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Closed measure zero sets

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Abstract

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We study the relationship between the σ-ideal generated by closed measure zero sets and the ideals of null and meager sets. We show that the additivity of the ideal of closed measure zero sets is not bigger than covering for category. As a consequence we get that the additivity of the ideal of closed measure zero sets is equal to the additivity of the ideal of meager sets.

1. Introduction

Let $\mathcal{M}$ and $\mathcal{N}$ denote the ideals of meager and null subsets of 2^ω respectively and let $\mathcal{E}$ be the σ-ideal generated by closed measure zero subsets of 2^ω. It is clear that $\mathcal{E}$ is a proper subideal of $\mathcal{M} \cap \mathcal{N}$.

For an ideal $\mathcal{J}$ of subsets of 2^ω define

1. $\mathbf{add}(\mathcal{J}) = \min\{|\mathcal{A}|: \mathcal{A} \subseteq \mathcal{J} \;\&\; \bigcup \mathcal{A} \notin \mathcal{J}\}$,
2. $\mathbf{cov}(\mathcal{J}) = \min\{|\mathcal{A}|: \mathcal{A} \subseteq \mathcal{J} \;\&\; \bigcup \mathcal{A} = 2^\omega\}$,
3. $\mathbf{unif}(\mathcal{J}) = \min\{|X|: X \subseteq 2^\omega \;\&\; X \notin \mathcal{J}\}$ and
4. $\mathbf{cof}(\mathcal{J}) = \min\{|\mathcal{A}|: \mathcal{A} \subseteq \mathcal{J} \;\&\; \forall B \in \mathcal{J}\; \exists A \in \mathcal{A}\; B \subseteq A\}$.

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We can further generalize these definitions and put for a pair of ideals $\mathcal{I} \subseteq \mathcal{J}$,

1. $\mathbf{add}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{A}|: \mathcal{A} \subseteq \mathcal{I} \;\&\; \bigcup \mathcal{A} \notin \mathcal{J}\}$,
2. $\mathbf{cof}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{A}|: \mathcal{A} \subseteq \mathcal{J} \;\&\; \forall B \in \mathcal{I}\ \exists A \in \mathcal{A}\ B \subseteq A\}$.

Let $\mathcal{I}_0$ be the ideal of finite subsets of 2^ω. Note that $\mathbf{cov}(\mathcal{I}) = \mathbf{cof}(\mathcal{I}_0, \mathcal{J})$, $\mathbf{unif}(\mathcal{J}) = \mathbf{add}(\mathcal{I}_0, \mathcal{J})$, $\mathbf{add}(\mathcal{J}) = \mathbf{add}(\mathcal{J}, \mathcal{J})$ and $\mathbf{cof}(\mathcal{J}) = \mathbf{cof}(\mathcal{J}, \mathcal{J})$.

The goal of this paper is to study the relationship between the cardinals defined above for the ideals $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{E}$. We will show that $\mathbf{add}(\mathcal{M}) = \mathbf{add}(\mathcal{E})$ and $\mathbf{cof}(\mathcal{M}) = \mathbf{cof}(\mathcal{E})$.

It will follow from the inequalities $\mathbf{add}(\mathcal{E}, \mathcal{N}) \leqslant \mathbf{cov}(\mathcal{M})$ and $\mathbf{cof}(\mathcal{E}, \mathcal{N}) \geqslant \mathbf{unif}(\mathcal{M})$ which will be proved in Section 3.

Finally in the last section we will present some consistency results—we will show the $\mathbf{cov}(\mathcal{E})$ may not be equal to $\max\{\mathbf{cov}(\mathcal{N}), \mathbf{cov}(\mathcal{M})\}$ and similarly $\mathbf{unif}(\mathcal{E})$ does not have to be equal to $\min\{\mathbf{unif}(\mathcal{M}), \mathbf{unif}(\mathcal{N})\}$.

For $f, g \in \omega^\omega$ let $f \leqslant^* g$ be the ordering of eventual dominance.

Recall that $\mathfrak{b}$ is the size of the smallest unbounded family in ω^ω and $\mathfrak{d}$ is the size of the smallest dominating family in ω^ω.

Throughout this paper we use the standard notation.

μ denotes the standard product measure on 2^ω. For a tree $T \subseteq 2^{<\omega}$ let $[T]$ be the set of branches of T. If T is finite (or has terminal nodes) then $[T]$ denotes the clopen subset of 2^ω determined by maximal nodes of T. Let $m(T) = \mu([T])$ in both cases.

If $s \in T \subseteq 2^{<\omega}$ then $T[s] = \{t: s^\frown t \in T\}$ where $s^\frown t$ denotes the concatenation of s and t. ZFC* always denotes some finite fragment of ZFC sufficiently big for our purpose.

We will conclude this section with several results concerning the cardinal invariants defined above.

Theorem 1.1 (Miller [8]).
(1) $\mathbf{add}(\mathcal{M}) = \min\{\mathbf{cov}(\mathcal{M}), \mathfrak{b}\}$ *and* $\mathbf{cof}(\mathcal{M}) = \max\{\mathbf{unif}(\mathcal{M}), \mathfrak{d}\}$,
(2) $\mathbf{add}(\mathcal{E}, \mathcal{M}) \leqslant \mathfrak{b}$ *and* $\mathbf{cof}(\mathcal{E}, \mathcal{M}) \geqslant \mathfrak{d}$. *In particular* $\mathbf{add}(\mathcal{E}) \leqslant \mathfrak{b}$ *and* $\mathbf{cof}(\mathcal{E}) \geqslant \mathfrak{d}$,
(3) $\mathbf{cov}(\mathcal{M}) \leqslant \mathbf{add}(\mathcal{E}, \mathcal{N})$ *and* $\mathbf{unif}(\mathcal{M}) \geqslant \mathbf{cof}(\mathcal{E}, \mathcal{N})$. □

We will also use the combinatorial characterizations of cardinals $\mathbf{cov}(\mathcal{M})$ and $\mathbf{unif}(\mathcal{M})$.

Theorem 1.2 (Bartoszynski [2]). (1) $\mathbf{cov}(\mathcal{M})$ *is the size of the smallest family* $F \subseteq \omega^\omega$ *such that*

$$\forall g \in \omega^\omega\ \exists f \in F\ \forall^\infty n\, f(n) \neq g(n).$$

(2) $\mathbf{unif}(\mathcal{M})$ *is the size of the smallest family* $F \subseteq \omega^\omega$ *such that*

$$\forall g \in \omega^\omega\ \exists f \in F\ \exists^\infty n\, f(n) = g(n). \qquad \square$$

2. Combinatorics

In this section we will prove several combinatorial lemmas which will be needed later. The following theorem uses the technique from [3].

Theorem 2.1. *Suppose that* $\{F_\eta : \eta < \lambda < \mathbf{add}(\mathscr{E}, \mathscr{N})\}$ *is a family of closed measure zero sets. Then there exists a partition of* ω *into intervals* $\{\bar{I}_n : n \in \omega\}$ *and a sequence* $\{T_n : n \in \omega\}$ *such that for all* n, $T_n \subseteq 2^{\bar{I}_n}$, $|T_n| \cdot 2^{-|\bar{I}_n|} \leq 2^{-n}$ *and*

$$\bigcup_{\eta<\lambda} F_\eta \subseteq \{x \in 2^\omega : \exists^\infty n\, x \restriction \bar{I}_n \in T_n\}.$$

Furthermore, we can require that

$$\forall \eta < \lambda\, \exists^\infty n\, F_\eta \restriction \bar{I}_n \subseteq T_n$$

where $F_\eta \restriction \bar{I}_n = \{s \in 2^{\bar{I}_n} : \exists x \in F_\eta\, x \restriction \bar{I}_n = s\}$.

