals), and}$$
$$C \text{ is of minimal cardinality such that } (H(\aleph_1), \in, C) \models \phi \text{ and}$$
$$(\mathrm{inv}^{\phi})^{V_1} = \lambda > |C| \geqslant (\mathrm{inv}^{\phi})^{V_2},$$

then we have that C is not covered by any set in V_1 of cardinality less than λ.

Remark on changing $\mathfrak{d}$, $\mathrm{cof}(\mathscr{M})$ *and* $\mathrm{cof}(\mathscr{N})$: Assume that for some first-order sentence $\phi = \phi(\in)$, where $\in$ is a two place predicate, we have that

$$\forall xyz \in H(\aleph_1) \quad (\phi(x,y) \wedge \phi(y,z) \rightarrow \phi(x,z)) \wedge$$
$$\forall x \in H(\aleph_1)\ \exists y \in H(\aleph_1)\ \phi(x,y).$$

Then we define for $B \subseteq H(\aleph_1)$, $B \in V$:

$$\mathrm{inv}^{V}_{\phi,B} = \min\{|A| \mid \text{for all } x \in B \text{ exists } y \in A \text{ such that } (H(\aleph_1), \in) \models \phi(x,y)\}.$$

Note that $\mathfrak{d}$, $\mathrm{cof}(\mathscr{M})$ and $\mathrm{cof}(\mathscr{N})$ are characteristics of this type.

Now we have

Theorem 1.4. *If V_1 and V_2 are two models of* ZFC, *such that $V_1 \subseteq V_2$ and such that they have the same cofinalities and the same reals, and if $B \in V_1$, $B \subseteq H(\aleph_1)$, then*

$$\mathrm{inv}^{V_1}_{\phi,B} \leqslant \mathrm{inv}^{V_2}_{\phi,B}.$$

Corollary 1.5. *If V_1 and V_2 are two models of* ZFC, *$V_1 \subseteq V_2$ and they have the same cofinalities and the same reals then their dominating numbers and their cofinalities of the ideals of Lebesgue null sets and meagre sets coincide.*

Proof of Theorem 1.4. Given V_1 and V_2 and ϕ we carry out an induction over $\mathrm{inv}^{V_1}_{\phi,B}$ simultaneously for all $B \subseteq H(\aleph_1)$, $B \in V_1$.

If $\mathrm{inv}^{V_1}_{\phi,B} = 1$, then the premise $H(\aleph_1)^{V_1} = H(\aleph_1)^{V_2}$ and the requirements on ϕ immediately yield the claim.

Now suppose that the claim is proved for all ϕ, B such that $\mathrm{inv}^{V_1}_{\phi,B} < \kappa$ and that we have some ϕ, B such that $\mathrm{inv}^{V_1}_{\phi,B} = \kappa$.

Case 1: κ is regular in V_1 and hence in V_2. In this case, Blass' Proposition 2.3 of Mildenberger [14] applies. For completeness' sake we repeat the argument here: Suppose that $\mathrm{inv}^{V_2}_{\phi,B} = \mu \leqslant \kappa$.

Let $Z = \{z_\alpha \mid \alpha < \kappa\}$ witness $\mathrm{inv}^{V_1}_{\phi,B} = \kappa$, and $Z' = \{z'_\alpha \mid \alpha < \mu\}$ witness $\mathrm{inv}^{V_2}_{\phi,B} = \mu$. Since $\mathbb{R}^{V_2} \subseteq \mathbb{R}^{V_1}$, in V_2 there is a function $h : \mu \rightarrow \kappa$ such that for $\alpha < \kappa$,

$$H(\aleph_1) \models \phi(z'_\alpha, z_{h(\alpha)}).$$

If μ were less than κ, then range(h) would be bounded in κ, say by a bound $\beta \in \kappa$.

Then $\forall a \in \mathbb{R}^{V_1} \exists \alpha \in \mu \phi(a, z'_\alpha) \wedge \phi(z'_\alpha, z_{h(\alpha)})$. Hence $\{z_\alpha \mid \alpha \leqslant \beta\}$ were a witness for $\mathrm{inv}^{V_1}_{\phi,B} \leqslant \mathrm{card}(\beta) < \kappa$, which contradicts the premise.

Case 2: κ is singular in V_1 and hence in V_2.

Let $\kappa = \lim_{i \to \mathrm{cf}(\kappa)} \kappa_i$ and $\kappa_i < \kappa$.

Let $Z = \{z_\alpha \mid \alpha < \kappa\}$ witness $\mathrm{inv}^{V_1}_{\phi,B} = \kappa$.

Set

$$Z_i = \{z_\alpha \mid \alpha \in \kappa_i\}$$

and

$$B_i = \{b \in B \mid \exists z \in Z_i \phi(b, z)\}.$$

Now we have that

$$\mathrm{inv}^{V_1}_{\phi,B_i} \leqslant \kappa_i$$

and

$$\sup_{i \in \mathrm{cf}(\kappa)} \mathrm{inv}^{V_1}_{\phi,B_i} = \kappa.$$

The second equation is easy to see: If $\sup_{i \in \mathrm{cf}(\kappa)} \mathrm{inv}^{V_1}_{\phi,B_i} = \theta < \kappa$ then we would have that $\mathrm{inv}^{V_1}_{\phi,B} = \theta \cdot \mathrm{cf}(\kappa) < \kappa$.

By induction hypothesis

$$\mathrm{inv}^{V_1}_{\phi,B_i} \leqslant \mathrm{inv}^{V_2}_{\phi,B_i}.$$

Since any witness for the computation of $\mathrm{inv}^{V_2}_{\phi,B}$ is a union of witnesses of the computation of $\mathrm{inv}^{V_2}_{\phi,B_i}$, we get that $\mathrm{inv}^{V_2}_{\phi,B} \geqslant \sup\{\mathrm{inv}^{V_2}_{\phi,B_i} \mid i \in \mathrm{cf}(\kappa)\} = \kappa$. $\square$

2. Changing the uniformity of Lebesgue measure

In this and the next three sections, we show how to change unif$(\mathcal{N})$ (and cov$(\mathcal{M})$, which comes for free, because of the inequality cov$(\mathcal{M}) \leqslant$ unif$(\mathcal{N})$, see [7]) under our given side conditions. In this section, we start to define the forcings we are going to use and look at automorphisms of forcings. We carry out the proof of the changing procedure up to some point in the proof of item (ε) of our main Theorem 2.1 at which techniques about transferring information about ω-tuples of conditions (in [20] called "whispering") are needed. We try to give some motivation for this fact by proving a lemma about a pure Cohen situation (Lemma 2.12), of which a weakened analogue for iterations of partial random reals and small c.c.c. forcings will be used later. This weakened analogue is the statement $(**)_{\bar{Q}}$ introduced in Lemma 2.11 and proved only by the end of Section 5.

These technical parts are then carried through in Sections 3–5.

Theorem 2.1. *Assume that we have*
(a) $V_1 \subseteq V_2$, *both models of* ZFC, $({}^{\omega}V_1)^{V_2} \subseteq V_1$ [*and* (β) *or* ((γ) + (β)) *from Corollary* 1.2(A)],
(b) $C \in V_2$, $|C| < \lambda$, $C \subseteq \mu$, $\lambda \leqslant \mu$,
(c) $\forall B \in V_1$, *if* $V_1 \models |B| < \lambda$, *then* $\sup(C \setminus B) = \mu$),
(d) $\mathrm{cf}^{V_1}(\mu) > \aleph_0$ *and* $\mathrm{cf}^{V_1}(\lambda) > \aleph_0$,
(e) *in* V_1, *there are uncountable cardinals* $\chi \geqslant 2^{\mu}$ *and* κ *such that* $\kappa < \chi$ *and* $2^{\kappa} \geqslant \chi$.
Then for some c.c.c. P *in* V_1 *we have*
(α) $V_1 \models P$ *is a finite support iteration of* σ*-linked forcing notions*,
(β) P *is c.c.c. in* V_2, *and*
for $G \subset P$ *generic over* V_2 *we have*
(γ) $({}^{\omega}V_1[G])^{V_2[G]} \subseteq V_1[G]$, [*and* (β) *or* ((γ) + (β)) *from Corollary* 1.2(A)],
(δ) $\mathrm{unif}(\mathcal{N})^{\mathcal{V}[\mathcal{G}]} \leqslant |\mathcal{C}|^{\mathcal{V}[\mathcal{G}]}$,
(ε) $\mathrm{unif}(\mathcal{N})^{\mathcal{V}[\mathcal{G}]} \geqslant \lambda$.

Proof. We work in V_1 (and often write V instead of V_1). For $\chi \geqslant 2^{\mu}$ we let $g_\chi : \chi \to [\mu]^{<\lambda}$ increasing with χ, that is for $\chi \leqslant \chi'$ we have that $g_{\chi'} \upharpoonright \chi = g_\chi$, and

$$\forall B \in [\mu]^{<\lambda} \exists^{\chi} \alpha < \chi g_\chi(\alpha) = B.$$

For $\xi < \mu$ let

$$E_\xi = E_\xi^\chi = \{\alpha < \chi \mid \xi \notin g_\chi(\alpha)\}$$

and

$$A^\chi_{\chi+\xi} = E^\chi_\xi \cup [\chi, \chi + \xi).$$

We take μ and λ as in the premises of Theorem 2.1. We also fix $\kappa \geqslant \aleph_1$ and some $\chi \geqslant 2^{\mu}$ as above such that $\mathrm{cf}(\chi) > \mu$ (used in Lemma 2.11) and $2^{\kappa} \geqslant \chi$ and such that $\kappa < \chi$ (for our special iteration where all Q_α of cardinality $< \kappa$ are already countable, $\kappa \leqslant \chi$ would suffice, see at Lemma 5.2 and the remarks in Lemma 2.11, if you like to work with weaker premises). Note for use in Theorem 5.5: the definition of g_χ and E_ξ, $A^\chi_{\chi+\xi}$ makes sense also if $2^{\kappa} < \chi$.

Definition 2.2. (1) $\mathscr{K}$ is the class of sequences

$$\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{\tau}_\beta \mid \alpha \leqslant \alpha^*, \beta < \alpha^* \rangle$$

satisfying
(A) $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leqslant \alpha^*, \beta < \alpha^* \rangle$ is a finite support iteration of c.c.c. forcings. We call $\alpha^* = \lg(\bar{Q})$ the length of $\bar{Q}$, and P_{α^*} is the limit.
(B) $\underset{\sim}{\tau}_\alpha \subseteq \mu_\alpha < \kappa$ is a name of the generic of $\underset{\sim}{Q}_\alpha$, i.e. over V^{P_α} from $G_{\underset{\sim}{Q}_\alpha}$ we can compute $\underset{\sim}{\tau}_\alpha$ and vice versa.

(C) $A_\alpha \subseteq \alpha$.

(D) $\underset{\sim}{Q}_\alpha$ is a P_α-name of a c.c.c. forcing notion that is computable from $\langle \underset{\sim}{r}_\gamma[\underset{\sim}{G}_{P_\alpha}] \mid \gamma \in A_\alpha \rangle$.

(E) $\alpha^* \geqslant \chi$ and for $\alpha < \chi$ we have that $Q_\alpha = ({}^\omega 2, \lhd)$ (the Cohen forcing) and $\mu_\alpha = \aleph_0$ (identify ${}^{<\omega}2$ with ω).

(F) For each $\alpha < \alpha^*$ one of the following holds (and the case is determined in V).

(α) $|\underset{\sim}{Q}_\alpha| < \kappa$, $|A_\alpha| < \kappa$ and (just for notational simplicity) the set of elements of $Q_\alpha = \underset{\sim}{Q}_\alpha[G_{P_\alpha}]$ is $\mu_\alpha < \kappa$ (but the order not necessarily the order of the ordinals) and Q_α is separative (i.e. $\alpha \Vdash \beta \in \underset{\sim}{G}_{Q_\alpha} \Leftrightarrow Q_\alpha \models \beta \leqslant \alpha$).

(β) $Q_\alpha = \text{Random}^{V[\underset{\sim}{r}_\gamma[G_{P_\alpha}] \mid \gamma \in A_\alpha]}$ and $|A_\alpha| \geqslant \kappa$.

(2) For the proof of Theorem 2.1 we shall be using the following instance of (1): For χ, μ, A^χ_α as above we define a finite support iteration:

$$\bar{Q}^\chi = \langle P^\chi_\alpha, \underset{\sim}{Q}^\chi_\beta, A^\chi_\beta, \aleph_0, \underset{\sim}{r}_\beta \mid \alpha \leqslant \chi + \mu, \beta < \chi + \mu \rangle,$$

$P^\chi = P^\chi_{\chi+\mu}$. For $\alpha < \chi$ we let $Q^\chi_\alpha = ({}^{<\omega}2, \lhd)$, the Cohen forcing. For $\alpha = \chi + \xi$, $\xi < \mu$, we let

$$\underset{\sim}{Q}^\chi_\alpha = \text{Random}^{V[\underset{\sim}{r}^\chi_\beta \mid \beta \in A^\chi_\alpha]},$$

where $\underset{\sim}{r}^\chi_\beta$ is $\underset{\sim}{Q}^\chi_\beta$-generic over V^{P_β}.

Thus, the $\bar{Q}^\chi$ from (b) is a member of $\mathscr{K}$ (and of Shelah [20, Definition 2.2; 18, Definition 1.4]) of a special form: $A_\alpha = \emptyset$ if $\alpha < \chi$, and $A^\chi_{\chi+\xi} = E_\xi \cup [\chi, \chi + \xi)$ for $\xi < \mu$.

The reader may wonder why we do not really fix χ. The reason is that in Section 5 we use a Löwenheim Skolem argument and work simultaneously with χ, χ^+, χ^{++}, $\ldots, \chi^{+(n-1)}$, n the size of some heart of a Δ-system, in order to expand $\bar{Q}^\chi$ to a richer structure that will be used for the proof of part (ε) of Theorem 2.1.

The Lebesgue measure is denoted by Leb and for a tree $T \subseteq 2^{<\omega}$ we define $\lim(T) = \{f \in 2^\omega \mid \forall n \in \omega f \upharpoonright n \in T\}$. Similar to Shelah [20, 2.2], we specify dense suborders of Random and call them Random again.

Definition 2.3. (a) $\text{Random}^{V[\underset{\sim}{r}_\alpha \mid \alpha \in A]} = \{p \mid$ there is in V a Borel function $\mathscr{B}^p = \mathscr{B}$ with variables ranging among $\{\text{true}, \text{false}\}$ and range perfect subtrees r of ${}^{<\omega}2$ with $\text{Leb}(\lim(r)) > 0$ such that $\forall \eta \in r \text{Leb}(\lim r^{[\eta]} > 0)$ (where $r^{[\eta]} = \{\nu \in r \mid \nu \trianglelefteq \eta \vee \eta \trianglelefteq \nu\}$) and there are pairs $(\gamma_\ell, \zeta_\ell)$ for $\ell \in \omega$, such that $\gamma_\ell \in A$, $\zeta_\ell \in \omega$, and such that $p = \mathscr{B}^p$ ((truth value $(\zeta_\ell \in \underset{\sim}{r}_{\gamma_\ell}))_{\ell \in \omega})\}$.

(b) In this case we let $\text{supt}(p) = \{\gamma_\ell \mid \ell \in \omega\}$.

(c) $P'_\alpha = \{p \in P_\alpha \mid \forall \gamma \in \text{dom}(p), \text{if } |A_\gamma| < \kappa, \text{then } p(\gamma) \in \mu_\gamma$ (not just a name for a member of μ_γ), and if $|A_\gamma| \geqslant \kappa$, then $p(\gamma) \in \text{Random}^{V[\underset{\sim}{r}_\delta \mid \delta \in A_\gamma]}\}$.

(d) For $A \subseteq \alpha$, we set

$$P'_A = \{p \in P_\alpha | \mathrm{dom}(p) \subseteq A \wedge \forall \gamma (\gamma \in \mathrm{dom}(p) \to \mathrm{supt}(p(\gamma)) \subseteq A)\}.$$

(e) $A \subseteq \alpha$ is called $\bar{Q}$-closed or called $\langle A_\gamma \mid \gamma \in \alpha^* \rangle$-closed if

$$\forall \alpha \in A\ (|A_\alpha| < \kappa \to A_\alpha \subseteq A).$$

So, in our situation of Definition 2.2, where all non-empty A_α have size $\chi \geqslant \kappa$, any $A \subseteq \chi + \mu$ is $\langle A_\alpha \mid \alpha < \chi + \mu \rangle$-closed.

Fact 2.4. *Let $\bar{Q}^\chi$ be in $\mathscr{K}$ from Definition 2.2.*

(1) *If $\alpha \leqslant \chi + \mu$ and $\underset{\sim}{X}$ is a P_α-name of a subset of $\theta < \chi + \mu$ then there is a set $A \subseteq \alpha$ such that $|A| \leqslant \theta$ and $\Vdash_{P_\alpha}$ "$\underset{\sim}{X} \in V[\underset{\sim}{\tau}_\gamma \mid \gamma \in A]$". Moreover, for each $\zeta < \theta$ there is in V a Borel function $\mathscr{B}_\zeta(x_0, x_1, \ldots)$ with domain and range the set $\{\mathrm{true}, \mathrm{false}\}$ and $\gamma_\ell \in A$, $\zeta_\ell < \mu_\ell$ for $\ell \in \omega$ such that*

$$\Vdash_{P_\alpha} \quad \text{“}\zeta \in \underset{\sim}{X} \text{ iff } \mathrm{true} = \mathscr{B}_\zeta((\mathit{truth\ value}(\zeta_\ell \in \underset{\sim}{\tau}_{\gamma_\ell}[G_{Q_{\gamma_\ell}}]))_{\ell \in \omega})\text{”}.$$

(2) *For $\bar{Q} \in \mathscr{K}$ and $A \subseteq \alpha$ every real in $V[\underset{\sim}{\tau}_\gamma \mid \gamma \in A]$ has the form*

$$(\mathscr{B}_n((\mathit{truth\ value}(\zeta_\ell \in \underset{\sim}{\tau}_{\gamma_\ell}[G_{Q_{\gamma_\ell}}]))_{\ell \in \omega}))_{n \in \omega}$$

with $\mathscr{B}_n$ as in (1), *and "true" interpreted by* 1 *and "false" interpreted by* 0.

Proof. (1) Let $\underset{\sim}{X}$ be a name for a subset of θ. Let ρ be a regular cardinal, and let the relation $<^*_\rho$ be a well-ordering of $H(\rho)$ such that $x \in y$ implies that $x <^*_\rho y$. Take ρ such that $(\bar{Q}, \theta, \underset{\sim}{X}) \in H(\rho)$; let M be an elementary submodel of $\mathscr{H}(\rho) = (H(\rho), \in, <^*_\rho)$ to which $\{\bar{Q}, \underset{\sim}{X}, \theta\}$ belongs and such that $\theta \subseteq H(\rho)$.

Thus, $\Vdash_{P_{\alpha^*}}$ "$M[G_{P_{\alpha^*}}] \cap H(\rho) = M$". Since $V^{P_\alpha} = V[\underset{\sim}{\tau}_\beta \mid \beta \in \alpha]$ we have that $M[\underset{\sim}{G}_{P_{\alpha^*}}] = M[\langle \tau_\beta \mid \beta \in \alpha \cap M \rangle]$. So $X \in M[\langle \tau_\beta \mid \beta \in \alpha \cap M \rangle]$, and we may choose a name for $\underset{\sim}{X}$ of the form $\underset{\sim}{X} = \{(\zeta, p) \mid \zeta \in \mu, p \in C_\zeta\}$, where C_ζ is a maximal antichain in $V[\underset{\sim}{\tau}_\gamma \mid \gamma \in \alpha \cap M]$ and from that we can build a Borel function $\mathscr{B}_\zeta$ in V such that

$$\Vdash_{P_\alpha} \quad \text{“}\zeta \in \underset{\sim}{X} \Leftrightarrow \mathscr{B}_\zeta(\langle \mathrm{truth\ value}(\xi_\ell \in \underset{\sim}{\tau}_{\beta_\ell}) | \ell \in \omega \rangle) = 1\text{”},$$

where all the $\beta_\ell \in \alpha \cap M$.

Hence we have that $\Vdash_{P_\alpha}$ "$\underset{\sim}{X} \in V[\underset{\sim}{\tau}_\gamma \mid \gamma \in M \cap \alpha]$".

(2) is a special case of (1) with $\theta = \omega$. We may clue the $\mathscr{B}_n$, $n \in \omega$, together to one Borel function is this case, and write all the arguments into all $\mathscr{B}_n$. $\square$

We are going to combine the techniques of Shelah [20, 18]. We use automorphisms of P_{α^*} that stem from permutations of $\lg(\bar{Q}) = \alpha^*$.

Definition 2.5. (1) For $\bar{Q} \in \mathscr{K}$ of the special form of Definition 2.2 Part (2), $\alpha < \alpha^*$, we let

$$\begin{aligned} \mathrm{AUT}(\bar{Q} \restriction \alpha) = \{ f : \alpha \to \alpha \mid f \text{ is bijective, and,} \\ (\forall \beta \in \alpha)(\forall \gamma \in [\chi, \alpha)) \\ ((\beta < \chi \leftrightarrow f(\beta) < \chi) \wedge (\beta \in A_\gamma \leftrightarrow f(\beta) \in A_{f(\gamma)}))\}. \end{aligned}$$

(2) We let for $f : \alpha \to \alpha$ the function $\hat{f} : P'_\alpha \to P'_\alpha$ be defined by $p_1 = \hat{f}(p_0)$ if $\mathrm{dom}(p_1) = \{f(\beta) \mid \beta \in \mathrm{dom}(p_0)\}$, $p_1(f(\beta)) = \mathscr{B}^\beta_{p_0}((\text{truth value}(f(\zeta_\ell) \in \underset{\sim}{\tau}_{f(\gamma_\ell)}))_{\ell\in\omega})$, where $p_0(\beta) = \mathscr{B}^\beta_{p_0}((\text{truth value}(\zeta_\ell \in \underset{\sim}{\tau}_{\gamma_\ell}))_{\ell\in\omega})$. (Here, we write $\mathscr{B}$ for $(\mathscr{B}_\zeta)_{\zeta\in\mu}$ when $Q_\beta = \mu$.)

We can also naturally extend $\hat{f}$ onto the set of all P'_α-names and name this extension $\hat{f}$ as well.

Now we have for $\bar{Q} \in \mathscr{K}$.

Lemma 2.6 (cf. Shelah [18, Fact 1.6 parts (4) and (5)]). (1) *For $f \in \mathrm{AUT}(\bar{Q} \restriction \alpha)$ we have that $\hat{f}$ is an automorphism of P'_α.*

(2) *Let $\otimes_{(\bar{Q},A)}$ be the following*

$$\otimes_{(\bar{Q},A)} \quad \begin{array}{l} \textit{For every } \alpha \in A \cap [\chi, \chi + \mu) \textit{ and for every countable} \\ B \subseteq \alpha \textit{ there is some } f \in \mathrm{AUT}(\bar{Q} \restriction \alpha) \textit{ such that} \\ f \restriction (A \cap B) = id, \qquad f''(B) \subseteq A, \qquad f''(B \cap A_\alpha) \subseteq A \cap A_\alpha. \end{array}$$

If A is $\bar{Q}$-closed and $\otimes_{(\bar{Q},A)}$, then $P'_A \lessdot P'_{\mathrm{lg}(\bar{Q})}$, and $\forall q \in P'_{\mathrm{lg}(\bar{Q})}$ we have
(a) *$q \restriction A \in P'_A$,*
(b) *$P'_{\mathrm{lg}(\bar{Q})} \models q \restriction A \leqslant q$,*
(c) *if $q \restriction A \leqslant p \in P'_A$, then $q' = p \cup q \restriction (\mathrm{lg}(\bar{Q}) \setminus A)$ belongs to $P'_{\mathrm{lg}(\bar{Q})}$ and is the lub of p, q.*

Proof. (1) is easy. (2) is carried out as in [18], but since we promised to write the proofs in a self-contained style, we write down a proof here.

We prove by induction on $\beta \leqslant \mathrm{lg}(\bar{Q})$ that for $A' = A \cap \beta$ and $q \in P'_\beta$, clauses (a)–(c) hold.

In successor stages $\beta = \alpha + 1$, if $\alpha \notin A$ or $A_\alpha = \emptyset$ it is trivial. So assume that $\alpha \in A$ and $A_\alpha \neq \emptyset$. By induction hypothesis, $P'_{A\cap\alpha} \lessdot P_\alpha$ and the analogues of (a)–(c) hold for stage α. It is enough to show

(*) if in $V^{P'_{A\cap\alpha}}$, $\mathscr{I}$ is a maximal antichain in $\mathrm{Random}^{V^{P'_{A\cap\alpha\cap A_\alpha}}}$, then in $V^{P'_\alpha}$ the set $\mathscr{I}$ is a maximal antichain in $\mathrm{Random}^{V^{P'_{A_\alpha}}}$.

By the c.c.c. this is equivalent to

$(*)'$ if $\zeta^* < \omega_1$, $\{p_\zeta \mid \zeta < \zeta^*\} \subseteq P'_{A\cap(\alpha+1)}$, $p \in P'_{A\cap\alpha}$, and $p \Vdash_{P'_{A\cap\alpha}}$ "$\{p_\zeta(\alpha) \mid \zeta < \zeta^*$ and $p_\zeta \restriction \alpha \in G_{P'_{A\cap\alpha}}\}$ is a predense subset of $\mathrm{Random}^{V^{P'_{A\cap\alpha\cap A_\alpha}}}$",

then $p \Vdash_{P'_\alpha}$ "$\{p_\zeta(\alpha) \mid \zeta < \zeta^*$ and $p_\zeta \restriction \alpha \in G_{P'_\alpha}\}$ is a predense subset of $\mathrm{Random}^{V^{P'_{A_\alpha}}}$".

