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Sweet & sour and other flavours of ccc forcing notions

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Abstract. We continue developing the general theory of forcing notions built with the use of *norms on possibilities*, this time concentrating on ccc forcing notions and classifying them.

Contents

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0. Introduction

The present paper has three themes. First, we continue the research started started in Judah, Rosłanowski and Shelah [8] and Rosłanowski and Shelah [15], [16], and we investigate the method of *norms on possibilities* in the context of ccc forcing notions, getting a number of constructions of nicely definable ccc forcings. Most of them belong to the class of nep–forcing notions of Shelah [19], [20], [21] (giving yet more examples to which the general theory developed there can be applied).

The second theme of the paper is a part of the general program "how special are random and Cohen forcing notions (or: the respective ideals)". Kunen (see [11, Question 1.2]) suspected that the null ideal and the meager ideal on 2^{ω} can be somehow characterized by their combinatorial properties, but in [16] we constructed σ–ideals (or rather forcing notions) that have nice properties, however are different from the two. (But see also Kechris and Solecki [10] and Solecki [28] for results in the opposite direction.) Shelah [25] shows that the two forcing notions may occupy special positions in the realm of nicely definable forcing notions. In this realm we may classify forcing notions using the methods of [19], [21] and, for example, declare that very Souslin (or generally ω–nw–nep) ccc forcing notions (see 1.3.1) are really nice. Both the Cohen forcing notion and the random forcing notion and their FS iterations (and nice subforcings) are all ccc ω–nw–nep, and [22, Problem 4.24] asked if we have more examples. It occurs that our method relatively easily results in very Souslin ccc forcing notions (see 1.3.4(3), 1.5.8(2), 1.5.11, 1.5.15(3)).

The third theme is *sweet & sour* and it is related to one of the most striking differences between the random and the Cohen forcing notions which appears when we consider the respective regularity properties of projective set. In [24], Shelah proved that the Lebesgue measurability of $\mathbf{\Sigma}^1_3$ sets implies ω_1 is inaccessible in $\mathbf{L}$, while one can construct (in ZFC) a forcing notion $\mathbb{P}$ such that $\mathbf{V}^{\mathbb{P}} \models$ "projective subsets of $\mathbb{R}$ have the Baire property". The latter construction involved a strong version of ccc, so called "sweetness" (see 4.1.2). The heart of the former result is that the composition of two Amoeba for measure forcing notions is sour (see 4.3.2) over random. Also from a sequence of ω_1 reals we can define a non-measurable set, but not one without the Baire Property.

It seems that sweet–sour properties of forcing notions could be used to classify them as either close to Cohen or as more random–like. Again, our methods result in examples for both cases. Surprisingly, there are sour examples which may appear to be not so much different from the sweet ones - see 4.4.3.

Let us postpone the discussion of the general context of this paper till *Epilogue*, when we can easier refer to the definitions and notions discussed in the paper. (But the curious reader may start reading this paper from the last section.)

We try to make this work self contained, citing the most important definitions and results from [15], [16] whenever needed. However, at least superficial familiarity with those papers could be of some help in reading this paper.

0.1. The content of the paper

Like in [15], the basic intention of this paper is to present "the general theory" rather than particular examples. Therefore, we extract those properties of an example we want to construct which are responsible for the fact that it works and we separate "the general theory" from its applications. But to make the paper more readable, in most cases, we sacrifice generality for clarity.

In the first section we uniformize and generalize the constructions of [8] and [16]. We investigate the complexity of the resulting forcing notions as well as properties like "adding unbounded reals", "preserving unbounded families", etc.

The next section introduces more ways in which creatures (or tree–creatures) can be used to build ccc forcing notions. We discuss mixtures with randoms, some relatives of the Universal Meager forcing notion, as well as as "artificial" modifications of previously introduced forcings.

The third part formalizes definitions of σ–ideals corresponding to our forcing notions.

The following section discusses sweet–sour properties of our forcing notions. We recall the notions of sweetness and introduce yet another sweet property, and we show that very often our constructions are (somewhat) sweet. However, there are exceptions to this rule. So we define some strong negations of sweetness (sourness) and we show how our schema may end up with very sour results.

Finally, the last section is (in some sense) a continuation of the introduction. We discuss the results of the paper and formulate some problems.

0.2. Notation

Most of our notation is standard and compatible with that of classical textbooks on Set Theory (like Bartoszyński and Judah [1]). However in forcing we keep the convention that *a stronger condition is the larger one*.

Basic Notation: In this paper $\mathbf{H}$ will stand for a function with domain ω and such that $(\forall m \in \omega)(|\mathbf{H}(m)| \geq 2)$. We usually assume that $0 \in \mathbf{H}(m)$ (for all $m \in \omega$); if it is not the case then we fix an element of $\mathbf{H}(m)$ and we use it whenever appropriate notions refer to 0. Moreover we demand $\mathbf{H} \in \mathcal{H}(\omega_1)$ (i.e., $\mathbf{H}$ is hereditarily countable).

More Notation:

(1) $\mathbb{R}^{\geq 0}$ stands for the set of non-negative reals. The integer part of a real $r \in \mathbb{R}^{\geq 0}$ is denoted by $\lfloor r \rfloor$.
(2) For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\mathrm{lh}(\eta)$.

(3) A *tree* is a family T of finite sequences such that for some $\mathrm{root}(T) \in T$ we have

$$(\forall \nu \in T)(\mathrm{root}(T) \trianglelefteq \nu) \quad \text{and} \quad \mathrm{root}(T) \trianglelefteq \nu \trianglelefteq \eta \in T \ \Rightarrow\ \nu \in T.$$

For a tree T, the family of all ω–branches through T is denoted by $[T]$, and we let

$$\max(T) \stackrel{\mathrm{def}}{=} \{\nu \in T : \text{ there is no } \rho \in T \text{ such that } \nu \vartriangleleft \rho\}.$$

If η is a node in the tree T then

$$\begin{array}{ll} \mathrm{succ}_T(\eta) & = \{\nu \in T : \eta \vartriangleleft \nu \ \&\ \mathrm{lh}(\nu) = \mathrm{lh}(\eta) + 1\} \ \text{ and} \\ T^{[\eta]} & = \{\nu \in T : \eta \trianglelefteq \nu\}. \end{array}$$

(4) The quantifiers $(\forall^\infty n)$ and $(\exists^\infty n)$ are abbreviations for

$$(\exists m \in \omega)(\forall n > m) \quad \text{and} \quad (\forall m \in \omega)(\exists n > m),$$

respectively.

(5) For a set X, $[X]^{\leq\omega}$, $[X]^{<\omega}$ and $\mathcal{P}(X)$ will stand for families of countable, finite and all, respectively, subsets of the set X. The family of k-element subsets of X will be denoted by $[X]^k$. The set of all finite sequences with values in X is called $X^{<\omega}$ (so domains of elements of $X^{<\omega}$ are integers). The collection of all *finite partial* functions from ω to X is $X^{\underline{\omega}}$.

(6) For a relation R (a set of ordered pairs), $\mathrm{dom}(R)$ stands for the domain of R and $\mathrm{rng}(R)$ denotes the range of R.

(7) The Cantor space 2^ω and the Baire space ω^ω are the spaces of all functions from ω to 2, ω, respectively, equipped with the natural (Polish) topology.

(8) For $f, g \in \omega^\omega$ we write $f <^* g$ ($f \leq^* g$, respectively) whenever $(\forall^\infty n \in \omega)(f(n) < g(n))$ ($(\forall^\infty n \in \omega)(f(n) \leq g(n))$, respectively).

(9) For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a dot above (e.g. $\dot{\tau}$, $\dot{X}$). The weakest element of $\mathbb{P}$ will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one, and that there is no other condition equivalent to it).

0.3. Some classical examples of ccc forcing notions

Many classical ccc forcing notions for the reals are covered by the general methods introduced in this paper. Let us recall some of those forcing notions and point to which cases of our method they belong.

Often, a forcing notion for the Reals fits the following pattern. A condition determines an initial segment of the generic real $\dot{W}$ and it puts some restrictions on possible further extensions of the initial segment. When we pass to a stronger condition we extend the determined part of the generic real and we put more restrictions on possible extensions. To guarantee that the forcing notion satisfies the ccc we require that the set of allowed values for $\dot{W}(n)$ is "large" (and "large" sets are supposed to behave somewhat like a filter). Let us look at two most popular examples: the Hechler forcing notion $\mathbb{D}$ and Miller's Eventually Different Real forcing notion $\mathbb{E}$.

A condition in $\mathbb{D}$ is a pair (s, f) where $s \in \omega^{<\omega}$ and $f \in \omega^{\omega}$ are such that $s \subseteq f$; the order of $\mathbb{D}$ is defined by

$$(s, f) \leq_{\mathbb{D}} (s', f') \quad \text{if and only if} \quad s \subseteq s' \text{ and } (\forall n < \omega)(f(n) \leq f'(n)).$$

A condition in $\mathbb{E}$ is a pair (s, F) where $s \in \omega^{<\omega}$ and $\emptyset \neq F \in [\omega^{\omega}]^{<\omega}$; the order of $\mathbb{E}$ is defined by

$$(s, F) \leq_{\mathbb{E}} (s', F') \quad \text{if and only if } s \subseteq s',\ F \subseteq F' \text{ and } (\forall n \in [\mathrm{lh}(s), \mathrm{lh}(s')))(\forall f \in F)(s'(n) \neq f(n)).$$

In both examples the part of the generic real decided by a condition is the first coordinate s. Let us look at the restrictions on possible extensions: in $\mathbb{D}$, a condition (s, f) says that (basically) the generic real $\dot{W}_{\mathbb{D}} \in \omega^{\omega}$ extends s so that it never gets below f, i.e., $\dot{W}_{\mathbb{D}}(n) \geq f(n)$ for all n. Thus, the possible values of $\dot{W}_{\mathbb{D}}(n)$ (from point of view of $(s, f) \in \mathbb{D}$) are elements of the set $\{m \in \omega : m \geq f(n)\}$. Similarly, in the case of $\mathbb{E}$, a condition (s, F) in this forcing decides that the generic real $\dot{W}_{\mathbb{E}} \in \omega^{\omega}$ extends s and satisfies $\dot{W}_{\mathbb{E}}(n) \notin \{f(n) : f \in F\}$.