Proof. Note that if the sequences $\{\bar{I}_n : n \in \omega\}$ and $\{T_n : n \in \omega\}$ satisfy the above conditions then the set $\{x \in 2^\omega : \exists^\infty n\, x \restriction \bar{I}_n \in T_n\}$ has measure zero.

For $\eta < \lambda$ and $n \in \omega$ define

$$F_\eta^n = \{x \in 2^\omega : \exists s \in 2^n\, s^\frown x \restriction (\omega - n) \in F_\eta\}.$$

By the assumption there exists a measure zero set $H \subseteq 2^\omega$ such that

$$\bigcup_{\eta<\lambda} \bigcup_{n\in\omega} F_\eta^n \subseteq H.$$

Lemma 2.2 (Oxtoby [10]). *There exists a sequence of finite sets* $\langle H_n : n \in \omega \rangle$ *such that* $H_n \subseteq 2^n$, $\sum_{n=1}^\infty |H_n| \cdot 2^{-n} < \infty$ *and* $H \subseteq \{x \in 2^\omega : \exists^\infty n\, x \restriction n \in H_n\}$.

Proof. Since H has measure zero there are open sets $\langle G_n : n \in \omega \rangle$ covering H such that $\mu(G_n) < 2^{-n}$ for $n \in \omega$. Represent each set G_n as a disjoint union of open basic intervals

$$G_n = \bigcup_{m=1}^\infty [s_m^n] \quad \text{for } n \in \omega.$$

Let $H_n = \{s \in 2^n : s = s_l^k$ for some $k, l \in \omega\}$ for $n \in \omega$. It follows that $\sum_{n=1}^\infty |H_n| \cdot 2^{-n} \leq \sum_{n=1}^\infty \mu(G_n) \leq 1$. If $x \in H$ then $x \in \bigcap_{n\in\omega} G_n$. Therefore $x \restriction n \in F_n$ must hold for infinitely many n. □ (Lemma 2.2)

Therefore

$$\bigcup_{n<\lambda} \bigcup_{n\in\omega} F_\eta^n \subseteq \{x \in 2^\omega : \exists^\infty n\, x \restriction n \in H_n\}.$$

For every $n < \lambda$ define an increasing sequence $\langle k_n^\eta : n \in \omega \rangle$ as follows: $k_0^\eta = 0$ and for $n \in \omega$,

$$k_{n+1}^\eta = \min\left\{m : F_\eta^{k_n^\eta} \subseteq \bigcup_{j=k_n^\eta}^m [H_j]\right\}.$$

Since sets F_η^n are compact this definition is correct.

We will need an increasing sequence $\langle k_n : n \in \omega \rangle$ such that

$$\forall \eta < \lambda \, \exists^\infty n \, \exists m \, k_{2n} < k^\eta_m < k^\eta_{m+1} < k_{2n+1}$$

and

$$2^{k_n} \cdot \sum_{j=k_{n+1}}^{\infty} \frac{|H_j|}{2^j} \leq \frac{1}{2^n}.$$

To construct such a sequence we will use the following lemma:

Lemma 2.3. *Suppose that $M \models \mathrm{ZFC}^*$ and $|M| < \mathfrak{d}$. Then there exists a function $g \in \omega^\omega$ such that either*

$$\forall f \in M \cap \omega^\omega \, \exists^\infty n \, \exists m \, g(2n) < f(m) < f(m+1) < g(2n+1)$$

or

$$\forall f \in M \cap \omega^\omega \, \exists^\infty n \, \exists m \, g(2n+1) < f(m) < f(m+1) < g(2n+2).$$

Proof. Let $g \in \omega^\omega$ be an increasing function such that $g \not\leq^* f$ for $f \in M \cap \omega^\omega$. We will show that g has the required properties.

Suppose not. Let $f_1, f_2 \in M \cap \omega^\omega$ be such that for all n,

$$|[g(2n), g(2n+1)] \cap \mathrm{ran}(f_1)| \leq 1 \quad \text{and}$$

$$|[g(2n+1), g(2n+2)] \cap \mathrm{ran}(f_2)| \leq 1.$$

We will get a contradiction by constructing a function $f \in M \cap \omega^\omega$ which dominates g.

Define $f(0) = f_1(0) > g(0)$ and $f(1) = f_2(0) > g(1)$. Let $l_1 = \min\{l : f_1(l) > f_2(1)\}$ and put $f(2) = f_1(l_1)$. Now $f(2) > g(2)$ since $f_2(1) > g(2)$. Let $l_2 = \min\{l : f_2(l) > f_1(l_1 + 1)\}$ and let $f(3) = f_2(l_2) > g(3)$ since $f_1(l_1 + 1) > g(3)$. And so on . . . .

In general define the sequence $\langle l_n : n \in \omega \rangle$ as $l_0 = 0$ and

$$l_{2n+1} = \min\{l : f_1(l) > f_2(l_{2n} + 1)\}$$

and

$$l_{2n+2} = \min\{l : f_2(l) > f_1(l_{2n+1} + 1)\}.$$

Let

$$f(n+1) = \begin{cases} f_1(l_n) & \text{if } n \text{ is even,} \\ f_2(l_n) & \text{if } n \text{ is odd.} \end{cases}$$

It is clear that $f \in M$. Easy induction shows that f dominates g. Contradiction.

□ (Lemma 2.3)

To get the desired sequence $\langle k_n : n \in \omega \rangle$ take a model $M \models \mathrm{ZFC}^*$ containing $\langle H_n : n \in \omega \rangle$ and $\{F_\eta : \eta < \lambda\}$. Since $\lambda < \mathbf{add}(\mathscr{C}, \mathscr{N}) \leq \mathfrak{d}$ we can assume that $|M| < \mathfrak{d}$. Apply the above lemma to get a function g and define $k_n = g(n)$ for $n \in \omega$. It is clear that this is the sequence we are looking for.

Now define for $n \in \omega$,

$$\bar{I}_n = [k_{2n-1}, k_{2n+1}]$$

and

$$T_n = \{s \in 2^{\bar{I}_n} : \exists j \in [k_{2n}, k_{2n+1}]\ \exists t \in H_j\ s \restriction \bar{I}_n = t \restriction \bar{I}_n\}.$$

Note that for every n,

$$\frac{|T_n|}{2^{|\bar{I}_n|}} \leq 2^{k_n} \cdot \sum_{j=k_{2n}}^{k_{2n+1}} \frac{|H_j|}{2^j} \leq \frac{1}{2^n}.$$

To finish the proof fix $\eta < \lambda$ and $k \in \omega$. By the construction there exists $n > k$ and $m \in \omega$ such that

$$k_{2n} < k_m^\eta < k_{m+1}^\eta < k_{2n+1}.$$

Suppose that $s \in F_\eta \restriction \bar{I}_n$. Then there exists $x \in F_\eta^{k_m^\eta}$ such that $s \subseteq x$. Furthermore, there exists $j \in [k_m^\eta, k_{m+1}^\eta)$ such that $x \restriction j \in H_j$. It follows that $s \in T_n$.

□ (Theorem 2.1)

Now we will prove another combinatorial lemma describing the structure of closed measure zero sets.

Let $\{I_n : n \in \omega\}$ be a partition of ω into disjoint intervals such that $|I_n| > n$.