Assume that $(*)'$ fails. So we can find q such that

$$p \leqslant q \in P'_\alpha,$$

$$q \Vdash_{P'_\alpha} \text{“}\{p_\zeta(\alpha) | \zeta < \zeta^* \text{ and } p_\zeta \restriction \alpha \in G_{P'_\alpha}\}$$

$$\text{is not a predense subset of } \mathrm{Random}^{V^{P'_{A_\alpha}}}\text{”}.$$

So for some $G_{P'_{A_\alpha}}$-name $\underset{\sim}{r}$

$q \Vdash_{P'_{A_\alpha}}$ "$\underset{\sim}{r} \in \mathrm{Random}^{V^{P'_{A_\alpha}}} (= \underset{\sim}{Q}_\alpha)$ and is incompatible with every $p_\zeta(\alpha) \in \underset{\sim}{Q}_\alpha$". Possibly increasing q w.l.o.g. $\underset{\sim}{r} = \mathscr{B}((\text{truth value}(\eta_\gamma \in \underset{\sim}{\tau}_\gamma))_{\gamma\in w})$ with a suitable countable $w \subseteq A_\alpha$. Now we choose

$$B = \mathrm{dom}(q) \cup \bigcup_{\zeta<\zeta^*} \mathrm{dom}(p_\zeta \restriction \alpha) \cup \bigcup \{\mathrm{supt}(q(\beta)) | \beta \in \mathrm{dom}(q)\}$$
$$\cup \bigcup \{\mathrm{supt}(p_\zeta(\beta)) | \beta \in \mathrm{dom}(p_\zeta \restriction \alpha) \text{ and } \zeta < \zeta^*\} \cup w.$$

Since B is a countable subset of α and since we have $\otimes_{(\bar{Q},A)}$ there is an $f \in \mathrm{AUT}(\bar{Q} \restriction \alpha)$ such that

$$f \restriction (B \cap A) = \text{the identity},$$
$$f''(B) \subseteq A,$$
$$f''(B \cap A_\alpha) \subseteq A \cap A_\alpha.$$

As $\hat{f}$ is a automorphism of P'_α and is the identity on $P_{A\cap B}$ we have that

$$\hat{f}(p) = p,$$
$$\hat{f}(p_\zeta) = p_\zeta,$$
$$p \leqslant \hat{f}(q) \in P'_{A\cap\alpha},$$
$$\hat{f}(\underset{\sim}{r}) = \mathscr{B}((\text{truth value}(\eta_\gamma \in \underset{\sim}{\tau}_{f(\gamma)}))_{\gamma\in w}),$$
$$f''(w) \subseteq f''(B \cap A_\alpha) \subseteq A \cap A_\alpha,$$
$$\text{hence } \Vdash_{P'_\alpha} \hat{f}(\underset{\sim}{r}) \in \mathrm{Random}^{V^{P'_{A\cap A_\alpha}}},$$
$$\hat{f}(q) \Vdash_{P'_{A\cap\alpha}} \text{“in } \underset{\sim}{Q}_\alpha,\ \hat{f}(\underset{\sim}{r}) \text{ and } p_\zeta(\alpha) \text{ are incompatible for } \zeta < \zeta^*$$

and thus get a contradiction to the fact that we started with a maximal antichain. □

Lemma 2.7. *For $A = E_\xi \cup [\chi, \chi+\xi)$, and for $\bar{Q}$ as in Definition 2.2 Part (2), we have that $\otimes_{(\bar{Q},A)}$ is true.*

Proof. Let $\alpha \in A$ and $B \subseteq \alpha$ be countable. W.l.o.g., we treat here the case when $\alpha \geqslant \chi$. We have to show that there in an f such that

$f : \alpha \to \alpha$ bijective,

$f \restriction \chi : \chi \to \chi$ bijective,

$\forall \beta, \gamma < \alpha \ (\beta \in A_\gamma \leftrightarrow f(\beta) \in A_{f(\gamma)})$.

(These first three items ensure that $f \in \mathrm{AUT}(\bar{Q} \restriction \alpha)$, and next we write the conditions in $\otimes_{(\bar{Q},A)}$:)

$$f \restriction ((E_\xi \cap B) \cup ([\chi, \xi) \cap B)) = id,$$

$$f''(B) \subseteq E_\xi \cup [\chi, \alpha),$$
$$\forall \alpha \in [\chi, \chi+\xi) \ f''(B \cap (E_{\alpha-\chi} \cup [\chi, \alpha))) \subseteq (E_\xi \cap E_{\alpha-\chi}) \cup [\chi, \alpha).$$

Next, we require that the f preserves slightly more

$f \restriction [\chi, \alpha) = id$ and hence

$\forall \beta \in [\chi, \alpha] \ f \restriction E_{\beta-\chi} : E_{\beta-\chi} \to E_{\beta-\chi}$.

So, f has to map $(B \backslash E_\xi) \cap E_{\alpha-\chi}$ into $E_\xi \cap E_{\alpha-\chi}$ and $((B \backslash E_\xi) \backslash E_{\alpha-\chi}) \cap \chi$ into $E_\xi \backslash E_{\alpha-\chi}$.

For $\gamma \in \chi$, $\alpha' \in \xi + 1$ we write $tp_{\alpha'}(\gamma) = \{\beta \in \alpha' \mid \gamma \in E_\beta\} = \{\beta \in \alpha' \mid g(\gamma) \not\ni \beta\}$. All subsets $T \subseteq \alpha'$ such that $|\alpha' \backslash T| < \lambda$ are realised as the type of χ elements because for each $B \in [\mu]^{<\lambda}$ we have χ many γ such that $g_\chi(\gamma) = B$. Since $\alpha - \chi < \xi$, the relation E_ξ does not play a rôle in $tp_{\alpha+1-\chi}(\gamma)$ and so we have that for all such $\alpha + 1 - \chi$-types T

$$\begin{aligned} &|\{\gamma | tp_{\alpha+1-\chi}(\gamma) = T\}| \\ &\quad = |\{\gamma | tp_{\alpha+1-\chi}(\gamma) = T \wedge \gamma \in E_\xi\}| \\ &\quad = |\{\gamma | tp_{\alpha+1-\chi}(\gamma) = T \wedge \gamma \notin E_\xi\}| = \chi. \end{aligned}$$

Hence, there is a bijection f' of χ preserving the $(\alpha + 1 - \chi)$-types and being the identity on $(E_\xi \cap B) \cup [\chi, \alpha)$ but mapping $(B \cap \chi) \backslash E_\xi$ into E_ξ. Then $f = f' \cup id_{[\chi,\alpha)}$ is as required. $\square$

Now we return to the conclusion of Theorem 2.1.

(γ) If $G \subseteq P$ is generic over V_2, then

$V_1[G]$ and $V_2[G]$ have the same reals, indeed $({}^\omega V_1[G])^{V_2[G]} \subseteq V_1[G]$,

$V_1[G]$ and $V_2[G]$ have the same cardinals if (V_1, V_2) have,

$V_1[G]$ and $V_2[G]$ have the same cofinality function if (V_1, V_2) have.

Since Cohen forcing and random forcing are σ-linked, the proof of Theorem 1.1 applies here as well. $\square$

Next we show

(δ') $V_2 \models \Vdash_{P_{\chi+\mu}}$ "$\{\tau_{\chi+i} \,|\, i \in C\}$ is not null".

Proof. Let $\underset{\sim}{\eta} \in V_2$ be a $P_{\chi+\mu}$-name for a Borel null set. Since $({}^\omega V_1)^{V_2} \subseteq V_1$ we may assume that $\underset{\sim}{\eta} \in V_1$. By 2.4(2), for some Borel function $\mathscr{B} \in V_1$ for some countable

$X=\{x_\ell \mid \ell \in \omega\} \subseteq \chi, Y=\{y_\ell \mid \ell \in \omega\} \subseteq \mu,\ \zeta_\ell, \ell \in \omega,\ \zeta'_\ell, \ell \in \omega$, we have that

$$\underset{\sim}{\eta} = \mathscr{B}((\text{truth value}(\zeta_\ell \in \underset{\sim}{\tau}_{x_\ell}))_{\ell\in\omega}, (\text{truth value}(\zeta'_\ell \in \underset{\sim}{\tau}_{\chi+y_\ell}))_{\ell\in\omega}).$$

Let $i(*)<\mu$ be such that $i(*)>\sup(Y)$. (Here we use that $\mathrm{cf}^{V_1}(\mu)>\aleph_0$.) Since $\mathrm{cf}^{V_1}(\lambda)>\aleph_0$, we have that $B:=\bigcup_{\xi\in X} g_\chi(\xi)\in([\mu]^{<\lambda})^{V_1}$. Since $\sup(C\backslash B)=\mu$, there is some $i\geqslant i(*)$, $i\in C\backslash B$. We claim, that $r_{\chi+i}$ is random (in the sense of V_1 and hence also in the sense of V_2 as Random and all maximal (countable) antichains of the random forcing are the same in V_1 and in V_2) over an extension of V_1, in which $\underset{\sim}{N}[G]$ has a name. Then the proof will be finished, because then $r_{\chi+i}\notin \underset{\sim}{N}[G]$ in $V_1[G]$ and also in $V_2[G]$. By our construction, we have

$$\tau_{\chi+i} \quad \text{is the Random}^{V[\underset{\sim}{\tau}_\alpha \mid \alpha\in E_i \vee \chi\leqslant\alpha<\chi+i]}\text{-generic over } V_1^{P_{\chi+i}}.$$

Since $i\in C\backslash B$, we have that $\forall \xi\in X\ g_\chi(\xi)\neq i$, hence $\forall\xi\in X \xi\in E_i$, so $X\subseteq E_i$. Moreover $\chi+Y\subseteq[\chi,\chi+i)$, as $i\geqslant i(*)\geqslant\sup(Y)$. Since, by Lemmas 2.6 and 2.7, $P_{A_{\chi+i}} \lessdot P_{\lg(\bar{Q})}$ the name $\underset{\sim}{N}$ is evaluated in the right manner in $V_1^{P_{A_{\chi+i}}}$. Thus the claim is proved. □

(δ) $V_2[G]\models \mathrm{unif}(\mathscr{N})\leqslant|\mathscr{C}|$. This follows from (δ′).

Now comes the part whose proof will be finished only at the end of Section 5.

(ε) $V_1[G]\models \mathrm{unif}(\mathscr{N})\geqslant\lambda$.

Proof. Suppose the contradiction. In V_1 there is $i(*)<\lambda$ and $p\in P_{\chi+\mu}$ such that

$$p \Vdash_{P_{\chi+\mu}} \text{“}\underset{\sim}{\eta}_i \in {}^{\omega}2 \text{ for } i<i(*) \wedge \{\underset{\sim}{\eta}_i \mid i<i(*)\} \text{ is not null”}.$$

A name of a real in $V_1[G]$ is given by

$$\underset{\sim}{\eta}_i = \mathscr{B}_i((\text{truth value}(\zeta_{i,\ell} \in \underset{\sim}{r}_{j_{i,\ell}}))_{\ell\in\omega})$$

for suitable $\langle \zeta_{i,\ell}, j_{i,\ell} \mid \ell\in\omega\rangle$, $\zeta_{i,\ell}\in\omega$, $j_{i,\ell}\in\chi+\mu$.

We set

$$X=\{j_{i,\ell} \mid i\in i(*), \ell\in\omega\}\cap\chi,$$

$$Y=\{j_{i,\ell} \mid i\in i(*), \ell\in\omega\}\cap[\chi,\chi+\mu).$$

We show the main point.

In $V_1[G]$, $({}^{\omega}2)^{V[\{\underset{\sim}{\tau}_\xi \mid \xi\in X\cup Y\}]}$ is a Lebesgue null set.

Since $\exists^\chi \alpha\ g_\chi(\alpha)=Y-\chi$ we can fix such an $\alpha\in\chi\backslash X$ that is not in E_ξ for every $\xi\in Y-\chi$. It is important to note that therefore the premises of Lemma 2.8 and or Lemma 2.11 can be fulfilled for our any X,Y as above, with a suitable choice of α.

Lemma 2.8. *In $V_1^{P_{\alpha*}}$, the set $({}^{\omega}2)^{V_1[\tau_\xi \mid \xi\in X\cup Y]}$ has Lebesgue measure 0, and a witness for a definition for a measure zero superset can be found in $V^{P_{\alpha+1}}$ (a forcing name is already in V^{P_α}) for any $\alpha\in\chi\backslash X$ that is not in E_ξ for every $\xi\in Y-\chi$.*

Proof. *Explanation*: This proof will be finished only with the proof of Lemma 2.11, which will, as we already mentioned, only be finished by the end of Section 5. The proof of this lemma requires reworking of almost the whole [20]. The lemma is also stated in [18, 1.11, 1.12], where a proof assuming the knowledge of Shelah [20] is given.

First, we introduce some paradigm null sets (see also [20, 2.4, 2.5]):

Definition 2.9. (1) Suppose that $\bar{a} = \langle a_\ell \mid \ell \in \omega \rangle$ and $\bar{n} = \langle n_\ell \mid \ell \in \omega \rangle$ are such that for $\ell \in \omega$

(a) $a_\ell \subseteq {}^{n_\ell}2$,
(b) $n_\ell < n_{\ell+1} < \omega$,
(c) $|a_\ell|/2^{n_\ell} > 1 - 1/10^\ell$.

Then we set $N[\bar{a}] = \{\eta \in {}^\omega 2 \mid \exists^\infty \ell \;\; \forall \nu \in a_\ell \, \nu \not\trianglelefteq \eta\}$.

(2) For $\bar{a}$ as above and $n \in \omega$, we let $\mathrm{tree}_n(\bar{a}) = \{\nu \in {}^{<\omega}2 \mid n_\ell \geqslant \max(n, \lg(\nu)) \rightarrow \nu \restriction n_\ell \in a_\ell\}$.

Then $N[\bar{a}] = {}^\omega 2 \setminus \bigcup_{n \in \omega} \lim \mathrm{tree}_n(\bar{a})$ and $\mathrm{Leb}(N[\bar{a}]) = 0$. The definitions $N[\bar{a}]$ and $\lim \mathrm{tree}_n(\bar{a})$ may be intepreted in any model V such that $\bar{a} \in V$. We indicate the model of set theory in which we evaluate them by superscripts.

Definition 2.10. For $\beta < \chi$ we identify Q_β, the Cohen forcing, with

$$\left\{ \langle (a_\ell, n_\ell) \mid \ell < k \rangle \mid k \in \omega, \; n_\ell < n_{\ell+1} < \omega, \; a_\ell \subseteq {}^{n_\ell}2, \frac{|a_\ell|}{2^{n_\ell}} > 1 - \frac{1}{10^\ell} \right\}.$$

If G_{Q_β} is Q_β-generic, let

$$\bar{a}^\beta = \underset{\sim}{\bar{a}}^\beta[G_{Q_\beta}] = \{(\ell, a) \mid \exists k \geqslant \ell + 1 \exists \langle (a_j, n_j) \mid j < k \rangle \in G_{Q_\beta} \exists j < k (\ell, a) = (j, a_j)\}$$

and define $\underset{\sim}{\bar{n}}^\beta[G_{Q_\beta}]$ analogously. We let $\underset{\sim}{\bar{a}}^\beta = \langle \underset{\sim}{a}^\beta_\ell \mid \ell \in \omega \rangle$ and $\underset{\sim}{\bar{n}}^\beta = \langle \underset{\sim}{n}^\beta_\ell \mid \ell \in \omega \rangle$ be the names for the corresponding objects.

Lemma 2.11. *If* $\beta \in \chi \setminus X$ *is such that* $\forall \xi \in Y - \chi \beta \notin E_\xi$, *then*

$$({}^\omega 2)^{V[r_\xi \mid \xi \in X \cup Y]} \subseteq (N[\bar{a}^\beta])^{V[G]}.$$

Proof (*Beginning*). In this section, we shall only show that

$$(**)_{\bar{Q}} \qquad \begin{array}{l} \text{in } V[G], \text{ for } E \in [\chi]^{\kappa^+} \text{ we have} \\ \bigcap_{\beta \in E} \mathrm{tree}_{\ell^*}(\bar{a}^\beta) \text{ does not contain a perfect tree} \end{array}$$

is a sufficient condition for Lemma 2.11. For certain members $\bar{Q}$ of $\mathscr{K}$, $(**)_{\bar{Q}}$ will be proved in the next three sections. Let $\beta \in \chi \setminus X$ be such that $\forall \xi \in Y - \chi \beta \notin E_\xi$.

We show by induction on $\gamma \geqslant \chi$ that

$$(**)_{\bar{Q}\restriction\gamma} \quad \begin{array}{l}\text{in } V^{P_\gamma}, \text{ for } E \in [\chi]^{\kappa^+} \text{ we have} \\ \bigcap_{\beta\in E} \mathrm{tree}_{\ell^*}(\bar{a}^\beta) \text{ does not contain a perfect tree}\end{array}$$

implies

$$(*)_{\bar{Q}\restriction\gamma} \quad \begin{array}{l}\forall X \subseteq \chi \forall Y \subseteq [\chi, \chi+\mu) \\ \forall \beta \in \chi \setminus X (\forall \xi \in Y - \chi \beta \notin E_\xi \to ({}^\omega 2)^{V[r_\xi \mid \xi \in (X\cup Y)\cap\gamma]} \subseteq (N[\bar{a}^\beta])^{V^{P_\gamma}}).\end{array}$$

Preliminary remarks: Assuming $\neg(*)_{\bar{Q}\restriction\gamma}$ we get a P_γ-name $\underset{\sim}{b}$ referring only to r_ξ, $\xi \in (X \cup Y) \cap \gamma$ such that

$$p \Vdash_{P_\gamma} \underset{\sim}{b} \notin N[\underset{\sim}{\bar{a}}^\beta].$$

Since $\forall \xi \in Y - \chi \beta \notin E_\xi$, we have for all $\xi' = \chi + \xi \in Y$, $\beta \notin E_\xi \cup [\chi, \chi+\mu) = A^\chi_{\xi'}$. Since all $r_{\xi'}$, $\xi' \in Y$ are $\mathrm{Random}^{V^{P_{A_{\xi'}}}}$-generic there are automorphisms $f_\zeta \in AUT(\bar{Q})$, $\zeta \in \chi$, leaving $\underset{\sim}{b}$ and every point from $[\chi, \chi+\mu)$ fixed and moving β to $\beta_\zeta \notin \{\beta_{\zeta'} \mid \zeta' < \zeta\}$. Hence, we get

$$p_\zeta = \hat{f}_\zeta(p) \Vdash_{P_\gamma} \underset{\sim}{b} \notin \bigcup_{\zeta\in\chi} N[\underset{\sim}{\bar{a}}^{\beta_\zeta}]$$

for $\chi \geqslant \kappa^+$ pairwise different β_ζ's.

Now we start the induction.

For $\gamma = \chi$ the proof is easy, because $({}^\omega 2)^{V[r_\xi \mid \xi \in (X\cup Y)\cap\chi]}$ contains only Cohen reals: if there is one real $\underset{\sim}{b}[G_\gamma]$ not in $(\bigcup_{\zeta\in\kappa^+} N[\bar{a}^{\beta_\zeta}])^{V^{P_\gamma}}$, then this real is Cohen and gives rise to a perfect tree of Cohen reals not in $(\bigcup_{\zeta\in\kappa^+} N[\bar{a}^{\beta_\zeta}])^{V^{P_\gamma}}$. So we have that $\neg(*)_{\bar{Q}\restriction\gamma}$ implies $\neg\ (**)_{\bar{Q}\restriction\gamma}$.

Now let $\gamma > \chi$ be a limit. Assuming $\neg(*)_{\bar{Q}\restriction\gamma}$ we get a P_γ-name $\underset{\sim}{b}$ referring only to r_ξ, $\xi \in (X \cup Y) \cap \gamma$ such that

$$p \Vdash_{P_\gamma} \underset{\sim}{b} \notin N[\underset{\sim}{\bar{a}}^\beta].$$

By automorphisms leaving $\underset{\sim}{b}$ and moving β to β_ζ and p to p_ζ we get

$$p_\zeta \Vdash_{P_\gamma} \underset{\sim}{b} \notin \bigcup_{\zeta\in\chi} N[\underset{\sim}{\bar{a}}^{\beta_\zeta}]$$

for χ pairwise different β_ζ's.

Because of the induction hypothesis we may assume that $p \Vdash_{P_\gamma} \underset{\sim}{b} \notin V^{P_\delta}$ for $\delta < \gamma$, and hence by the properties of c.c.c. iterations that $\mathrm{cf}(\gamma) = \aleph_0$.

So for each $\zeta < \chi$ there are p_ζ, m_ζ such that

$$p \leqslant p_\zeta \in P_\gamma, \quad p_\zeta \Vdash \underset{\sim}{b} \in \lim \mathrm{tree}_{m_\zeta}(\underset{\sim}{\bar{a}}^{\beta_\zeta}).$$

By properties of c.c.c. forcing notions $\langle\{\zeta<\chi \mid p_\zeta \in P_\delta\} \mid \delta\in\gamma\rangle$ is an increasing sequence of subsets of χ of length $\gamma\leqslant\mu$. In the beginning on the proof of Theorem 2.1 we chose $\mu<\chi$. So for some $\gamma_1<\gamma$ there is $E\in[\chi]^{\kappa^+}$ such that $p_\zeta\in P_{\gamma_1}$ for $\zeta\in E$ and $m_\zeta=m$ for $\zeta\in E$. Note that for all but $<\kappa^+$ of the ordinals $\eta\in E$ we have that

$$p_\eta \Vdash |\{\zeta\in E | p_\zeta \in G_{P_{\gamma_1}}\}|=\kappa^+.$$

Fix such an η, and let $G_{P_{\gamma_1}}$ be P_{γ_1}-generic over V so that $p_\eta\in G_{P_{\gamma_1}}$. In $V[G_{P_{\gamma_1}}]$, let $E'=\{\zeta\in E \mid p_\zeta\in G_{P_{\gamma_1}}\}$, so $|E'|=\kappa^+$. Let $T^*=\bigcap_{\zeta\in E'}\mathrm{tree}_m(\bar{a}^{\beta_\zeta})$. In V^{P_γ}, T^* is a subtree of ${}^{<\omega}2$ and by $(**)_{\bar{Q}\restriction\gamma}$, T^* contains no perfect subtree. Hence $\lim(T^*)$ is countable, so absolute: T^* is a P_{γ_1}-name and $(\lim(T^*))^{V[G_{P_\gamma}]}=(\lim(T^*))^{V[G_{P_{\gamma_1}}]}$. But $p_\eta\Vdash \underset{\sim}{b}\in\lim(T^*)$, hence $p_\eta\Vdash\underset{\sim}{b}\in V^{P_{\gamma_1}}$, a contradiction.

Assume now that $\gamma=\delta+1$ and that $\neg(*)_{\bar{Q}\restriction\gamma}$. Choose $p_\zeta=p'_\zeta * \underset{\sim}{q}_\delta(\zeta)$ as in the preliminary remark such that $p_\zeta\in P_\delta$, $\underset{\sim}{q}_\delta(\zeta)\in\underset{\sim}{Q}_\delta$, and additionally such that the $\underset{\sim}{q}_\delta(\zeta)$ all coincide (because we may assume that f_ζ, chosen as in the preliminary remarks, does not move δ), say that all $\underset{\sim}{q}_\delta(\zeta)=\underset{\sim}{q}_\delta$. Choose E, p_η, G_{P_γ} analogous to the above. We have $E'=\{\zeta\in E \mid p'_\zeta\in G_{P_\delta}\}=\{\zeta\in E \mid p'_\zeta * \underset{\sim}{q}_\gamma\in G_{P_\delta}\}$, and similarly to the above, together with $(**)_{\bar{Q}\restriction\gamma}$ we get the contradiction $p_\eta\Vdash\underset{\sim}{b}\in V^{P_\delta}$.

Since we have covered the cases $\gamma=\chi$ and $\gamma>\chi$ limit and $\gamma>\chi$ successor, we have finished the proof that $(**)_{\bar{Q}}$ implies the statement in Lemma 2.11.

Our proof of $(**)_{\bar{Q}}$ will in some parts be similar to Shelah [20]. However, the difference to Shelah [20] is that the our $A^\chi_\alpha, \alpha\in[\chi,\chi+\mu)$ (from Definition 2.2, Part 2) are large in cardinality, namely the same as the iteration length, and hence some techniques of Shelah [20] are not applicable here. We also take the technique of automorphisms of $\bar{Q}$ taken from Shelah [18], and additionally, like there as well, we are going to work $\bar{Q}^\chi$ for many χ's at the same time. Tomek Bartoszyński [1] gives a simplified exposition of some of the results of Shelah [20], that the reader might want to consult first.

The proof of Lemma 2.11 will be finished only at the end of Section 5.

In the next lemma, which stems from Winfried Just, we show $(**)_{\bar{Q}}$ in the special case that all the p_ζ are Cohen. It serves as a motivation for the rest of our work: it shows that the main point is to get something similar to the premise no. 3 of Just's lemma for the partial random conditions. We may (and later do) weaken the conclusion of Just's lemma: Instead of requiring the intersection to be empty we derive only that the intersection does not contain a perfect tree, that is $(**)_{\bar{Q}}$.