Both $\mathbb{D}$ and $\mathbb{E}$ could be represented as follows. A condition is a sequence $(s, A_0, A_1, A_2, \ldots)$, where $s \in \omega^{<\omega}$ and $\emptyset \neq A_n \subseteq \omega$, and the sets A_n are "large" in appropriate sense (see later). The order is such that

$$\begin{aligned}(s, A_0, A_1, A_2, \ldots) \leq (s', A'_0, A'_1, A'_2, \ldots) \quad &\text{if and only if} \\ s \subseteq s',\ (\forall n \in [\mathrm{lh}(s), \mathrm{lh}(s')))(s'(n) \in A_{n-\mathrm{lh}(s)}) &\text{ and} \\ (\forall n < \omega)(A'_n \subseteq A_{k+n}),\ \text{where } k = \mathrm{lh}(s') - \mathrm{lh}(s)&.\end{aligned}$$

The difference between $\mathbb{D}$ and $\mathbb{E}$ comes from deciding the meaning of "large": for the former forcing we just demand that all sets A_n are co-finite, in the case of $\mathbb{E}$ we demand that the sets $\omega \setminus A_n$ are finite and their sizes have a common upper bound. Thus the two forcing notions fit the scheme of subsection 1.1 for local creating pairs (so no $\Sigma^{\perp}$; in relation with $\mathbb{E}$ see Example 1.5.9).

In a similar way we may look at the Mathias forcing with an ultrafilter. Let D be an ultrafilter on ω. A condition in the forcing notion $\mathbb{M}_D$ is a pair (u, B) such that $u \in [\omega]^{<\omega}$, $B \in D$ and $\max(u) < \min(B)$. The order of $\mathbb{M}_D$ is such that

$$(u, B) \leq (u', B') \quad \text{if and only if} \quad u \subseteq u',\ B' \subseteq B, \text{ and } u' \setminus u \subseteq B.$$

We may think of this forcing notion as adding a real $\dot{W}_{\mathbb{M}_D} \in 2^{\omega}$ and then treat conditions in the forcing as sequences $(s, A_0, A_1, A_2, \ldots)$, where $s \in 2^{<\omega}$, $A_n \in \{\{0\}, 2\}$ and "large" means "equal to 2", but here we demand also that A_n's are large on a set from the ultrafilter. This type of constructions is considered in the first part of subsection 2.2.

But we have also ccc forcing notions with conditions of a different nature: forcing with trees. Let us recall the "Laver with an ultrafilter" forcing notion. Suppose that D is an ultrafilter on ω. A condition in $\mathbb{L}_D$ is a tree $T \subseteq \omega^{<\omega}$ with root $\mathrm{root}(T)$

and such that if $\nu \in T$ then the set $\{n : \nu^\frown\langle n\rangle\}$ is in D. Thus again, we may think that a condition $T \in \mathbb{L}_D$ gives us a finite part of the generic real $\dot{W}_{\mathbb{L}_D}$ (it is $\mathrm{root}(T)$) and then it gives some possibilities for the successive values of $\dot{W}_{\mathbb{L}_D}(n)$, but this time the possibilities for $\dot{W}_{\mathbb{L}_D}(n)$ depend on $\dot{W}_{\mathbb{L}_D}\restriction n$. Still, if $\nu \in T$ (so it may be the initial segment of $\dot{W}_{\mathbb{L}_D}$), then the set of all possible values for $\dot{W}_{\mathbb{L}_D}(\mathrm{lh}(\nu))$ is "large" - it belongs to the ultrafilter D. Note also that conditions $T \in \mathbb{L}_D$ may be thought of as systems $\langle A_\nu : \nu \in T\rangle$ where $A_\nu = \mathrm{succ}_T(\nu)$ (and our demand is that each A_ν belongs to D). This is exactly the setting of subsection 1.2. (The particular case of $\mathbb{L}_D$ is also covered by the construction of subsection 2.1.)

Yet another type of classical ccc forcing notions is represented by the Universal Meager forcing $\mathbb{UM}$. A condition in $\mathbb{UM}$ is a pair (N, T) where $N < \omega$ and $T \subseteq 2^{<\omega}$ is a tree such that $[T]$ is a nowhere dense subset of 2^ω. The extension relation is defined by

$$(N,T) \leq_{\mathbb{UM}} (N',T') \quad \text{if and only if} \quad N \leq N',\ T \subseteq T', \ \text{ and}$$
$$T' \cap 2^{\leq N} = T \cap 2^{\leq N}.$$

The generic object here is a nowhere dense subtree of $2^{<\omega}$, and to really reflect all the properties of this type of forcing notions, in subsection 2.3 we introduce another variation of our method: tree forcings determined by universality parameters. (For the representation of $\mathbb{UM}$ as one of those forcing notions see Example 2.4.7.)

1. Building Souslin ccc forcing notions

In this section we will review methods for building ccc forcing notions announced or present in some form in [8], [15], and [16].

1.1. Glue and cut

Here we re-present the method of building ccc forcing notions with use of (semi–) creating triples from [16]. We will slightly modify the definitions loosing some generality. However, we will gain more direct connection to the method of [15] and (hopefully) a better clarity of arguments. Note that the main difference is that here we do not worry about "the permutation invariance" of our forcing notions, so the creatures get back their $m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}$ (and they are like those of [15]).

Definition 1.1.1. *Let* $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$.

(1) (See [15, Def. 1.1.1, 1.2.1]) A creature for $\mathbf{H}$ *is a triple*

$$t = (\mathbf{nor}, \mathbf{val}, \mathbf{dis}) = (\mathbf{nor}[t], \mathbf{val}[t], \mathbf{dis}[t])$$

such that $\mathbf{nor} \in \mathbb{R}^{\geq 0}$, $\mathbf{dis} \in \mathcal{H}(\omega_1)$*, and for some integers* $m^t_{\mathrm{dn}} < m^t_{\mathrm{up}} < \omega$

$$\emptyset \neq \mathbf{val} \subseteq \{\langle u, v\rangle \in \prod_{i<m^t_{\mathrm{dn}}} \mathbf{H}(i) \times \prod_{i<m^t_{\mathrm{up}}} \mathbf{H}(i) : u \vartriangleleft v\}.$$

The set of all creatures for $\mathbf{H}$ *will be denoted by* $\mathrm{CR}[\mathbf{H}]$*, and for* $m_0 < m_1 < \omega$ *we let* $\mathrm{CR}_{m_0,m_1}[\mathbf{H}] = \{t \in \mathrm{CR}[\mathbf{H}] : m^t_{\mathrm{dn}} = m_0 \ \&\ m^t_{\mathrm{up}} = m_1\}$.

(2) *(See [15, Def. 1.1.4, 1.2.2, 1.2.5]) Let* $K \subseteq \mathrm{CR}[\mathbf{H}]$. *We say that a function* $\Sigma : [K]^{<\omega} \longrightarrow \mathcal{P}(K)$ *is a* composition operation on K *whenever the following conditions are satisfied.*

(a) *If* $\mathcal{S} \in [K]^{<\omega}$ *and* $\Sigma(\mathcal{S}) \neq \emptyset$, *then for some enumeration* $\mathcal{S} = \{t_0, \ldots, t_k\}$ *we have* $m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}}$ *for all* $i < k$ *[from now on, whenever we write* $\Sigma(t_0, \ldots, t_k)$, *we mean the enumeration in which* $m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}}$*], and*

(b) *if* $s \in \Sigma(t_0, \ldots, t_k)$, *then* $m^s_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$ *and* $m^s_{\mathrm{up}} = m^{t_k}_{\mathrm{up}}$, *and*

(c) $t \in \Sigma(t)$ *for each* $t \in K$, $\Sigma(\emptyset) = \emptyset$, *and*

(d) *[*transitivity*] if* $s_i \in \Sigma(t^i_0, \ldots, t^i_{k_i})$ *(for* $i \leq n$*), then*

$$\Sigma(s_0, \ldots, s_n) \subseteq \Sigma(t^i_j : i \leq n \;\&\; j \leq k_i),$$

(e) *[*niceness & smoothness*] if* $s \in \Sigma(t_0, \ldots, t_k)$, $m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}}$ *(for* $i < k$*), then* $\mathrm{dom}(\mathbf{val}[t_0]) = \mathrm{dom}(\mathbf{val}[s])$ *and*

$$(\forall \langle u, v\rangle \in \mathbf{val}[s])(\forall i \leq k)(\langle v\restriction m^{t_i}_{\mathrm{dn}}, v\restriction m^{t_i}_{\mathrm{up}}\rangle \in \mathbf{val}[t_i]).$$

(3) *(See [16, Def. 1.1]) A function* $\Sigma^\perp : K \longrightarrow [K]^{<\omega} \setminus \{\emptyset\}$ *is called* a decomposition operation on K *if for each* $t \in K$*:*

(a)$^\perp$ *if* $\mathcal{S} \in \Sigma^\perp(t)$, *then for some enumeration* $\mathcal{S} = \{s_0, \ldots, s_k\}$ *we have* $m^{s_i}_{\mathrm{up}} = m^{s_{i+1}}_{\mathrm{dn}}$ *(for* $i < k$*) [from now on, if we write* $\{s_0, \ldots, s_k\} \in \Sigma^\perp(t)$, *we mean the enumeration in which* $m^{s_i}_{\mathrm{up}} = m^{s_{i+1}}_{\mathrm{dn}}$*], and*

(b)$^\perp$ *if* $\{s_0, \ldots, s_k\} \in \Sigma^\perp(t)$ *then* $m^{s_0}_{\mathrm{dn}} = m^t_{\mathrm{dn}}$, $m^{s_k}_{\mathrm{up}} = m^t_{\mathrm{up}}$,

(c)$^\perp$ $\{t\} \in \Sigma^\perp(t)$,

(d)$^\perp$ *[*transitivity*] if* $\mathcal{S} = \{s_0, \ldots, s_k\} \in \Sigma^\perp(t)$ *and* $\mathcal{S}_i \in \Sigma^\perp(s_i)$ *(for* $i \leq k$*), then* $\mathcal{S}_0 \cup \ldots \cup \mathcal{S}_k \in \Sigma^\perp(t)$,

(e)$^\perp$ *if* $\{s_0, \ldots, s_k\} \in \Sigma^\perp(t)$, $m^{s_i}_{\mathrm{up}} = m^{s_{i+1}}_{\mathrm{dn}}$ *(for* $i < k$*), then*

$$\begin{aligned}&\mathrm{dom}(\mathbf{val}[t]) = \mathrm{dom}(\mathbf{val}[s_0]) \;\; \textit{and}\\ &\quad(\forall i < k)\big(\mathrm{rng}(\mathbf{val}[s_i]) \subseteq \mathrm{dom}(\mathbf{val}[s_{i+1}])\big),\end{aligned}$$

and

$$\begin{aligned}&\{\langle u, v\rangle : u \in \mathrm{dom}(\mathbf{val}[s_0]) \;\&\; u \lhd v \;\&\\ &\quad(\forall i \leq k)(\langle v\restriction m^{s_i}_{\mathrm{dn}}, v\restriction m^{s_i}_{\mathrm{up}}\rangle \in \mathbf{val}[s_i])\} \subseteq \mathbf{val}[t].\end{aligned}$$

(4) *If* $K \subseteq \mathrm{CR}[\mathbf{H}]$ *and* Σ *is a composition operation on* K, *then* (K, Σ) *is called* a creating pair for $\mathbf{H}$. *If, additionally,* $\Sigma^\perp$ *is a decomposition operation on* K, *then* $(K, \Sigma, \Sigma^\perp)$ *is called* a $\otimes$–creating triple for $\mathbf{H}$.