For $n < m$ let

$$Seq_{n,m} = \{s : \mathrm{dom}(s) \subseteq [n, m]\ \&\ \forall j \in \mathrm{dom}(s)\ s(j) \in I_j\}.$$

For every $s \in Seq_{n,m}$ define

$$C_s = \left\{t : \mathrm{dom}(t) = \bigcup_{j=n}^{m} I_j\ \&\ \forall j \in \mathrm{dom}(s)\ t(s(j)) = 0\right\}.$$

For $k, j \in \omega$ let

$$C_k^j = \begin{cases} \{t \in 2^{I_j} : t(k) = 0\} & \text{if } k \in I_j, \\ 2^{I_j} & \text{otherwise.} \end{cases}$$

Note that we can identify the set C_s with $\prod_{j=n}^m C_{s(j)}^j$ in the following way:

$$t \in C_s \leftrightarrow \exists \langle t_n, t_{n+1}, \ldots, t_m \rangle \in \prod_{j=n}^{m} C_{s(j)}^j\ t = t_n {}^\frown t_{n+1} {}^\frown \cdots {}^\frown t_m.$$

Fix $n < m$ and let $I = I_n \cup I_{n+1} \cup \cdots \cup I_m$. Suppose that $T \subseteq 2^I$ is a finite tree such that

1. $\forall s \in T\ \exists t \in T\ (s \subseteq t\ \&\ |t| = |I|)$,
2. $m(T) \leq \frac{1}{4}$.

Lemma 2.4. *Suppose that for some $s \in Seq_{n,m}$, $C_s = \prod_{j=n}^m C_{s(j)}^j \subseteq T$. Then there exists $k \in [n, m)$ and $t \in T \cap \prod_{j=1}^{k-1} C_{s(j)}^j$ (if $k = n$ then $t = \emptyset$) such that*

$$\forall t' \in C_{s(k)}^k\ m(T[t^\frown t']) > \left(1 + \frac{1}{2^k}\right) \cdot m(T[t]).$$

Proof. Suppose not. We build by induction a sequence $\langle t_j : j \in [n, m-1]\rangle$ such that $t_j \in C^j_{s(j)}$ and $m(T[t_j{}^\frown t_{j+1}]) \leq (1 + 2^{-j}) \cdot m(T[t_j])$ for $j < m$.

After $m-1$ many steps we get that

$$m(T[t_n{}^\frown t_{n+1}{}^\frown \cdots {}^\frown t_{m-1}]) \leq m(T) \cdot \prod_{j=n}^{m-1} \left(1 + \frac{1}{2^j}\right) < \frac{1}{2}.$$

Therefore there is $t_m \in C^m_{s(m)} - T[t_n{}^\frown t_{n+1}{}^\frown \cdots {}^\frown t_{m-1}]$. This is a contradiction since

$$t = t_n{}^\frown t_{n+1}{}^\frown \cdots {}^\frown t_m \in C_s - T. \qquad \square$$

Suppose that $t \in T$ and $|t| = |\bigcup_{j=n}^{k} I_j|$ for some $k \in [n, m)$. Let

$$S_t^{k+1} = \left\{ l \in I_{k+1} \colon \forall t' \in C_l^{k+1}\ m(T[t^\frown t']) > \left(1 + \frac{1}{2^k}\right) \cdot m(T[t]) \right\}.$$

Note that the sets $\{C_l^{k+1} \colon l \in I_{k+1}\}$ are independent. Therefore the set

$$\bigcup_{l \in S_t^{k+1}} \bigcup_{t' \in C_l^{k+1}} T[t^\frown t']$$

has measure at least

$$(1 - 2^{-|S_t^{k+1}|}) \cdot \left(1 + \frac{1}{2^k}\right) \cdot m(T[t]).$$

Since this set is included in $T[t]$ we get

$$(1 - 2^{-|S_t^{k+1}|}) \cdot \left(1 + \frac{1}{2^k}\right) \leq 1.$$

Therefore

$$|S_t^{k+1}| \leq k + 1.$$

Let $S^{k+1} = \{l \in I_{k+1} \colon \exists t \in T\ l \in S_t^{k+1}\}$. Then

$$|S^{k+1}| \leq (k+1) \cdot \prod_{j=n}^{k} 2^{|I_j|}.$$

Also if $t = \emptyset$ then define

$$S_\emptyset^n = \left\{ l \in I_n \colon \forall t' \in C_l^n\ m(T[t']) > \left(1 + \frac{1}{2^n}\right) \cdot m(T) \right\}.$$

Similarly we get $|S_\emptyset^n| \leq n + 1$.

Note that in particular we get that the size of S^k does not depend on the size of I_k.

Combining 2.4 with the observations above we get the following:

Lemma 2.5. *Suppose that $I = I_n \cup I_{n+1} \cup \cdots \cup I_m$ and $T \subseteq 2^I$ such that $m(T) < \frac{1}{4}$. Then there exists a sequence $\langle S^k : k \in [n, m] \rangle$ such that*

(1) $S^k \subseteq I_k$ *for* $k \in [n, m]$,

(2) $|S^k| \leqslant (k+1) \cdot \prod_{j=n}^{k-1} 2^{|I_j|}$ *for* $k \in (n, m]$ *and* $|S^n| \leqslant n+1$,

(3) *for every* $s \in Seq_{n,m}$, *if* $C_s \subseteq T$ *then there exists* $k \in [n, m]$ *such that* $s(k) \in S^k$. $\square$

We conclude this section with a theorem of Miller which gives an upper bound for $\mathbf{cov}(\mathscr{E}, \mathscr{N})$. We will prove it here for completeness.

Theorem 2.6 (Miller [8]). $\mathbf{add}(\mathscr{E}, \mathscr{N}) \leqslant \mathfrak{b}$ *and* $\mathbf{cof}(\mathscr{E}, \mathscr{N}) \geqslant \mathfrak{b}$.

Proof. Suppose that $H \subseteq 2^\omega$ is a measure zero set. Using 2.2, we can find a sequence $\langle H_n : n \in \omega \rangle$ such that $H_n \subseteq 2^n$, $\sum_{n=1}^\infty |H_n| \cdot 2^{-n} \leqslant \frac{1}{4}$ and

$$H \subseteq \{x \in 2^\omega : \exists^\infty n\, x \restriction n \in H_n\}.$$

Define for $n \in \omega$,

$$f_H(n) = \min\left\{m : \sum_{j=m}^{\infty} \frac{|H_j|}{2^j} < \frac{1}{4^n}\right\}.$$

Suppose that $f \in \omega^\omega$ is an increasing function. Let

$$G_f = \{x \in 2^\omega : \forall n\, x(f(n)) = 0\}.$$

Clearly G_f is a closed measure zero set.

Lemma 2.7. *If* $f_H \leqslant^* f$ *then* $G_f \nsubseteq H$.

Proof. Suppose that $f_H \leqslant^* f$. Without loss of generality we can assume that $f_H(n) < f(n)$ for all n. For $n \in \omega$ define

$$\bar{H}_n = \{s \in 2^{f_H(n+1)} : \exists j \in [f_H(n), f_H(n+1))\, \exists t \in H_j\, s \restriction j = t\}.$$

Note that for all n,

$$[\bar{H}_n] = \bigcup_{j=f_H(n)}^{f_H(n+1)} [H_j] \quad \text{and} \quad m(\bar{H}_n) \leqslant 4^{-n}.$$

By compactness, if $G_f \subseteq H$ then for some n,

$$G_f \subseteq \bigcup_{j=1}^{f_F(n+1)} [H_j] = \bigcup_{j \leqslant n} [\bar{H}_j].$$

We will show that this inclusion fails for every n which will give a contradiction.