Lemma 2.12 (Winfried Just [12]). *Suppose that $\{p_\zeta \mid \zeta\in Z\}$ is a set of conditions in $P_{\chi+\mu}$ such that*
1. *Z is infinite.*
2. *$\{\mathrm{dom}(p_\zeta) \mid \zeta\in Z\}$ forms a Δ-system with root u.*
3. *$\exists q\ \forall\zeta\in Z p_\zeta\restriction u=q$.*
4. *$\beta_\zeta\in\mathrm{dom}(p_\zeta)\setminus u$ for all ζ, $p_\zeta(\beta_\zeta)$ is Cohen.*
5. *$\exists k^*, n^*$ such that $\forall\zeta\in Z$, if $p_\zeta(\beta_\zeta)=\langle(n^\zeta_\ell, a^\zeta_\ell) \mid \ell\in k_\zeta\rangle$ then $k_\zeta=k^*$ and $n^\zeta_{k_\zeta-1}=n^*$.*

We set $\underset{\sim}{E} = \{\zeta \in Z \mid p_\zeta \in G\}$. *Then we have for every* $\ell^* \in \omega$ *that*

$$q \Vdash \bigcap_{\zeta \in \underset{\sim}{E}} \lim \operatorname{tree}_{\ell^*}(\bar{\underset{\sim}{a}}^{\beta_\zeta}) = \emptyset.$$

Proof. Suppose the contradiction. Then there exist some ℓ^* and some $q_1 \geqslant q$ and some name $\underset{\sim}{b}$ for an infinite branch such that

$$q_1 \Vdash \underset{\sim}{b} \in \bigcap_{\zeta \in \underset{\sim}{E}} \lim \operatorname{tree}_{\ell^*}(\bar{\underset{\sim}{a}}^{\beta_\zeta}).$$

Let $n > \max\{k^* - 1, n^*\}$ and such that $2^{-n} < 10^{-k^*}$. There are some $r \geqslant q_1$ and some v such that

$$r \Vdash \underset{\sim}{b} \restriction n = v.$$

Now take some ζ such that $\operatorname{dom}(p_\zeta) \cap \operatorname{dom}(r) = u$. Since Z is infinite and all conditions are bounded in size by k^*, n^*, such a ζ exists. Finally, we set $n^\zeta_{k^*} = n$ and $a^\zeta_n = 2^n \setminus \{v\}$ and

$$p^+_\zeta = p_\zeta \restriction (\operatorname{dom}(p_\zeta) \setminus \{\beta_\zeta\}) \cup \{(\beta_\zeta, \langle n^\zeta_\ell, a^\zeta_\ell \mid \ell \leqslant n \rangle)\}.$$

Since $v \notin a^\zeta_n$, we get

$$p^+_\zeta \Vdash \underset{\sim}{b} \in \lim \operatorname{tree}_{\ell^*}(\bar{\underset{\sim}{a}}^{\beta_\zeta}) \rightarrow \underset{\sim}{b} \restriction n \neq v.$$

However, p^+_ζ and r are compatible. Contradiction. □

3. About finitely additive measures

In order to prove the existence of a condition $p^\otimes$ that forces that many of the p_ℓ's (where the p_ℓ, $\ell \in \omega$ are the first ω of some thinned out part of the p_ζ from Lemma 2.11) are in G_{α^*} we use names $(\underset{\sim}{\Xi}^t_\alpha)_{t \in \mathscr{T},\ \alpha \in \chi+\mu}$ for finitely additive measures. We shall have that for every $\alpha < \chi + \mu$, $\Vdash_{P_\alpha}$ "$\underset{\sim}{\Xi}^t_\alpha$ is a finitely additive measure on $\mathscr{P}(\omega)$". The superscript t ranges over some set of blueprints (see Definition 4.1) and indicates the type of the ω conditions p_ℓ that are taken care of by $\underset{\sim}{\Xi}^t_\alpha$, and there are some coherence requirements regarding different α's. The $\underset{\sim}{\Xi}^t_\alpha$ are an item in the class of forcing iterations $\mathscr{K}^3$ that we are going to define in Definition 4.2. Certain members of $\mathscr{K}$ can be expanded to members of $\mathscr{K}^3$, and these expandible members of $\mathscr{K}$ are the notions of forcing for which we show $(**)_{\bar{Q}}$ is Sections 4 and 5.

For the expansion of a $\bar{Q}$ in $\mathscr{K}$ to a member of $\mathscr{K}^3$ some requirements linking the A_α and the $\underset{\sim}{\Xi}^t_\alpha$ need to be fulfilled (called "whispering" in [20, Definition 2.11(i)]). By increasing the A_α these can be satisfied. Another way is to use the requirements only at finitely many points that are determined at a later stage in a proof. We shall work according this latter method: In our case, where we have also automorphisms as in Fact 2.4, we shall first specify som $\langle p_\ell \mid \ell \in \omega \rangle$, and only thereafter we shall define sufficiently many $\underset{\sim}{\Xi}^t_\alpha$ (see Theorem 5.5).

Anyway, the "sufficiently many $\underset{\sim}{\Xi}^t_\alpha$" need the same lemmas about extensions of finitely additive measures to longer iterations that are also used to proof that our class $\mathscr{K}^3$ of forcings has enough members. These will be Lemmas 4.5–4.7.

This short section collects some facts about finitely additive measures, that can be presented separately before we return to the iterated forcings in $\mathscr{K}$ and come to the mentioned lemmas. All statements of this section, however only few of their proofs, can also be found in [20].

Definition 3.1. (1) $\mathscr{M}$ is the set of functions Ξ from some Boolean subalgebra P of $\mathscr{P}(\omega)$ including the finite sets to $[0,1]_{\mathbb{R}}$ such that

- $\Xi(\emptyset)=0$, $\Xi(\omega)=1$,
- Ξ is finitely additive, that is: If $Y, Z \in P$ are disjoint, then $\Xi(Y \cup Z) = \Xi(Y) + \Xi(Z)$,
- $\Xi(\{n\})=0$ for $n \in \omega$.

Members of $\mathscr{M}$ are called partial finitely additive measures.

(2) $\mathscr{M}^{\text{full}}$ is the set of $\Xi \in \mathscr{M}$ whose domain is $\mathscr{P}(\omega)$, and the members of $\mathscr{M}^{\text{full}}$ are called finitely additive measures.

(3) We write "$\Xi(A)=a$" (or $>a$ or whatever) if $A \in \operatorname{dom}(\Xi)$ and $\Xi(A)=a$ (or $>a$ or whatever).

For extending finitely additive measures we are going to use

Theorem 3.2 (Hahn Banach). *Suppose that Ξ is a partial finitely additive measure on a algebra P and that $X \notin P$. Let $a \in [0,1]$ be such that*

$$\sup\{\Xi(A) \mid A \subseteq X, A \in P\} \leqslant a \leqslant \inf\{\Xi(B) \mid B \supseteq X, B \in P\}.$$

Then there exists a finitely additive measure Ξ^* extending Ξ and such that $\Xi^*(X)=a$.

Proposition 3.3. *Let α^* be an ordinal. Assume that $\Xi_0 \in \mathscr{M}$ and that for $\alpha < \alpha^*$, $A_\alpha \subseteq \omega$ and $0 \leqslant a_\alpha \leqslant b_\alpha \leqslant 1$, a_α, b_α reals. Then we have that*

- (1) $\Rightarrow$ (2)
- (2) $\Rightarrow$ ((3.A) *with all* $b_\alpha = 1$)
- (3.A) $\Leftrightarrow$ (3.B),

where

(1) *If $A^* \in \operatorname{dom}(\Xi_0)$, $\Xi_0(A^*)>0$ and $n \in \omega$ and $\alpha_0 < \cdots < \alpha_{n-1} < \alpha^*$ then $A^* \cap \bigcap_{\ell<n} A_{\alpha_\ell} \neq \emptyset$.*

(2) *$\forall \varepsilon > 0$, $\forall A^* \in \operatorname{dom}(\Xi_0)$ such that $\Xi_0(A^*)>0$, $n \in \omega$, $\alpha_0 < \cdots < \alpha_{n-1} < \alpha^*$ we can find a finite non-empty $u \subseteq A^*$ such that for $\ell \in n$*

$$a_{\alpha_\ell} - \varepsilon \leqslant \frac{|A_{\alpha_\ell} \cap u|}{|u|}.$$

(3.A) *There is* $\Xi \in \mathscr{M}^{\text{full}}$ *extending* Ξ_0 *such that* $\forall \alpha < \alpha^* \, \Xi(A_\alpha) \in [a_\alpha, b_\alpha]$.
(3.B) *for all* $\varepsilon > 0$, *for all* $k \in \omega$, *for all* $\langle A^*_0, \ldots, A^*_{m-1}\rangle$ *partition of* ω *and* $A^*_i \in \mathrm{dom}(\Xi_0)$ *such that* $\Xi_0(A^*_i) > 0$, $n \in \omega$, $\alpha_0 < \cdots < \alpha_{n-1} < \alpha^*$ *we can find a finite non-empty* $u \subseteq \omega \setminus k$ *such that for* $\ell \in n$ *and* $i \in m$

$$a_{\alpha_\ell} - \varepsilon \leqslant \frac{|A_{\alpha_\ell} \cap u|}{|u|} \leqslant b_{\alpha_\ell} + \varepsilon,$$

$$\Xi_0(A^*_i) - \varepsilon \leqslant \frac{|A^*_i \cap u|}{|u|} \leqslant \Xi_0(A^*_i) + \varepsilon.$$

Proof. (1) $\Rightarrow$ (2): Given $\varepsilon, A^*, \alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ we take $k \in A^* \cap \bigcap_{\ell < n} A_{\alpha_\ell}$ and $u = \{k\}$.

(2) $\Rightarrow$ (3.B) with $b_\alpha = 1$: Given $\varepsilon, k, A^*_0, \ldots, A^*_{m-1}$, pairwise disjoint with positive Ξ_0 measure, $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ then we can find finite u_i, $i < m$ such that

$$u_i \subseteq \omega \setminus k,$$

$$u_i \subseteq A^*_i,$$

$$\frac{|u_i|}{|\bigcup_{i \in m} u_i|} \in (\Xi_0(A^*_i) - \varepsilon, \Xi_0(A^*_i) + \varepsilon),$$

$$a_{\alpha_\ell} - \varepsilon \leqslant \frac{|A_{\alpha_\ell} \cap u_i|}{|u_i|}.$$

It is now easy to check that $u = \bigcup_{i<m} u_i$ is as required.

(3.B) $\Rightarrow$ (3.A): This is the special case of a symmetrized variant of Fact 3.6 with $a^\alpha_\ell = 1$ iff $\ell \in A_\alpha$ and $a^\alpha_\ell = 0$ else. This is the most important implication. Its proof is not circular, it just more economic to do Definition 3.4, Proposition 3.5, and Fact 3.6 first.

(3.A) $\Rightarrow$ (3.B): Fix ε' such that $2\ell m \varepsilon' \leqslant \varepsilon$. We put for $i < m$ and $\ell < n$ the first

$$\left\lceil \frac{\Xi(A^*_i \cap A_{\alpha_\ell})}{\varepsilon'} \right\rceil$$

elements of $A^*_i \cap A_{\alpha_\ell}$ into u (and nothing else). It is important to see that the tasks for the different A_{α_ℓ} can be simultaneously fulfilled. Best look for each $i < m$ at the atoms in the Boolean algebra generated by the $A_{\alpha_\ell} \cap A^*_i$, $\ell < n$.

For a real x, $\lceil x \rceil$ is the least integer greater than or equal x. Then it is an easy computation that the $|A^*_i \cap u|/|u|$ and the $|A_{\alpha_\ell} \cap u|/|u|$ are in the right intervals of width 2ε. $\square$

In order to convey information to later stages of our forcing iteration, we are going to use averages. These are integrals of functions from ω to with respect to finitely additive measures. If the average of some function is large then we can go back to some finite subset of ω where the function takes large values.

Definition 3.4. (1) For $\Xi \in \mathscr{M}^{\text{full}}$ and a sequence $\bar{a} = \langle a_\ell \mid \ell \in \omega \rangle$ of reals in $[0,1]_{\mathbb{R}}$ (or just $\sup_{\ell \in \omega} |a_\ell| < \infty$) we let

$$\mathrm{Av}_\Xi(\bar{a}) = \sup \left\{ \sum_{k<k^*} \Xi(A_k) \inf(\{a_\ell \mid \ell \in A_k\}) \mid \langle A_k \mid k < k^* \rangle \text{ is a partition of } \omega \right\}$$

$$= \inf \left\{ \sum_{k<k^*} \Xi(A_k) \sup(\{a_\ell \mid \ell \in A_k\}) \mid \langle A_k \mid k < k^* \rangle \text{ is a partition of } \omega \right\}.$$

(Think of $A_k = \{\ell \mid a_\ell \in [(k/2^n, (k+1)/2^n)\}$ and $n \to \infty$, then it is easy to see that both are equal.)

(2) For $\Xi \in \mathscr{M}$, $A \subseteq \omega$ such that $\Xi(A) > 0$ define $\Xi_A(B) = \Xi(A \cap B)/\Xi(A)$ and $\mathrm{Av}_\Xi(\langle a_k \mid k \in B \rangle) = \mathrm{Av}_{\Xi_B}(\langle a'_k \mid k \in \omega \rangle)$ with

$$a'_k = \begin{cases} a_k & \text{if } k \in B, \\ 0 & \text{if } k \notin B. \end{cases}$$

Proposition 3.5. *Assume that* $\Xi \in \mathscr{M}^{\text{full}}$ *and* $a^i_\ell \in [0,1]_{\mathbb{R}}$ *for* $i < i^* \in \omega$, $\ell \in \omega$, $B \subseteq \omega$, $\Xi(B) > 0$ *and* $\mathrm{Av}_{\Xi_B}(\langle a^i_\ell \mid \ell < \omega \rangle) = b_i$ *for* $i < i^*$, $m^* < \omega$ *and lastly* $\varepsilon > 0$. *Then for some finite* $u \subseteq B \setminus m^*$ *we have*: *if* $i < i^*$ *then*

$$b_i - \varepsilon < \frac{\sum\{a^i_\ell \mid \ell \in u\}}{|u|} < b_i + \varepsilon.$$

Proof. Let $j^* \in \omega$ and $\langle B_j \mid j < j^* \rangle$ be a partition of B such that for every $i < i^*$ we have

$$\left(\sum_{j<j^*} \sup\{a^i_\ell \mid \ell \in B_j\} \Xi(B_j) \right) - \left(\sum_{j<j^*} \inf\{a^i_\ell \mid \ell \in B_j\} \Xi(B_j) \right) < \frac{\varepsilon}{2}.$$

Now choose k^* large enough such that there are k_j satisfying $k^* = \sum_{j<j^*} k_j$ and for $j < j^*$,

$$\left| \frac{k_j}{k^*} - \frac{\Xi(B_j)}{\Xi(B)} \right| < \frac{\varepsilon}{2}.$$

Let $u_j \subseteq B_j \setminus m^*$, $|u_j| = k_j$ for $j < j^*$. Now let $u = \bigcup_{j<j^*} u_j$ and calculate

$$\sum_{\ell \in u} \frac{a^i_\ell}{|u|} = \sum_{j<j^*} \sum_{\ell \in u_j} \frac{a^i_\ell}{|u|} \leqslant \sum_{j<j^*} \sup\{a^i_\ell \mid \ell \in B_j\} \frac{k_j}{k^*}$$

$$\leqslant \sum_{j<j^*} \sup\{|a^i_\ell \mid \ell \in B_j\} \left(\frac{\Xi(B_j)}{\Xi(B)} + \frac{\varepsilon}{2j^*} \right) \leqslant b_i + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = b_i + \varepsilon,$$

$$\sum_{\ell\in u}\frac{a^i_\ell}{|u|}=\sum_{j<j^*}\sum_{\ell\in u_j}\frac{a^i_\ell}{|u|}\geqslant\sum_{j<j^*}\inf\{a^i_\ell\mid\ell\in B_j\}\frac{k_j}{k^*}$$
$$\geqslant\sum_{j<j^*}\inf\{|a^i_\ell\mid\ell\in B_j\}\left(\frac{\Xi(B_j)}{\Xi(B)}-\frac{\varepsilon}{2j^*}\right)\geqslant b_i-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}=b_i-\varepsilon. \qquad\square$$

Fact 3.6. *Assume that Ξ is a partial finitely additive measure and $\bar{a}^\alpha=\langle a^\alpha_k\mid k\in\omega\rangle$ is a sequence of reals for $\alpha<\alpha^*$ such that $\limsup_{k\to\omega}|a^\alpha_k|<\infty$ for each α. Then* (B) $\Rightarrow$ (A).

(A) *There is $\Xi^*\supseteq\Xi$, $\Xi^*\in\mathscr{M}^{\mathrm{full}}$ such that $\mathrm{Av}_{\Xi^*}(\bar{a}^\alpha)\geqslant b_\alpha$ for $\alpha<\alpha^*$.*

(B) *For every partition $\langle B_0,\ldots,B_{m^*-1}\rangle$ of ω with $B_m\in\mathrm{dom}(\Xi)$ and $\varepsilon>0$, $k^*>0$ and $\alpha_0<\cdots<\alpha_{n-1}<\alpha^*$ there is a finite $u\in\omega\backslash k^*$ such that*
 (i) $\Xi(B_m)-\varepsilon<|B_m\cap u|/|u|<\Xi(B_m)+\varepsilon$,
 (ii) $\frac{1}{|u|}\sum_{k\in u}a^{\alpha_\ell}_k>b_{\alpha_\ell}-\varepsilon$ *for* $\ell<n$.

Proof. We take

$$\varDelta=[\{\text{partitions }\langle B_0,\ldots,B_{m^*-1}\rangle\text{ of }\mathrm{dom}(\Xi)\}\times(0,1]\times\omega\times[\alpha^*]^{<\omega}]^{<\omega}$$

and take a filter $\mathscr{F}\subseteq\mathscr{P}(\varDelta)$ such that for each

$$\bar{c}\in\{\text{partitions }\langle B_0,\ldots,B_{m^*-1}\rangle\text{ of }\mathrm{dom}(\Xi)\}\times(0,1]\times\omega\times[\alpha^*]^{<\omega}$$

we have that

$$\{F\in\varDelta\mid\bar{c}\in F\}\in\mathscr{F}.$$

For each $F\in\varDelta$ we choose $u(F)$ fulfilling the tasks (B) simultaneously for all $\bar{c}\in F$, i.e. (i) and (ii) of (B) hold for $u(F)=u$, $\bar{c}(0)=\langle B_0,\ldots,B_{m^*-1}\rangle$, $\bar{c}(1)=\varepsilon$, $\bar{c}(2)=k^*$, $\bar{c}(3)=\{\alpha_0,\ldots,\alpha_{n-1}\}$.

Then we take an ultrafilter $\mathscr{U}\supseteq\mathscr{F}$ and set for A in the algebra $\mathscr{A}$ generated by $\{\{k\mid a^\alpha_k\in[q,q']\}\mid\alpha<\alpha^*,0\leqslant q\leqslant q'\leqslant1\}\cup\mathrm{dom}(\Xi)$

$$\Xi^*(A)=\text{the standard part of }\left(\left\langle\frac{|u(F)\cap A|}{|u(F)|}\,\middle|\,F\in\varDelta\right\rangle\Big/\mathscr{U}\right).$$

By the Hahn Banach theorem, there is an extension of Ξ^* to $\mathscr{P}(\omega)$. $\square$

An important application of Proposition 3.3 (and the hard part thereof, which is only proved in Fact 3.6) is

Claim 3.7. *Suppose that Q_1,Q_2 are forcing notions in V, $\Xi_0\in\mathscr{M}^{\mathrm{full}}$ in V, $\Vdash_{Q_\ell}$ "$\underset{\sim}{\Xi}_\ell$ is a finitely additive measure extending Ξ_0 for $\ell=1,2$,". Then $\Vdash_{Q_1\times Q_2}$ "there is a finitely additive measure extending $\underset{\sim}{\Xi}_1$ and $\underset{\sim}{\Xi}_2$ (and hence $\underset{\sim}{\Xi}_0$)".*

Proof. We are going to show, that $\Vdash_{Q_1\times Q_2}$ "$\underset{\sim}{\Xi}_1$ (in the rôle of Ξ_0 of Proposition 3.3) and $\{A^*_\alpha \mid A^*_\alpha \in V^{Q_2} \cap \mathscr{P}(\omega)\}$ (in the rôle of $\langle A^*_\alpha \mid \alpha<\alpha^*\rangle$ of 3.3) fulfil (3.B) of Proposition 3.3".

First, we show that

$$\Vdash_{Q_1\times Q_2} \mathrm{dom}(\underset{\sim}{\Xi}_1)\cap \mathrm{dom}(\underset{\sim}{\Xi}_2) = \mathrm{dom}(\Xi_0) = \check{V}\cap\mathscr{P}(\omega).$$

So assume that we have an Q_1-name $\underset{\sim}{X}$ and a Q_2-name $\underset{\sim}{Y}$ such that $\Vdash_{Q_1\times Q_2} \underset{\sim}{X}=\underset{\sim}{Y}$.

Let $Z=\{n\in\omega \mid \exists p\in Q_1\ p\Vdash_{Q_1} n\in\underset{\sim}{X}\}$. The set Z is in V and $\Vdash_{Q_1} \underset{\sim}{X}\subseteq Z$. It is easy to see that $\Vdash_{Q_2} Z\subseteq \underset{\sim}{Y}$. So we get

$$\Vdash_{Q_1\times Q_2} \underset{\sim}{X}\subseteq Z\subseteq \underset{\sim}{Y}=\underset{\sim}{X}$$

and our first claim is proved.

Now we check (3.B). Let ε, k, $\langle A^*_i\in V^{Q_1} \mid i<m\rangle$ a partition of ω and α_ℓ, $\ell<n$ be given. W.l.o.g. the $A_{\alpha_\ell}\in V^{Q_2}$ are a partition of ω as well.

If for some i,ℓ

$$\Vdash_{Q_1\times Q_2} \underset{\sim}{A}^*_i\cap \underset{\sim}{A}_{\alpha_\ell} \text{ is finite,}$$

then A^*_i and A_{α_ℓ} can be separated by some $A\in V$. This is shown in a manner similar to the proof of the first claim.

We choose a separator $A^{i,\ell}\in V$ for each i,ℓ such that $\Vdash_{Q_1\times Q_2} \underset{\sim}{A}^*_i\cap\underset{\sim}{A}_{\alpha_\ell}$ is finite and let A^j, $j<j^*$ be the partition of ω in V that is generated by all the $A^{i,\ell}$.

Then, we set $\varepsilon'=\varepsilon/mnj^*$ and put for each i,ℓ,j such that

$$\Vdash_{Q_1\times Q_2} \underset{\sim}{A}^*_i\cap\underset{\sim}{A}_{\alpha_\ell}\cap\check{A}^j \text{ is infinite,}$$

in the forcing extension $V^{Q_1\times Q_2}$, the first

$$\left\lceil \frac{\Xi_1(A^*_i\cap A^j)\times\Xi_2(A_{\alpha_\ell}\cap A^j)}{\varepsilon'\times\Xi_0(A^j)}\right\rceil$$

elements of $A^*_i\cap A_{\alpha_\ell}\cap A^j$ (and no further points) into u. $\square$

4. The first part of the proof of $(**)_{\bar{Q}}$: introduction of $\mathscr{K}^3$

In order to prove $(**)_{\bar{Q}}$, we need that for suitable $\bar{Q}=\langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \underset{\sim}{\tau}_\beta, \mu_\beta, \mid \beta<\lg(\bar{Q}),\ \alpha\leqslant\lg(\bar{Q})\rangle$ from $\mathscr{K}$ (see Definition 2.2) we have almost (in the sense explained in the proof of Theorem 5.5) an expansion of the form

$$\bar{Q}^{\exp} = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \underset{\sim}{\tau}_\beta, \mu_\beta, \eta_\beta, (\underset{\sim}{\Xi}^t_\alpha)_{t\in\mathscr{T}} \mid \beta<\lg(\bar{Q}),\ \alpha\leqslant\lg(\bar{Q})\rangle$$

such that $\bar{Q}^{\exp}$ is in a special class $\mathscr{K}^3$, which we shall define in Definition 4.2.

In order to introduce $\mathscr{K}^3$, we shall first define and (try to) explain the set $\mathscr{T}$ of blueprints (Definition 4.1). For each blueprint t and $\alpha<\alpha^*$ the $\underset{\sim}{\Xi}^t_\alpha$ will be P_α-name for some finitely additive measure on $\mathscr{P}(\omega)$ that conveys some information about ω-tuples $\langle p_k \mid k\in\omega\rangle$ of conditions that fit well to the blueprint t, from stage α to later stages in the iteration.

Let us tell more about the ideas of the proof of $(**)_{\bar{Q}}$: In Lemma 2.12, if the p_ζ are not all Cohen, the premise 3 is hard to fulfil. Think of κ^+ many p_ζ being given, so that we can do many thinning out procedures and have them similar, i.e. similar partial random conditions and Cohen conditions. Then we keep only the first ω of the ζ's and the first ω conditions $\langle p_\zeta \mid \zeta\in\omega\rangle$. We try to strengthen them a little bit (to p'_ζ) and then get that the strengthened conditions allow to define one condition $p^\otimes \geqslant p^*$ such that

$$p^\otimes \Vdash \text{“} \bigcap_{\zeta\in\underset{\sim}{E}=\{\zeta \mid p'_\zeta\in\underset{\sim}{G}\}} \operatorname{tree}_{\ell^*}(\bar{\underset{\sim}{a}}^{\alpha_\zeta}) \text{ has finitely many branches”}$$

and hence cannot contain a perfect tree. There are some requirements on $\langle p_\zeta \mid \zeta\in\omega\rangle$, as they have to predict some probabilities about the branches of the $\operatorname{tree}_{\ell^*}(\bar{\underset{\sim}{a}}^{\alpha_\zeta})$ and about the subset of the $\{p'_\zeta \mid \zeta\in\omega\}$, that lies in G.

The technical means to allow these predictions is the use of finitely additive measures and the properties (e)–(i) in the definition of $\mathscr{K}^3$. These items in the definition have long premises by themselves. However, the premises are sufficiently often fulfilled if we start with κ^+ many p_ζ, thin out, and choose an appropriate $t\in\mathscr{T}$.

We embark with the definition of a blueprint t. The set of all blueprints is denoted by $\mathscr{T}$. The reader may think that t describes some relevant information about the chosen tuples $\langle p_\zeta \mid \zeta\in\omega\rangle$. Later, it will turn out that sequences described by the same t are compatible forcing conditions (though we have finite supports and are not interested in taking the union of countably many conditions). This will be used in Lemma 4.8.

In the case of iterations where all Cohen forcings are just those forcings in an initial segment of the iteration (as in Definition 2.2, Part (2)), we can dispense with the parameter $\mathbf{m}$ in the next definition. This simplification is not worthwhile because the generality allows another application of the method. In Section 6, we shall work with a type of iteration where Cohens are added cofinally often.

However, we could simplify Definition 4.2 slightly and leave out (f) there in the special case that the f_ζ of Lemma 2.11 move only one α in the Cohen part and leave the indices at which partial randoms are attached fixed. We do not simplify because we hope for future applications.