(5) *If* $t_0, \ldots, t_n \in K$ *are such that* $m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}}$ *(for* $i < n$*) and* $w \in \mathrm{dom}(\mathbf{val}[t_0])$, *then we let*

$$\begin{aligned}&\mathrm{pos}(w, t_0, \ldots, t_n)\\ &\quad\stackrel{\mathrm{def}}{=} \{v \in \prod_{j < m^{t_n}_{\mathrm{up}}} \mathbf{H}(j) : w \lhd v \;\&\; (\forall i \leq n)(\langle v\restriction m^{t_i}_{\mathrm{dn}}, v\restriction m^{t_i}_{\mathrm{up}}\rangle \in \mathbf{val}[t_i])\}.\end{aligned}$$

Definition 1.1.2. *Let $(K, \Sigma, \Sigma^{\perp})$ be a $\otimes$–creating triple for $\mathbf{H}$. We say that*

(1) $\Sigma^{\perp}$ *is* trivial *if* $\Sigma^{\perp}(t) = \big\{\{t\}\big\}$ *for each* $t \in K$*;*
(2) (K, Σ) *is* simple *if* $\Sigma(\mathcal{S})$ *is non-empty for singletons only; if additionally* $\Sigma^{\perp}$ *is trivial, then we say that* $(K, \Sigma, \Sigma^{\perp})$ *is* simple*;*
(3) K *(or* (K, Σ) *or* $(K, \Sigma, \Sigma^{\perp})$*) is* local *if* $m^t_{\rm up} = m^t_{\rm dn} + 1$ *for each creature* $t \in K$ *(so then necessarily* $(K, \Sigma, \Sigma^{\perp})$ *is simple);*
(4) K *is* forgetful *if for every creature* $t \in K$ *we have*

$$[\langle u, v\rangle \in \mathbf{val}[t] \ \&\ w \in \prod_{i<m^t_{\rm dn}} \mathbf{H}(i)] \ \Rightarrow\ \langle w, w^\frown v{\restriction}[m^t_{\rm dn}, m^t_{\rm up})\rangle \in \mathbf{val}[t];$$

(5) K *is* full *if* $\mathrm{dom}(\mathbf{val}[t]) = \prod\limits_{i<m^t_{\rm dn}} \mathbf{H}(i)$ *for each* $t \in K$*.*

Definition 1.1.3 (See [15, Def. 1.1.7, 1.2.6], [16, Def. 1.3]). *Let $(K, \Sigma, \Sigma^{\perp})$ be a $\otimes$–creating triple for $\mathbf{H}$ and let $\mathcal{C}(\mathbf{nor})$ be a property of ω–sequences of creatures from K (so $\mathcal{C}(\mathbf{nor})$ can be thought of as a subset of K^{ω}). We define a forcing notion $\mathbb{Q}^*_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^{\perp})$ as follows.*

A condition *in $\mathbb{Q}^*_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^{\perp})$ is a sequence $p = (w^p, t^p_0, t^p_1, t^p_2, \dots)$ such that*

(a) $t^p_i \in K$ *and* $m^{t^p_i}_{\rm up} = m^{t^p_{i+1}}_{\rm dn}$ *(for* $i < \omega$*),*
(b) $w \in \mathrm{dom}(\mathbf{val}[t^p_0])$ *and* $\langle t^p_0, t^p_1, t^p_2, \dots\rangle \in \mathcal{C}(\mathbf{nor})$*,*
(c) $\mathrm{pos}(w^p, t^p_0, \dots, t^p_i) \subseteq \mathrm{dom}(\mathbf{val}[t^p_{i+1}])$ *for each* $i < \omega$*.*

$\mathbb{Q}^*_{\emptyset}(K, \Sigma, \Sigma^{\perp})$ is defined similarly, but we skip the demand "$\langle t^p_0, t^p_1, \dots\rangle \in \mathcal{C}(\mathbf{nor})$" in clause (b) above (or we just let $\mathcal{C}(\mathbf{nor}) = K^{\omega}$; it is perhaps unfortunate to use $\emptyset$ in this context, but that notation was established in [15]).

The relation $\leq$ *on $\mathbb{Q}^*_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^{\perp})$ is given by:* $\quad p \leq q \quad$ *if and only if $(w^q, t^q_0, t^q_1, t^q_2, \dots)$ can be obtained from $(w^p, t^p_0, t^p_1, t^p_2, \dots)$ by applying finitely many times the following operations (describing the operations, we say what are the results of applying the operation to a condition $(w, t_0, t_1, t_2, \dots) \in \mathbb{Q}^*_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^{\perp})$).*

Deciding the value *for $(w, t_0, t_1, t_2, \dots)$: $\quad$ a result of this operation is a condition $(w^*, t_n, t_{n+1}, t_{n+2}, \dots) \in \mathbb{Q}^*_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^{\perp})$ such that $w^* \in \mathrm{pos}(w, t_0, \dots, t_{n-1})$ for some $n < \omega$.*

Applying Σ *to $(w, t_0, t_1, t_2, \dots)$: $\quad$ a result of this operation is a condition $(w, t^*_0, t^*_1, t^*_2, \dots) \in \mathbb{Q}^*_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^{\perp})$ such that for some increasing sequence $0 = n_0 < n_1 < n_2 < \dots < \omega$, for each $i < \omega$, we have $t^*_i \in \Sigma(t_{n_i}, \dots, t_{n_{i+1}-1})$.*

Applying $\Sigma^{\perp}$ *to $(w, t_0, t_1, t_2, \dots)$: $\quad$ a result of this operation is a condition $(w, t^*_0, t^*_1, t^*_2, \dots) \in \mathbb{Q}^*_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^{\perp})$ such that for some increasing sequence $0 = n_0 < n_1 < n_2 < \dots < \omega$, for each $i < \omega$, we have $\{t^*_{n_i}, \dots, t^*_{n_{i+1}-1}\} \in \Sigma^{\perp}(t_i)$.*

Remark 1.1.4. In the definition of the relation $\leq$ on $\mathbb{Q}^*_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^\perp)$ we do require that the intermediate steps satisfy the norm condition $\mathcal{C}(\mathbf{nor})$. However, one may consider a variant when the intermediate stages are just from $\mathbb{Q}^*_\emptyset(K, \Sigma, \Sigma^\perp)$. Many properties of the resulting forcing notion will remain unchanged.

If $\Sigma^\perp$ is trivial we may omit it; note that then we are exactly in the setting of [15, §1.2].

Definition 1.1.5. *We will consider the following norm conditions $\mathcal{C}(\mathbf{nor})$:*

- *A sequence $\langle t_i : i < \omega \rangle$ satisfies $\mathcal{C}^\infty(\mathbf{nor})$ if $\lim_{i \to \infty} \mathbf{nor}[t_i] = \infty$*
  [the respective forcing notion is called $\mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$].
- *Let $\mathcal{F} \subseteq \omega^\omega$; a sequence $\langle t_i : i < \omega \rangle$ satisfies $\mathcal{C}^{\mathcal{F}}(\mathbf{nor})$ if*
  $$(\exists f \in \mathcal{F})(\forall^\infty i \in \omega)(\mathbf{nor}[t_i] \geq f(m^{t_i}_{\mathrm{dn}}))$$
  [the respective forcing notion is denoted $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma, \Sigma^\perp)$].
- *Let $f : \omega \times \omega \longrightarrow \omega$; a sequence $\langle t_i : i < \omega \rangle$ satisfies $\mathcal{C}^f(\mathbf{nor})$ if*
  $$(\forall n \in \omega)(\forall^\infty i \in \omega)(\mathbf{nor}[t_i] \geq f(n, m^{t_i}_{\mathrm{dn}}))$$
  [the respective forcing notion is denoted $\mathbb{Q}^*_f(K, \Sigma, \Sigma^\perp)$].

We will consider the norm conditions $\mathcal{C}^{\mathcal{F}}(\mathbf{nor})$, $\mathcal{C}^f(\mathbf{nor})$ only for h–closed families $\mathcal{F}$ and fast functions f, see 1.1.6 below. Later we will introduce more methods for building ccc forcing notions, including more norm conditions.

Definition 1.1.6. *(1) A function $f : \omega \times \omega \longrightarrow \omega$ is* fast *if*
$$(\forall k, \ell \in \omega)(f(k, \ell) \leq f(k, \ell + 1) \;\&\; 2 \cdot f(k, \ell) < f(k + 1, \ell)).$$
(2) A function $h : \omega \times \omega \longrightarrow \omega$ is regressive *if*
$$(\forall m \in \omega)\big((\forall k > 1)(1 \leq h(m, k) < k) \;\&\; (\forall k < \ell < \omega)(h(m, k) \leq h(m, \ell))\big).$$
*(3) Let $h : \omega \times \omega \longrightarrow \omega$. We say that a family $\mathcal{F} \subseteq \omega^\omega$ is h–*closed *if for every $f \in \mathcal{F}$ there is $f^* \in \mathcal{F}$ such that $(\forall^\infty n \in \omega)(f^*(n) \leq h(n, f(n)))$.*
*(4) A family $\mathcal{F} \subseteq \omega^\omega$ is $\geq^*$–*directed *if*
$$(\forall f_0, f_1 \in \mathcal{F})(\exists f^* \in \mathcal{F})(\forall^\infty n \in \omega)(f^*(n) \leq \min\{f_0(n), f_1(n)\}).$$
Similarly we define $\leq^*$–directed families (just reversing the inequality).

Remark 1.1.7. Let $f(n, m) = 2^{2^n}$ (for $n, m \in \omega$). Then the function f is fast and the norm conditions $\mathcal{C}^f(\mathbf{nor})$ and $\mathcal{C}^\infty(\mathbf{nor})$ agree (and thus $\mathbb{Q}^*_f(K, \Sigma) = \mathbb{Q}^*_\infty(K, \Sigma)$ for a local creating pair (K, Σ)). In practical applications, when we consider the norm condition $\mathcal{C}^f(\mathbf{nor})$, the function f is such that $f(n, m) < f(n, m + 1)$ (for all $n, m \in \omega$) and thus the norm condition $\mathcal{C}^f(\mathbf{nor})$ is stronger than $\mathcal{C}^\infty(\mathbf{nor})$.