Fix $n \in \omega$. Note that it is enough to find $s \in 2^{f_H(n+1)}$ such that $s(f(j)) = 0$ and $s \restriction f_H(j+1) \notin \bar{H}_j$ for $j \leqslant n$.

We will use the following simple construction.

Lemma 2.8. *Suppose that $n_1 < n_2 < n_3$ and that $T \subseteq 2^{[n_1, n_3]}$ is such that $m(T) = a < \frac{1}{2}$. For $l \in [n_2, n_3]$ let $C_l = \{s \in 2^{[n_2, n_3]}: s(l) = 0\}$. Then for every $l \in [n_2, n_3]$ there exists $s \in C_l$ such that the set $T[s] = \{t \in 2^{[n_1, n_2)}: t^\frown s \in T\}$ has measure $\leqslant 2a$.*

Proof. Fix $l \in [n_2, m_3]$ and choose s such that $m(T[s])$ is minimal.

If $T[s] = \emptyset$ we are done. Otherwise

$$m(T) \geqslant \tfrac{1}{2} \cdot m(T[s]).$$

It follows that $m(T[s]) \leqslant 2a$. □ (Lemma 2.8)

We will build by induction sequences $s_n, s_{n-1}, \ldots, s_0$ and sets $H'_n, H'_{n-1}, \ldots, H'_0$ such that for all $j \leqslant n$,

1. $\mathrm{dom}(s_j) = [f_H(j), f_H(j+1))$,
2. $H'_j \subseteq 2^{f_H(j+1)}$,
3. $m(H'_j[s_j]) \leqslant 2 \cdot m(H'_j)$.

Let $H'_n = \bar{H}_n$ and let $s_n \in 2^{[f_H(n), f_H(n+1))}$ be the sequence obtained by applying 2.8 to H'_n and $C_{f(n)}$.

Suppose that H'_{n-j} and s_{n-j} are already constructed. Let

$$H'_{n-j-1} = \bar{H}_{n-j-1} \cup H'_{n-j}[s_{n-j}]$$

and let s_{n-j-1} be the sequence obtained by applying 2.8 to H'_{n-j-1} and $C_{f(n-j-1)}$.

Let $s = s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_n$. Note that $s(f(j)) = 0$ for all $j \leqslant n$. We have to check that $s \restriction f_H(j+1) \notin \bar{H}_j$ for $j \leqslant n$. Suppose this is not true. Pick minimal j such that

$$s \restriction f_H(j+1) = s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_j \in \bar{H}_j.$$

By the choice of s_j we have

$$s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_{j-1} \in \bar{H}_{j-1} \cup \bar{H}_j[s_j].$$

Since j was minimal,

$$s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_{j-1} \in \bar{H}_j[s_j].$$

Proceeding like that we get that

$$s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_{j-2} \in \bar{H}_j[s_j][s_{j-1}].$$

Finally

$$s_0 \in \bar{H}_j[s_j][s_{j-1}] \cdots [s_1] \subseteq H'_0$$

which is a contradiction. □ (Lemma 2.7)

Now we are ready to finish the proof of the theorem. Suppose that $F \subseteq \omega^\omega$ is a dominating family which consists of increasing functions. Consider the set $\bigcup_{f \in F} G_f$. We claim that this set does not have measure zero. It follows from the fact that if H is a measure zero set then there exists $f \in F$ such that $f_H \leq^* f$. In particular $G_f \nsubseteq H$.

Similarly, if $\mathscr{B} \subseteq \mathcal{N}$ is a family of size $<\mathfrak{b}$ then there exists $f \in \omega^\omega$ such that

$$\forall H \in \mathfrak{b}\, f_H \leq^* f.$$

Thus $G_f \nsubseteq H$ for any $H \in \mathscr{B}$. □ (Theorem 2.6)

3. Cohen reals from closed measure zero sets

The goal of this section is to prove that $\mathbf{add}(\mathscr{E}, \mathcal{N}) = \mathbf{cov}(\mathcal{M})$. In fact we have the following:

Theorem 3.1. (1) $\mathbf{add}(\mathscr{E}, \mathcal{N}) = \mathbf{cov}(\mathcal{M})$. *In particular,* $\mathbf{add}(\mathscr{E}) = \mathbf{add}(\mathcal{M})$.

(2) $\mathbf{cof}(\mathscr{E}, \mathcal{N}) = \mathbf{unif}(\mathcal{M})$. *In particular,* $\mathbf{cof}(\mathscr{E}) = \mathbf{cof}(\mathcal{M})$.

Proof. Note that by 1.1 and 2.6, we get

$$\mathbf{add}(\mathcal{M}) = \min\{\mathbf{cov}(\mathcal{M}), \mathfrak{b}\} \leq \mathbf{add}(\mathscr{E}, \mathcal{N}) \leq \mathfrak{b}.$$

Therefore the equality $\mathbf{add}(\mathscr{E}) = \mathbf{add}(\mathcal{M})$ follows from the inequality $\mathbf{add}(\mathscr{E}, \mathcal{N}) \leq \mathbf{cov}(\mathcal{M})$.

Similarly, to show that $\mathbf{cof}(\mathscr{E}) = \mathbf{cof}(\mathcal{M})$ we have to check that $\mathbf{cof}(\mathscr{E}, \mathcal{N}) \geq \mathbf{unif}(\mathcal{M})$.

(1) $\mathbf{add}(\mathscr{E}, \mathcal{N}) \leq \mathbf{cov}(\mathcal{M})$.

By the first part of 1.2, it is enough to prove that for every family $F \subseteq \omega^\omega$ of size $< \mathbf{add}(\mathscr{E}, \mathcal{M})$ there exists a function $g \in \omega^\omega$ such that

$$\forall f \in F\, \exists^\infty n\, f(n) = g(n).$$

Fix a family F as above.

For every $f \in F$ let

$$f'(n) = \max\{f(i): i \leq n\} + 1 \quad \text{for } n \in \omega.$$

We will need two increasing sequences $\{m_n, l_n: n \in \omega\}$ such that

1. $m_0 = l_0 = 0$,
2. $l_{n+1} = l_n + 2^{m_n} \cdot (n+1)$,
3. $\forall f \in F\, \exists^\infty n\, m_{n+1} > f'(l_{n+1})^{l_{n+1}} + m_n$.

The existence of these sequences follows from the fact that $|F| < \mathfrak{d}$.

Let $I_n = [m_n, m_{n+1})$ and $J_n = [l_n, l_{n+1})$ for $n \in \omega$. Without loss of generality we can assume that $|I_n| = K_n^{|J_n|}$ for some $K_n \in \omega$. Thus we can identify elements of I_n with $K_n^{J_n}$.

For every $f \in F$ and $n \in \omega$ define $\vec{f}(n) = f \restriction J_n$. By the choice of sequences $\langle I_n, J_n: n \in \omega \rangle$ we have

$$\forall f \in F\, \exists^\infty n\, \vec{f}(n) \in I_n.$$

Using the notation from previous section, define for $f \in F$,

$$C_f = \bigcap_{n \in \omega} C_{\bar{f} \restriction n}.$$

Note that the sets C_f are closed sets of measure zero.

Since $|F| < \mathbf{add}(\mathscr{E}, \mathscr{N})$, the set $\bigcup_{f \in F} C_f$ has measure zero.