Definition 4.1. We fix a κ such that $2^\kappa\geqslant\chi$ (from 2.2). The set $\mathscr{T}$ of blueprints is the set of tuples

$$t=(w^t,\mathbf{n}^t,\mathbf{m}^t,\bar{\eta}^t,h^t_0,h^t_1,h^t_2,\bar{n}^t)$$

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such that

(a) $w^t \in [\kappa]^{\aleph_0}$. (What is the purpose? Think of the latter as $[\chi]^{\aleph_0}$ disguised. Suppose that $|\mathrm{dom}(p_\zeta)| = \mathbf{n}^t$ for all ζ, $\mathrm{dom}(p_\zeta) = \{\gamma^i_\zeta \mid i < \mathbf{n}^t\}$, $\langle \gamma^i_k \mid k \in \omega\rangle \in \chi^\omega$ for each fixed $i < \mathbf{n}^t$, but $\chi \leqslant 2^\kappa$ and we can fix an injection and keep as relevant information certain parts of κ coming from of certain $f \in 2^\kappa$. Look at the w^t in Subclaim 5.3.)

(b) $0 < \mathbf{n}^t < \omega$, $0 \leqslant \mathbf{m}^t \leqslant \mathbf{n}^t$. ($\mathbf{n}^t$ will be the cardinality of the heart of the Δ-system built from many p_ζ and $\mathbf{m}^t$ will be the cardinality of the part of the heart that is lying below χ.)

(c) $\bar{\eta}^t = \langle \eta^t_{n,k} \mid n < \mathbf{n}^t, k \in \omega\rangle$, $\eta^t_{n,k} \in {}^{w^t}2$. ($\eta^t_{n,k}$ codes the nth element of the support of p_k for $k \in \omega$ and these k are the first ω of the ζ.)

(d) h^t_0 is a partial function from $[0, \mathbf{n}^t)$ to κ.[3] ($\mathrm{dom}(h^t_0)$ is the part of those α in the heart of the Δ-system where $\underset{\sim}{Q}_\alpha$ is the Cohen forcing. In the somewhat simpler case of Definition 2.2, Part 2), this domain coincides with the part of the heart that lies below χ).

(e) h^t_2 is a function from $[0, \mathbf{n}^t) \setminus \mathrm{dom}(h^t_0)$ to ${}^{<\omega}2$. (Think of h^t_2 giving some information of a partial random condition attached at some point of the heart.)

(f) h^t_1 is a function from $[0, \mathbf{n}^t)$ into the rational interval $[0,1)_{\mathbb{Q}}$, such that $\{n \mid h^t_1(n) \neq 0\} \subseteq \mathrm{dom}(h^t_2)$. Furthermore, we have that $\sum_{n<\mathbf{n}^t} \sqrt{h^t_1(n)} < \frac{1}{10}$. (Think of h^t_1 giving some information about the Lebesgue measure of the limit of the partial random condition attached at some point of the heart intersected with $\mathrm{dom}(h^t_2)$.)

(g) $\eta^t_{n_1,k_1} = \eta^t_{n_2,k_2} \Rightarrow n_1 = n_2$. (This is some compatibility requirement, which is useful in 4.5.)

(h) For each $n < \mathbf{n}^t$ we have that $\langle \eta^t_{n,k} \mid k \in \omega\rangle$ is either constant or with no repetitions (that is: either in the heart of the system or among the moved parts of the domains of the $\langle p_k \mid k \in \omega\rangle$).

(i) $\bar{n}^t = \langle n^t_k \mid k \in \omega\rangle$ where $n^t_0 = 0$, $n^t_k < n^t_{k+1} < \omega$ and the sequence $\langle n^t_{k+1} - n^t_k \mid k \in \omega\rangle$ goes to infinity. (This last ingredient does not describe p_ℓ but is just an additional part handling the finitely additive measures $\underset{\sim}{\Xi}^t_\alpha$. The sequences $\bar{n}^t$ shall allow to compute intersections of sets of branches from lim tree, and for these computations (see Subclaim 5.3) the p_ℓ are grouped together for $\ell \in [n^t_k, n^t_{k+1})$.)

There are κ^ω many blueprints. (Remember we also require that $2^\kappa \geqslant \chi$, otherwise the choice of the η in the following definition would fail.)

Explanation: We continue the explanations begun in the parentheses in order to explain how the conditions shall work together:

As mentioned, $(**)_{\bar{Q}}$ follows from the fact that in $V^{P_{\alpha*}}$, if $E \in [\chi]^{\kappa^+}$ and $m \in \omega$, then $\bigcap_{\alpha \in E} \mathrm{tree}_m(\bar{a}^\alpha)$ is a tree with finitely many branches. Suppose some p forces the

[3] We do carry out the simplification suggested in a footnote in [20] and take κ instead of ${}^\omega\kappa$ here. This does not bring any disadvantages, because when choosing $\langle p_\zeta \mid \zeta \in \omega\rangle$ we have initially κ^+ many p_ζ, and hence can thin out such that for each ζ, $|\mathrm{dom}\, p_\zeta|$ is the same, say $\mathbf{n}^t$, and that for auch $n < \mathbf{n}^t$, the p^t_ζ (nth element of $\mathrm{dom}(p^t_\zeta)) = h^t_0(n)$ are independent of ζ, if they lie in some notion of forcing with conditions in some Q_α with $|Q_\alpha| < \kappa$.

contrary. We take $p_\zeta \geqslant p$ such that $p_\zeta \Vdash$ "$\beta_\zeta \in E$" for $\zeta \in \kappa$ and such that $\beta_\zeta \notin \{\beta_\xi \mid \xi < \zeta\}$.

We can assume that the p_ζ are in some given dense set (will be $\mathscr{I}_{\bar{\varepsilon}}$ of Lemma 5.1 in our case) and that the $\langle p_\zeta \mid \zeta \in \kappa^+\rangle$ form a Δ-system with some additional thinning demands, putting κ^+ many objects into less than κ many pigeonholes. (See our earlier remarks about working with κ^+ many ζ and the proof of Lemma 5.2.)

We assume that $\operatorname{dom}(p_\zeta) = \{\gamma_{\mathbf{n},\zeta} \mid \mathbf{n} < \mathbf{n}^t\}$, $\gamma_{\mathbf{n},\zeta}$ is increasing in $\mathbf{n}$ and $\gamma_{\mathbf{n},\zeta} < \chi$ iff $\mathbf{n} < \mathbf{m}^t$ and that β_ζ is one of the $\gamma_{\mathbf{n},\zeta}$. We let p'_ζ be p_ζ except that $p_\zeta(\beta_\zeta)$ is increased a little. It suffices to find some $p^\otimes \geqslant p$ such that $p^\otimes \Vdash$ "$\underset{\sim}{A} = \{\zeta \in \omega \mid p'_\zeta \in \underset{\sim}{G}\}$ is 'large enough' such that $\bigcap_{\zeta \in \underset{\sim}{A}} \operatorname{tree}_m(\underset{\sim}{\bar{q}}^{\beta_\zeta})$ has only finitely many branches".

The 'large enough' is interpreted in terms of a Ξ^t_α-measure.

The $\mathbf{n} < \mathbf{n}^t$ such that $Q_{\gamma_{\mathbf{n},\zeta}}$ is a forcing notion of cardinality $< \kappa$ (in our forcings, then it is just the Cohen forcing) do not cause problems because $h^t_0(\mathbf{n})$ tells us exactly what the condition is. Still there are many cases of such $\langle p_\zeta \mid \zeta \in \omega\rangle$ which fall into the same t, and we will get contradictory demands if $\gamma_{\mathbf{n}_1,\zeta_1} = \gamma_{\mathbf{n}_2,\zeta_2}$ and $\mathbf{n}_1 \neq \mathbf{n}_2$. But the w^t, $\bar{\eta}^t$ are built in order to prevent this. That is we have to assume that $2^\kappa \geqslant \chi$ in order to be able to choose $\langle \eta_\alpha \mid \alpha \in \chi\rangle$, $\eta_\alpha \in 2^\kappa$ with no repetitions and such that for $v \subseteq \chi$, $|v| \leqslant \aleph_0$ (in the applications, we shall have $v = \{\alpha_{\mathbf{n},\zeta} \mid \zeta \in \omega\}$) there is some $w = w^t \in [\kappa]^{\aleph_0}$ such that $\langle \eta_\alpha \restriction w \mid \alpha \in v\rangle$ is without repetitions.

So the blueprint t describes such a situation giving much information, though the number of blueprints is κ^ω.

If $Q_{\alpha_{\mathbf{n},\zeta}}$ is partial random, we get many different possibilities for $p_\zeta(\gamma_{\mathbf{n},\zeta})$, too many to apply a pigeonhole principle. We want that many of them will lie in the generic set. Using $(h^t_1(\mathbf{n}), h^t_2(\mathbf{n}))$ we know that in the interval $({}^\omega 2)^{[h^t_2(\mathbf{n})]}$ the set $\lim(p_\zeta(\gamma_{\mathbf{n},\zeta}))$ is of relative measure $\geqslant 1 - h^t_1(\mathbf{n})$. Still there are too many (possibly incompatible) $p_\zeta(\gamma_{\mathbf{n},\zeta})$ and finally, in Lemma 5.2 and Subclaim 5.3, the existence of many compatible candidates is ensured by the finitely additive measures.

The $\bar{n}^t = \langle n^t_k \mid k \in \omega\rangle$ are going to be used in the end of Section 5, where we show that $\{\zeta \mid p'_\zeta \in G\}$ is large by showing that for infinitely many k we have that

$$\frac{|\{\zeta \mid n^t_k \leqslant \zeta < n^t_{k+1} \text{ and } p'_\zeta \in G\}|}{n^t_{k+1} - n^t_k}$$

is large, say $> \varepsilon > 0$.

The n^t_k will be chosen such that they are increasing fast enough with k and $\langle p'_\zeta(\gamma_{\mathbf{n},\zeta}) \mid \zeta \in [n^t_k, n^t_{k+1})\rangle$ will be chosen such that for each $\varepsilon > 0$ there is some $s \in \omega$ such that for k large enough: if the above fraction is above ε then

$${}^k 2 \cap \bigcap \{\operatorname{tree}_m(\underset{\sim}{\bar{q}}^{\beta_\ell}) \mid n^t_k \geqslant \ell < n^t_{k+1} \text{ and } p'_\ell \in G\}$$

has $< s$ members, hence the tree has fewer than s branches.

Comment on simplifications: Now we finally define the kind of iteration we use for the proof of $(**)_{\bar{Q}}$. The reader who is longing for some simplification may omit condition (f) in Definition 4.2, Lemma 4.5 and Subclaim 5.3 and work just with conditions p_ζ that do not differ at any index in the iteration where a partial random real is attached to it, but only at those indices where a forcing of size less than κ is attached, or even work with p_ζ that differ only at $\beta_\zeta < \chi$ (from Lemma 2.11). A look at the beginning of Lemma 5.2, where the p_ζ and p'_ζ are chosen, and a look $AUT(\bar{Q})$ shows that the restriction to this simplified situation is always possible when forcing with a member of the restricted class described in Definition 2.2 Part 2.

Definition 4.2. $\mathscr{K}^3$ is the class of sequences

$$\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{\tau}_\beta, \eta_\beta, (\underset{\sim}{\Xi}^t_\alpha)_{t\in\mathscr{T}} \mid \alpha \leqslant \alpha^*, \beta < \alpha^* \rangle$$

(we write $\alpha^* = \lg(\bar{Q})$) such that

(a)

$$\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{\tau}_\beta, \mid \alpha \leqslant \alpha^*, \beta < \alpha^* \rangle$$

is in $\mathscr{K}$ from Definition 2.2.

(b) $\eta_\beta \in {}^\kappa 2$ and for $\beta < \alpha < \alpha^*$ we have that $\eta_\beta \neq \eta_\alpha$.

(c) $\mathscr{T}$ is the set of all blueprints, and $\underset{\sim}{\Xi}^t_\alpha$ is a P_α-name for a finitely additive measure in V^{P_α}, increasing with α.

(d) We say the $\langle \alpha_\ell \mid \ell \in \omega \rangle$ satisfies $(t, \mathbf{n})$ for $\bar{Q}$, if

(Think of p_ℓ being the first ω of the p_ζ and $\langle \alpha_\ell \mid \ell \in \omega \rangle = \langle \gamma_{\mathbf{n},\zeta} \mid \zeta \in \omega \rangle$, and in particular, $\langle \alpha_\ell \mid \ell \in \omega \rangle = \beta_\ell \mid \ell \in \omega$ from 2.10. (α_ℓ is for some $\mathbf{n}$ always the $\mathbf{n}$th element in $\mathrm{dom}(p_\ell)$.) Further think that the following items also mean that $\langle p_\ell \mid \ell \in \omega \rangle$ being sufficiently described by $t \in \mathscr{T}$.)

1. $\langle \alpha_\ell \mid \ell \in \omega \rangle \in V$,
2. $t \in \mathscr{T}$, $\mathbf{n} < \mathbf{n}^t$,
3. $\alpha_\ell < \alpha_{\ell+1} < \alpha^*$,
4. $\mathbf{n} < \mathbf{m}^t \Leftrightarrow \forall \ell (\alpha_\ell < \chi) \Leftrightarrow \exists \ell (\alpha_\ell < \chi)$ (the moved positions α_ℓ are in the Cohen part),
5. $\eta^t_{\mathbf{n},\ell} = \eta_{\alpha_\ell} \restriction w^t$. ($\eta_{\alpha_\ell}$ describes where α_ℓ really is, and $\eta^t_{\mathbf{n},\ell}$ describes a part of it of size ω. For a given t, the $\mathbf{n}$ such that $\bar{Q}$ satisfies $(t, \mathbf{n})$ is unique by Definition 4.1 (g).),
6. if $\mathbf{n} \in \mathrm{dom}(h^t_0)$ then $\mu_{\alpha_\ell} < \kappa$ and $\Vdash_{P_{\alpha_\ell}}$ "$|\underset{\sim}{Q}_{\alpha_\ell}| < \kappa$ and $(h^t_0(\mathbf{n}))(\ell) \in \underset{\sim}{Q}_{\alpha_\ell}$",
7. if $\mathbf{n} \in \mathrm{dom}(h^t_1)$ then $\mu_{\alpha_\ell} \geqslant \kappa$, so $\Vdash_{P_{\alpha_\ell}}$ "$\underset{\sim}{Q}_{\alpha_\ell}$ has cardinality $\geqslant \kappa$" (hence it is partial random),
8. if $\langle \eta^t_{\mathbf{n},k} \mid k \in \omega \rangle$ is constant, then $\forall \ell \, \alpha_\ell = \alpha_0$,
9. if $\langle \eta^t_{\mathbf{n},k} \mid k \in \omega \rangle$ is not constant, then $\forall \ell \, \alpha_\ell < \alpha_{\ell+1}$.

(e) If $\bar{\alpha} = \langle \alpha_\ell \mid \ell \in \omega \rangle$ satisfies $(t, \mathbf{n})$ for $\bar{Q}$, $\bigwedge_{\ell \in \omega} (\alpha_\ell < \alpha_{\ell+1})$, $\mathbf{n} \in \mathrm{dom}(h^t_0)$ and

$$C = \{k \in \omega \mid \forall \ell \in [n_k, n_{k+1}) h^t_0(\mathbf{n})(\ell) \in G_{Q_{\alpha_\ell}}\},$$

then

$$\Vdash_{P_{\alpha^*}} \underset{\sim}{\Xi}^t_{\alpha^*}(\underset{\sim}{C}) = 1.$$

(f) If $\bar{\alpha}=\langle\alpha_\ell \mid \ell\in\omega\rangle$ satisfies $(t,\mathbf{n})$ for $\bar{Q}$, $\bigwedge_{\ell\in\omega}(\alpha_\ell<\alpha_{\ell+1})$, $\mathbf{n}\in\mathrm{dom}(h_1^t)$, $\underset{\sim}{\bar{p}}=\langle\underset{\sim}{p}_\ell \mid \ell\in\omega\rangle$ is such that $\underset{\sim}{p}_\ell$ is a P_{α_ℓ}-name for a member of $\underset{\sim}{Q}_{\alpha_\ell}$, and for every ℓ,

$$(*)\quad \Vdash_{P_{\alpha_\ell}} 1-h_1^t(\mathbf{n})\leqslant\frac{\mathrm{Leb}(\{\eta\in{}^{\omega}2 \mid h_2^t(\mathbf{n})\lhd\eta\in\lim(\underset{\sim}{p}_\ell)\})}{2^{\lg(h_2^t(\mathbf{n}))}}$$

and if $\varepsilon>0$ is such that

$$C=\left\{k\in\omega \;\middle|\; \frac{|\{\ell\in[n_k^t,n_{k+1}^t) \mid \underset{\sim}{p}_\ell\in G_{Q_{\alpha_\ell}}\}|}{n_{k+1}^t-n_k^t}\geqslant(1-h_1^t(\mathbf{n}))(1-\varepsilon)\right\},$$

then

$$\Vdash_{P_{\alpha^*}} \underset{\sim}{\Xi}^t_{\alpha^*}(\underset{\sim}{C})=1.$$

(g) If $\bar{\alpha}=\langle\alpha_\ell \mid \ell\in\omega\rangle$ satisfies $(t,\mathbf{n})$ for $\bar{Q}$, $\bigwedge_{\ell\in\omega}\alpha_\ell=\alpha$, $\mathbf{n}\in\mathrm{dom}(h_1^t)$, $\underset{\sim}{r}$ and $\underset{\sim}{\bar{r}}=\langle\underset{\sim}{r}_\ell \mid \ell\in\omega\rangle$ are P_α-names for members of Q_α such that

(**)

in V^{P_α}: $\forall r'\in Q_\alpha$ if $r'\geqslant r$, then

$\mathrm{Av}_{\Xi^t{}_\alpha}\left(\langle a_k(r') \mid k\in\omega\rangle\right)\geqslant 1-h_1^t(\mathbf{n})$, where

$$a_k(r')=a_k(r',\bar{r})=\left(\sum_{\ell\in[n_k,n_{k+1})}\frac{\mathrm{Leb}(\lim(r')\cap\lim(r_\ell))}{\mathrm{Leb}(\lim(r'))}\right)\cdot\frac{1}{n_{k+1}^t-n_k^t},$$

then

$\Vdash_{P_{\alpha^*}}$ "if $\underset{\sim}{r}\in Q_\alpha$, then

$$1-h_1^t(\mathbf{n})\leqslant\mathrm{Av}_{\underset{\sim}{\Xi}^t_{\alpha^*}}\left(\left\langle\frac{|\{\ell\in[n_k^t,n_{k+1}^t) \mid \underset{\sim}{r}_\ell\in G_{Q_{\alpha_\ell}}\}|}{n_{k+1}^t-n_k^t}\;\middle|\; k\in\omega\right\rangle\right)\text{''}.$$

(h) $P'_{A_\alpha}\lessdot P_\alpha$.

(i) For $t\in\mathscr{T},\alpha\in\alpha^*$: If $\Vdash_{P_\alpha}|Q_\alpha|\geqslant\kappa$, then $\underset{\sim}{\Xi}^t_\alpha\restriction\mathscr{P}(\omega)^{V^{P_{A_\alpha}}}$ is a P_{A_α}-name.[4]

Definition 4.3. (1) For $\bar{Q}\in\mathscr{K}^3$ and for $\alpha^*<\lg(\bar{Q})$ let

$$\bar{Q}\restriction\alpha^*=\langle P_\alpha,\underset{\sim}{Q}_\beta,A_\beta,\mu_\beta,\underset{\sim}{\tau}_\beta,\eta_\beta,(\underset{\sim}{\Xi}^t_\alpha)_{t\in\mathscr{T}} \mid \alpha\leqslant\alpha^*,\beta<\alpha^*\rangle.$$

[4] This is where the information is whispered, showing that $\underset{\sim}{Q}_\alpha$, the random forcing over $V[\tau_\beta \mid \beta\in A_\alpha]$, behaves in the sense of Ξ^t_α instead of the Lebesgue measure in a certain sense generic: r_α hits sets of large Ξ^t_α measure.

(2) For $\bar{Q}^1, \bar{Q}^2 \in \mathscr{K}^3$ we say

$$\bar{\underset{\sim}{Q}}^1 < \bar{\underset{\sim}{Q}}^2 \quad \text{if } \bar{\underset{\sim}{Q}}^1 = \bar{\underset{\sim}{Q}}^2 \restriction \lg(\bar{\underset{\sim}{Q}}^1).$$

In the next three steps, we show that $\mathscr{K}^3$ is sufficiently rich: that is, if we have some $\bar{Q}$ in $\mathscr{K}^3$ then we can find an extension. The successor step and the limit step of cofinality ω require some work, whereas the limits of larger cofinality are easy because no new reals are introduced in these limit steps.

Fact 4.4. (1) *If* $\bar{Q} \in \mathscr{K}^3$, $\alpha \leqslant \lg(\bar{Q})$, *then* $\bar{Q} \restriction \alpha \in \mathscr{K}^3$.

(2) $(\mathscr{K}^3, \leqslant)$ *is a partial order.*

(3) *If a sequence* $\langle \bar{Q}^\zeta \mid \zeta < \delta \rangle$ *is increasing,* $\mathrm{cf}(\delta) > \aleph_0$, *then there is a unique* $\bar{Q} \in \mathscr{K}^3$ *which is the least upper bound,* $\lg(\bar{Q}) = \bigcup_{\zeta<\delta} \lg(\bar{Q}^\zeta)$ *and* $\bar{Q}^\zeta \leqslant \bar{Q}$ *for all* $\zeta < \delta$.

Proof. Easy. $\square$

Lemma 4.5. *Suppose that* $\bar{Q}_n < \bar{Q}_{n+1}$, $\bar{Q}_n \in \mathscr{K}^3$, $\alpha_n = \lg(\bar{Q}_n)$, $\delta = \sup(\alpha_n)$. *Then there is some* $\bar{Q} \in \mathscr{K}^3$ *such that* $\lg(\bar{Q}) = \delta$ *and* $\bar{Q}_n < \bar{Q}$ *for* $n \in \omega$.

Proof. We have to define $(\underset{\sim}{\Xi}^t_\delta)_{t \in \mathscr{T}}$, such that (e) and (f) of the definitions of $\mathscr{K}^3$ hold. Items (g) and (i) do not produce no new tasks in the limit steps, and we proved (h) in Lemmas 2.6 and 2.7.

So, we look again at (e) and (f) of Definition 4.2.

(e) If $\bar{\alpha} = \langle \alpha_\ell \mid \ell \in \omega \rangle$ satisfies $(t, \mathbf{n})$ for $\bar{Q}$, $\bigwedge_{\ell\in\omega}(\alpha_\ell < \alpha_{\ell+1})$, $\mathbf{n} \in \mathrm{dom}(h^t_0)$ and

$$C = \{k \in \omega \mid \forall \ell \in [n_k, n_{k+1}) h^t_0(\mathbf{n})(\ell) \in G_{Q_{\alpha_\ell}}\},$$

then

$$\Vdash_{P_{\alpha^*}} \underset{\sim}{\Xi}^t_{\alpha^*}(\underset{\sim}{C}) = 1.$$

(f) If $\bar{\alpha} = \langle \alpha_\ell \mid \ell \in \omega \rangle$ satisfies $(t, \mathbf{n})$ for $\bar{Q}$, $\bigwedge_{\ell\in\omega}(\alpha_\ell < \alpha_{\ell+1})$, $\mathbf{n} \in \mathrm{dom}(h^t_1)$, $\bar{\underset{\sim}{p}} = \langle \underset{\sim}{p}_\ell \mid \ell \in \omega \rangle$ is such that

$$(*) \quad \Vdash_{P_{\alpha_\ell}} 1 - h^t_1(\mathbf{n}) \leqslant \frac{\mathrm{Leb}(\{\eta \in {}^\omega 2 \mid h^t_2(\mathbf{n}) \lhd \eta \in \lim(\underset{\sim}{p} d_\ell)\})}{2^{\lg(h^t_2(\mathbf{n}))}}$$

and $\varepsilon > 0$ and

$$C = \left\{ k \in \omega \,\middle|\, \frac{|\{\ell \in [n^t_k, n^t_{k+1}) \mid \underset{\sim}{p}_\ell \in G_{Q_{\alpha_\ell}}\}|}{n^t_{k+1} - n^t_k} \geqslant (1 - h^t_1(\mathbf{n}))(1 - \varepsilon) \right\},$$

then

$$\Vdash_{P_{\alpha^*}} \underset{\sim}{\Xi}^t_{\alpha^*}(\underset{\sim}{C}) = 1.$$

By Theorem 3.2 it suffices to show

$\Vdash_{P_\delta}$ "if $\underset{\sim}{B} \in \bigcup_{\alpha<\delta} \mathrm{dom}(\underset{\sim}{\Xi}^t_\alpha) = \bigcup_{\alpha<\delta}(\mathscr{P}(\omega))^{V^{P_\alpha}}$ and $\Xi^t_\alpha(\underset{\sim}{B})>0$ and $j^* \in \omega$ and $\underset{\sim}{C}_j$, $j<j^*$, are sets from (e) or (f) (whose measure is required to be 1 there), then $\underset{\sim}{B} \cap \bigcap_{j<j^*} \underset{\sim}{C}_j \neq \emptyset$".