Proposition 1.1.8. *If $(K, \Sigma, \Sigma^\perp)$ is a $\otimes$–creating triple for $\mathbf{H}$, $\mathcal{C}(\mathbf{nor}) \subseteq K^\omega$, then $\mathbb{Q}^*_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^\perp)$ is a forcing notion (i.e., the relation $\leq$ of $\mathbb{Q}^*_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^\perp)$ is transitive).*

Definition 1.1.9 (See [15, Def. 1.2.4]). *Let $(K, \Sigma, \Sigma^{\perp})$ be a $\otimes$–creating triple for* $\mathbf{H}$. *We define* finite candidates *(*FC*) and* pure finite candidates *(*PFC*) with respect to $(K, \Sigma, \Sigma^{\perp})$:*

$$\begin{aligned}\mathrm{FC}(K, \Sigma, \Sigma^{\perp}) = \{(w, t_0, \ldots, t_n) : w \in \mathrm{dom}(\mathbf{val}[t_0]) \textit{ and for each } i \le n :\\ t_i \in K, m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}} \textit{ and } \mathrm{pos}(w, t_0, \ldots, t_i) \subseteq \mathrm{dom}(\mathbf{val}[t_{i+1}])\},\end{aligned}$$

$$\begin{aligned}&\mathrm{PFC}(K, \Sigma, \Sigma^{\perp})\\ &\quad = \{(t_0, \ldots, t_n) : (\exists w \in \mathrm{dom}(\mathbf{val}[t_0]))((w, t_0, \ldots, t_n) \in \mathrm{FC}(K, \Sigma, \Sigma^{\perp}))\}.\end{aligned}$$

We have a natural relation $\le$ on $\mathrm{FC}(K, \Sigma, \Sigma^{\perp})$ *(defined like in 1.1.3). [Note that $\Sigma, \Sigma^{\perp}$ have no influence on* $\mathrm{FC}(K, \Sigma, \Sigma^{\perp})$*, that is they are not present in the definition of finite candidates, and we could have written* $\mathrm{FC}(K, \Sigma)$ *or* $\mathrm{FC}(K)$*. However, they come to the game when the relation $\le$ on* $\mathrm{FC}(K, \Sigma, \Sigma^{\perp})$ *is considered.]*

A sequence $\langle t_0, t_1, t_2, \ldots\rangle$ of creatures from K is a pure candidate *with respect to $(K, \Sigma, \Sigma^{\perp})$ if*

$$(\forall i < \omega)(m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}}) \quad \textit{and}$$

$$(\exists w \in \mathrm{dom}(\mathbf{val}[t_0]))(\forall i < \omega)(\mathrm{pos}(w, t_0, \ldots, t_i) \subseteq \mathrm{dom}(\mathbf{val}[t_{i+1}])).$$

The set of pure candidates with respect to (K, Σ) is denoted by $\mathrm{PC}(K, \Sigma, \Sigma^{\perp})$*. The relation $\le$ on* $\mathrm{PC}(K, \Sigma, \Sigma^{\perp})$ *is defined naturally.*

For a norm condition $\mathcal{C}(\mathbf{nor})$ the family of $\mathcal{C}(\mathbf{nor})$–normed pure candidates *is*

$$\begin{aligned}&\mathrm{PC}_{\mathcal{C}(\mathbf{nor})}(K, \Sigma, \Sigma^{\perp})\\ &\quad = \{\langle t_0, t_1, \ldots\rangle \in \mathrm{PC}(K, \Sigma, \Sigma^{\perp}) : \langle t_0, t_1, \ldots, \rangle \textit{ satisfies } \mathcal{C}(\mathbf{nor})\}.\end{aligned}$$

Definition 1.1.10. *Let $(K, \Sigma, \Sigma^{\perp})$ be a $\otimes$–creating triple for* $\mathbf{H}$.

(1) For a condition $p \in \mathbb{Q}^*_{\emptyset}(K, \Sigma, \Sigma^{\perp})$ we let

$$\mathrm{POS}(p) \stackrel{\mathrm{def}}{=} \{u \in \bigcup_{n<\omega} \prod_{i<n} \mathbf{H}(i) : (\exists \ell < \omega)(\exists v \in \mathrm{pos}(w^p, t^p_0, \ldots, t^p_\ell))(u \trianglelefteq v)\}.$$

(2) For a finite candidate $c = (w, t_0, \ldots, t_k) \in \mathrm{FC}(K, \Sigma, \Sigma^{\perp})$ we define

$$\mathrm{POS}(c) \stackrel{\mathrm{def}}{=} \{u \in \bigcup_{n \le m^{t_k}_{\mathrm{up}}} \prod_{i<n} \mathbf{H}(i) : (\exists v \in \mathrm{pos}(w, t_0, \ldots, t_k))(u \trianglelefteq v)\}.$$

Proposition 1.1.11. *Suppose $(K, \Sigma, \Sigma^{\perp})$ is a $\otimes$–creating triple for* $\mathbf{H}$.

(1) If $p, q \in \mathbb{Q}^*_{\emptyset}(K, \Sigma, \Sigma^{\perp})$, $p \le q$ then $\mathrm{POS}(p) \subseteq \mathrm{POS}(q)$*, and* if $\mathrm{lh}(w^q) = m^{t^p_\ell}_{\mathrm{up}}$ *for some $\ell < \omega$,* then $w^q \in \mathrm{pos}(w^p, t^p_0, \ldots, t^p_\ell)$.

(2) The same holds if one replaces conditions from $\mathbb{Q}^*_{\emptyset}(K, \Sigma, \Sigma^{\perp})$ by finite candidates from $\mathrm{FC}(K, \Sigma, \Sigma^{\perp})$.

Proof. Note that each of the three operations described in 1.1.3 shrinks POS (remember 1.1.1(2e) and 1.1.1(3e$^\perp$)). $\square$

Definition 1.1.12 (See [16, Def. 2.1], [15, Def. 2.1.7]). *Assume that $(K, \Sigma, \Sigma^\perp)$ is a $\otimes$–creating triple for* $\mathbf{H}$.

(1) *We say that (K, Σ) (or $(K, \Sigma, \Sigma^\perp)$) is* linked *if for each $t_0, t_1 \in K$ such that $\mathbf{nor}[t_0], \mathbf{nor}[t_1] > 1$ and $m^{t_0}_{\rm dn} = m^{t_1}_{\rm dn}$, $m^{t_0}_{\rm up} = m^{t_1}_{\rm up}$, there is $s \in \Sigma(t_0) \cap \Sigma(t_1)$ with*

$$\mathbf{nor}[s] \geq \min\{\mathbf{nor}[t_0], \mathbf{nor}[t_1]\} - 1.$$

*Let $h : \omega \times \omega \longrightarrow \omega$. The pair (K, Σ) is said to be h–*linked *if for each $k > 1$, and creatures $t_0, t_1 \in K$ such that $\mathbf{nor}[t_0], \mathbf{nor}[t_1] \geq k$ and $m^{t_0}_{\rm dn} = m^{t_1}_{\rm dn}$, $m^{t_0}_{\rm up} = m^{t_1}_{\rm up}$, there is $s \in \Sigma(t_0) \cap \Sigma(t_1)$ with $\mathbf{nor}[s] \geq h(m^{t_0}_{\rm dn}, k)$.*

(2) *We say that (K, Σ) (or $(K, \Sigma, \Sigma^\perp)$) is* gluing *if it is full and for each $k < \omega$ there is $n_0 = n_0(k) < \omega$ such that for every $n \geq n_0$ and $(t_0, \dots, t_n) \in \mathrm{PFC}(K, \Sigma)$, there is $s \in \Sigma(t_0, \dots, t_n)$ such that*

$$\mathbf{nor}[s] \geq \min\{k, \mathbf{nor}[t_0], \dots, \mathbf{nor}[t_n]\}.$$

We say that (K, Σ) is straightforward gluing *if for every $(t_0, \dots, t_n) \in \mathrm{PFC}(K, \Sigma)$ there is $s \in \Sigma(t_0, \dots, t_n)$ such that*

$$\mathbf{nor}[s] \geq \min\{\mathbf{nor}[t_0], \dots, \mathbf{nor}[t_n]\}.$$

(3) *We say that $(K, \Sigma^\perp)$ (or $(K, \Sigma, \Sigma^\perp)$)* has the cutting property *if for every $t \in K$ with $\mathbf{nor}[t] > 1$ and an integer $m \in (m^t_{\rm dn}, m^t_{\rm up})$, there are $s_0, s_1 \in K$ such that*
(α) $m^{s_0}_{\rm dn} = m^t_{\rm dn}$, $m^{s_0}_{\rm up} = m = m^{s_1}_{\rm dn}$, $m^{s_1}_{\rm up} = m^t_{\rm up}$,
(β) $\mathbf{nor}[s_\ell] \geq \min\{\mathbf{nor}[t] - 1, m^t_{\rm dn}\}$ *(for $\ell = 0, 1$),*
(γ) $\{s_0, s_1\} \in \Sigma^\perp(K)$.

Definition 1.1.13. *A forcing notion $\mathbb{Q}$ is σ-n–*linked *if there is a partition $\langle A_i : i < \omega\rangle$ of $\mathbb{Q}$ such that*

$$\text{if} \quad q_0, \dots, q_{n-1} \in A_i,\ i \in \omega \quad \text{then} \quad (\exists q \in \mathbb{Q})(q_0 \leq q \ \&\ \dots\ \&\ q_{n-1} \leq q).$$

*We say that $\mathbb{Q}$ is σ-$*$–*linked *if it is σ-n–linked for every $n \in \omega$.*

Proposition 1.1.14. *Let $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$ and let $(K, \Sigma, \Sigma^\perp)$ be a $\otimes$–creating triple for* $\mathbf{H}$.

(1) *If $(K, \Sigma, \Sigma^\perp)$ is linked, gluing and has the cutting property, then the forcing notion $\mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$ is σ-$*$–linked.*
(2) *If $f : \omega \times \omega \longrightarrow \omega$ is fast and $(K, \Sigma, \Sigma^\perp)$ is local and linked, then the forcing notions $\mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$ and $\mathbb{Q}^*_f(K, \Sigma, \Sigma^\perp)$ are σ-$*$–linked.*
(3) *Assume that $h : \omega \times \omega \longrightarrow \omega$ is regressive and $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ is an h–closed family which is either countable, or $\geq^*$–directed. Suppose $(K, \Sigma, \Sigma^\perp)$ is local and h–linked. Then the forcing notion $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma, \Sigma^\perp)$ is σ-$*$–linked.*

Proof. Straightforward (and the proof of the first part is essentially the same as that of [16, Thm 2.4]; compare the proof of 1.3.4.1). $\square$

1.2. Tree–like conditions

Here we recall the setting of [15, §1.3] and [8]. Since in getting the ccc we will have to require that the tree–creating pair under considerations is local, we will restrict our attention to that case only. So our definitions here are much simpler than those in the general case, but we still try to keep the notation and flavour of the tree case of [15].

Definition 1.2.1. *Let* $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$.