By 2.1, there exist sequences $\langle \bar{I}_n, T_n : n \in \omega \rangle$ such that for all n, $T_n \subseteq 2^{\bar{I}_n}$, $|T_n| \cdot 2^{-|\bar{I}_n|} \leq 2^{-n}$ and

$$\forall f \in F\, \exists^\infty n\, C_f \restriction \bar{I}_n \subseteq T_n.$$

Moreover, without loss of generality we can assume that whenever $I_m \cap \bar{I}_n \neq \emptyset$ then $I_m \subseteq \bar{I}_n$ for $n, m \in \omega$.

We will build the function $g \in \omega^\omega$ we are looking for from the sequences $\langle T_n : n \in \omega \rangle$ and $\langle I_n : n \in \omega \rangle$.

For every n let $v_n \in \omega$ be such that

$$\bar{I}_n = I_{v_n} \cup I_{v_n+1} \cup \cdots \cup I_{v_{n+1}-1}.$$

Note that for $f \in F$ and $n \in \omega$,

$$C_f \restriction \bar{I}_n = C_{\bar{f} \restriction [v_n, v_{n+1})}.$$

Now we are ready to define function g. For every n we will define $g \restriction \bar{I}_n$ using the set T_n.

Fix $n \in \omega$ and consider the set $T_n \subseteq 2^{\bar{I}_n}$. By 2.5 there exists a sequence $\langle S^k : k \in [v_n, v_{n+1}) \rangle$ such that

1. $S^k \subseteq I_k$ for $k \in [v_n, v_{n+1})$,
2. $|S^k| \leq (k+1) \cdot \prod_{j=n}^{k-1} 2^{|I_j|}$ for $k \in (v_n, v_{n+1})$ and $|S^{v_n}| \leq n+1$,
3. for every $s \in Seq_{v_n, v_{n+1}-1}$, if $C_s \subseteq T$ then there exists $k \in [v_n, v_{n+1})$ such that $s(k) \in S^k$.

Note that for every $k \in [v_n, v_{n+1})$,

$$|S^k| \leq (k+1) \cdot \prod_{j=n}^{k-1} 2^{|I_j|} \leq (k+1) \cdot \prod_{j=n}^{k-1} 2^{m_{j+1}-m_j} \leq (k+1) \cdot 2^{m_j} \leq |J_k|.$$

We can view S^k as a subset of $K_k^{J_k}$ of size $\leq |J_k|$. For $k \in [v_n, v_{n+1})$ let $s^k \in K_k^{J_k}$ be such that

$$\forall t \in S^k\, \exists l \in J_k\, s^k(l) = t(l).$$

Define

$$g \restriction \bar{I}_n = s^{v_n} {}^\frown \cdots {}^\frown s^{v_{n+1}-1}.$$

Note that $g \restriction \bar{I}_n$ 'diagonalizes' all sets S^k for $k \in [v_n, v_{n+1})$.

Now we are ready to finish the proof. Suppose that $f \in F$. Therefore there exists infinitely many n such that

$$C_f \upharpoonright \bar{I}_n = C_{\vec{f} \upharpoonright [v_n, v_{n+1})} \subseteq T_n.$$

In particular there exists $k \in [v_n, v_{n+1})$ such that $\vec{f}(k) = f \upharpoonright J_k \in S^k$. Thus there exists $j \in J_k$ such that

$$f(j) = s^k(j) = g(j)$$

which finishes the proof of the first part of the theorem. Note that we only used the fact that $m(T_n) \leq \frac{1}{4}$ for $n \in \omega$.

(2) **unif**$(\mathcal{M}) \leq$ **cof**$(\mathscr{E}, \mathcal{M})$.

To prove this inequality we have to 'dualize' the above argument. Suppose that $\mathscr{B} \subseteq \mathcal{N}$ is a family of size λ witnessing that **cof**$(\mathscr{E}, \mathcal{N}) = \lambda$. We will construct a family $F \subseteq \omega^\omega$ of size λ such that

$$\forall f \in \omega^\omega \, \exists g \in F \, \exists^\infty n \, f(n) = g(n).$$

By 1.2, this will finish the proof.

Since **cof**$(\mathscr{E}, \mathcal{N}) \geq \mathfrak{b}$ we can find a family $G \subseteq \omega^\omega$ of size λ which is unbounded and consists of increasing functions.

Let $G = \{f_\eta : \eta < \lambda\}$ and $\mathscr{B} = \{H_\eta : \eta < \lambda\}$. Without loss of generality we can assume that

$$H_\eta = \{x \in 2^\omega : \exists^\infty n \, x \upharpoonright n \in H_n^\eta\}$$

where $\sum_{n=1}^\infty |H_n^\eta| \cdot 2^{-n} < \infty$. For every $\xi, \eta < \lambda$ and $n \in \omega$ define

$$\bar{I}_n^{\xi,\eta} = [f_\eta(2n-1), f_\eta(2n+1)]$$

and

$$T_n^{\xi,\eta} = \{s \in \bar{I}_n^{\xi,\eta} : \exists j \in [f_\eta(2n), f_\eta(2n+1)] \, \exists t \in H_j^\xi \, s \upharpoonright \bar{I}_n^{\xi,\eta} = t \upharpoonright \bar{I}_n^{\xi,\eta}\}.$$

Let

$$W = \{\langle \xi, \eta \rangle : \forall n \, |T_n^{\xi,\eta}| \cdot 2^{-|\bar{I}_n^{\xi,\eta}|} \leq 2^{-n}\}.$$

Arguing as in the proof of 2.1, we show that for every closed measure zero set $F \subseteq 2^\omega$ there exists $\langle \xi, \eta \rangle \in W$ such that

$$\exists^\infty n \, F \upharpoonright \bar{I}_n^{\xi,\eta} \subseteq T_n^{\xi,\eta}.$$

Let V be the set of triples $\langle \xi, \eta, \gamma \rangle \in \lambda^3$ such that $\langle \xi, \eta \rangle \in W$ and the partition $\langle [f_\gamma(n), f_\gamma(n+1)) : n \in \omega \rangle$ is finer than $\langle \bar{I}_n^{\xi,\eta} : n \in \omega \rangle$.

For every triple $\langle \xi, \eta, \gamma \rangle \in V$ let $g^{\xi,\eta,\gamma} \in \omega^\omega$ be the function g defined in the proof above.

Let

$$F = \{g^{\xi,\eta,\gamma} : \langle \xi, \eta, \gamma \rangle \in V\}.$$

We will show that this family has required properties. Suppose that $f \in \omega^\omega$. Find γ, $\delta < \lambda$ such that

1. $f_\delta(n+1) \geq f_\delta(n) + 2^{f_\gamma(n)} \cdot (n+1)$,
2. $\exists^\infty n\, f_\gamma(n+1) > f'(f_\delta(n+1))^{f_\delta(n+1)} + f_\gamma(n)$

where $f'(n) = \max\{f(1), \ldots, f(n)\} + 1$.

Define $I_n = [f_\gamma(n), f_\gamma(n+1))$ and $J_n = [f_\delta(n), f_\delta(n+1))$ for $n \in \omega$. As in the above part we have

$$\exists^\infty n \vec{f}(n) \in I_n.$$

Now we can find $\langle \xi, \eta \rangle \in W$ such that

$$\exists^\infty n\, C_f \restriction \bar{I}_n^{\xi,\eta} \subseteq T_n^{\xi,\eta}.$$

It follows that

$$\exists^\infty n\, f(n) = g^{\xi,\eta,\gamma}(n)$$

which finishes the proof. □

We conclude this section with two applications.