Towards a contradiction, assume $q \in P_\delta$ forces the negation. So possibly increasing q we have: for some $\underset{\sim}{B}$ and for some $j^* \in \omega$, for each $j<j^*$ we have $\varepsilon>0$, and $\mathbf{n}(j)<\mathbf{n}^t$, $\langle \alpha^j_\ell \mid \ell \in \omega\rangle$, $\langle \underset{\sim}{p}^j_\ell \mid \ell \in \omega\rangle$ involved in the definition of $\underset{\sim}{C}_j$ (in (e) or (f) of Definition 4.2), and q forces:

$$\underset{\sim}{B} \in \bigcup_{\alpha<\delta} \mathrm{dom}(\underset{\sim}{\Xi}^t_\alpha) = \bigcup_{\alpha<\delta} \mathscr{P}(\omega)^{V^{P_\alpha}},$$

$$\bigcup_{\alpha<\delta} \mathrm{dom}\left(\underset{\sim}{\Xi}\alpha^t(\underset{\sim}{B})\right) > 0,$$

$\underset{\sim}{C}_j$ comes from (e) or (f),

$$\underset{\sim}{B} \cap \bigcap_{j<j^*} \underset{\sim}{C}_j = \emptyset.$$

There is some $\alpha(*)<\delta$ such that $\underset{\sim}{B} \in \mathrm{dom}(\underset{\sim}{\Xi}^t_{\alpha(*)})$ is a $P_{\alpha(*)}$-name. The $\underset{\sim}{C}_j$ have $\mathbf{n}(j)<\mathbf{n}^t$, $\langle \alpha^j_\ell \mid \ell \in \omega\rangle$, $\langle \underset{\sim}{p}^j_\ell \mid \ell \in \omega\rangle$ as witnesses as required in (e) or (f) above. W.l.o.g. $q \in P_{\alpha(*)}$ and $q \in G_{P_{\alpha(*)}} \subseteq P_{\alpha(*)}$, $G_{P_{\alpha(*)}}$ generic over V.

We can find $k \in \underset{\sim}{B}[G_{P_{\alpha(*)}}]$ such that $\bigwedge_{j<j^*} \bigwedge_{\ell \in [n^t_k, n^t_{k+1})} (\alpha^j_\ell > \alpha(*))$ and moreover such that $n^t_{k+1} - n^t_k$ is large enough compared to $1/\varepsilon$, j^*, in order to allow us to apply the Tchebyshev inequality and the law of large numbers for $n^t_{k+1} - n^t_k$ random choices. (The n^t_k come from item (f) of the definition of a blueprint, and are not the $\mathbf{n}$.)

Let $\{\alpha^j_\ell \mid j<j^* \text{ and } \ell \in [n^t_k, n^t_{k+1})\}$ be listed as $\{\beta_m \mid m<m^*\}$, in increasing order (so $\beta_0>\alpha(*)$) (possibly $\alpha^{j_1}_{\ell_1} = \alpha^{j_2}_{\ell_2} \wedge (j_1,\ell_1) \neq (j_2,\ell_2)$). We now choose by induction on $m \leqslant m^*$ a condition $q_m \in P_{\beta_m}$ above q, increasing with m and such that $\mathrm{dom}(q_m) = \mathrm{dom}(q) \cup \{\beta_0, \beta_1, \ldots, \beta_{m-1}\}$. We stipulate $\beta_{m^*} = \delta$.

During this definition we throw a dice probability of success (i.e. $q \Vdash$ "$k \in \underset{\sim}{C}_j$" for $j<j^*$) is positive, and hence q_{m^*} will show that our assumption on q is false.

Case A: $m=0$. Let $q_0 = q$.

Case B: We are to choose q_{m+1} and for some $\mathbf{n}<\mathbf{n}^t$ we have $\mathbf{n} \in \mathrm{dom}(h^t_0)$ and γ and: if $j<j^*$ and $\ell \in \omega$ then $(\alpha^j_\ell = \beta_m \Rightarrow \mathbf{n}(j) = \mathbf{n} \wedge p^j_\ell = \gamma (= h^t_0(\mathbf{n}(j))(\ell)) \in Q_{\beta_m})$.

In this case $\mathrm{dom}(q_{m+1}) = \mathrm{dom}(q_m) \cup \{\beta_m\}$, and

$$q_{m+1}(\beta) = \begin{cases} q_m(\beta) & \text{if } \beta < \beta_m, \\ \gamma & \text{if } \beta = \beta_m. \end{cases}$$

The choice of (j,ℓ) is immaterial as for each β_m there is by the definition of "satisfying $(t,\mathbf{n})$ for $\bar{Q}$", item 5, a unique $\mathbf{n}<\mathbf{n}^t$, such that there is some ℓ such that

$\eta_{\beta_m} \restriction w^t = \eta^t_{\mathbf{n},\ell}$ and conditions (g) of Definition 4.1 and (d)8 of Definition 4.2 imply that if $\eta^t_{n,\ell}$ is not constant then $(\beta_m = \alpha^{i_1}_{\ell_1} = \alpha^{i_2}_{\ell_2} \rightarrow \ell_1 = \ell_2)$. Hence $\gamma = p^j_\ell$ is well defined.

Case C: We are to choose q_{m+1} and for some $\mathbf{n} < \mathbf{n}^t$ we have $\mathbf{n} \in \mathrm{dom}(h^t_1)$ and: if $j < j^*$ and $\ell \in \omega$ then $\alpha^j_\ell = \beta_m \Rightarrow \mathbf{n}(j) = \mathbf{n}$.

Work first in $V[G_{P_{\beta_m}}]$, $q_m \in G_{P_{\beta_m}}$, $G_{P_{\beta_m}}$ generic over V. The sets

$$\{\lim(\underset{\sim}{p}^j_\ell[G_{P_{\beta_m}}]) | \alpha^j_\ell = \beta_m, \ell \in [n^t_k, n^t_{k+1}), j < j^*)\}$$

are subsets of $({}^\omega 2)^{[h^t_2(\mathbf{n})]} = \{\eta \in {}^\omega 2 \mid h^t_2(\mathbf{n}) \lhd \eta\}$. We can define an equivalence relation E_m on $({}^\omega 2)^{[h^t_2(\mathbf{n})]}$:

$$\nu_1 E_m \nu_2 \quad \text{iff } (\forall (j,\ell) \text{ s.th. } \alpha^j_\ell = \beta_m : \nu_1 \in \lim(\underset{\sim}{p}^j_\ell[G_{P_{\beta_m}}]) \Leftrightarrow \nu_2 \in \lim(\underset{\sim}{p}^j_\ell[G_{P_{\beta_m}}])).$$

Clearly E_m has finitely many equivalence classes, call them $\langle Z^m_i \mid i < i^*_m \rangle$. All are Borel hence are measurable; w.l.o.g. $\mathrm{Leb}(Z^m_i) = 0 \leftrightarrow i \in [i^\otimes_m, i^*_m)$. For $i < i^\otimes_m$ there is $r = r_{m,i} \in \underset{\sim}{Q}_{\beta_m}[G_{P_{\beta_m}}]$ such that

$$\lim(\underset{\sim}{p}^j_\ell[G_{P_{\beta_m}}]) \supseteq Z^m_i \Rightarrow r \geqslant \underset{\sim}{p}^j_\ell[G_{P_{\beta_m}}],$$

$$\lim(\underset{\sim}{p}^j_\ell[G_{P_{\beta_m}}]) \cap Z^m_i = \emptyset \Rightarrow \lim(r) \cap \underset{\sim}{p}^j_\ell[G_{P_{\beta_m}}] = \emptyset.$$

We can also find a rational $a_{m,i} \in (0,1)_\mathbb{R}$ such that

$$a_{m,i} < \frac{\mathrm{Leb}(Z^m_i)}{2^{\lg}(h^t_2(\mathbf{n}))} < a_{m,i} + \frac{\varepsilon}{2 i^*_m}.$$

We can find $q'_m \in G_{P_{\beta_m}}$, $q_m \leqslant q'_m$ such that q'_m forces all this information (so for $\underset{\sim}{Z}^m_i$, $\underset{\sim}{r}_{m,i}$ we shall have names, but $a_{m,i}$, $i^\otimes_m$, i^*_m are actual objects). We then can find rationals $b_{m,i} \in (a_{m,i}, a_{m,i} + \varepsilon/2)$ such that $\sum_{i < i^\otimes_m} b_{m,i} = 1$.

Now we throw a dice old die choosing $i_m < i^\otimes_m$ with the probability of $i_m = i$ being $b_{m,i}$, and finally we choose q_{m+1} as follows:

$$\mathrm{dom}(q_{m+1}) = \mathrm{dom}(q_m) \cup \{\beta_m\},$$

$$q_{m+1} = \begin{cases} q'_m(\beta) & \text{if } \beta < \beta_m, \\ \underset{\sim}{r}_{m,i_m} & \text{if } \beta = \beta_m. \end{cases}$$

This covers all cases. Basic probability computation (for $n^t_{k+1} - n^t_k$ independent experiments, using $(*)$ of (f)) shows that for each j coming from clause (f), by the law of large numbers the probability of success, i.e. having $q_{m+1} \Vdash_{P_\delta} k \in \underset{\sim}{C}_j \cap \underset{\sim}{B}$, is $> (1 - 1/j^*)(1 - \varepsilon^{-2}(n^t_{k+1} - n^t_k)^{-1})$. For j coming from clause (e) we surely succeed. $\square$

In the following lemma, the whispering conditions (i) of Definition 4.2 are crucial for building $\mathscr{K}^3$.

Lemma 4.6. (1) *Assume that*
(a) $\bar{Q} \in \mathscr{K}^3$, $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{\tau}_\beta, \eta_\beta, (\underset{\sim}{\Xi}^t_\alpha)_{t\in\mathscr{T}} \mid \alpha \leqslant \alpha^*, \beta < \alpha^* \rangle$,
(b) $A \subseteq \alpha^*$, $\kappa \leqslant |A|$,
(c) $\eta \in ({}^\kappa 2)^V \setminus \{\eta_\beta \mid \beta \in \alpha\}$,
(d) $P_A \lessdot P_{\alpha^*}$, $\underset{\sim}{Q}_{\alpha^*}$ *is the* P_{α^*}*-name from Definition 2.2* (F)(β) *and*

$$\text{if } t \in \mathscr{T} \text{ then } \underset{\sim}{\Xi}^t_{\alpha^*} \upharpoonright V^{P_A} \text{ is a } P_A\text{-name.}$$

Then there is $\bar{Q}^+ = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\beta, \mu_\beta, \underset{\sim}{\tau}_\beta, \eta_\beta, (\underset{\sim}{\Xi}^t_\alpha)_{t\in\mathscr{T}} \mid \alpha \leqslant \alpha^*+1,\ \beta < \alpha^*+1 \rangle$ *from* $\mathscr{K}^3$, *extending* $\bar{Q}$ *such that* $A_{\alpha^*} = A$, $\eta_{\alpha^*} = \eta$.

(2) *If clauses* (a)–(c) *of part* (1) *hold then we can find* A' *such that* $A \subseteq A' \subseteq \alpha^*$, $|A'| \leqslant (|A| +$ *number of blueprints*$)^{\aleph_0}$ *such that* $\bar{Q}$, A', η *satisfy* (a)–(d).

Proof. (1) As before the problem is to define $\underset{\sim}{\Xi}^t_{\alpha^*+1}$. We have to satisfy clause (g) of Definition 4.2 for each fixed $t \in \mathscr{T}$. Let $\mathbf{n}^*$ be the unique $\mathbf{n} < \mathbf{n}^t$ such that $\eta \upharpoonright w^t = \eta^t_{\mathbf{n},\ell}$ for some $\ell \in \omega$. If $\mathbf{n}^* \in \mathrm{dom}(h^t_0)$ or if $\langle \eta^t_{\mathbf{n}^*,\ell} \mid \ell \in \omega \rangle$ is not constant or if there is no such $\mathbf{n}^*$ then we have nothing to do.

So assume that $\alpha_\ell = \alpha^*$ for $\ell \in \omega$ and that $\eta^t_{\mathbf{n}^*,\ell} = \eta \upharpoonright w^t$ for $\ell \in \omega$. Let Γ be the set of all pairs $(\underset{\sim}{r}, \langle \underset{\sim}{r}_\ell \mid \ell \in \omega \rangle)$ which satisfy the assumption $(**)$ of Definition 4.2(g). In $V^{P_{\alpha^*+1}}$ we have to choose $\underset{\sim}{\Xi}^t_{\alpha^*+1}$ taking care of all these obligations.

We work in $V^{P_{\alpha^*}}$. By assumption (d), which says that $\underset{\sim}{\Xi}^t_{\alpha^*} \upharpoonright P_A$ (hence in particular the $\underset{\sim}{\Xi}_{\alpha^*}(X)$, where X is built from the $\underset{\sim}{r}$, $\underset{\sim}{r}_\ell$) is a P_A-name, and by Claim 3.7 it suffices to prove it for $\underset{\sim}{\Xi}^t_{\alpha^*+1} \upharpoonright (P_A * \underset{\sim}{Q})$ (as $\underset{\sim}{\Xi}_1$ there) and for $\underset{\sim}{\Xi}^t_{\alpha^*+1} \upharpoonright P_{\alpha^*}$ (as $\underset{\sim}{\Xi}_2$ there) separately, and for the latter there is nothing to prove.

By Fact 3.6 it is enough to prove condition (B) of Fact 3.6. So suppose that fails. Then there are $\langle B_m \mid m < m^* \rangle$, a partition of ω from V^{P_A} such that $\underset{\sim}{\Xi}^t_{\alpha^*}(B_m) > 0$ for $m < m^*$ and $(\underset{\sim}{r}^i, \langle \underset{\sim}{r}^i_\ell \mid \ell \in \omega \rangle) \in \Gamma$ and $\mathbf{n}(i) = \mathbf{n}^* < \mathbf{n}^t$ for $i < i^* < \omega$ and $\varepsilon^* > 0$, $k^* \in \omega$ and $r \in \underset{\sim}{Q}_{\alpha^*}$ which forces that there is no finite $u \subseteq \omega \setminus k^*$ with (i) and (ii) of Fact 3.6(B). W.l.o.g. r forces that $\underset{\sim}{r}^i \in G_{Q_\alpha}$ for $i < i^*$, otherwise we ignore such an $\underset{\sim}{r}^i$. So $r \geqslant r^i$ for $i < i^*$.

By our assumption $(**)$ of Definition 4.2(g) we have that for each $i < i^*$ and $r' \geqslant r$,

$$\mathrm{Av}_{\Xi^t_{\alpha^*}}(\langle a^i_k(r') \mid k \in \omega \rangle) \geqslant 1 - h^t_1(\mathbf{n}),$$

where

$$a^i_k(r') = \frac{1}{n^t_{k+1} - n^t_k} \sum_{\ell \in n^t_k, n^t_{k+1}} \frac{\mathrm{Leb}(\lim(r') \cap \lim(r^i_\ell))}{\mathrm{Leb}(\lim(r'))}.$$

Now V^{P_A} plays the rôle of the ground model (V in Fact 3.6) and $\mathrm{Random}^{V[\tau_\alpha \mid \alpha \in A]} = \mathrm{Random}^{V^{P_A}}$ is the full random forcing over this ground model. So by Fact 3.6 is suffices to prove

Lemma 4.7. *Assume that Ξ is a finitely additive measure, $\langle B_0,\dots,B_{m^*-1}\rangle$ a partition of $\omega, \Xi(B_m)=a_m$, $i^*<\omega$ and r, $r^i_\ell\in\mathrm{Random}$ for $i<i^*$, $\ell\in\omega$ are such that*
($*$) *for every $r'\in\mathrm{Random}$ such that $r'\geqslant r$ and for every $i<i^*$ we have*

$$\mathrm{Av}_\Xi(\langle a^i_k(r')\,|\,k\in\omega\rangle)\geqslant b_i,$$

where

$$a^i_k(r')=\frac{1}{n^t_{k+1}-n^t_k}\sum_{\ell=n^t_k}^{n^t_{k+1}-1}\frac{\mathrm{Leb}(\lim(r')\cap\lim(r^i_\ell))}{\mathrm{Leb}(\lim(r'))}.$$

Then for each $\varepsilon>0$, $k^*\in\omega$ there is a finite $u\subseteq\omega\setminus k^*$ and $r'\geqslant r$ such that
(1) *$a_m-\varepsilon<|u\cap B_m|/|u|<a_m+\varepsilon$, for $m<m^*$,*
(2) *for each $i<i^*$ we have*

$$\frac{1}{|u|}\sum_{k\in u}\frac{|\{\ell\,|\,n^t_k\leqslant\ell<n^t_{k+1}\text{ and } r'\geqslant r^i_\ell\}|}{n^t_{k+1}-n^t_k}\geqslant b_i-\varepsilon.$$

Proof. Let for $i<i^*$, $m<m^*$:

$$c_{i,m}(r')=\mathrm{Av}_{\Xi\restriction B_m}(\langle a^i_k(r')\,|\,k\in B_m\rangle)\in[0,1]_{\mathbb{R}}.$$

So clearly

$$b_i\leqslant\mathrm{Av}_\Xi(\langle a^i_k(r')\,|\,k\in\omega\rangle)=\sum_{m<m^*}\mathrm{Av}_{\Xi\restriction B_m}(\langle a^i_k(r')\,|\,k\in B_m\rangle)\cdot\Xi(B_m)$$
$$=\sum_{m<m^*}c_{i,m}(r')\cdot a_m.$$

Since for each $z\in\omega\setminus\{0\}$ there are only finitely many equivalence classes in the equivalence relation E_z where

$$\langle c_{i,m}\,|\,i<i^*,m<m^*\rangle E_z\langle c'_{i,m}\,|\,i<i^*,m<m^*\rangle$$

iff

$$(\text{for } z'<z,i<i^*,m<m^*)c_{i,m}\in\left[\frac{z'}{z},\frac{z'+1}{z}\right)\leftrightarrow c'_{i,m}\in\left[\frac{z'}{z},\frac{z'+1}{z}\right),$$

we have that there is a condition r^*_z such that each class is either dense above r^*_z or does not appear above r^*_z.

We apply this with some $z\geqslant 1/\varepsilon$ and get an $r^*\geqslant r$ and a sequence $\langle c_{i,m}\,|\,i<i^*, m<m^*\rangle$ such that
(a) $c_{i,m}\in[0,1]_{\mathbb{R}}$,
(b) $\sum_{m<m^*}c_{i,m}\cdot a_m\geqslant b_i$,
(c) for every $r'\geqslant r^*$ there is $r''\geqslant r'$ such that

$$(\forall i<i^*)(\forall m<m^*)[c_{i,m}-\varepsilon<c_{i,m}(r'')<c_{i,m}+\varepsilon].$$

Let $k^* \in \omega$ be given. We now choose $s^* \in \omega$ large enough and try to choose by induction on $s \leqslant s^*$ a condition $r_s \in$ Random and natural numbers (m_s, k_s) (flipping coins along the way) such that

$$r_0 = r^*,$$

$$r_{s+1} \geqslant r_s$$

$$c_{i,m} - \varepsilon < c_{i,m}(r_s) < c_{i,m} + \varepsilon \quad \text{for } i < i^*, m < m^*,$$

$$k_s > k^*, k_{s+1} > k_s,$$

$$k_s \in B_{m_s}.$$

In stage s, given r_s we define r_{s+1}, i_s, m_s, k_s as follows: We choose $m_s < m^*$ randomly with the probability of m_s being m being a_m. Next, we can find a finite set $u_s \subseteq B_{m_s} \setminus \max\{k^* + 1, k_{s_1} + 1 \mid s_1 < s\}$ such that

(+) if $i < i^*$ then $c_{i,m_s} - \varepsilon/2 < \frac{1}{|u_s|} \sum_{k \in u_s} a^i_k(r_s) < c_{i,m_s} + \varepsilon/2$.

We define an equivalence relation $\mathbf{e}_s$ on $\lim(r_s)$ by

$\eta_1 \mathbf{e}_s \eta_2$ iff $(\forall i < i^*)$ $(\forall k \in u_s)$ $(\forall \ell \in [n^t_k, n^t_{k+1}))$ $[\eta_1 \in \lim(r^i_\ell) \leftrightarrow \eta_2 \in \lim(r^i_\ell)]$.

The number of equivalence classes is finite. If $Y \in \lim(r_s)/\mathbf{e}_s$ satisfies $\mathrm{Leb}(Y) > 0$ choose $r_{s,Y} \in$ Random such that $\lim(r_{s,Y}) \subseteq Y$. Now choose r_{s+1} among $\{r_{s,Y} \mid Y \in \lim(r_s)/\mathbf{e}_s$ and $\mathrm{Leb}(Y) > 0\}$ with the probability of $r_{s+1} = r_{s,Y}$ being $\mathrm{Leb}(Y)$. Lastly, choose $k_s \in u_s$ with all $k \in u_s$ having the same probability.

Now the expected value (in the probability space of the flipping coins), assuming that $m_s = m$ of

$$\frac{1}{n^t_{k+1} - n^t_k} \times |\{\ell \mid n^t_k \leqslant \ell < n^t_{k+1} \text{ and } r_{s+1} \geqslant r^i_\ell\}|$$

belongs to the interval $(c_{i,m} - \varepsilon/2, c_{i,m} + \varepsilon/2)$ because the expected value of

$$\frac{1}{|u_s|} \sum_{k \in u_s} \frac{1}{n^t_{k+1} - n^t_k} \times |\{\ell \mid n^t_k \leqslant \ell < n^t_{k+1} \text{ and } r_{s+1} \geqslant r^i_\ell\}|$$

belongs to this interval (which is straightforward).

Let $r' = r_{s^*}$, $u = \{k_s \mid s \leqslant s^*\}$. Hence the expected value of

$$\frac{1}{|u|} \sum_{k \in u} \frac{1}{n^t_{k+1} - n^t_k} \times |\{\ell \mid n^t_k \leqslant \ell < n^t_{k+1} \text{ and } r' \geqslant r^i_\ell\}|$$

is $\geqslant \sum_{m < m^*} a_m (c_{i,m} - \varepsilon/2) \geqslant b_i - \varepsilon/2$.

As s^* is large enough with high probability (though just positive probability suffices), the $(r_{s^*}, \{k_s \mid s \leqslant s^*\})$ are as required for (r', u). Note: We do not know the variance, but we have an upper bound for it not depending on s. There is also a strong law of large numbers that does not require a bound on the variance (see [3]). □

Ad 4.6, Part 2: The proof is an easy counting argument, just enrich A successively such that everything required becomes an P_A-name. □

Remark. We do not use Lemma 4.6(2) in our work, nor do we need here that the number of blueprints is small compared to χ (which is important in [20]), because we shall never use that $\mathscr{K}^3$ is not empty. In Subclaim 5.3, Lemma 5.4 we need only small parts of the properties of elements in $\mathscr{K}^3$. So we shall keep the parts needed in mind and, in Theorem 5.5 we shall show that an arbitrary member $\bar{Q}$ of the subclass of $\mathscr{K}$ given in Definition 2.2 Part (2) behaves similar to a member of $\mathscr{K}^3$ as far as $(**)_{\bar{Q}}$ is concerned.

The following is needed later to show that sufficiently often the clause (g) of Definition 4.2 is not trivial, that is, the premise $(**)$ there holds.

Lemma 4.8. *Assume*
(a) Ξ *is a finitely additive measure on* ω *and* $b \in (0,1]_{\mathbb{R}}$,
(b) $n^t_k < \omega$ *for* $k \in \omega$, $n^t_k < n^t_{k+1}$, *and* $\lim(n^t_{k+1} - n^t_k) = \infty$,
(c) $r^*, r_\ell \in \mathrm{Random}$ *are such that* $(++)$ $(\forall \ell \in \omega)[\mathrm{Leb}(\lim(r^*) \cap \lim(r_\ell))/\mathrm{Leb}(\lim(r^*)) \geqslant b]$.
Then for some $r^\otimes \geqslant r^*$ *we have that*
$\otimes(r^\otimes)$ *for every* $r' \geqslant r^\otimes$ *we have* $\mathrm{Av}_\Xi(\langle a(r',k) \mid k \in \omega\rangle) \geqslant b$ *where* $a_k(r') = a(r',k) = a_k(\lim(r'))$ *and for* $X \subseteq 2^\omega$ *we have that*

$$a_k(X) = \frac{1}{n^t_{k+1} - n^t_k} \sum_{\ell \in n^t_k, n^t_{k+1}} \frac{\mathrm{Leb}(X \cap \lim(r_\ell))}{\mathrm{Leb}(X)}.$$

Proof. Let

$$\mathscr{I} = \{r \in \mathrm{Random} \mid r \geqslant r^*, \text{ and } \mathrm{Av}_\Xi(\langle a_k(r') \mid k \in \omega\rangle) < b\}.$$

If $\mathscr{I}$ is not dense above r^* there is some ${}^\otimes \geqslant r^*$ (in Random) such that for every $r \geqslant r^\otimes$, $r \notin \mathscr{I}$, so $r^\otimes$ is as required.

So suppose that $\mathscr{I}$ is dense above r^*. We take a maximal antichain $\{s_i : i \leqslant i^*\} \subseteq \mathscr{I}$. Because $\mathscr{I}$ is dense above r^* we have that $\{s_i : i \leqslant i^*\}$ is a maximal antichain above r^*. Hence $\mathrm{Leb}(\lim(r^*)) = \sum_{i<i^*} \mathrm{Leb}(\lim(s_i))$. Since Random has the c.c.c. we have that i^* is countable and we assume that $i^* \leqslant \omega$.

For any $j < i^*$ let $s^j = \bigcup_{i \in j} s_i$. Note that $\lim(\bigcup_{m<i} s_m) = \bigcup_{m<i} \lim(s_m)$ and

$$a_k(s^j) = a_k\left(\bigcup_{m<i} s_m\right) = \sum_{i<j} \frac{\mathrm{Leb}(s_i)}{\mathrm{Leb}(\bigcup_{m<j} s_m)} a_k(s_i).$$

Hence we compute

$$\begin{aligned}\mathrm{Av}_\Xi(\langle a_k(s^j) \mid k \in \omega\rangle) &= \mathrm{Av}_\Xi\left(\left\langle a_k\left(\bigcup_{m<j} s_m\right) \mid k \in \omega\right\rangle\right)\\ &= \sum_{i<j} \frac{\mathrm{Leb}(s_i)}{\mathrm{Leb}(\bigcup_{m<j} s_m)} \times \mathrm{Av}_\Xi(\langle a_k(s_i) \mid k \in \omega\rangle)\end{aligned}$$

$$\leqslant \frac{\mathrm{Leb}(s_0)}{\mathrm{Leb}(\bigcup_{i<j} s_i)}(b-\varepsilon) + \sum_{0<i<j} \frac{\mathrm{Leb}(s_i)}{\mathrm{Leb}(\bigcup_{m<j} s_m)} \cdot b$$
$$= b - \mathrm{Leb}(\lim(s_0)) \cdot \varepsilon,$$

where $\varepsilon = b - \mathrm{Av}_{\Xi}(\langle a_k(s_0) \mid k \in \omega\rangle)$, so $\varepsilon > 0$.