(1) A local tree–creature for $\mathbf{H}$ *is a triple*

$$t = (\mathbf{nor}, \mathbf{val}, \mathbf{dis}) = (\mathbf{nor}[t], \mathbf{val}[t], \mathbf{dis}[t])$$

such that $\mathbf{nor} \in \mathbb{R}^{\geq 0}$, $\mathbf{dis} \in \mathcal{H}(\omega_1)$, *and for some sequence* $\eta \in \prod\limits_{i<n} \mathbf{H}(i)$, $n < \omega$, *we have*

$$\emptyset \neq \mathbf{val} \subseteq \{\langle \eta, \nu \rangle : \eta \vartriangleleft \nu \in \prod_{i \leq n} \mathbf{H}(i)\}.$$

For a tree–creature t we let $\mathrm{pos}(t) \stackrel{\mathrm{def}}{=} \mathrm{rng}(\mathbf{val}[t])$.
The set of all local tree–creatures for $\mathbf{H}$ *will be denoted by* $\mathrm{LTCR}[\mathbf{H}]$, *and for* $\eta \in \bigcup\limits_{n<\omega} \prod\limits_{i<n} \mathbf{H}(i)$ *we let* $\mathrm{LTCR}_\eta[\mathbf{H}] = \{t \in \mathrm{LTCR}[\mathbf{H}] : \mathrm{dom}(\mathbf{val}[t]) = \{\eta\}\}$.

(2) Let $K \subseteq \mathrm{LTCR}[\mathbf{H}]$. *We say that a function* $\Sigma : K \longrightarrow \mathcal{P}(K)$ *is a* local tree composition on K *whenever the following conditions are satisfied.*
(a) If $t \in \mathrm{LTCR}_\eta[\mathbf{H}]$, $\eta \in \prod\limits_{i<n} \mathbf{H}(i)$, $n < \omega$, *then* $\Sigma(t) \subseteq \mathrm{LTCR}_\eta[\mathbf{H}]$.
(b) If $s \in \Sigma(t)$ *then* $\mathbf{val}[s] \subseteq \mathbf{val}[t]$.
*(c) [*transitivity*] If* $s \in \Sigma(t)$ *then* $\Sigma(s) \subseteq \Sigma(t)$.

(3) If $K \subseteq \mathrm{LTCR}[\mathbf{H}]$ *and* Σ *is a local tree composition operation on* K *then* (K, Σ) *is called* a local tree–creating pair for $\mathbf{H}$. *We may forget the adjective* local *as other cases will not be considered in the present paper.*

Definition 1.2.2. *Let* (K, Σ) *be a (local) tree–creating pair for* $\mathbf{H}$.

(1) We define the forcing notion $\mathbb{Q}_1^{\mathrm{tree}}(K, \Sigma)$ *as follows.*
A condition *is a system* $p = \langle t_\eta : \eta \in T \rangle$ *such that*
(a) $T \subseteq \bigcup\limits_{n\in\omega} \prod\limits_{i<n} \mathbf{H}(i)$ *is a non-empty tree with* $\max(T) = \emptyset$,
(b) $t_\eta \in \mathrm{LTCR}_\eta[\mathbf{H}] \cap K$ *and* $\mathrm{pos}(t_\eta) = \mathrm{succ}_T(\eta)$ *(for* $\eta \in T$*),*
(c)$_1$ *for every* $\eta \in [T]$ *we have:*

the sequence $\langle \mathbf{nor}[t_{\eta\restriction k}] : \mathrm{lh}(\mathrm{root}(T)) \leq k < \omega \rangle$ *diverges to infinity.*

The order *is given by:*
$\langle t^1_\eta : \eta \in T^1 \rangle \leq \langle t^2_\eta : \eta \in T^2 \rangle$ *if and only if*
$T^2 \subseteq T^1$ *and* $t^2_\eta \in \Sigma(t^1_\eta)$ *for each* $\eta \in T^2$.
If $p = \langle t_\eta : \eta \in T \rangle$, *then we write* $\mathrm{root}(p) = \mathrm{root}(T)$, $T^p = T$, $t^p_\eta = t_\eta$ *etc.*

(2) Similarly, we define forcing notions $\mathbb{Q}^{\rm tree}_{\mathcal{F}}(K,\Sigma)$ *for a family* $\mathcal{F}\subseteq\omega^\omega$ *and* $\mathbb{Q}^{\rm tree}_{f}(K,\Sigma)$ *for a function* $f:\omega\times\omega\longrightarrow\omega$*, replacing the condition* (c)$_1$ *by* (c)$_{\mathcal{F}}$, (c)$_f$*, respectively, where:*

(c)$_{\mathcal{F}}$ $(\exists f\in\mathcal{F})(\exists N<\omega)(\forall\eta\in T)({\rm lh}(\eta)\geq N\ \Rightarrow\ \mathbf{nor}[t_\eta]\geq f({\rm lh}(\eta)))$,

(c)$_f$ $(\forall n\in\omega)(\exists N<\omega)(\forall\eta\in T)({\rm lh}(\eta)\geq N\ \Rightarrow\ \mathbf{nor}[t_\eta]\geq f(n,{\rm lh}(\eta)))$.

(3) If $p\in\mathbb{Q}^{\rm tree}_x(K,\Sigma)$ *then, for* $\eta\in T^p$*, we let* $p^{[\eta]}=\langle t^p_\nu:\nu\in(T^p)^{[\eta]}\rangle$.

Definition 1.2.3. *Assume that* (K,Σ) *is a tree–creating pair for* $\mathbf{H}$.

(1) We say that (K,Σ) *is* linked *if for each* $\eta\in\bigcup\limits_{n<\omega}\prod\limits_{i<n}\mathbf{H}(i)$ *and tree–creatures* $t_0,t_1\in K\cap{\rm LTCR}_\eta[\mathbf{H}]$ *with* $\mathbf{nor}[t_0],\mathbf{nor}[t_1]>1$*, there is* $s\in\Sigma(t_0)\cap\Sigma(t_1)$ *such that* $\mathbf{nor}[s]\geq\min\{\mathbf{nor}[t_0],\mathbf{nor}[t_1]\}-1$.

(2) Let $h:\omega\times\omega\longrightarrow\omega$*. The pair* (K,Σ) *is* h*–*linked *if for each* $t_0,t_1\in K\cap{\rm LTCR}_\eta[\mathbf{H}]$ *such that* $\mathbf{nor}[t_0],\mathbf{nor}[t_1]\geq k$*,* $k>1$*, there is* $s\in\Sigma(t_0)\cap\Sigma(t_1)$ *with* $\mathbf{nor}[s]\geq h({\rm lh}(\eta),k)$.

Proposition 1.2.4. *Let* $\mathbf{H}:\omega\longrightarrow\mathcal{H}(\omega_1)$ *and let* (K,Σ) *be a local tree–creating pair for* $\mathbf{H}$.

(1) If $f:\omega\times\omega\longrightarrow\omega$ *is fast and* (K,Σ) *is linked, then the forcing notions* $\mathbb{Q}^{\rm tree}_1(K,\Sigma)$ *and* $\mathbb{Q}^{\rm tree}_f(K,\Sigma)$ *are* σ*-*$*$*–linked.*

(2) Assume that $h:\omega\times\omega\longrightarrow\omega$ *is regressive and a family* $\mathcal{F}\subseteq(\omega\setminus 2)^\omega$ *is* h*–closed and either countable, or* $\geq^*$*–directed. Suppose* (K,Σ) *is* h*–linked. Then the forcing notion* $\mathbb{Q}^{\rm tree}_{\mathcal{F}}(K,\Sigma)$ *is* σ*-*$*$*–linked.*

Proof. Straightforward. □

1.3. The complexity of our forcing notions

Definition 1.3.1. *(1) A forcing notion* $(\mathbb{P},\leq_{\mathbb{P}})$ *is* Souslin *(*Borel*, respectively) if* $\mathbb{P}$*,* $\leq_{\mathbb{P}}$ *and the incompatibility relation* $\perp_{\mathbb{P}}$ *are* Σ^1_1 *(Borel, respectively) subsets of* $\mathbb{R}$ *and* $\mathbb{R}\times\mathbb{R}$.

(2) A forcing notion $(\mathbb{P},\leq_{\mathbb{P}})$ *is* very Souslin ccc *(*very Borel ccc*, respectively), if it is Souslin (Borel, resp.), satisfies the ccc and the notion*

$$\text{“ } \langle r_n:n<\omega\rangle \textit{ is a maximal antichain ”}$$

is Σ^1_1 *(Borel, resp.)*

On Souslin forcing notions and their applications see Judah and Shelah [5] and Goldstern and Judah [3] (the results of these two and many other papers on the topic are presented in Bartoszyński and Judah [1]). A systematic treatment of definable forcing notions is presented in [19], [20] (note that very Souslin ccc forcing notions are ω–nw–nep). Here we are going to show that the forcing notions built according to the schemes presented above typically are Borel ccc and (sometimes) even very Borel ccc. Thus we have tools for constructing new ccc ω–nw–nep forcing notions (the only examples known before were those coming from random forcing, the Cohen forcing and their FS iterations; see [22, §4] for a discussion of this topic). Note that, by Shelah [21], ccc ω–nw–nep forcing notions cannot add dominating reals. Thus the forcing notions that are covered by 1.4.4 cannot be represented as very Souslin ccc forcing notions.

Definition 1.3.2. *A $\otimes$–creating triple $(K, \Sigma, \Sigma^\perp)$ for* $\mathbf{H}$ *is* regular *if the following condition is satisfied.*

($\boxdot$) *Assume $(w, t_0, \dots, t_n), (u, s_0, \dots, s_m) \in \mathrm{FC}(K, \Sigma, \Sigma^\perp)$ are such that*
- $m^{t_\ell}_{\mathrm{dn}} < m^{s_0}_{\mathrm{dn}} < m^{t_\ell}_{\mathrm{up}} \le m^{s_0}_{\mathrm{up}}$ *for some $\ell \le n$,*
- $\mathbf{nor}[t_\ell] \ge 3$ *(for the ℓ as above), and*
- $(w, t_0, \dots, t_n) \le (u, s_0, \dots, s_m)$, $m^{s_m}_{\mathrm{up}} \le m^{t_n}_{\mathrm{up}}$, *and* $\mathbf{nor}[s_0] \ge 3$.

Then there are t', t'' such that $\{t', t''\} \in \Sigma^\perp(t_\ell)$, $m^{t'}_{\mathrm{up}} = m^{s_0}_{\mathrm{dn}} = m^{t''}_{\mathrm{dn}}$, $\mathbf{nor}[t''] \ge 2$ and $u \in \mathrm{pos}(w, t_0, \dots, t_{\ell-1}, t')$.

Definition 1.3.3. *Let $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$ and let (K, Σ) be either a creating pair for $\mathbf{H}$ or a (local) tree–creating pair for $\mathbf{H}$. We say that (K, Σ) is* really finitary *if the following conditions are satisfied:*

(a) $\mathbf{H}(n)$ is finite for all $n < \omega$ (so $\mathbf{val}[t]$ is finite for all $t \in K$), and
(b) for each $n \in \omega$, the set $\{t \in K : \mathrm{rng}(\mathbf{val}[t]) \subseteq \prod\limits_{i<n} \mathbf{H}(i)\}$ is finite.