In [9] it is proved that:

Theorem 3.2 (Miller [9]). **add**$(\mathcal{N}) \leq \mathfrak{b}$ *and* **cof**$(\mathcal{N}) \geq \mathfrak{d}$. □

Theorem 3.3 (Bartoszynski [1], Raisonnier, Stern [11]). **add**$(\mathcal{N}) \leq$ **add**$(\mathcal{M})$ *and* **cof**$(\mathcal{N}) \geq$ **cof**$(\mathcal{M})$.

Proof. We have

$$\mathbf{add}(\mathcal{N}) \leq \min\{\mathfrak{b}, \mathbf{add}(\mathscr{E}, \mathcal{N})\} = \min\{\mathfrak{b}, \mathbf{cov}(\mathcal{M})\} = \mathbf{add}(\mathcal{M}).$$

Similarly

$$\mathbf{cof}(\mathcal{N}) \geq \max\{\mathfrak{d}, \mathbf{cof}(\mathscr{E}, \mathcal{N})\} = \max\{\mathfrak{d}, \mathbf{unif}(\mathcal{M})\} = \mathbf{cof}(\mathcal{M}). \quad \square$$

Also we get another proof of the main result from [4]:

Theorem 3.4 (Bartoszynski, Judah [4]). cf(**cov**$(\mathcal{M})) \geq$ **add**$(\mathcal{N})$.

Proof. Clearly cf(**add**$(\mathscr{E}, \mathcal{N})) \geq$ **add**$(\mathcal{N})$. □

4. Cardinals cov($\mathscr{E}$) and unif($\mathscr{E}$)

In this section we will prove some results concerning the covering number of $\mathscr{E}$. Most of the results are implicit in [3] and [5].

Let us start with the following easy observation.

Lemma 4.1. (1) *Every null set can be covered by* $\mathfrak{d}$ *many closed null sets.*
(2) *Every null set of size* $<\mathfrak{b}$ *can be covered by a null set of type* F_σ.

Proof. Suppose that G is a null subset of 2^ω. As in 2.2, we can assume that

$$G = \{x \in 2^\omega : \exists^\infty n\ x \restriction n \in F_n\}$$

where $\sum_{n=1}^\infty |F_n| \cdot 2^{-n} < \infty$. For every $x \in G$ let $f_x \in \omega^\omega$ be an increasing enumeration of the set $\{n \in \omega : x \restriction n \in F_n\}$. For a strictly increasing function $f \in \omega^\omega$ let

$$G_f = \{x \in 2^\omega : \forall^\infty n\ \exists m \in [n, f(n)]\ x \restriction m \in F_m\}.$$

It is clear that for every $f \in \omega^\omega$ the set $G_f \subseteq G$ is a measure zero set of type F_σ. Notice also that if $f_x \leqslant^* f$ then $x \in G_f$.

(1) Let $F \subseteq \omega^\omega$ be a dominating family of size $\mathfrak{d}$ which consists of increasing functions. Then by the above remarks

$$G = \bigcup_{f \in F} G_f.$$

(2) Suppose that $X \subseteq G$ is a set of size $<\mathfrak{b}$. Let f be an increasing function which dominates all functions $\{f_x : x \in X\}$. Then $X \subseteq G_f$. $\square$

As a corollary we get:

Theorem 4.2. (1) *If* $\mathbf{cov}(\mathcal{M}) = \mathfrak{d}$ *then* $\mathbf{cov}(\mathcal{E}) = \max\{\mathbf{cov}(\mathcal{M}), \mathbf{cov}(\mathcal{N})\}$.
(2) *If* $\mathbf{unif}(\mathcal{N}) = \mathfrak{b}$ *then* $\mathbf{unif}(\mathcal{E}) = \min\{\mathbf{unif}(\mathcal{M}), \mathbf{unif}(\mathcal{N})\}$.

Proof. Since $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$ we have

$$\mathbf{cov}(\mathcal{E}) \geqslant \max\{\mathbf{cov}(\mathcal{M}), \mathbf{cov}(\mathcal{N})\}$$

and

$$\mathbf{unif}(\mathcal{E}) \leqslant \min\{\mathbf{unif}(\mathcal{M}), \mathbf{unif}(\mathcal{N})\}.$$

By the previous lemma

$$\max\{\mathbf{cov}(\mathcal{M}), \mathfrak{d}\} \geqslant \mathbf{cov}(\mathcal{E})$$

and

$$\mathbf{unif}(\mathcal{E}) \geqslant \min\{\mathbf{unif}(\mathcal{N}), \mathfrak{b}\}$$

which finishes the proof. $\square$

Suppose that $f \in \omega^\omega$ and $\sum_{n=1}^\infty 2^{-f(n)} < \infty$. Define

$$\varphi \in \Sigma_f \leftrightarrow \varphi \in ([\omega]^{<\omega})^\omega \,\&\, \forall n \left(\varphi(n) \subseteq 2^{f(n)} \,\&\, \frac{|\varphi(n)|}{2^{f(n)}} \leqslant \frac{1}{4^n}\right)$$

and

$$\varphi \in \Pi_f \leftrightarrow \varphi \in ([\omega]^{<\omega})^\omega \,\&\, \forall n\; \varphi(n) \subseteq 2^{f(n)} \,\&\, \exists^\infty n \frac{|\varphi(n)|}{2^{f(n)}} \leqslant \frac{1}{4^n}$$

and let $\chi_f = \prod_{n=1}^\infty 2^{f(n)}$.

Notice that $\Sigma_f \subseteq \Pi_f$.

For $\varphi \in \Sigma_f \cup \Pi_f$ define define set $H_\varphi \subseteq 2^\omega$ as follows:

Let $k_n = 1 + 2 + \cdots + f(n)$ for $n \in \omega$. Identify natural numbers $\leqslant 2^{f(n)}$ with 0–1 sequences of length $f(n)$ and define

$$H_\varphi = \{x \in 2^\omega : \forall^\infty n\, x \upharpoonright [k_n, k_{n+1}) \in \varphi(n)\}.$$

Note that

$$\mu(H_\varphi) \leqslant \prod_{n=m}^\infty \mu(\{x \in 2^\omega : x \upharpoonright [k_n, k_{n+1}) \in \varphi(n)\})$$
$$\leqslant \sum_{m=1}^\infty \prod_{n=m}^\infty \frac{|\varphi(n)|}{2^{f(n)}} = 0.$$

For $x \in 2^\omega$, define $h_x(n) = x \upharpoonright [k_n, k_{n+1})$ for $n \in \omega$. Clearly h_x corresponds to an element of χ_f.

Finally we have

$$x \in H_\varphi \leftrightarrow \forall^\infty n\, h_x(n) \in \varphi(n).$$

Theorem 4.3. *Suppose that $C \in \mathscr{E}$. Then there exists $f \in \omega^\omega$ and $\varphi \in \Sigma_f$ such that $C \subseteq H_\varphi$.*

Proof. Suppose that $C \subseteq 2^\omega$ is a null set of type F_σ. Represent C as $\bigcup_{n\in\omega} C_n$ where $\langle C_n : n \in \omega\rangle$ is an increasing family of closed sets of measure zero. Define sequence $\langle k_n : n \in \omega\rangle$ as follows: $k_0 = 0$ and

$$k_{n+1} = \min\left\{m > k_n : \exists T_n \subseteq 2^m \left(C_n \subseteq [T_n] \,\&\, \frac{|T_n|}{2^m} \leqslant \frac{1}{4^{k_n}}\right)\right\}.$$

Let $I_n = [k_n, k_{n+1})$ and $J_n = \{s \upharpoonright I_n : s \in T_n\}$ for $n \in \omega$. We can see that for all $n \in \omega$

$$\frac{|J_n|}{2^{|I_n|}} \leqslant 2^{k_n} \cdot \frac{1}{4^{k_n}} \leqslant \frac{1}{2^n}.$$

We also have

$$F \subseteq \{x \in 2^\omega : \forall^\infty n\, x \upharpoonright I_n \in J_n\} = H_\varphi$$

where $f(n) = |I_n|$ and $\varphi(n) = J_n$ for all n. By the above remarks $\varphi \in \Sigma_f$. □

For an increasing function $g \in \omega^\omega$ define $g^* \in \omega^\omega$ as $g^*(0) = 0$ and $g^*(n+1) = g(g^*(n) + 1)$.