Now let j be large enough such that $\mathrm{Leb}(\lim(r^*) \setminus \lim(s^j))/\mathrm{Leb}(\lim(r^*)) < \mathrm{Leb}(\lim(s_0)) \cdot \varepsilon$. Then

$$\mathrm{Av}_{\Xi}(\langle a_k(r^*) \mid k \in \omega\rangle)$$
$$= \frac{\mathrm{Leb}(\lim(r^*) \setminus \lim(s^j))}{\mathrm{Leb}(\lim(r^*))} \cdot \mathrm{Av}_{\Xi}(\langle a_k(\lim(r^*) \setminus \lim(s^j)) \mid k \in \omega\rangle)$$
$$+ \frac{\mathrm{Leb}(\lim(s^j))}{\mathrm{Leb}(\lim(r^*))} \cdot \mathrm{Av}_{\Xi}(\langle a_k(\lim(s^j)) \mid k \in \omega\rangle)$$
$$\leqslant \frac{\mathrm{Leb}(\lim(r^*) \setminus \lim(s^j))}{\mathrm{Leb}(\lim(r^*))} \cdot 1 + \frac{\mathrm{Leb}(\lim(s^j))}{\mathrm{Leb}(\lim(r^*))} \cdot (b - \mathrm{Leb}(\lim(s_0)) \cdot \varepsilon)$$
$$< \mathrm{Leb}(\lim(s_0)) \cdot \varepsilon + (b - \mathrm{Leb}(\lim(s_0)) \cdot \varepsilon) = b$$

contradicting assumption (c). $\square$

Lemma 4.5 took care of the successor step in the case of $|A| \geqslant \kappa$. We close this section with the successor step for $|A| < \kappa$ (which means empty A for the iterations from Definition 2.2 Part (2). Everything in this section applies to Definition 2.2 Part (1). Only at the end of the next section we shall make use of the particularly good additional features of the narrower class in Definition 2.2 Part (2): Small forcing conditions, orderly separation between Cohen part and random part, etc.

Claim 4.9. *Assume that*
(a) $\bar{Q} \in \mathscr{K}^3$, $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\alpha, \mu_\beta, \underset{\sim}{\tau}_\beta, \eta_\beta, (\underset{\sim}{\Xi}^t_\alpha)_{t \in \mathscr{T}} \mid \alpha \leqslant \alpha^*, \beta < \alpha^*\rangle$,
(b) $A \subseteq \alpha^*$, $\kappa > |A|$, *and* $\hat{\mu} < \kappa$,
(c) $\eta \in ({}^\kappa 2)^V \setminus \{\eta_\beta \mid \beta \in \alpha\}$,
(d) $\underset{\sim}{Q}$ *is the* P_{α^*}*-name for a forcing notion with set of elements* $\hat{\mu}$, *and is definable in* $V[\langle \tau_\beta \mid \beta \in A\rangle]$ *from* $\langle \tau_\beta \mid \beta \in A\rangle$ *and parameters from* V.

Then there is

$$\bar{Q}^+ = \langle P_\alpha, \underset{\sim}{Q}_\beta, A_\alpha, \mu_\beta, \underset{\sim}{\tau}_\beta, \eta_\beta, (\underset{\sim}{\Xi}^t_\alpha)_{t \in \mathscr{T}} \mid \alpha \leqslant \alpha^* + 1, \beta < \alpha^* + 1\rangle$$

from $\mathscr{K}^3$, *extending* $\bar{Q}$ *such that* $\underset{\sim}{Q}_{\alpha^*} = \underset{\sim}{Q}$, $A_{\alpha^*} = A$, $\eta_{\alpha^*} = \eta$, $\mu_{\alpha^*} = \hat{\mu}$.

Proof. Definition 4.2 gives no requirements on the $\underset{\sim}{\Xi}^t_{\alpha^*+1}$. $\square$

5. The last part of the proof of $(**)_{\bar{Q}}$

In this section we shall finish the proof of $(**)_{\bar{Q}}$ for $\mathscr{K}^3$, and then we shall finish the proof of Lemma 2.11 and Theorem 2.1.

We give an outline of the proof of $(**)_{\bar{Q}}$ for $\mathscr{K}^3$: We assume that we have a counterexample $p^*, \underaccent{\tilde}{T}$ (for a perfect tree $\subseteq (\bigcap_{\zeta \in \underaccent{\tilde}{E}} \lim \mathrm{tree}_m(a^\zeta))^{V[G]}$), m (for the tree_m), $\underaccent{\tilde}{E}$ to it. We thin out the p_ζ that are forced to be in $\underaccent{\tilde}{E}$. Thus, we get a in some sense indiscernible set of conditions. Some features the first ω of these indiscernibles are described well by a blueprint $t \in \mathscr{T}$, and this description allows us to define some $p^\otimes \geqslant p^*$ such that $p^\otimes$ forces that $T = \underaccent{\tilde}{T}[G]$ cannot be a perfect tree because the subset $A \subseteq \underaccent{\tilde}{E}[G]$ over which we build the intersection is 'too large', and thus we have a contradiction. Having Ξ^t_α-measure non-zero ensures infinity, and indeed the measure Ξ^t_α will lead to the notion of 'too large' that we are going use (see Lemma 5.2 and Subclaim 5.3).

Then we show $(**)_{\bar{Q}}$ for the members of the subclass of $\mathscr{K}$ that is given in Definition 2.2 Part (2). We start looking for finitely additive measures only after p_ζ, $\zeta \in \omega$ and $t \in \mathscr{T}$ (remember: $\mathscr{T}$ is the set of blueprints for κ from Definition 4.1) are chosen and do it only for one suitable t. We want to have some $\Xi^t_{\alpha^*}$ that satisfies just the requirements in Definition 4.2 (with true premises in (e)–(g) for our chosen $\langle \alpha_\ell \mid \ell \in \omega \rangle$!) that speak about our p'_ζ, in order to jump into the proofs of Lemma 5.2 and of Subclaim 5.3, which work with $\mathscr{K}^3$, and go on like there.

It turns out that only requirements about $p'_\zeta(\chi + \gamma_n)$, $n < n^* \in \omega$, n^* the size of the part of the heart of a Δ-system lying above χ, are relevant. We shall look at $\bar{Q}^\chi$ for several χ (and the same $\kappa, \mu, \gamma_0, \ldots, \gamma_{n^*-1}$) and use a Löwenheim Skolem argument to provide the $(\Xi^t_{\underaccent{\tilde}{\gamma}_n})_{n<n^*, t \in \mathscr{T}}$ good for these requirements. Besides some elementary embedding, we shall use the automorphisms for the $\bar{Q}$ from Definition 4.2, Part (2) in order to make sufficiently many instances of (e), (g), (i) of Definition 4.2 true. (We already mentioned that (f) is ad libitum.)

Lemma 5.1. *Suppose that $\bar{\varepsilon} = \langle \varepsilon_\ell \mid \ell \in \omega \rangle$ is a sequence of positive reals and that $\bar{Q} \in \mathscr{K}^3$ has length α. Recall that P'_α was defined in Definition* 2.3(c). *Then the following $\mathscr{I}_{\bar{\varepsilon}} \subseteq P_\alpha$ is dense:*

$$\begin{aligned}\mathscr{I}_{\bar{\varepsilon}} = \{p \in P'_\alpha \mid & \text{ there are } m \text{ and } a_\ell, v_\ell \text{ for } \ell < m \text{ such that}\\ & \text{(a) } \mathrm{dom}(p) = \{\alpha_0, \ldots, \alpha_{m-1}\},\ \alpha_0 < \alpha_1 < \cdots < \alpha_{m-1} < \alpha,\\ & \text{(b) if } |Q_{\alpha_\ell}| < \kappa, \text{ then } p(\alpha_\ell) \text{ is an ordinal},\\ & \text{(c) if } |Q_{\alpha_\ell}| \text{ is partial random, then } \Vdash_{P_{\alpha_\ell}} \text{“}p(\alpha_\ell) \subseteq ({}^{\omega}2)^{[v_\ell]}\\ & \qquad \text{and } \mathrm{Leb}(\lim(p(\alpha_\ell))) \geqslant (1-\varepsilon_\ell)/2^{\lg(v_\ell)}\text{”}\}.\end{aligned}$$

Proof. By induction on α for all possible $\bar{\varepsilon}$. Use the Lebesgue density theorem [15]. □

Lemma 5.2. *If $P_\alpha = \lim(\bar{Q})$, $\alpha = \lg(\bar{Q})$ and $\bar{Q} \in \mathscr{K}^3$, then $(**)_{\bar{Q}}$ from Lemma 2.11 holds.*

Proof. Suppose that $p^* \Vdash_{P_\alpha}$ "$\underset{\sim}{T}, m, \underset{\sim}{E}$ form a counterexample to $(**)_{\bar{Q}}$". Let $\bar{\varepsilon} = \langle \varepsilon_\ell \mid \ell \in \omega \rangle$ be such that $\varepsilon_\ell \in (0,1)_{\mathbb{R}}$ and such that $\sum_{\ell\in\omega} \sqrt{2\varepsilon_\ell} < 1/10$. For each $\zeta < \kappa^+$ let $p'_\zeta \geqslant p_\zeta \geqslant p^*$ be such that $p'_\zeta \in \mathscr{I}_{\bar{\varepsilon}}$ is witnessed by $\langle v^\zeta_\alpha \mid \alpha \in \mathrm{dom}(p'_\zeta) \wedge |Q_\alpha| \geqslant \kappa \rangle$ and

$$p'_\zeta \Vdash_{P_\alpha} \text{“}\beta_\zeta \text{ is the } \zeta\text{th element such that } \underset{\sim}{T} \subseteq N[\underset{\sim}{\bar{a}}^{\beta_\zeta}]\text{”}.$$

Call the p'_ζ now p_ζ again. By thinning out we may assume that there are $i^*, v_0, v_1, \Delta, z, \gamma^\zeta_i, v_i, s^*$ such that

(1) $\mathrm{dom}(p_\zeta) = \{\gamma^\zeta_i \mid i < i^*\}$ with γ^ζ_i increasing with i, let $v^\zeta_0 = \{i < i^* \mid |Q_{\gamma^\zeta_i}| < \kappa\}$, then $v^\zeta_0 = v_0$ is fixed for all ζ, $v_1 = i^* \setminus v_0$,

(2) $\mathrm{dom}(p_\zeta)(\zeta < \kappa^+)$ form a Δ-system with heart $\Delta \subseteq \mathrm{dom}(p^*)$,

(3) $\beta_\zeta \in \mathrm{dom}(p_\zeta)$, $\beta_\zeta = \gamma^\zeta_z$ for a fixed $z < i^*$,

(4) $(\mathrm{dom}(p_\zeta), \Delta, \chi, <)$ are isomorphic for $\zeta < \kappa^+$,

(5) if $i \in v_0$, then $p_\zeta(\gamma^\zeta_i) = \gamma_i$ for $\zeta < \kappa^+$,

(6) if $i \in v_1$, then $v^\zeta_{\gamma^\zeta_i} = v_i$ (recall $v^\zeta_{\gamma^\zeta_i} \in {}^{<\omega}2$ is given by the definition of $\mathscr{I}_{\bar{\varepsilon}}$),

(7) $p_\zeta(\beta_\zeta) = s^*$ for $\zeta < \kappa^+$, $s^* = \langle (n_\ell, a_\ell) \mid \ell < m^* \rangle$, w.l.o.g. $m^* > m$ (where m is from the counterexample to $(**)_{\bar{Q}}$) and $m^* > 10$ (this is a similar but not the same as in Lemma 2.12),

(8) for each $i < i^*$ the sequence $\langle \gamma^\zeta_i \mid \zeta \in \kappa^+ \rangle$ is constant or strictly increasing,

(9) the sequence $\langle \beta_\zeta \mid \zeta \in \kappa^+ \rangle$ is with no repetitions (since, if p_{ζ_1}, p_{ζ_2} are compatible and $\zeta_1 < \zeta_2 < \chi$, then $\beta_{\zeta_1} \neq \beta_{\zeta_2}$).

Now we keep only the first ω conditions p_ζ, $\zeta < \omega$. For every such ζ let $p'_\zeta \geqslant p_\zeta$ be such that $\mathrm{dom}(p'_\zeta) = \mathrm{dom}(p_\zeta)$, $p'_\zeta(\gamma) = p_\zeta(\gamma)$ except for $\gamma = \beta_\zeta$ in which case we extend $p_\zeta(\beta_\zeta) = s^*$ in the following way.

We put $\lg(p'_\zeta(\beta_\zeta)) = \lg(s^*) + 1 = m^* + 1$ and set $p'_\zeta(\beta_\zeta) = s^* \frown \langle (j^0_\zeta, a_\zeta) \rangle$.

Before we define (j^0_ζ, a_ζ) we choose an increasing sequence of integers $\bar{s} = \langle s_\ell \mid \ell \in \omega \rangle$, $s_0 = 0$, such that

$$s_{k+1} - s_k = |(2^{j_k})^{(2^{j_k}(1-8^{-m^*}))}|,$$

where

$$j^* = 3n_{m^*-1} + 1$$

(recall from (7) that n_{m^*-1} is the first coordinate of the last pair in s^*) and we let $j_k = j^* + k!!$ and let $j^0_\zeta = j_k$ when $\zeta \in [s_k, s_{k+1})$. Now for $\zeta \in [s_k, s_{k+1})$ define a_ζ such that

$$\{a_\zeta \mid \zeta \in [s_k, s_{k+1})\} = [{}^{j_k}2]^{2^{j_k}(1-8^{-m^*})}.$$

For $\varepsilon^* > 0$ we define a P_α-name by

$$\underset{\sim}{A}_{\varepsilon^*} = \left\{ k \in \omega \,\middle|\, \frac{|\{\zeta \in [s_k, s_{k+1}) \mid p'_\zeta \in \underset{\sim}{G}_{P_\alpha}\}|}{s_{k+1} - s_k} > \varepsilon^* \right\}.$$

For the proof of Lemma 5.2 we need

Subclaim 5.3. *There is a condition $p^{\otimes} \geqslant p^*$ that forces that for some $\varepsilon^* > 0$ the set $\underset{\sim}{A}_{\varepsilon^*}$ is infinite.*

Explanation: The $p^{\otimes}$ is an analogue to the premise no. 3 of Just's Lemma 2.12. The condition $p^{\otimes}(\gamma)$ is roughly spoken "as compatible as possible with many, in the sense of the $\Xi^t_\gamma(A_{\varepsilon^*}) > 0$, of the $\langle p^t_\zeta(\gamma) \mid \zeta \in \omega\rangle$". The coding with the $\eta^t_{\mathbf{n},\zeta}$ and the $\eta_\gamma \restriction w^t$, w^t from (5.1), ensures that $p^{\otimes}$ is well defined by the definition below.

Proof. We may choose any $\varepsilon^* < 1 - \sum_{\ell\in\omega} \sqrt{2\varepsilon_\ell}$ (where the $\bar{\varepsilon} = \langle \varepsilon_\ell \mid \ell \in \omega\rangle$ was chosen at the beginning of Lemma 5.2). First, we define a suitable blueprint $t \in \mathscr{T}$,

$$t = (w^t, \mathbf{n}^t, \mathbf{m}^t, \bar{\eta}^t, h^t_0, h^t_1, h^t_2, \bar{n}^t).$$

We let

$$w^t = \{\min\{\beta \in \kappa \mid \eta_{\gamma^{\zeta(1)}_{i(1)}}(\beta) \neq \eta_{\gamma^{\zeta(2)}_{i(2)}}(\beta)\} \mid \zeta(1), \zeta(2) < \omega$$

and

$$i(1), i(2) < i^* \quad \text{and} \quad \gamma^{\zeta(1)}_{i(1)} \neq \gamma^{\zeta(2)}_{i(2)}\}, \tag{5.1}$$

where the η_α come from the definition of $\mathscr{K}^3$. (w^t is well defined because η is injective.)

Let $\mathbf{n}^t = i^*$, $\mathrm{dom}(h^t_0) = v_0$, $\mathrm{dom}(h^t_1) = \mathrm{dom}(h^t_2) = v_1$ and $n^t_\ell = s_\ell$.

We set $\eta^t_{\mathbf{n},\zeta} = \eta_{\gamma^\zeta_{\mathbf{n}}} \restriction w^t$. Note that the $\eta^t_{\mathbf{n},\zeta}$ satisfy the requirements from 4.1(g) and (h): By Lemma 5.2 item (4), we have that $\gamma^\zeta_{\mathbf{n}} = \gamma^{\zeta'}_{\mathbf{n}'}$ implies $\mathbf{n} = \mathbf{n}'$. Hence we have that $\eta^t_{\mathbf{n},\zeta} = \eta^t_{\mathbf{n}',\zeta'}$ implies that $\eta_{\gamma^\zeta_{\mathbf{n}}} \restriction w^t = \eta_{\gamma^{\zeta'}_{\mathbf{n}'}} \restriction w^t$ and hence by the definition of w^t, that $\gamma^\zeta_{\mathbf{n}} = \gamma^{\zeta'}_{\mathbf{n}'}$ and hence $\mathbf{n} = \mathbf{n}'$.

If $\mathbf{n} \in v_0$, then $h^t_0(\mathbf{n})(\ell) = \gamma_{\mathbf{n}}$ so it is constant independent of ℓ.

If $\mathbf{n} \in v_1$ then $h^t_1(\mathbf{n}) = \varepsilon_{\mathbf{n}}$ and $h^t_2(\mathbf{n}) = v_{\mathbf{n}}$. Finally, we set $\mathbf{m}^t = \max\{k \mid \forall \zeta\ \gamma^\zeta_k < \chi\} + 1$.

Note that by our choice of t, $\langle \gamma^\zeta_{\mathbf{n}} \mid \zeta \in \omega\rangle$ satisfies $(t, \mathbf{n})$ for $\bar{Q}$ for every $\mathbf{n} < i^*$.

We now define a condition $p^{\otimes}$ such that it will be in P_α, $\mathrm{dom}(p^{\otimes}) = \Delta$, $p^* \leqslant p^{\otimes}$. Remember that $\mathrm{dom}(p^*) \subseteq \Delta$, because for each ζ we have that $p^* \leqslant p_\zeta$. If $\gamma \in \Delta$ then for some $\mathbf{n} < \mathbf{n}^t$, we have that $\bigwedge_{\zeta\in\omega} \gamma^\zeta_{\mathbf{n}} = \gamma$.

Case: $\mathbf{n} \in v_0$. If $\mathbf{n} \in v_0$ we let $p^{\otimes}(\gamma) = h^t_0(\mathbf{n})$, so in V^{P_γ}

$$p^{\otimes} \Vdash_{Q_\gamma} \text{“}\underset{\sim}{\Xi}^t_{\gamma+1}(\{\zeta \in \omega \mid h^t_0(\mathbf{n}) \in G_{Q_\gamma}\}) = 1 \quad \text{if } \mathbf{n} \in \mathrm{dom}(h^t_0)\text{”}.$$

Case: $\mathbf{n} \in v_1$. If $\mathbf{n} \in v_1$, then we define a P_γ-name for a member of Q_γ as follows. Consider $\underset{\sim}{r}^{\mathbf{n}}_\zeta = \underset{\sim}{p}^t_\zeta(\gamma)$ for $\zeta < \omega$. Let $\underset{\sim}{r} = \underset{\sim}{p}^*(\gamma) \cap ({}^\omega 2)^{[h^t_2(\mathbf{n})]}$ if $\gamma \in \mathrm{dom}(p^*)$ and otherwise we let $\underset{\sim}{r}$ be just $({}^\omega 2)^{[h^t_2(\mathbf{n})]}$. Now the premise (c) (++) of Lemma 4.8 is true with $b = 1 - 2\varepsilon_{\mathbf{n}}$. Thus by Lemma 4.8 there is some $r^*_\gamma \geqslant r$ such that for every $r' \geqslant r^*_\gamma$ in Q_γ we have that

$$\mathrm{Av}_{\Xi^t_\alpha}(\langle a^{\mathbf{n}}_k(r') \mid k \in \omega\rangle) \geqslant 1 - 2h^t_1(\mathbf{n}) = 1 - 2\varepsilon_{\mathbf{n}},$$

where

$(**)_{r',\bar{\varepsilon}}$

$$a^{\mathbf{n}}_k(r') = \frac{1}{n^t_{k+1} - n^t_k} \sum_{\ell \in [n^t_k, n^t_{k+1})} \frac{\mathrm{Leb}(\lim(r') \cap \lim(r^{\mathbf{n}}_\ell))}{\mathrm{Leb}(\lim(r'))}.$$

Since $\langle \gamma^\zeta_{\mathbf{n}} \mid \zeta \in \omega \rangle$ is constant since, by $(**)_{r',\bar{\varepsilon}}$ the assumption $(**)$ of condition (g) of Definition 4.2 holds, we get that in V^{P_γ}

$$r^*_\gamma \Vdash_{Q_\gamma} \text{“}\mathrm{Av}_{\underset{\sim}{\Xi}^t_{\gamma+1}}\left(\left\langle \frac{|\ell \in [n^t_k, n^t_{k+1}) \mid p_\ell(\gamma) \in \underset{\sim}{G}_{Q_\gamma}|}{n^t_{k+1} - n^t_k} \,\middle|\, k \in \omega \right\rangle\right) \geqslant 1 - 2\varepsilon_{\mathbf{n}}\text{”}.$$

For every $\varepsilon' > 0$ we have: if $\mathrm{Av}_\Xi(\langle a_k \mid k \in \omega \rangle) \geqslant 1 - \varepsilon'$ then for every $\varepsilon > 0$ such that $\varepsilon + \varepsilon' < 1$,

$$\Xi(\{\ell \mid a_\ell \leqslant 1 - \varepsilon' - \varepsilon\})\ (1 - \varepsilon' - \varepsilon) \\ + \Xi(\{\ell \mid a_\ell > 1 - \varepsilon' - \varepsilon\}) 1 \geqslant \mathrm{Av}_\Xi(\langle a_\ell \mid \ell \in \omega \rangle) \geqslant 1 - \varepsilon'$$

and hence

$$\Xi(\{\ell \mid a_\ell \leqslant 1 - \varepsilon' - \varepsilon\}) \leqslant \frac{\varepsilon'}{\varepsilon' + \varepsilon}.$$

Now we put $\varepsilon' = 2\varepsilon_{\mathbf{n}}$ and get for every $\varepsilon > 0$

$$r^*_\gamma \Vdash_{Q_\gamma} \text{“}\underset{\sim}{\Xi}^t_{\gamma+1}\left\{ k \in \omega \,\middle|\, \frac{|\ell \in [n^t_k, n^t_{k+1}) \mid p_\ell(\gamma) \in \underset{\sim}{G}_{Q_\gamma}|}{n^t_{k+1} - n^t_k} \leqslant 1 - 2\varepsilon_{\mathbf{n}} - \varepsilon \right\} \\ \leqslant \frac{2\varepsilon_{\mathbf{n}}}{2\varepsilon_{\mathbf{n}} + \varepsilon}\text{”}.$$

We take $\varepsilon = \sqrt{2\varepsilon_{\mathbf{n}}} - 2\varepsilon_{\mathbf{n}}$ and thus get

$$r^*_\gamma \Vdash_{Q_\gamma} \text{“}\underset{\sim}{\Xi}^t_{\gamma+1}\left\{ k \in \omega \,\middle|\, \frac{|\ell \in [n^t_k, n^t_{k+1}) \mid p_\ell(\gamma) \in \underset{\sim}{G}_{Q_\gamma}|}{n^t_{k+1} - n^t_k} \leqslant 1 - \sqrt{2\varepsilon_{\mathbf{n}}} \right\} \leqslant \sqrt{2\varepsilon_{\mathbf{n}}}\text{”}.$$

So there is a P_γ-name $\underset{\sim}{r}^*_\gamma$ of such a condition. In this case let $p^\otimes(\gamma) = \underset{\sim}{r}^*_\gamma$. So we have finished the definition of $p^\otimes$, and it clearly has the right domain.

(Notice for later generalisation: Property (g) is used here only for γ in the heart of a Δ-system. Moreover, in order to establish (g) for γ as in Lemma 4.6, property (i) is needed only for γ.)

Now suppose that $\mathbf{n} < \mathbf{n}^t$ is such that $\gamma^\zeta_{\mathbf{n}} \notin \Delta$. (Note that this case can be avoided by an appropriate choice of p'_ζ, see our earlier remarks on simplifications.) Define $\bar{\beta} = \langle \beta_\zeta \mid \zeta \in \omega \rangle$, $\beta_\zeta = \gamma^\zeta_{\mathbf{n}}$, $r^{\mathbf{n}}_\zeta = p'_\zeta(\gamma^\zeta_{\mathbf{n}})$. Then $\bar{\beta}$ satisfies $(t, \mathbf{n})$ for P_α. If $\mathbf{n} \in v_1$, by our assumption that $p'_\zeta(\gamma) \in \mathscr{I}_{\bar{\varepsilon}}$ and $\varepsilon_{\mathbf{n}} = h^t_1(\mathbf{n})$, we get that the premise of clause (f) of Definition 4.2 is fulfilled, hence in V^{P_α}.