Theorem 1.3.4. *Let $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$.*

(1) Let $(K, \Sigma, \Sigma^\perp)$ be a $\otimes$–creating triple for $\mathbf{H}$ such that K is countable.
   (a) If $(K, \Sigma, \Sigma^\perp)$ is regular, linked, gluing and has the cutting property, then the forcing notion $\mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$ is Souslin ccc.
   (b) If $f : \omega \times \omega \longrightarrow \omega$ is fast and $(K, \Sigma, \Sigma^\perp)$ is local and linked then $\mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$ and $\mathbb{Q}^*_f(K, \Sigma, \Sigma^\perp)$ are Borel ccc.
   (c) Assume that $h : \omega \times \omega \longrightarrow \omega$ is a regressive function and $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ is a countable h–closed family which is $\ge^*$–directed. If $(K, \Sigma, \Sigma^\perp)$ is local and h–linked, then $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma, \Sigma^\perp)$ is Borel ccc.

(2) Assume that (K, Σ) is a local tree–creating pair for $\mathbf{H}$ and K is countable.
   (a) If $f : \omega \times \omega \longrightarrow \omega$ is fast and (K, Σ) is linked, then $\mathbb{Q}^{\mathrm{tree}}_f(K, \Sigma)$ is Borel ccc.
   (b) Suppose that $h : \omega \times \omega \longrightarrow \omega$ is regressive and $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ is a countable h–closed family which is $\ge^*$–directed. If (K, Σ) is h–linked, then $\mathbb{Q}^{\mathrm{tree}}_{\mathcal{F}}(K, \Sigma)$ is Borel ccc.

(3) If in 1(c) and 2(b) above the pair (K, Σ) is really finitary, then the respective forcing notions are very Borel ccc.

Proof. 1(a) Let $\mathcal{X} = (\bigcup\limits_{n<\omega} \prod\limits_{i<n} \mathbf{H}(i)) \times K^\omega$ be equipped with the product topology (of countably many countable discrete spaces). So $\mathcal{X}$ is a Polish space and it should be clear that $\mathbb{Q}^*_\emptyset(K, \Sigma, \Sigma^\perp)$, $\mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$ are its Borel subsets. To express "$p \le q$" we have to say that *there is* a sequence $p = p_0, \dots, p_n = q$ of elements of $\mathbb{Q}^*_\emptyset(K, \Sigma, \Sigma^\perp)$ such that p_{i+1} is obtained from p_i by one of the operations described in 1.1.3. Each of these operations corresponds to a Borel subset of $\mathcal{X} \times \mathcal{X}$, so easily $\le_{\mathbb{Q}^*_\emptyset(K,\Sigma,\Sigma^\perp)}$, $\le_{\mathbb{Q}^*_\infty(K,\Sigma,\Sigma^\perp)}$ are Σ^1_1 subsets of $\mathcal{X} \times \mathcal{X}$. The main difficulty is to show that the incompatibility relation $\perp_{\mathbb{Q}^*_\infty(K,\Sigma,\Sigma^\perp)}$ is a Σ^1_1 subset of $\mathcal{X} \times \mathcal{X}$. But this follows from the following observation (note that this is the place where we use the assumption that $(K, \Sigma, \Sigma^\perp)$ is regular).

Claim 1.3.4.1. *Conditions* $p, q \in \mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$ *are compatible if and only if there are* $N, \ell, m < \omega$, $t'_0, t'_1, t''_0, t''_1 \in K$ *and* u *such that*

- $m^{t^p_\ell}_{\rm dn} \le N < m^{t^p_\ell}_{\rm up}$, $m^{t^q_m}_{\rm dn} \le N < m^{t^q_m}_{\rm up}$,
- $\{t'_0, t'_1\} \in \Sigma^\perp(t^p_\ell)$, $\{t''_0, t''_1\} \in \Sigma^\perp(t^q_m)$, $m^{t'_1}_{\rm dn} = m^{t''_1}_{\rm dn} = N$,
- $\mathbf{nor}[t'_1] \ge 2$, $\mathbf{nor}[t''_1] \ge 2$,
- $u \in \mathrm{pos}(w^p, t^p_0, \ldots, t^p_{\ell-1}, t'_0) \cap \mathrm{pos}(w^q, t^q_0, \ldots, t^q_{m-1}, t''_0)$,
- $(\forall n > \ell)(\mathbf{nor}[t^p_n] \ge 2)$ *and* $(\forall n > m)(\mathbf{nor}[t^q_n] \ge 2)$.

(If $N = m^{t^p_\ell}_{\rm dn}$ *then* t'_0 *is not present; similarly on the* q *side.)*

Proof of the claim. First assume that conditions $p, q \in \mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$ are compatible and let $r \in \mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$ be stronger than both p and q. Passing to a stronger condition we may demand that if ℓ, m are such that

$$m^{t^p_\ell}_{\rm dn} \le m^{t^r_0}_{\rm dn} < m^{t^p_\ell}_{\rm up}, \qquad m^{t^q_m}_{\rm dn} \le m^{t^r_0}_{\rm dn} < m^{t^q_m}_{\rm up}$$

then $m^{t^p_\ell}_{\rm up} \le m^{t^r_0}_{\rm up}$, $m^{t^q_m}_{\rm up} \le m^{t^r_0}_{\rm up}$ and that $\mathbf{nor}[t^r_0] \ge 5$ and

$$m^{t^p_n}_{\rm up} \ge \mathrm{lh}(w^r) \quad \Rightarrow \quad \mathbf{nor}[t^p_n] \ge 5, \quad \text{and} \quad m^{t^q_n}_{\rm up} \ge \mathrm{lh}(w^r) \quad \Rightarrow \quad \mathbf{nor}[t^q_n] \ge 5.$$

Now we may apply the regularity of $(K, \Sigma, \Sigma^\perp)$ (see 1.3.2) to get $t'_0, t'_1, t''_0, t''_1 \in K$ such that

$$\begin{array}{l}\{t'_0, t'_1\} \in \Sigma^\perp(t^p_\ell), \quad \{t''_0, t''_1\} \in \Sigma^\perp(t^q_m), \quad \mathbf{nor}[t'_1] \ge 2, \quad \mathbf{nor}[t''_1] \ge 2 \quad \text{and}\\ u = w^r \in \mathrm{pos}(w^p, t^p_0, \ldots, t^p_{\ell-1}, t'_0) \cap \mathrm{pos}(w^q, t^q_0, \ldots, t^q_{m-1}, t''_0).\end{array}$$

Put $N = \mathrm{lh}(w^r)$ and check that all demands are satisfied.

For the other implication suppose that $N, \ell, m, t'_0, t'_1, t''_0, t''_1$ and u are as in the second statement. Just for simplicity, let us also assume that (K, Σ) is straightforward gluing (see the second half of 1.1.12(2); the case of "gluing" requires a small change in the choice of $\bar{n}, \bar{k}$ below). Choose increasing sequences $\bar{n} = \langle n_i : i < \omega\rangle$ and $\bar{k} = \langle k_i : i < \omega\rangle$ such that $n_0 > \ell + 5$, $k_0 > m + 5$ and

- $(\forall n \ge n_i)(\mathbf{nor}[t^p_n] \ge i + 5)$ and $(\forall n \ge k_i)(\mathbf{nor}[t^q_n] \ge i + 5)$, and
- $m^{t^q_{k_i}}_{\rm dn} \le m^{t^p_{n_i}}_{\rm dn} < m^{t^q_{k_i}}_{\rm up}$, $m^{t^q_{k_i+1}}_{\rm up} < m^{t^p_{n_{i+1}}}_{\rm dn}$.

Apply the cutting property to choose (for each $i < \omega$) $s'_i, s''_i \in K$ such that

$$\begin{array}{l}\{s'_i, s''_i\} \in \Sigma^\perp(t^q_{k_i}), \quad m^{s'_i}_{\rm dn} = m^{t^q_{k_i}}_{\rm dn}, \quad m^{s''_i}_{\rm dn} = m^{t^p_{n_i}}_{\rm dn}, \quad \text{and}\\ \quad \mathbf{nor}[s'_i], \mathbf{nor}[s''_i] \ge i + 4.\end{array}$$

(If $m^{t^p_{n_i}}_{\rm dn} = m^{t^q_{k_i}}_{\rm dn}$ then the s'_i is not present.) Next use gluing to choose r_i, s_i so that

$$\begin{array}{l}r_0 \in \Sigma(t'_1, t^p_{\ell+1}, \ldots, t^p_{n_0-1}), \ s_0 \in \Sigma(t''_1, t^q_{m+1}, \ldots, t^q_{k_0-1}, s'_0)\\ r_{i+1} \in \Sigma(t^p_{n_i}, \ldots, t^p_{n_{i+1}-1}), \ \ s_{i+1} \in \Sigma(s''_i, t^q_{k_i+1}, \ldots, t^q_{k_{i+1}-1}, s'_{i+1}),\\ \mathbf{nor}[r_i], \mathbf{nor}[s_i] \ge i + 2.\end{array}$$

Since (K, Σ) is linked we may choose $t_i \in \Sigma(r_i) \cap \Sigma(s_i)$ such that $\mathbf{nor}[t_i] \geq i+1$. Now look at $(u, t_0, t_1, \dots)$. It is a condition in $\mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$ stronger than both p and q. □

1(b,c) and 2(a,b) Similarly (and much easier).

3. Let $h \in \omega^\omega$ be a regressive function and let $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ be a countable h–closed family which is $\geq^*$–directed. Suppose that (K, Σ) is a local, h–linked and really finitary creating pair (because of the "local" $\Sigma^\perp$ can be omitted as it is trivial). We are going to show that "being a (countable) pre-dense subset of $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$" is a Borel property.