Lemma 4.4. *Suppose that $f, g \in \omega^\omega$ are increasing functions and $\varphi \in \Sigma_f$.*

(1) *If $f \leqslant^* g$ then there exists $\psi \in \Sigma_{g^*}$ such that $H_\varphi \subseteq H_\psi$.*

(2) *If $g \nleqslant^* f$ then there exists $\psi \in \Pi_{g^*}$ such that $H_\varphi \subseteq H_\psi$.*

Proof. Let $I_n = [f(n), f(n+1))$ and $I_n^* = [g^*(n), g^*(n+1))$ for $n \in \omega$. Note that if $f \nleqslant^* g$ then

$$\forall^\infty n\, \exists m\, I_n \subseteq I_n^*$$

and if $g \nleqslant^* f$ then

$$\exists^\infty n\, \exists m\, I_m \subseteq I_n^*.$$

Define

$$\psi(n) = \begin{cases} \{s \in 2^{I_n^*} : \exists m\, (I_m \subseteq I_n^* \,\&\, s \restriction I_m \in \varphi(m))\} & \text{if } \exists m\, I_m \subseteq I_n^*, \\ 2^{I_n^*} & \text{otherwise.} \end{cases}$$

It follows that $\psi \in \Sigma_{g^*}$ in the first case and $\psi \in \Pi_{g^*}$ in the second case. Moreover, the inclusion $H_\varphi \subseteq H_\psi$ is an immediate consequence of the above definition. □

As a consequence we get:

Theorem 4.5. *Suppose that $\{F_\xi : \xi < \kappa\}$ is a family of elements of $\mathscr{E}$.*

(1) *If $\kappa < \mathfrak{b}$ then there exists a function $g \in \omega^\omega$ and a family $\{\varphi_\xi : \xi < \kappa\} \subseteq \Sigma_g$ such that $F_\xi \subseteq H_{\varphi_\xi}$ for $\xi < \kappa$.*

(2) *If $\kappa < \mathfrak{d}$ then there exists a function $g \in \omega^\omega$ and a family $\{\varphi_\xi : \xi < \kappa\} \subseteq \Pi_g$ such that $F_\xi \subseteq H_{\varphi_\xi}$ for $\xi < \kappa$.* □

The following fact follows immediately from 4.5.

Theorem 4.6. *If $\mathbf{cov}(\mathscr{E}) < \mathfrak{d}$ then there exists $f \in \omega^\omega$ such that $\mathbf{cov}(\mathscr{E})$ is equal to the size of the smallest family $\Psi \subseteq \Pi_f$ such that*

$$\forall h \in \chi_f\, \exists \psi \in \Psi\, \forall^\infty n\, h(n) \in \psi(n). \qquad \square$$

As a corollary we get the following:

Theorem 4.7 (Miller). *If $\mathbf{cov}(\mathscr{E}) \leqslant \mathfrak{d}$ then* $\mathrm{cf}(\mathbf{cov}(\mathscr{E})) > \aleph_0$.

Proof. Suppose that $\mathrm{cf}(\mathbf{cov}(\mathscr{E})) = \aleph_0$. Since $\mathfrak{d}$ has uncountable cardinality we have $\mathbf{cov}(\mathscr{E}) < \mathfrak{d}$. By 4.6 under this assumption there exists $g \in \omega^\omega$ such that $\mathbf{cov}(\mathscr{E})$ is the size of the smallest family $\Psi \subseteq \Pi_g$ such that

$$\forall h \in \chi_g\, \exists \psi \in \Psi\, \forall^\infty n\, h(n) \in \psi(n).$$

Assume that Ψ is the smallest family having above properties and let $\{\Psi_n : n \in \omega\}$ be an increasing family such that $\Psi = \bigcup_{n \in \omega} \Psi_n$ and $|\Psi_n| < |\Psi|$ for all $n \in \omega$.

By the assumption for every $m \in \omega$ there exists a function $h_m \in \chi_g$ such that

$$\forall m\, \forall \psi \in \Psi_m\, \exists^\infty n\, h_m(n) \notin \psi(n).$$

For $\psi \in \Psi$ define $k_0^\psi = 0$ and for $n \in \omega$

$$k_{n+1}^\psi = \min\{m > k_n^\psi \colon \forall j \leq n\, \exists i \in [k_n^\psi, m)\, h_j(i) \notin \psi(i)\}.$$

Since $|\Psi| < \mathfrak{b}$ we can find an increasing function $r \in \omega^\omega$ such that

$$\forall \psi \in \Psi\, \exists^\infty n\, k_n^\psi \leq r(n).$$

Let $h = h_1 \upharpoonright [r^*(0), r^*(1))^\frown h_2 \upharpoonright [r^*(1), r^*(2))^\frown h_3 \upharpoonright [r^*(2), r^*(3))^\frown \cdots$.

Fix $\psi \in \Psi$. By the assumption about r we have

$$\exists^\infty n\, \exists m > n\, r^*(n) < k_m^\psi < k_{m+1}^\psi < r^*(n+1).$$

But this means that

$$\exists i \in [r(n), r(n+1))\, h_{n+1}(i) = h(i) \notin \psi(i).$$

Since ψ is an arbitrary element of Ψ it finishes the proof. □

5. Consistency results

The goal of this section is to show that $\mathbf{cov}(\mathscr{E}) > \max\{\mathbf{cov}(\mathcal{N}), \mathbf{cov}(\mathcal{M})\}$ and $\mathbf{unif}(\mathscr{E}) < \min\{\mathbf{unif}(\mathcal{N}), \mathbf{unif}(\mathcal{M})\}$ are both consistent with ZFC. We use the technique developed in [7].

Lemma 5.1. *Suppose that $\mathscr{P}$ is a notion of forcing satisfying ccc. Let $\dot{C}$ be a $\mathscr{P}$-name for an element of $\mathscr{E}$.*

(1) *If $\mathscr{P}$ does not add dominating reals then there exists $f \in \omega^\omega \cap \mathbf{V}$ and a $\mathscr{P}$-name $\dot{\varphi}$ such that $\Vdash_{\mathscr{P}} \dot{\varphi} \in \Pi_f$ and $\Vdash_{\mathscr{P}} \dot{C} \subseteq H_{\dot{\varphi}}$.*

(2) *If $\mathscr{P}$ is ω^ω-bounding then there exists $f \in \omega^\omega \cap \mathbf{V}$ and a $\mathscr{P}$-name $\dot{\varphi}$ such that $\Vdash_{\mathscr{P}} \dot{\varphi} \in \Sigma_f$ and $\Vdash_{\mathscr{P}} \dot{C} \subseteq H_{\dot{\varphi}}$.*

Proof. Follows immediately from 4.4. □

Definition 5.2. Suppose that $N \Vdash \mathrm{ZFC}^*$. A function $x \in 2^\omega$ is called *N-big* iff

$$x \notin \bigcup(\mathscr{E} \cap N).$$

We say that a partial ordering $\mathscr{P}$ satisfying ccc is *good* if for every model $N \prec H(\chi)$ and every filter G which is $\mathscr{P}$-generic over $\mathbf{V}$, if $x \in 2^\omega$ is N-big then x is $N[G]$-big.