For each $\varepsilon > 0$

$$\Vdash_{P_\alpha} \text{“}\underset{\sim}{\Xi}^t_\alpha\left(\left\{k \,\middle|\, \frac{|\{\ell \in [n^t_k, n^t_{k+1}) \colon p_\ell(\gamma^\ell_{\mathbf{n}}) \in \underset{\sim}{G}_{\gamma^\ell_{\mathbf{n}}}\}|}{n^t_{k+1} - n^t_k} \geqslant (1-\varepsilon_{\mathbf{n}})(1-\varepsilon)\right\}\right) = 1\text{”}.$$

Putting both cases of $\mathbf{n} \in v_1$ (the one with $\gamma^\zeta_{\mathbf{n}} \in \Delta$ and the latter, complementary one) together and assuming that $p^\otimes \in G$ we get in V^{P_α} for every $\mathbf{n} \in v_1$

$$\sqrt{2\varepsilon_{\mathbf{n}}} \geqslant \Xi^t_\alpha\left(\left\{k \in \omega \,\middle|\, 1 - \sqrt{2\varepsilon_{\mathbf{n}}} \geqslant \frac{|\{\ell \mid n^t_k \leqslant \ell < n^t_{k+1} \text{ and } r^{\mathbf{n}}_\ell \in G_{P_\alpha}\}|}{n^t_{k+1} - n^t_k}\right\}\right).$$

Let

$$\underset{\sim}{A}'_{\varepsilon^*} = \{k \in \omega \mid \text{ if } \zeta \in [n^t_k, n^t_{k+1}) \text{ and } i \in v_0 \text{ then } p_\zeta \restriction \{\gamma^\zeta_i\} \in \underset{\sim}{G}_{P_\alpha}\}.$$

Then, by Definition 4.2(e), $\Xi^t_\alpha(A'_{\varepsilon^*}) = 1$.
So

$$\begin{aligned} &A_{\varepsilon^*} \cup (\omega \setminus A'_{\varepsilon^*}) \\ &\supseteq \left\{k \in \omega \,\middle|\, \text{if } \mathbf{n} \in v_1 \text{ then } \frac{|\{\ell \mid n^t_k \leqslant \ell < n^t_{k+1} \text{ and } r^{\mathbf{n}}_\ell \in G_{P_\alpha}\}|}{n^t_{k+1} - n^t_k} \geqslant 1 - \sqrt{2\varepsilon_{\mathbf{n}}}\right\} \\ &= \omega \Big\backslash \bigcup_{\mathbf{n} \in v_1} \left\{k \in \omega \,\middle|\, \frac{|\{\ell \mid n^t_k \leqslant \ell < n^t_{k+1} \text{ and } r^{\mathbf{n}}_\ell \in G_{P_\alpha}\}|}{n^t_{k+1} - n^t_k} < 1 - \sqrt{2\varepsilon_{\mathbf{n}}}\right\}. \end{aligned}$$

Hence $\Xi^t_\alpha(A_{\varepsilon^*} \cup (\omega \setminus A'_{\varepsilon^*})) \geqslant 1 - \sum_{\mathbf{n} \in v_0} \sqrt{2\varepsilon_{\mathbf{n}}} \geqslant \varepsilon^* > 0$, but

$$\Xi^t_\alpha(\omega \setminus A'_{\varepsilon^*}) = 1 - \Xi^t_\alpha(A'_{\varepsilon^*}) = 1 - 1 = 0,$$

hence necessarily A_{ε^*} is infinite. □

Let $p^\otimes$ be as in Subclaim 5.3. Let G_{P_α} be a generic subset of P_α to which $p^\otimes$ belongs. So $A = \underset{\sim}{A}_{\varepsilon^*}[G]$ be infinite. For $k \in A$, let $b_k = \{\zeta \in [s_k, s_{k+1}) \mid p'_\zeta \in G\}$. We know that $|b_k| > (s_{k+1} - s_k)\varepsilon^*$. Let $\underset{\sim}{T}[G] = T$.

If $k \in A$, then there are $(s_{k+1} - s_k)\varepsilon^*$ many $\zeta \in [s_k, s_{k+1})$ such that $p'_\zeta \in G$ and $p'_\zeta \Vdash \underset{\sim}{T} \cap {}^{j_k}2 \subseteq a_\zeta$, hence $T \cap {}^{j_k}2 \subseteq \bigcap_{\zeta \in b_k} a_\zeta$ as $\lg(s^*) = m^* > m$. To reach a contradiction it is enough to show that for infinitely many $k \in A$ there is a bound on the size of $T \cap {}^{j_k}2$ which does not depend on k.

Now $|b_k|/(s_{k+1} - s_k)$ is at most the probability that if we choose a subset of ${}^{j_k}2$ with $2^{j_k}(1 - 8^{-m^*})$ elements, it will include $T \cap {}^{j_k}2$. If $k \in A$ (and these are infinitely many k, because A is infinite) this probability has a lower bound ε^* not depending on k, and this implies that $\langle |T \cap {}^{j_k}2| \mid k \in \omega\rangle$ is bounded and that hence T is finite.

More formally, for a fixed $k \in \omega$ we have

$$\begin{aligned}
|b_k| &= |\{a_\zeta \mid \zeta \in [s_k, s_{k+1}),\ \zeta \in b_k\}| \\
&\leqslant |\{a_\zeta \mid \zeta \in [s_k, s_{k+1}),\ T \cap {}^{j_k}2 \subseteq a_\zeta\}| \\
&\leqslant |\{a \overset{j_k}{\subseteq} 2 \mid T \cap {}^{j_k}2 \subseteq a \text{ and } |a| = 2^{j_k}(1 - 8^{-m^*})\}| \\
&= |\{a \overset{j_k}{\subseteq} 2 \backslash (T \cap {}^{j_k}2) \mid |a| = 2^{j_k} \times 8^{-m^*}\}| \\
&= \binom{2^{j_k} - |T \cap {}^{j_k}2|}{2^{j_k} \cdot 8^{-m^*}}.
\end{aligned}$$

By definition we have that $s_{k+1} - s_k = \binom{2^{j_k}}{2^{j_k} \cdot (1 - 8^{-m^*})} = \binom{2^{j_k}}{2^{j_k} \cdot 8^{-m^*}}$.

Hence

$$\frac{|b_k|}{s_{k+1} - s_k} \leqslant \frac{\binom{2^{j_k} - |T \cap {}^{j_k}2|}{2^{j_k} \cdot 8^{-m^*}}}{\binom{2^{j_k}}{2^{j_k} \cdot 8^{-m^*}}} = \prod_{i < |T \cap {}^{j_k}2|} \left(1 - \frac{2^{j_k} 8^{-m^*}}{2^{j_k} - i}\right).$$

Let $i_k(*) = \min(|T \cap {}^{j_k}2|, 2^{j_k - 1})$, so

$$\begin{aligned}
\varepsilon^* \leqslant \frac{|b_k|}{s_{k+1} - s_k} &\leqslant \prod_{i < |T \cap {}^{j_k}2|} \left(1 - \frac{2^{j_k} 8^{-m^*}}{2^{j_k} - i}\right) \\
&\leqslant \prod_{i < i_k(*)} \left(1 - \frac{2^{j_k} 8^{-m^*}}{2^{j_k}}\right) = (1 - 8^{-m^*})^{i_k(*)}.
\end{aligned}$$

So we can find a bound on $i_k(*)$ not depending on k:

$$i_k(*) \leqslant \frac{\log(\varepsilon^*)}{\log(1 - 8^{-m^*})}.$$

Remember $m^* > 10$, so $1 - 8^{-m^*} \in (0,1)_{\mathbb{R}}$. So for k large enough,

$$|T \cap {}^{j_k}2| = i_k(*) \leqslant \frac{\log(\varepsilon^*)}{\log(1 - 8^{-m^*})}.$$

This finishes the proof. □

So, how do we get a proof of Lemma 2.11 from Lemma 5.2? We have to show that our members of $\mathscr{K}$ as defined in Definition 2.2, Part (2) behave like members of $\mathscr{K}^3$ at sufficiently many points in the domain of the iteration, that is we have to define suitable $\underset{\sim}{\Xi}^t_\alpha$ and η.

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Now we shall look at several iteration lengths χ at the same time. Recall the definitions of g_χ, E^χ_ξ, $A^\chi_{\chi+\xi}$ from the beginning of the proof of Theorem 2.1.

For $\bar{Q}=\bar{Q}^\chi$ as in Definition 2.2, Part (2) we set $\bar{Q}^\chi = P^\chi = P_\chi$ (of length $\chi+\mu$!); for $A \subseteq \chi+\mu$, we let $P'_A = P'_{\chi,A}$.

Recall our choice of memories from the beginning of the proof of Theorem 2.1: $g_\chi : \chi \to [\mu]^{<\lambda}$ such that $g_\chi \subseteq g_{\chi'}$ for $\chi<\chi'$ and such that every point has χ preimages uner g_χ. From the g_χ's we defined

$$\text{for } \xi \in \mu \; E^\chi_\xi = \{\alpha < \chi \mid \xi \notin g_\chi(\alpha)\},$$

$$A^\chi_{\chi+\xi} = E^\chi_\xi \cup [\chi, \chi+\xi).$$

We have that $A^\chi_{\chi+\xi} \cap \chi = A^{\chi'}_{\chi'+\xi} \cap \chi$.

First we need the following.

Lemma 5.4. (1) *If* $\xi \leqslant \gamma < \mu$ *then in* $\bar{Q}^\chi$

(a) $P'_{(\chi \cap A_{\chi+\gamma}) \cup [\chi,\chi+\xi)} = P'_{E^\chi_\gamma \cup [\chi,\chi+\xi)} \lessdot P'_{\chi+\xi}$.

(b) *If* $q \in P'_{\chi+\xi}$ *and* $q \restriction (E_\gamma \cup [\chi,\chi+\xi)) \leqslant p \in P'_{E_\gamma \cup [\chi,\chi+\xi)}$ *then*

$$p \cup q \restriction (\lg Q) \setminus (E_\gamma \cup [\chi,\chi+\xi)) \in P'_{\chi+\xi}$$

is the least upper bound of p *and* q.

(2) *If* $\chi \leqslant \chi'$, *then*

$$P'_{\chi',\chi \cup [\chi',\chi'+\mu)} \lessdot P'_{\chi',\chi'+\mu}$$

and $P'_{\chi,\chi+\mu}$ *is isomorphic to* $P'_{\chi',\chi \cup [\chi',\chi'+\mu)}$ *by* $\hat{h}$ *where* $h = h^{\chi,\chi'}$ *is the canonical mapping, i.e.* $h : \chi+\mu \to \chi'+\mu$ *be the identity below* χ *and* $h(\chi+\alpha) = \chi'+\alpha$ *for* $\alpha<\mu$.

Proof. (1) By Lemmas 2.6 and 2.7. For (2): Like in Lemma 2.7, it is easy to see that $P'_{\chi',\chi \cup [\chi',\chi'+\xi)} \lessdot P'_{\chi',\chi' \cup [\chi',\chi'+\xi)}$ as enough types (see Lemma 2.7) are realised in χ. □

Theorem 5.5. *For* $\bar{Q}^\chi$ *as in Definition* 2.2, *Part* (2) *we have that* $(**)_{\bar{Q}^\chi}$ *holds.*

Proof. Given $p^*, \underset{\sim}{T}, m, \underset{\sim}{E}$ as in Lemma 5.2, we choose $\bar{\varepsilon}$ and p'_ζ as in Lemma 5.2 (at the end of Lemma 5.2), t as in Subclaim 5.3. We let $w_\zeta = \mathrm{dom}(p'_\zeta)$, and w be the heart of the Δ-system. Note that we may choose p'_ζ such that $w_\zeta \setminus \chi = w \setminus \chi$, which allows us to avoid Definition 4.2(f). We now do so. We even might choose p'_ζ such that $w_\zeta \setminus \{\beta_\zeta\} = w$, but this does not lead to a further simplification.

Let

$$w \setminus \chi = \{\chi + \gamma_n \mid n \in n^*\}, \quad \gamma_0 < \gamma_1 < \cdots < \gamma_{n^*-1}.$$

We can replace χ by χ^{+k} using $\bar{E}^{\chi^{+k}}$ and thus (by Lemma 5.4) get counterexamples to $(**)_{\bar{Q}^{\chi^{+k}}}$ with the same t, $\bar{\varepsilon}$, and with $h^{\chi,\chi^{+k}}(p'_\zeta)$,

$$h^{\chi,\chi^{+k}}{''}(w) \setminus \chi^{+k} = \{\chi^{+k} + \gamma_n \mid n \in n^*\}, \quad \gamma_0 < \gamma_1 < \cdots < \gamma_{n^*-1},$$

and with $A^{\chi^{+k+1}}_{\chi^{+k+1}+\gamma} \cap \chi^{+k} = A^{\chi^{+k}}_{\chi^{+k}+\gamma} \cap \chi^{+k}$ for $\gamma < \mu$.

Now, fixing $\langle \gamma_n \mid n < n^* \rangle$ and $\bar{\varepsilon}$, we prove by induction on $n < n^*$ that for every $k \in \omega$ ($k \leqslant n^*$ would suffice), for $\bar{Q}^{\chi^{+k}}$ and for $\gamma_0, \ldots, \gamma_{n^*-1}$, α, and $\langle p'_\ell \mid \ell \in \omega \rangle$ as above, we can find a suitable modifications $P(n)$ of our original forcing P^χ and $P(n)^{\chi^{+k}}_{\chi^{+k}+\gamma_n+1}$-names for a finitely additive measures $(\underset{\sim}{\Xi}^t_{\chi^{+k}+\gamma_n+1})_{t \in \mathscr{T}}$ such that

- demand (e) of Definition 4.2 holds for $\langle \alpha_\ell \mid \ell \in \omega \rangle = \langle f_n \circ \cdots \circ f_0 \circ h^{\chi,\chi^{+k}}(\gamma^\ell_i) \mid \ell \in \omega \rangle$, $i < i^*$ (from Lemma 5.2(1), only the part before χ is considered). The f_i are the "shuffling" maps coming from the Löwenheim Skolem argument below and such that
- (f) and (g) of Definition 4.2 hold for every $n < n^*$ for $\langle \alpha_\ell \mid \ell \in \omega \rangle = \langle \chi^{+k} + \gamma_n \mid \ell \in \omega \rangle$ (so α_ℓ is constant) and thus to get the next step in the iteration according to Lemma 4.6, and
- though Definition 4.2(b) is not fulfilled for $\alpha^* = \chi^{+k} + \mu$, $k \geqslant 1$, the original $\eta_\beta \in {}^\kappa 2$ are still strong enough to code the arguments of $f_n \circ \cdots \circ f_0 \circ h^{\chi,\chi^{+k}}(p'_\zeta)$, $\zeta \in \omega$, according to the (5.1) in Subclaim 5.3. Look at the γ^ζ_i to be treated there and at $f_0, \ldots, f_{n^*-1}$ and at $h^{\chi,\chi^{+k}}$, how they shift the supports of the p'_ζ.

Then we can carry out the proof of Lemma 5.2 and of Subclaim 5.3. In the end we shall first show $(**)_{P(n^*)^\chi}$ for some modified $P(n^*)^\chi$ and mapped p'_ζ, however with the same μ, same $\gamma_0, \ldots, \gamma_{n^*-1}$, and possibly modified $\beta_\zeta, \underset{\sim}{T}, t$. Thereafter, we shall read the automorphisms and bijections in the reverse direction in order to get $(**)_{\bar{Q}^\chi}$.

In order to prove the claim "for all $k \in \omega$, $\bar{Q}^{\chi^{+k}}$ can be extended by $(\underset{\sim}{\Xi}^t_\alpha)_{\alpha \in \chi^{+k}+\gamma_n, t \in \mathscr{T}}$ respecting the whispering conditions at $\chi^{+k}+\gamma_0, \ldots, \chi^{+k}+\gamma_n$ and such that $\langle \alpha_\ell \mid \ell \in \omega \rangle = \langle \chi^{+k} + \gamma_n \mid \ell \in \omega \rangle$ satisfies $(t, \mathbf{n}_n)$ (for the same fixed $t \in \mathscr{T}$, $n < \mathbf{n}^*$, with $\mathbf{n}_n = |\Delta \cap \chi| + n$, not depending on k) (let us call this: stage $n+1$)" , we shall use "for all $k \in \omega$, $\bar{Q}^{\chi^{+k+1}}$ can be extended by $(\underset{\sim}{\Xi}^t_\alpha)_{\alpha \in \chi^{+k+1}+\gamma_n}$ respecting the whispering conditions at $\chi^{+k+1} + \gamma_0$, $\ldots, \chi^{+k+1}+\gamma_{n-1}$ and such that $\langle \alpha_\ell \mid \ell \in \omega \rangle = \langle \chi^{+k+1}+\gamma_n \mid \ell \in \omega \rangle$ satisfies $(t, \mathbf{n}_n)$ for $n < n^*$ (let us call this stage n)", a Löwenheim and Skolem argument and the uniqueness of $\mathbf{n}$ in (d) of Definition 4.2.

To carry out the induction: For the stage $n = 0$, $k \in \omega$ ($k = n^*$ would suffice, because we need to be able to descend n^* steps in the k's) we stipulate that $\gamma_{-1} + 1 = 0$ and just let $\underset{\sim}{\Xi}^t_{\chi^{+k}}$ be a $P_{\chi^{+k}}$-name for a finitely additive measure on ω such that condition (e) of Definition 4.2 is fulfilled for the blueprint t and the interesting instances of $\langle \alpha_\zeta \mid \zeta \in \omega \rangle$. In the step from stage n to stage $n+1$, for χ^{+k}, we apply the induction hypothesis to $\gamma_0 < \cdots < \gamma_{n-1}$ and χ^{+k+1} and $\langle f^{k+2}_{n-1} \circ \cdots \circ f^{k+2+n-1}_0 \circ h^{\chi,\chi^{+k+1+n}}(p'_\zeta) \mid \zeta \in \omega \rangle$, (the f^j_i are got from the induction hypothesis, see below, where we get f^{k+1}_n) and thus we get a $P^{\chi^{+k+1}}_{\chi^{+k+1}+\gamma_{n-1}+1}$-names $(\underset{\sim}{\Xi}^t_{\chi^{+k+1}+\gamma_{n-1}+1})_{t \in \mathscr{T}}$ for finitely additive measures as required, i.e. the whispering conditions hold for $A^{\chi^{+k+1}}_{\chi^{+k+1}+\gamma_m}$, $m < n$.

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Though we only have $2^\kappa \geqslant \chi$, the injective coding of the indices in the iteration length $\chi + \mu$ by $\eta_{\text{index}} \in 2^\kappa$ works not only for the original $\bar{Q}$ but also for $f_{n-1}^{k+2} \circ \cdots \circ f_0^{k+1+n} \circ h^{\chi,\chi^{+k+1+n}}{}''(\bar{Q})$, which is isomorphic to a complete suborder of $\bar{Q}^{\chi^\kappa}$.
There is a $P^{\chi^{+k+1}}_{\chi^{+k+1}+\gamma_n}$-name $\underset{\sim}{\Xi}^t_{\chi^{+k+1}+\gamma_n}$ for a finitely additive measure on ω extending $\underset{\sim}{\Xi}^t_{\chi^{+k+1}+\gamma_{n-1}+1}$: this is proved as in Lemmas 4.5 and 4.6, because there are no "whispering tasks" (i) of Definition 4.2 about the $A^{\chi^{+k+1}}_{\chi^{+k+1}+\gamma_n}$ in the stretch between $\chi^{+k+1} + \gamma_{n-1} + 1$ and $\chi^{+k+1} + \gamma_n$ and no new instances of (g) of Definition 4.2 as well.

Now we come to the crucial step from $\chi^{+k+1} + \gamma_n$ to $\chi^{+k} + \gamma_n + 1$. Let

$$M_0 \prec M_1 \prec (H(\psi), \in, <^*_\psi),$$

where $\psi = \beth_2(\chi^{+\omega})^+$.

For abbreviation, set $f' = f_{n-1}^{k+2} \circ \cdots \circ f_0^{k+2+n-1} \circ h^{\chi,\chi^{+k+n+1}}$, and we use f' also for the function which arises by putting hats over all objects on the right-hand side.

$(*)_1$ the objects $\langle \gamma_0, \ldots, \gamma_{n^*-1} \rangle$, $\langle g_{\chi^{+l}} \mid l \in \omega \rangle$, $\langle h^{\chi^{+k},\chi^{+k+1}} \mid k \in \omega \rangle$, $\mu, \chi, \langle f'(p'_\zeta) \mid \zeta < \omega \rangle$, $\langle \bar{Q}^{\chi^{+k}} \mid k \in \omega \rangle, \langle P^{\chi^{+k}}_{n-1} \mid k \in \omega \rangle (\underset{\sim}{\Xi}^t_{\chi^{+k+1}+\gamma_n})_{t \in \mathscr{T}}$, $f'(\underset{\sim}{T}) = \mathscr{B}(\langle \text{truth value}(f'(\delta_\ell) \in \underset{\sim}{\tau}_{f'(\gamma_\ell)}) \mid \ell \in \omega \rangle)$ belong to M_0.

$(*)_2$ $\|M_0\| = \|M_1\| = \chi^{+k}$, $\chi^{+k} + 1 \subseteq M_0$, $M_0 \in M_1$, ${}^{\max(\mu,\kappa)}(M_0) \subseteq M_0$, ${}^{\max(\mu,\kappa)}(M_1) \subseteq M_1$.

Claim. *There is an injective function f_n^{k+1} from $(\chi^{+k+1} + \gamma_n + 1) \cap M_1$ to $\chi^{+k} + \gamma_n + 1$ such that*

(a) $f_n^{k+1}(\chi^{+k+1} + \gamma) = \chi^{+k} + \gamma$ *for* $\gamma \leqslant \gamma_n$,

(b) f_n^{k+1} *maps* $(\chi^{+k+1} + \gamma_n) \cap M_0$ *onto* $A^{\chi^{+k}}_{\chi^{+k}+\gamma_n}$ *and*

(c) $g_{\chi^{+k}}(f_n^{k+1}(\alpha)) \cap \gamma_n = g_{\chi^{+k+1}}(\alpha) \cap \gamma_n$ *for* $\alpha \in \lambda^{+k+1} \cap M_1$, *i.e. for* $\gamma \in \gamma_n$, $\alpha \in \lambda^{+k+1} \cap M_1 : (f_n^{k+1}(\alpha) \notin A^{\chi^{+k}}_{\chi^{+k}+\gamma} \leftrightarrow \alpha \notin A^{\chi^{+k+1}}_{\chi^{+k+1}+\gamma})$.

Proof. Since $M_0 \in M_1$ we have that $|\chi^{+k+1} \cap (M_1 \setminus M_0)| = |\chi^{+k+1} \cap M_1| = |\chi^{+k+1} \cap M_0|$, and considering types as in the proof of Lemma 2.7 we get for any $c \in {}^{n+1}2$, with $E^0 = E$, $E^1 = \chi^{+k} \setminus E$,

$$\left| M_1 \cap \bigcap_{m < n+1} (E^{\chi^{+k+1}}_{\gamma_m})^{c(m)} \right| = \chi^{+k},$$

$$\left| \bigcap_{m < n+1} (E^{\chi^{+k}}_{\gamma_m})^{c(m)} \right| = \chi^{+k},$$

$$\left| M_0 \cap \bigcap_{m < n+1} (E^{\chi^{+k}}_{\gamma_m})^{c(m)} \right| = \chi^{+k}$$

and

$$|M_0 \cap \chi^{+k+1}| = \chi^{+k}.$$

Hence, we can find an f_n^{k+1} fulfilling the requirements (a)–(c). Hence the claim is proved. □

Now we change the forcing orders accordingly: We set $P(0)^{\chi^{+k}} = P^{\chi^{+k}}$. As in Definition 2.5 we can define a structure $P(n)^{\chi^{+k}}$ by

$$\widehat{f_n^{k+1}} : (P(n-1)^{\chi^{+k+1}}) \cap M_1 \cong P(n)^{\chi^{+k}}$$

and can extend $\widehat{f_n^{k+1}}$ onto the space of $(P(n-1)^{\chi^{+k+1}}) \cap M_1$-names.

From $f_n^{k+1} \circ f' \circ h^{\chi,\chi^{+k+n+1}}(\chi + \gamma_m)) = \chi^{+k} + \gamma_m$ we get that $\langle \alpha_\ell \mid \ell \in \omega \rangle = \langle \chi^{+k} + \gamma_m \mid \ell \in \omega \rangle$ still satisfies $(t, \mathbf{n}_m)$ (see Definition 4.2(d)) for $P(n)^{\chi^{+k}}$ for every $m \leqslant n^*$. Moreover, $f_n^{k+1} \circ f' \circ h^{\chi,\chi^{+k+n+1}}(\chi + \gamma_n) = \chi^{+k} + \gamma_n$ is the argument where $\langle f_n^{k+1} \circ f' \circ h^{\chi,\chi^{+k+n+1}}(p'_\zeta) \mid \zeta \in \omega \rangle$ is treated as in Lemma 5.2.

Now we prove that $P(n)^{\chi^{+k}}$ satisfies the conditions at $\gamma_0, \gamma_1, \ldots, \gamma_n$:

First, for $m = n$, we have that $\underset{\sim}{\Xi}^t_{\chi^{k+1}+\gamma_n}$ is in M_1 a $P(n-1)^{\chi^{+k+1}} \cap M_1$-name, and its restriction to $\mathscr{P}(\omega)^{V^{P(n-1)^{\chi^{+k+1}}_{\chi^{+k+1}+\gamma_n}}} \cap M_i$ is a $P(n-1)^{\chi^{+k+1}}_{\chi^{+k+1}+\gamma_n} \cap M_i$-name. We get that $\widehat{f_n^{k+1}}(\underset{\sim}{\Xi}^t_{\chi^{k+1}+\gamma_n}) \restriction M_1 =: \underset{\sim}{\Xi}^t_{\chi^k+\gamma_n+1} (= \underset{\sim}{\Xi}^t$ in the next paragraphs) is as required: We write only f for f_n^{k+1} in the proof of this claim so that the notation be slightly less clumsy.

We show that it is a $P(n)^{\chi^{+k}}_{\chi^{+k}+\gamma_n+1}$-name for a finitely additive measure on ω such that its restriction to $\mathscr{P}(\omega)$ in $V^{P(n)^{\chi^{+k}}_{A^{\chi^{+k}}_{\chi^{+k}+\gamma_n}}}$ is a $P(n)^{\chi^{+k}}_{A^{\chi^{+k}}_{\chi^{+k}+\gamma_n}}$-name, so condition (i) of Definition 4.2 is satisfied: Let $\underset{\sim}{A}$ be a $P(n)^{\chi^{+k}}_{A^{\chi^{+k}}_{\chi^{+k}+\gamma_n}}$-name:

$$\hat{f}(\underset{\sim}{\Xi}^t)(\underset{\sim}{A}) = \hat{f}(\underset{\sim}{\Xi}^t)(\hat{f}_n(\hat{f}^{-1}(\underset{\sim}{A}))),$$

where $\hat{f}^{-1}(\underset{\sim}{A}) \in M_0$.