Let $\mathcal{X} = (\bigcup\limits_{n<\omega} \prod\limits_{i<n} \mathbf{H}(i)) \times K^\omega$, $\mathcal{X}^\omega$ and $\mathcal{Y} = \mathcal{P}(\mathrm{FC}(K, \Sigma))$ be equipped with the natural (product) Polish topologies (note that $\mathrm{FC}(K, \Sigma)$ is a countable set). For $\bar{p} = \langle p_n : n < \omega \rangle \in \mathcal{X}^\omega$, $\bar{p} \subseteq \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$, $w \in \bigcup\limits_{m<\omega} \prod\limits_{i<m} \mathbf{H}(i)$ and $f \in \mathcal{F}$ we define

$$N^{\bar{p}}(n) = \min\{m^{t_i^{p_n}}_{\mathrm{dn}} : (\forall j \geq i)(\mathbf{nor}[t_j^{p_n}] \geq 2)\},$$

and

$$\begin{aligned}\mathcal{T}^{\bar{p}}_{w,f} = \{(w, t_0, \dots, t_k) \in \mathrm{FC}(K, \Sigma) : (\forall i \leq k)(\mathbf{nor}[t_i] \geq f(m^{t_i}_{\mathrm{dn}})) \text{ and} \\ (\forall n < \omega)(N^{\bar{p}}(n) \leq m^{t_k}_{\mathrm{up}} \ \Rightarrow \ \mathrm{pos}(w, t_0, \dots, t_k) \cap \mathrm{POS}(p_n) = \emptyset)\}.\end{aligned}$$

Note that $(\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma))^\omega$ is a Borel subset of $\mathcal{X}^\omega$ and the functions

$$\bar{p} \mapsto N^{\bar{p}} : (\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma))^\omega \longrightarrow \omega^\omega \quad \text{and} \quad \bar{p} \mapsto \mathcal{T}^{\bar{p}}_{w,f} : (\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma))^\omega \longrightarrow \mathcal{Y}$$

are Borel. Now, each $\mathcal{T}^{\bar{p}}_{w,f}$ is essentially a finitary tree, so

$$\mathcal{T}^{\bar{p}}_{w,f} \text{ is well founded} \quad \text{if and only if} \quad \mathcal{T}^{\bar{p}}_{w,f} \text{ is finite.}$$

Consequently, for each w and f, the set

$$\{\bar{p} \in (\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma))^\omega : \mathcal{T}^{\bar{p}}_{w,f} \text{ is well founded}\}$$

is Borel. Since there are countably many possibilities for w and f, we easily finish the proof using the following observation.

Claim 1.3.4.2. *Let $\bar{p} = \langle p_n : n < \omega \rangle \in (\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma))^\omega$. Then*
$\bar{p}$ is pre-dense in $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ if and only if
for each $w \in \bigcup\limits_{m<\omega} \prod\limits_{i<m} \mathbf{H}(i)$ and $f \in \mathcal{F}$ the tree $\mathcal{T}^{\bar{p}}_{w,f}$ is well–founded.

Proof of the claim. Suppose that, for some w and f, the tree $\mathcal{T}^{\bar{p}}_{w,f}$ has an ω–branch and let $q = (w, t_0, t_1, \dots)$ be such a branch. Necessarily $q \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ (as witnessed by f). If follows from the definition of $\mathcal{T}^{\bar{p}}_{w,f}$ that $\mathrm{POS}(q) \cap \mathrm{POS}(p_n)$ is finite for each $n \in \omega$ and therefore $q \perp_{\mathbb{Q}^*_{\mathcal{F}}(K,\Sigma)} p_n$ (remember 1.1.11).

Now assume that $\bar{p}$ is not pre-dense in $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ and let $q \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ be a condition incompatible with all p_n. We may demand that for some $f \in \mathcal{F}$ we have $(\forall i \in \omega)(\mathbf{nor}[t_i^q] \geq f(m^{t_i^q}_{\mathrm{dn}}))$. It should be clear that q determines an ω–branch in the tree $\mathcal{T}^{\bar{p}}_{w^q,f}$ (remember that (K, Σ) is h–linked). □

Similarly we deal with the respective variant of 2(b). □

1.4. Unbounded and dominating reals

Lemma 1.4.1. *Let* $\mathbf{V} \subseteq \mathbf{V}^*$ *be universes of* ZFC*. *Assume that* $\langle f_i : i < \omega \rangle \in \mathbf{V}$ *and* $g \in \mathbf{V}^*$ *are such that*

(a) $g \in \omega^\omega$, $f_i \in \omega^\omega$, $f_{i+1} <^* f_i$ *for all* $i \in \omega$,
(b) $(\forall i \in \omega)(\exists^\infty k \in \omega)(g(k) < f_i(k))$,
(c) if $h \in \omega^\omega \cap \mathbf{V}$ *is such that* $(\forall i \in \omega)(h <^* f_i)$, *then* $h <^* g$.

Then $\omega^\omega \cap \mathbf{V}$ *is bounded in* $\mathbf{V}^*$.

Proof. It follows from the assumptions (a), (b) that we may find an infinite set $K = \{k_0, k_1, k_2, \dots\} \in \mathbf{V}^* \cap [\omega]^\omega$ such that for each $i \in \omega$ we have

$(*)$ $f_0(k_i) > f_1(k_i) > \cdots > f_i(k_i) > g(k_i)$.

Let $\varphi \in \omega^\omega \cap \mathbf{V}^*$ be such that $(\forall n \in \omega)(|K \cap (n, \varphi(n))| > 2^{n+1})$. We claim that the function φ dominates $\omega^\omega \cap \mathbf{V}$, i.e.

$$(\forall f \in \omega^\omega \cap \mathbf{V})(\forall^\infty n \in \omega)(f(n) < \varphi(n)).$$

If not, then we may choose an increasing sequence $\langle n_i : i < \omega \rangle \in \mathbf{V}$ of integers such that $n_0 = 0$ and

(i) $(\exists^\infty i \in \omega)(|(n_i, n_{i+1}) \cap K| > 2^{n_i})$,
(ii) $(\forall i \in \omega)(\forall n \geq n_{i+1})(f_{i+1}(n) < f_i(n))$.

Define $h \in \omega^\omega \cap \mathbf{V}$ by $h \restriction [n_i, n_{i+1}) = f_i \restriction [n_i, n_{i+1})$ (for $i \in \omega$). It follows from (ii) that $h <^* f_i$ for each $i \in \omega$, so we may apply the assumption (c) to conclude that $h <^* g$. But look at the clauses $(*)$ and (i) above. Whenever $|(n_i, n_{i+1}) \cap K| > 2^{n_i}$, there is $\ell \in \omega$ such that $k_\ell \in (n_i, n_{i+1})$, $\ell > i$ and

$$f_0(k_\ell) > \cdots > f_i(k_\ell) = h(k_\ell) > \cdots > f_\ell(k_\ell) > g(k_\ell),$$

so easily we get a contradiction. $\square$

Definition 1.4.2. *Let* (K, Σ) *be a creating pair or a (local) tree–creating pair.*

(1) (See [15, Def. 5.1.6]) We say that (K, Σ) *is* reducible *if for each* $t \in K$ *with* $\mathbf{nor}[t] \geq 3$, *there is* $s \in \Sigma(t)$ *such that* $\frac{\mathbf{nor}[t]}{2} \leq \mathbf{nor}[s] \leq \mathbf{nor}[t] - 1$.
(2) The pair (K, Σ) *is* normal *if it is reducible, linked and if*
$(\boxplus)$ *for each* $s, t \in K$:

$$\mathbf{nor}[s] < \mathbf{nor}[t] \quad \Rightarrow \quad (\forall u \in \mathrm{dom}(\mathbf{val}[t]))(\exists v)(\langle u, v \rangle \in \mathbf{val}[t] \setminus \mathbf{val}[s]).$$

(3) A creating pair (K, Σ) *is* semi–normal *if it is linked, and for each* $n \in \omega$ *and* $t \in K$ *such that* $\mathbf{nor}[t] > 2^{n+2}$, $m^t_{\mathrm{up}} - m^t_{\mathrm{dn}} > 2^{2^{n+4}}$, *there is a sequence* $\langle s_\ell : \ell \leq n \rangle \subseteq K$ *satisfying*
(α) $s_0 = t$, $s_{\ell+1} \in \Sigma(s_\ell)$, $2^{n+1-\ell} < \mathbf{nor}[s_{\ell+1}] \leq 2^{n-\ell+2}$ *(for* $\ell < n$*), and*

(β) *if* $s \in K$, $m^s_{\rm dn} = m^t_{\rm dn}$, $m^s_{\rm up} = m^t_{\rm up}$ *and* $\mathbf{nor}[s] > 2^{n-\ell+3}$, $\ell < n$, *then*

$$(\forall u \in \mathrm{dom}(\mathbf{val}[s]))(\exists v)(\langle u, v\rangle \in \mathbf{val}[s] \setminus \mathbf{val}[s_{\ell+1}]).$$

(4) A $\otimes$*–creating triple* $(K, \Sigma, \Sigma^\perp)$ *(or just* (K, Σ)*) is* super–gluing *if it is gluing and for every* $s_0, \ldots, s_k \in K$ *and* $N \in \omega$ *such that* $m^{s_i}_{\rm up} \le m^{s_{i+1}}_{\rm dn}$ *for* $i < k$ *and* $m^{s_k}_{\rm up} \le N$*, there is* $s \in K$ *satisfying:*

- $m^s_{\rm dn} = m^{s_0}_{\rm dn}$, $m^s_{\rm up} = N$, $\mathrm{dom}(\mathbf{val}[s]) = \mathrm{dom}(\mathbf{val}[s_0])$*, and*
- $\mathbf{nor}[s] \ge \min\{\mathbf{nor}[s_i] : i \le k\} - 1$*, and*
- $(\forall \langle u, v\rangle \in \mathbf{val}[s])(\forall i \le k)(\langle v{\restriction}m^{s_i}_{\rm dn}, v{\restriction}m^{s_i}_{\rm up}\rangle \in \mathbf{val}[s_i])$.

Remark 1.4.3. Note that "normal" implies "semi–normal". What we really need in the proofs of 1.4.4(1,2) is semi–normality (or rather a suitable variant of it). However, the normality is more natural and only in the case of $\otimes$–creating triples (which are gluing and have the cutting property) the natural norms are semi–normal but not normal; see **Examples** at the end of this section.

Theorem 1.4.4. *(1) Let* $f : \omega \times \omega \longrightarrow \omega$ *be a fast function, and let* (K, Σ) *be a local creating pair (a local tree–creating pair, respectively). Assume that* (K, Σ) *is normal and* $\mathbb{Q}^*_f(K, \Sigma) \ne \emptyset$ *(*$\mathbb{Q}^{\rm tree}_f(K, \Sigma) \ne \emptyset$*, resp.). Then the forcing notion* $\mathbb{Q}^*_f(K, \Sigma)$ *(*$\mathbb{Q}^{\rm tree}_f(K, \Sigma)$*, resp.) adds a dominating real.*

(2) If (K, Σ) *is a normal (local) tree–creating pair and* $\mathbb{Q}^{\rm tree}_1(K, \Sigma) \ne \emptyset$*, then the forcing notion* $\mathbb{Q}^{\rm tree}_1(K, \Sigma) \ne \emptyset$ *adds a dominating real.*

(3) Assume that $(K, \Sigma, \Sigma^\perp)$ *is a semi–normal* $\otimes$*–creating triple which is super–gluing and has the cutting property (and* $\mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp) \ne \emptyset$*). Then the forcing notion* $\mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp)$ *adds a dominating real.*

Proof. In all cases we will use Lemma 1.4.1 for functions $f_i \in \prod\limits_{n\in\omega}(n+1)$ defined by $f_i{\restriction}[0, i) \equiv 0$, $f_i(n) = n - i$ for $n \ge i$ (for $i \in \omega$) and a suitably chosen name $\dot g$ for a function in $\prod\limits_{n\in\omega}(n+1)$.