Let B denote the random real forcing.

Theorem 5.3. *B is good.*

Proof. Suppose that x is N-big. Let $\dot{C} \in N$ be a B-name for an element of $\mathscr{E}$. Since B is ω^ω-bounding, by 5.1, we can find a function $f \in \omega^\omega \cap N$ and a B-name $\dot{\varphi} \in N$ for an element of Σ_f such that $\Vdash_B \dot{C} \subseteq H_{\dot{\varphi}}$.

For $s \in 2^{f(n)}$ define $B_{n,s} = [\![s \in \dot{\varphi}(n)]\!]_B$. Let

$$\varphi(n) = \left\{ s \colon \mu(B_{n,s}) \geq \frac{1}{2^n} \right\} \quad \text{for } n \in \omega.$$

Note that since

$$\Vdash_B \frac{|\dot{\varphi}(n)|}{2^{f(n)}} \leq \frac{1}{4^n}$$

we get that

$$\frac{|\varphi(n)|}{2^{f(n)}} \leq \frac{1}{2^n} \quad \text{for } n \in \omega.$$

Suppose that $p \Vdash_B \forall n \geq m \; x \restriction n \in \dot{\varphi}(n)$. Find k such that $\mu(p) \geq 2^{-k}$. Since x is N-big there exists $n \geq k$ such that $\hat{s} = x \restriction I_n \notin \varphi(n)$. In particular, $\mu(B_{n,\hat{s}}) < 2^{-k}$. Let $q = p - B_{n,\hat{s}}$. It is clear that

$$q \Vdash_B x \restriction I_n \notin \dot{\varphi}(n)$$

which gives a contradiction. $\square$

Lemma 5.4. (1) *If $\mathscr{P}$ and $\mathscr{Q}$ are good forcing notations then $\mathscr{P} * \mathscr{Q}$ is good.*

(2) *if $\{\mathscr{P}_\alpha, \dot{\mathscr{Q}}_\alpha \colon \alpha < \delta\}$ is a finite support iteration such that*

(a) $\Vdash_\alpha \dot{\mathscr{Q}}_\alpha$ *is good,*

(b) $\Vdash_\alpha$ "$\omega^\omega \cap \mathbf{V}$ *is unbounded*",

then $\mathscr{P}_\delta = \lim_{\alpha<\delta} \mathscr{P}_\alpha$ is good.

Proof. The first part is obvious. We will prove the second part by induction on δ. Without loss of generality we can assume that δ is a limit ordinal. Suppose that the lemma is true for $\alpha < \delta$. Let $N \prec H(\chi)$ be a model and let $\dot{C}$ be a $\mathscr{P}_\delta$-name for an element of $\mathscr{E} \cap N$. It is well known that under the assumptions $\mathscr{P}_\delta$ does not add dominating reals. Therefore there exists $f \in \omega^\omega \cap N$ and a $\mathscr{P}_\delta$-name $\dot{\varphi}$ for an element of Π_f such that

$$\Vdash_\delta \dot{C} \subseteq H_{\dot{\varphi}}.$$

Assume that x is N-big and suppose that for some $p \in \mathscr{P}_\delta$,

$$p \vDash_\delta \forall n > n_0 \; x \restriction [f(n), f(n+1)) \in \dot{\varphi}(n).$$

Define a sequence $\langle p_n \colon n \in \omega \rangle$, $\langle k_n \colon n \in \omega \rangle \in N$ and $\varphi \in \Pi_f$ such that

1. $p = p_0 \leq p_1 \leq p_2 \leq \cdots,$
2. $p_{n+1} \Vdash_\delta \forall j \leq k_n \; \dot{\varphi}(j) = \varphi(j),$
3. $p_{n+1} \Vdash_\delta \exists j \in [k_n, k_{n+1}] \; |\varphi(j)| \cdot 2^{f(j)} \leq 4^{-j}.$

Since x is N-big there exists $m > n_0$ such that $x \restriction [f(m), f(m+1)) \notin \varphi(m)$. Therefore $p_m \Vdash x \restriction [f(m), f(m+1)) \notin \dot{\varphi}(m)$. In particular,

$$p \nVdash x \restriction [f(m), f(m+1)) \notin \dot{\varphi}(m)$$

which is a contradiction. □

Theorem 5.5. *It is consistent with* ZFC *that*

$$\mathbf{unif}(\mathscr{E}) < \min\{\mathbf{unif}(\mathcal{N}), \mathbf{unif}(\mathcal{M})\}.$$

Proof. Let $\mathscr{P}_{\omega_2}$ be a finite support iteration of length ω_2 of random real forcing. Let G be a $\mathscr{P}_{\omega_2}$-generic filter over a model $\mathbf{V} \models \mathrm{GCH}$. Since $\mathscr{P}_{\omega_2}$ adds random and Cohen reals we have $\mathbf{V}[G] \models \mathbf{unif}(\mathcal{M}) = \mathbf{unif}(\mathcal{N}) = \aleph_2$. We will show that $\mathbf{V}[G] \models \mathbf{unif}(\mathscr{E}) = \aleph_1$. It is enough to show that $\mathbf{V}[G] \models 2^\omega \cap V \notin \mathscr{E}$.

Suppose that $C \in V[G] \cap \mathscr{E}$. Let $\dot{C}$ be a $\mathscr{P}_{\omega_2}$-name for C. Let $N < H(\chi)$ be a countable model containing $\dot{C}$ and $\mathscr{P}_{\omega_2}$. Since N is countable, there exists $x \in 2^\omega \cap \mathbf{V}$ which is N-big. By 5.4, x is also $N[G]$-big. In particular $x \notin C$. □

Theorem 5.6. *It is consistent with* ZFC *that*

$$\mathbf{cov}(\mathscr{E}) > \max\{\mathbf{cov}(\mathcal{N}), \mathbf{cov}(\mathcal{M})\}.$$

Proof. Let $\mathscr{P}_{\omega_1}$ be a finite support iteration of length ω_1 of random real forcing. Let G be a $\mathscr{P}_{\omega_1}$-generic filter over a model $\mathbf{V} \models \mathbf{cov}(\mathscr{E}) = \aleph_2$.

It is clear that $\mathbf{V}[G] \models \mathbf{cov}(\mathcal{N}) = \mathbf{cov}(\mathcal{M}) = \aleph_1$. We will show that $\mathbf{V}[G] \models \mathbf{cov}(\mathscr{E}) = \aleph_2$.

Suppose that $\{C_\xi : \xi < \omega_1\} \subseteq \mathbf{V}[G] \cap \mathscr{E}$. Let $\dot{C}_\alpha$ be a $\mathscr{P}_{\omega_1}$-name for C_α. Let $N < H(\chi)$ be a model of size $\aleph_1$ containing all names $\dot{C}_\alpha$ and $\mathscr{P}_{\omega_1}$. Since $\mathbf{V} \models \mathbf{cov}(\mathscr{E}) > \aleph_1$ there exists $x \in 2^\omega \cap \mathbf{V}$ which is N-big. By 5.4, x is also $N[G]$-big. In particular, $x \notin \bigcup_{\xi<\omega_1} C_\xi$. □

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