Hence

$$\hat{f}(\underset{\sim}{\Xi}^t)(\hat{f}_n(\hat{f}^{-1}(\underset{\sim}{A}))) = \hat{f}(\underset{\sim}{\Xi}^t(\hat{f}^{-1}(\underset{\sim}{A})))$$

and where $\underset{\sim}{\Xi}^t(\hat{f}^{-1}(\underset{\sim}{A})) \in M_0$. Hence $\hat{f}(\underset{\sim}{\Xi}^t(\hat{f}^{-1}(\underset{\sim}{A})))$ is an $f''(M_0 \cap (\chi^{+k+1} + \gamma_n)) = A^{\chi^{+k}}_{\chi^{+k}+\gamma_n}$-name.

For $m < n$ the claim that $\underset{\sim}{\Xi}^t_{\chi^k+\gamma_m+1} := \underset{\sim}{\Xi}^t_{\chi^k+\gamma_n+1} \restriction (\mathscr{P}(\omega) \text{ in } V^{P(n)^{\chi^{+k}}_{\chi^{+k}+\gamma_m+1}})$ is a $P(n)^{\chi^{+k}}_{\chi^{+k}+\gamma_m+1}$-name for a finitely additive measure on ω such that its restriction to $\mathscr{P}(\omega)$ in $V^{P(n)^{\chi^{+k}}_{A^{\chi^{+k}}_{\chi^{+k}+\gamma_m}}}$ is a $P(n)^{\chi^{+k}}_{A^{\chi^{+k}}_{\chi^{+k}+\gamma_m}}$-name, follows from $f''_n(A^{\chi^{+k+1}}_{\chi^{+k+1}+\gamma_m}) = A^{\chi^{+k}}_{\chi^{+\kappa}+\gamma_m}$ for $m < n$.

Hence we have $\underset{\sim}{\Xi}^t_{\chi^{+k+1}+\gamma_m+1}$, which are $P(n)^{\chi^{+k}}_{\chi^{+k}+\gamma_m+1}$-names respecting the whispering conditions 4.2(i) at $\chi^{+k} + \gamma_0, \ldots, \chi^{+k} + \gamma_n$ (which where needed in the premises of Lemma 4.6(1)), and the inductive proof is finished.

Now we perform the induction with starting point $h^{\chi,\chi^{+n^*}}(P)$ and get $f_0^{n^*}, f_1^{n^*-1}, \dots, f_n^{n^*-n}, \dots, f_{n^*-1}^1$ and $k := f_{n^*-1}^1 \circ \cdots \circ f_0^{n^*}$, $f := k \circ h^{\chi,\chi^{+n^*}}$. After n^* induction steps, we have that the mapped forcing $\hat{k}''P^{\chi^{+n^*}} = P(n^*)^\chi$ is expanded by measures $\Xi^t_{\chi+\gamma_n+1}$, $n \leqslant n^*$.

So the proofs of Lemma 5.2 and Subclaim 5.3 go through for the modified forcing and the mapped objects: $\hat{f}(\underset{\sim}{T})$, $\hat{f}(p'_\zeta)$, $\hat{f}(t)$ (blueprints), $\langle \hat{f}(\gamma_i^\zeta) \mid i < i^* \rangle$ (the domain of $\hat{f}(p'_\zeta)$). Hence the proofs of Lemma 5.2 and of Subclaim 5.3 show that there is no perfect tree in the intersection of the mapped trees. So $\hat{f}(\underset{\sim}{T})$ is not perfect in the generic extension $V^{P(n^*)^\chi}$.

We have that $h^{\chi,\chi^{+n^*}}$ is a complete embedding, and that in each step $P(n)^{\chi^k}$ is isomoprhic to $P(n-1)^{\chi^{+k+1}} \cap M_1$, which is is a complete suborder of $P(n-1)^{\chi^{+k+1}}$ (because $M_1 \prec H_\psi$ and all antichains are countable and ${}^\omega M_1 \subseteq M_1$.) Being a perfect tree is absolute for ZFC models and hence $n^* + 1$ applications of Kunen [13, VII, Lemma 13] the condition

$$p \Vdash_{P(n^*)^\chi} \text{“}\hat{f}(\underset{\sim}{T}) \text{ isnot perfect in the generic extension } V^{P(n^*)^\chi}\text{”}$$

implies that some condition in G forces that $\underset{\sim}{T}$ is not a perfect tree in V^P. Thus $(**)_{\bar{Q}}$ is also proved for the original $\bar{Q}$. □

6. The case of $\mathrm{cf}(\mu) = \omega$

In this section, we show a version of Theorem 2.1 for the case of $\mathrm{cf}(\mu) = \omega$. The main technical point is: the part of the iteration as in Definition 2.2, Part (2) lying before χ and the part thereafter now are going to take shifts ω_1 often.

This means a slight increase of the complexity of our notation. We are going to rework the previous three sections and benefit from the fact that we did some (but not all) work for the class of forcings of Definition 2.2, Part (1). We shall often only hint to the parallels and give an informal description of the modifications and strengthenings.

Theorem 6.1. *In Theorem* 2.1, *we can replace* $(\mathrm{cf}(\mu) > \aleph_0$ *and* $\sup(C) = \mu)$ *by*

$$\mathrm{cf}^{V_1}(\mu) = \omega, \textit{ and there is some } \lambda \textit{ such that}$$

$$\omega_1 \leqslant |C|^{V_2} < \lambda < \mu,$$

$$\mathrm{cf}^{V_1}(\lambda) \geqslant \omega_1$$

and

$$\forall B \in V_1 \quad (|B|^{V_1} < \lambda \rightarrow C \not\subseteq B).$$

Proof. We first give an outline: We define a member of $\mathscr{K}$ (of Definition 2.2) that we are going to use. Then (after adapting Lemmas 2.6 and 2.7) we get the items (α) to (δ) of the conclusion of Theorem 2.1 and of Theorem 6.1. For item (ε), we begin with the analogon of the end of Section 2. Then we slightly modify the blueprints. Again we can deal with automorphisms of the iteration length. We take those automorphisms moving only some element α within one of our ω_1 intervals $[\chi\cdot\gamma, \chi\cdot(\gamma+1))$. So we basically do the old proof in some interval of the longer iteration. We use that we never required that there are only partial random forcings after χ.

We take $\chi \geqslant 2^\mu$ and κ such that $2^\kappa \geqslant \chi$. Then we define

$$\bar{Q}^\chi = \langle P^\chi_\alpha, \underset{\sim}{Q}_\beta, A^\chi_\beta, \mu_\beta, \underset{\sim}{\tau}_\beta \mid \beta < \chi\cdot\omega_1, \alpha \leqslant \chi\cdot\omega_1 \rangle \in \mathscr{K}$$

as follows.

We take for $\chi \leqslant \chi'$

$$\begin{aligned} &g_{\chi,\omega_1} : \chi\cdot\omega_1 \to (\mu\times\omega_1)^{<\lambda}, \\ &g_{\chi,\omega_1}(\chi\gamma+\xi) = \emptyset \quad \text{for } \mu \leqslant \xi < \chi, \\ &g_{\chi',\omega_1}(\chi'\gamma+\xi) = g_{\chi,\omega_1}(\chi\gamma+\xi) \quad \text{for } \xi < \chi,\ \gamma\in\omega_1 \\ &\forall\gamma\in\omega_1\ \forall B\in(\mu\times\omega_1)^{<\lambda}\ \exists^{\chi'}\alpha\in[\chi'\cdot\gamma, \chi'\cdot(\gamma+1))g_{\chi',\omega_1}(\alpha) = B. \end{aligned}$$

For $\alpha = \chi\gamma+\xi$, $\gamma\in\omega_1$, $\xi\in\chi$ we set

$$A^\chi_\alpha = \begin{cases} \emptyset & \text{if } \gamma = 0 \text{ or } \xi > \mu, \\ \{\beta < \chi\gamma \mid (\xi,\gamma)\notin g_{\chi,\omega_1}(\beta)\} & \text{else,} \end{cases}$$

$$\underset{\sim}{Q}_\alpha = \begin{cases} ({}^{\omega}2, \lhd), & \text{if} A^\chi_\alpha = \emptyset, \\ \text{Random}^{V[\underset{\sim}{\tau}_\beta \mid \beta\in A_\alpha]^\chi}, & \text{else.} \end{cases}$$

We adopt Fact 2.4 as follows.

Definition 6.2. For $\bar{Q}\in\mathscr{K}$ of the special form of Theorem 6.1, $\alpha < \chi\cdot\omega_1$, we let

$$\begin{aligned} \mathrm{AUT}(\bar{Q}^\chi \restriction \alpha) = \{f : \alpha\to\alpha \mid\ & f \text{ is bijective, and,} \\ & (\forall\beta,\delta\in\alpha) \\ & ((|\underset{\sim}{Q}_\alpha| < \kappa \leftrightarrow |\underset{\sim}{Q}_{f(\alpha)}| < \kappa) \\ & \wedge (\beta\in A_\delta \leftrightarrow f(\beta)\in A_{f(\delta)}))\}. \end{aligned}$$

Then we have that $\hat{f}$ is an automorphisms of P_α and of P'_α (from Definition 3.2(c)), and Fact 2.5 holds for $\mathscr{K}$.

Now we get the analogues of Lemma 2.6 and of Lemma 2.7 (consider types, similarly to there) and are ready to prove

(δ') $V_2 \models \Vdash_{P_{\chi\cdot\omega_1}}$ "$\{\tau_{\chi\cdot\gamma+i} \mid i\in C,\ \gamma\in\omega_1\}$ is not null".

Proof. Let $\underset{\sim}{N}$ be a $P_{\chi\cdot\omega_1}$-name for a Borel null set. Hence for some Borel function $\mathscr{B}\in V_1$ and for some countable

$$X=\{x_\ell \mid \ell\in\omega\}\subseteq\chi\cup\bigcup_{\gamma\in\omega_1\setminus\{0\}}[\chi\cdot\gamma+\mu,\chi\cdot(\gamma+1)),$$

$$Y=\{y_\ell|\ell\in\omega\}\subseteq\bigcup_{\gamma\in\omega_1\setminus\{0\}}[\chi\cdot\gamma,\chi\cdot\gamma+\mu),$$

$\zeta_\ell,\ \ell\in\omega,\ \zeta'_\ell,\ \ell\in\omega$, we have that

$$\underset{\sim}{N}=\mathscr{B}((\text{truth value}(\zeta_\ell\in\underset{\sim}{\tau}_{x_\ell}))_{\ell\in\omega},(\text{truth value}(\zeta'_\ell\in\underset{\sim}{\tau}_{y_\ell}))_{\ell\in\omega}).$$

Let $i(*)<\omega_1$ be such that $\chi\cdot i(*)>\sup(Y)$. Since $\mathrm{cf}^{V_1}(\lambda)>\aleph_0$, we have that $B:=\bigcup_{\xi\in X\cup Y}g_{\chi,\omega_1}(\xi)\in([\mu\times\omega_1]^{<\lambda})^{V_1}$.

Since $C\setminus\pi_\mu(B)\neq\emptyset$, there is some $i\in\mu,\ i\in C\setminus\pi_\mu(B)$. We claim, that $\tau_{\chi\cdot i(*)+i}$ is random over a universe, in which $\underset{\sim}{N}[G]$ has a name. (Moreover regarding V_1 and V_2, the same remarks as in the proof of (δ') of Theorem 2.1 apply.) Then the proof will be finished, because then $\tau_{\chi\cdot i(*)+i}\notin\underset{\sim}{N}[G]$ in $V_2[G]$. By our construction, we have

$$\tau_{\chi\cdot i(*)+i}\text{ is the Random}^{V[\underset{\sim}{\tau}_\alpha|\alpha\in A^\chi_{\chi\cdot i(*)+i}]}\text{-generic over }V^{P_{\chi\cdot i(*)+i}}.$$

Since $i\in C\setminus\pi_\mu(B)$, we have that $\forall\xi\in X\cup Y\ \forall\gamma\in\omega_1$ that $g_\chi(\xi)\not\ni(i,\gamma)$, hence $\forall\xi\in X\cup Y\ \xi\in A_{\chi\cdot\gamma+i}$, so $X\cup Y\subseteq A^\chi_{\chi\cdot i(**)+i}$. Since $P_{A_{\chi\cdot i(*)+i}}\lessdot P_{\lg(\bar{Q})}$ the name $\underset{\sim}{N}$ is evaluated in the right manner in $V^{P_{A_{\chi\cdot i(*)+i}}}$. Thus the claim is proved. $\square$

(δ) $V_2[G]\models\mathrm{unif}\mathscr{N}\leqslant|C|$.
This follows from (δ').
(ε) $V_1[G]\models\mathrm{unif}(\mathscr{N})\geqslant\lambda$.

Again the item (ε) will be the longest part. However, it is almost the same as our previous work. Put all the β_ζ of an analogue of Lemma 2.11 into one $[\chi\cdot\gamma+\mu,\chi\cdot(\gamma+1))$. Also the extension of χ to χ' now can be done either only in the relevant interval where the α_ζ lie, or just all over, thus leading to $h^{\chi,\chi'}$.

More explicit, we start as in the corresponding proof in Theorem 2.1: Suppose that (ε) is not true. In V_1 there is $i(*)<\lambda$ and $p\in P_{\chi\cdot\omega_1}$ such that

$$p\Vdash_{P_{\chi\cdot\omega_1}}\text{“}\underset{\sim}{\eta}_i\in{}^\omega2\text{ for }i<i(*)\wedge\{\underset{\sim}{\eta}_i\mid i<i(*)\}\text{ is not null”}.$$

A name of a real in $V_1[G]$ is given by

$$\underset{\sim}{\eta}_i=\mathscr{B}_i(\langle\text{ truth value }(\zeta_{i,\ell}\in\underset{\sim}{r}_{j_{i,\ell}})\mid\ell\in\omega\rangle)$$

for suitable $\langle\zeta_{i,\ell},j_{i,\ell}\mid\ell\in\omega\rangle,\ \zeta_{i,\ell}\in\omega,\ j_{i,\ell}\in\chi+\mu$.

We set

$$X = \{j_{i,\ell} \mid i \in i(*), \ell \in \omega\} \cap (\chi \cup \bigcup \{\chi \cdot \gamma + \mu, \chi \cdot (\gamma + 1)) \mid \gamma \in \omega_1 \setminus \{0\}\},$$

$$Y = \{j_{i,\ell} \mid i \in i(*), \ell \in \omega\} \cap \bigcup \{\chi \cdot \gamma, \chi \cdot \gamma + \mu) \mid \gamma \in \omega_1 \setminus \{0\}\}$$

We show the main point

In $V_1[G]$, $({}^{\omega}2)^{V[\{\tau_\xi \mid \xi \in X \cup Y\}]}$ is a Lebesgue null set.

Since $\exists^{\chi}\alpha\ g_{\chi,\omega_1}(\alpha) = \{(\gamma, y) \mid \chi \cdot \gamma + y \in Y\}$ we can fix such an $\alpha \in (\chi \cdot \omega_1) \setminus X$ that is not in $A^{\chi}_{\chi\cdot\gamma+y}$ for $\chi \cdot \gamma + y \in Y$.

Lemma 6.3 (See Lemma 2.8). *In $V_1^{P_{\alpha^*}}$, the set $({}^{\omega}2)^{V_1[\tau_\xi \mid \xi \in X \cup Y]}$ has Lebesgue measure 0, and a witness for a definition for a measure zero superset can be found in $V^{P_{\alpha+1}}$ for $\alpha \in \chi \setminus X$ that is not in E_ξ for every $\xi \in Y - \chi$.*

Now proceed through the analogues of Sections 2 and 3. In the definition of a blueprint we allow $\mathbf{m}^t$ and $\mathbf{n}^t$ to indicate in which intervals $[\chi \cdot \gamma, \chi \cdot (\gamma + 1))$ the heart of the delta system (intersected with the Cohen parts for $\mathbf{m}^t$) lies, hence $\mathbf{m}^t, \mathbf{n}^t \in [\omega_1]^{<\omega}$ and $\mathbf{m}^t \subseteq \mathbf{n}^t$ in general not as an initial segment, but inserted according to the type of the heart. (The old $\mathbf{n}^t$ would be just the length of our new $\mathbf{n}^t$.)

Then we modify Definition 4.2 as follows: In (d) (2) we say $n < |\mathbf{n}^t|$ and in (d) (4) we say

if $n < \mathrm{dom}(\mathbf{m}^t) \Leftrightarrow \forall \ell(\alpha_\ell \in [\chi \cdot \mathbf{m}^t(n), \chi \cdot \mathbf{m}^t(n) + \mu)) \Leftrightarrow \exists \ell(\alpha_\ell \in [\chi \cdot \mathbf{m}^t(n), \chi \cdot \mathbf{m}^t(n) + \mu))$, and

if $n < \mathrm{dom}(\mathbf{n}^t) \setminus \mathrm{dom}(\mathbf{m}^t) \Leftrightarrow \forall \ell(\alpha_\ell \in [\chi \cdot \mathbf{m}^t(n) + \mu, \chi \cdot (\mathbf{m}^t(n) + 1)) \Leftrightarrow \exists \ell(\alpha_\ell \in [\chi \cdot \mathbf{m}^t(n) + \mu, \chi \cdot (\mathbf{m}^t(n) + 1))$.

The rest of Section 4 shows that the new $\mathscr{K}^3$ has the desired members. In Subclaim 5.3, the choice of the blueprint has to be modified accordingly. Thus we get $(**)_{\bar{Q}}$ for the modified class $\mathscr{K}^3$.

Since the analogue of Lemma 2.7 holds, we also get analogues to Lemma 5.4 and to Theorem 5.5 and hence can finish the proof of Theorem 6.1. □

7. Getting the premises of Theorems 1.1 and 2.1

In this section, we discuss how to get the bare set-theoretic premises of Theorems 1.2 and 2.1.

If we do not insist on (V_1, V_2) having the same cardinals but just require $({}^{\omega}V_1)^{V_2} \subseteq V_1$, then we can get the situation in the premise of Theorem 1.2 for example as follows.

Take for V_1 any model of ZFC and let $\aleph_1 \leqslant v < v'$ be regular cardinals in V_1. We extend V_1 by forcing with $P = (\{f \mid f : v \to v', |\mathrm{dom}(f)|^{V_1} \leqslant \aleph_0\}, \subseteq)$. Since P is ω-closed we have that $({}^{\omega}V_1)^{V_2} \subseteq V_1$. We set

$$N = \{(\mu, \mu') \in V_1 \mid \exists f \in V_2 f : \mu \overset{\text{cofinal}}{\to} \mu', \mu, \mu' \text{ regular in } V_1, \mu < \mu'\}.$$

Let $\lambda = \min(\pi_0(N))$, where π_0 denotes the projection onto the first coordinate. Then we have that $\mathrm{cf}^{V_2}(\lambda)$ is uncountable. Let $\mu' = \mu'(\lambda)$ a minimal witness that $\lambda \in \pi_0(N)$ and let $f \in V_2$, $f : \lambda \overset{\text{cofinal}}{\to} \mu'$. Let $C = \mathrm{range}(f) \in V_2$. Then $|C|^{V_2} = |\lambda|^{V_2} = \lambda$. Let $\mathscr{I} \in V_2$ be the set of all bounded subsets of C. For any $B \in V_1$ such that $|B|^{V_1} < \mu'$ we have that $B \cap C$ is not cofinal in μ'.

If we allow cofinalities to be changed, there is the following constellation with consistency strength $\exists\kappa\ o(\kappa) = \omega_1$: Gitik [18] shows that assuming $\exists\kappa\ o(\kappa) = \omega_1$ there is some V (got with a preparatory forcing) such that in V, there is a regular cardinal $\kappa > \omega_1$ and a notion of forcing P that adds a cofinal sequence of length ω_1 to κ and does not add any countable sequences and does not add any bounded subsets of κ. Now we have $V_1 = V$, $V_2 = V^P$, C = the range of the new cofinal sequence, $\mu = \kappa$, $\lambda = \aleph_1$, $\mathscr{I} = \{C' \subseteq \kappa \mid C' \in V_2, |C'| < \aleph_1\}$.

In order to get (V_1, V_2) with the same cofinality function, we take a model announced in the “Added in proof” in Gitik [9]:

Theorem 7.1 (Gitik). *Assume that there is a measurable κ of Mitchell order $\kappa^{++} + \theta$, θ regular and $\theta \geqslant \omega_1$. Then the singular cardinal hypothesis can be violated in the following manner: There is some model V such that $2^\kappa = \kappa^+$ in V and such that there is a notion of forcing P such that P does not change cofinalities above κ and such that in V^P, κ is a singular strong limit, $\aleph_0 < \mathrm{cf}(\kappa) = \theta$, $2^\kappa = \kappa^{++}$ and such that $\forall x(x \in V^P \wedge x \subseteq \mathrm{Ord} \wedge |x|^{V^P} < \kappa^+ \to \exists y \in V(y \in \mathrm{Ord} \wedge |y|^{V^P} < \kappa^+ \wedge x \subseteq y))$.*

Remark. By Gitik and Mitchell [10] the lower bound for the consistency strength is of such a failure of SCH is between $\exists\kappa\ o(\kappa) = \kappa^{++}$ and $\exists\kappa\ o(\kappa) = \kappa^{++} + \theta$, and if $\theta > \aleph_1$ then the strength is $o(\kappa) = \kappa^{++} + \theta$.

Theorem 7.2. *Suppose that we have that $2^\kappa = \kappa^+$ in V and that there is a notion of forcing P such that P does not change cofinalities above κ and such that in V^P, κ is a singular strong limit, $\aleph_0 < \mathrm{cf}(\kappa) = \theta$, $2^\kappa = \kappa^{++}$ and such that $\forall x(x \in V^P \wedge x \subseteq \mathrm{Ord} \wedge |x|^{V^P} < \kappa^+ \to \exists y \in V(y \in \mathrm{Ord} \wedge |y|^{V^P} < \kappa^+ \wedge x \subseteq y))$.*

Then there are V_1, V_2 such that

(1) $V \subseteq V_1 \subseteq V_2 \subseteq V[G]$,

(2) $(H(\kappa))^{V_1} = (H(\kappa))^{V_2} = (H(\kappa))^{V[G]}$,

(3) $({}^{<\theta}V_1)^{V_2} \subseteq V_1$,

(4) *V_1 and V_2 have the same cofinality function,*

(5) *in V_2 there is a subset C of κ of size θ such that C is not covered by any set in V_1 of size less than κ.*

Proof. Let $A = H(\kappa)^{V[G]}$.

By the “cov versus pp (= pseudo-power) theorem” [17, II, 5.4] we get that $\mathrm{pp}(\kappa) = 2^\kappa = \kappa^{++}$ in V_2, and hence by the definition of pp there is a $\langle \kappa_i \mid i < \theta \rangle \in V[G]$ be a sequence of regular cardinals cofinal in κ and an ideal I on θ containing all the bounded sets in θ such that $\mathrm{tcf}(\prod \kappa_i / I) = \kappa^{++}$. That means: there is a $<_I$-cofinal scale

$\langle f_\alpha \mid \alpha \in \kappa^{++} \rangle$ in V_2, i.e. for $\alpha < \beta \in \kappa^{++}$ we have

$$f_\alpha : \theta \to \kappa,$$
$$f_\alpha(\gamma) \in \kappa_\gamma \quad \text{for } \gamma \in \theta,$$
$$f_\alpha <_I f_\beta \quad \text{for } \alpha < \beta \in \kappa^{++}$$
$$\forall g \in \prod_{i \in \theta} \kappa_i \ \exists \alpha \in \kappa^{++} \ g <_I f_\alpha,$$

where $f <_I g$ iff $\{i < \theta \mid f(i) \geqslant g(i)\} \in I$. (By [17, VIII, Section 1] that there is even a scale with respect to the ideal J_θ^{bd} of the bounded subsets of θ.)

We set

$$V_1 = V[A, \langle \kappa_i \mid i < \theta \rangle].$$

Then we have that there is some $f_\alpha \in V^P$ that $<_I$-dominates V_1:

Proof. In V, in the subalgebra P' of the Gitik algebra P that is generated by $H(\kappa)^{V[G]} \cup \{\langle \kappa_i \mid i < \theta \rangle\}$ there are only $\leqslant \kappa^+$ elements (since the Gitik algebra P hat the κ^+-c.c.) and it has the κ^+ c.c. Hence there are only κ^+ many P'-names for subsets of κ in V, so we have that in $V_1 = V^{P'}$, $2^\kappa = \kappa^+$.

Since $C_\alpha = \{f \in {}^\theta\kappa \cap V_1 \mid f \not\leq_I f_\alpha\}$ is decreasing, of length κ^{++} and has empty intersection, there is some $\alpha < \kappa^{++}$ such that $C_\alpha = \emptyset$ and hence f_α that $<_I$-dominates ${}^\theta\kappa \cap V_1$.

We fix such an f_α and set

$$V_2 = V_1[f_\alpha].$$

For C we take range(f_α). Now all the items claimed in Theorem 7.2 are true.

We give a proof of item 5, the others are easier. We show that range$(f_\alpha) = C$ is a set in V_2 that is not covered by any set B in V_1 of size less than κ.

Suppose the contrary: $B \supseteq C$, $B \in V_1$ and $|B| < \kappa$. We show that these premises imply $f_\alpha \in V_1$. We have that $\langle \sup(B \cap \kappa_i) \mid i < \theta \rangle \in V_1$. Since $|B| < \kappa$, there is some $\theta_0 < \theta$ such that for $i > \theta_0$ we have that $\sup(B \cap \kappa_i) < \kappa_i$.

We set

$$g(i) = \begin{cases} \sup(B \cap \kappa_i) + 1 & \text{if } i > \theta_0, \\ 0 & \text{else.} \end{cases}$$

But we have that $f_\alpha(\gamma) < g(\gamma)$ for $\gamma > \theta_0$. Since that latter is in V_1 and since I contains all the bounded subsets of θ and is proper, this is a contradiction to f_α being $<_I$-unbounded and hence to being $<_I$-dominating over V_1.

Remark. Unboundedness with respect to $<_I$ instead of being dominating w.r.t. $<_I$ would suffice for the proof of item 5 and all other items.

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