(1) We consider the case when (K, Σ) is a local creating pair only.

Let $p \in \mathbb{Q}^*_f(K, \Sigma)$. Using the normality of (K, Σ), choose an increasing sequence $\langle m_n : n < \omega\rangle \subseteq \omega$ and a sequence $\langle s^\ell_n : \ell \le n,\ n < \omega\rangle \subseteq K$ such that for each $n \in \omega$ and $\ell < n$:

(a) $s^0_n = t^p_{m_n}$, $\mathbf{nor}[t^p_{m_n}] > f(n+2, m^{t^p_{m_n}}_{\rm dn})$,

(b) $s^{\ell+1}_n \in \Sigma(s^\ell_n)$, $f(n-\ell+1, m^{s^{\ell+1}_n}_{\rm dn}) < \mathbf{nor}[s^{\ell+1}_n] \le f(n-\ell+2, m^{s^{\ell+1}_n}_{\rm dn})$.

Let $\dot W$ be the name for $\mathbb{Q}^*_f(K, \Sigma)$–generic real, i.e.

$$\Vdash_{\mathbb{Q}^*_f(K,\Sigma)} \dot W = \bigcup\{w^q : q \in \Gamma_{\mathbb{Q}^*_f(K,\Sigma)}\}$$

(see [15, Def. 1.1.13, Prop. 1.1.14]). Let $\dot g$ be a $\mathbb{Q}^*_f(K, \Sigma)$–name for a function in $\prod\limits_{n\in\omega}(n+1)$ defined by

$$p \Vdash \text{“ } (\forall n \in \omega)(\forall \ell \le n)(\dot g(n) = \ell \Leftrightarrow \langle \dot W{\restriction}m^{s^\ell_n}_{\rm dn}, \dot W{\restriction}m^{s^\ell_n}_{\rm up}\rangle \in \mathbf{val}[s^\ell_n] \setminus \mathbf{val}[s^{\ell+1}_n]) \text{ ”}$$

(if $\langle \dot W{\restriction}m^{s^n_n}_{\rm dn}, \dot W{\restriction}m^{s^n_n}_{\rm up}\rangle \in \mathbf{val}[s^n_n]$ then $\dot g(n) = n$).

Claim 1.4.4.1. (α) $p \Vdash_{\mathbb{Q}^*_f(K,\Sigma)} (\forall i \in \omega)(\exists^\infty n \in \omega)(\dot g(n) < f_i(n))$,
(β) *Assume that $h \in \omega^\omega$ is such that $h <^* f_i$ for all $i \in \omega$. Then $p \Vdash$ " $h <^* \dot g$ ".*

Proof of the claim. (α) Let $i \in \omega$ and let $p_0 \geq p$, $N \in \omega$. Take $n > N+i$ so that for some $k < \omega$ we have $t^{p_0}_k \in \Sigma(t^p_{m_n})$ and $\mathbf{nor}[t^{p_0}_k] > f(i+3, m^{t^{p_0}_k}_{\mathrm{dn}})$. Since (K,Σ) is normal, we may find $w \in \mathrm{pos}(w^{p_0}, t^{p_0}_0, \ldots, t^{p_0}_k)$ such that $\langle w{\restriction}m^{t^{p_0}_k}_{\mathrm{dn}}, w{\restriction}m^{t^{p_0}_k}_{\mathrm{up}}\rangle \notin \mathbf{val}[s^{n-i}_n]$ (remember $\mathbf{nor}[s^{n-i}_n] \leq f(i+3, m^{s^{n-i}_n}_{\mathrm{dn}})$). Clearly the condition $q = (w, t^{p_0}_{k+1}, t^{p_0}_{k+2}, \ldots)$ forces that $\dot g(n) < n - i = f_i(n)$.

(β) Let $h \in \omega^\omega$ be such that $h <^* f_i$ for all $i \in \omega$ and let $p_0 \geq p$. Let ℓ be such that $t^{p_0}_0 \in \Sigma(t^p_\ell)$ (so $t^{p_0}_k \in \Sigma(t^p_{k+\ell})$ for each k). We may assume that if $m_n \geq \ell$ then $h(n) < n-5$ and that $(\forall k < \omega)(\mathbf{nor}[t^{p_0}_k] > f(5, m^{t^{p_0}_k}_{\mathrm{dn}}))$. For each $k \in \omega$ choose t_k as follows:

- if $k + \ell \notin \{m_n : n \in \omega\}$, then $t_k = t^{p_0}_k$,
- if $k + \ell = m_n$, $n \in \omega$, then $t_k \in \Sigma(t^{p_0}_k) \cap \Sigma(s^{h(n)+1}_n)$ is such that $\mathbf{nor}[t_k] \geq \min\{\mathbf{nor}[t^{p_0}_k], \mathbf{nor}[s^{h(n)+1}_n]\} - 1$

(remember that (K,Σ) is linked). Since $\mathbf{nor}[s^{h(n)+1}_n] > f(n - h(n) + 1, m^{s^0_n}_{\mathrm{dn}})$ we easily see that $q = (w^{p_0}, t_0, t_1, \ldots) \in \mathbb{Q}^*_f(K,\Sigma)$, and clearly it is a condition stronger than p_0. As $q \Vdash_{\mathbb{Q}^*_f(K,\Sigma)} (\forall n \in \omega)(m_n \geq \ell \Rightarrow \dot g(n) > h(n))$, the claim follows. $\square$

Now, the first clause of the theorem is an immediate consequence of 1.4.4.1 and 1.4.1.

(2) The proof is similar to the one above. Let $p \in \mathbb{Q}^{\mathrm{tree}}_1(K,\Sigma)$. Choose fronts F_n of T^p (for $n \in \omega$) such that for each n:

- $(\forall \eta \in F_{n+1})(\exists \nu \in F_n)(\nu \lhd \eta)$,
- $(\forall \eta \in F_n)(\mathbf{nor}[t^p_\eta] > 2^{n+2})$

(clearly possible; see [15, Prop. 1.3.8]). For each $n \in \omega$ and $\eta \in F_n$ choose a sequence $\langle s^\ell_\eta : \ell \leq n\rangle \subseteq K$ such that

$$s^0_\eta = t^p_\eta, \quad s^{\ell+1}_\eta \in \Sigma(s^\ell_\eta) \quad \text{and} \quad 2^{n-\ell+1} < \mathbf{nor}[s^{\ell+1}_\eta] \leq 2^{n-\ell+2}.$$

Let $\dot W$ be the name for the $\mathbb{Q}^{\mathrm{tree}}_1(K,\Sigma)$–generic real and let $\dot g$ be a $\mathbb{Q}^{\mathrm{tree}}_1(K,\Sigma)$–name for a real in $\prod\limits_{n\in\omega}(n+1)$ such that (the condition p forces that) if $\eta = \dot W{\restriction}m \in F_n$ (for some $m, n \in \omega$) and $\dot W{\restriction}(m+1) \in \mathrm{pos}(s^\ell_\eta) \setminus \mathrm{pos}(s^{\ell+1}_\eta)$, then $\dot g(n) = \ell$.

Claim 1.4.4.2. (α) $p \Vdash_{\mathbb{Q}^{\mathrm{tree}}_1(K,\Sigma)} (\forall i \in \omega)(\exists^\infty n \in \omega)(\dot g(n) < f_i(n))$,
(β) *If $h \in \omega^\omega$ is such that $h <^* f_i$ for all $i \in \omega$, then $p \Vdash_{\mathbb{Q}^{\mathrm{tree}}_1(K,\Sigma)}$ " $h <^* \dot g$ ".*

Proof of the claim. (α) Like 1.4.4.1(α).

(β) Let $q \geq p$. We may assume that for some $m > 2$ we have: $\mathrm{root}(q) \in F_m$, $\mathbf{nor}[t^q_\nu] > 8$ for all $\nu \in T^q$, and $h(n) < n-5$ for all $n \geq m$. We build inductively a tree $T \subseteq T^q$ and a system $\langle t_\eta : \eta \in T\rangle$ as follows. We declare that $\mathrm{root}(q) = \mathrm{root}(T) \in T$. Suppose we have declared that $\eta \in T$. If $\eta \notin \bigcup_{n\in\omega} F_n$, then we let $t_\eta = t^q_\eta$ and we declare $\mathrm{pos}(t_\eta) \subseteq T$. If $\eta \in F_n$ for some $n \geq m$, then we choose $t_\eta \in \Sigma(t^q_\eta) \cap \Sigma(s^{h(n)+1}_\eta)$ such that $\mathbf{nor}[t_\eta] \geq \min\{\mathbf{nor}[t^q_\eta], 2^{n-h(n)+1}\} - 1$, and we declare $\mathrm{pos}(t_\eta) \subseteq T$.

Finally, we let $q^* = \langle t_\eta : \eta \in T\rangle$ and we notice that $q^* \Vdash h <^* \dot{g}$. $\square$

(3) Let $p \in \mathbb{Q}^*_\infty(K,\Sigma,\Sigma^\perp)$. By "gluing", we may assume that $m^{t^p_k}_{\mathrm{up}} - m^{t^p_k}_{\mathrm{dn}} > 2^{2^{k+4}}$ for each $k < \omega$. Using semi–normality we may choose an increasing sequence $\langle m_n : n < \omega\rangle$ and a sequence $\langle s^\ell_n : \ell \leq n,\ n < \omega\rangle \subseteq K$ such that

(a) $s^0_n = t^p_{m_n}$, $\mathbf{nor}[t^p_{m_n}] > 2^{n+2}$, $s^{\ell+1}_n \in \Sigma(s^\ell_n)$, $2^{n-\ell+1} < \mathbf{nor}[s^{\ell+1}_n] \leq 2^{n-\ell+2}$ (for $\ell < n < \omega$),

(b) if $t \in K$, $m^t_{\mathrm{dn}} = m^{s^0_n}_{\mathrm{dn}}$, $m^t_{\mathrm{up}} = m^{s^0_n}_{\mathrm{up}}$, and $\mathbf{nor}[t] > 2^{n-\ell+3}$, $\ell < n < \omega$, then

$$(\forall u \in \mathrm{dom}(\mathbf{val}[t]))(\exists v)(\langle u, v\rangle \in \mathbf{val}[t] \setminus \mathbf{val}[s^{\ell+1}_n]).$$

Now we define the name $\dot{g}$ like before, so for $n \in \omega$ and $\ell < n$,

$$p \Vdash_{\mathbb{Q}^*_\infty(K,\Sigma,\Sigma^\perp)} \text{“}\ \dot{g}(n) = \ell \ \Leftrightarrow\ \langle \dot{W}{\restriction}m^{s^\ell_n}_{\mathrm{dn}}, \dot{W}{\restriction}m^{s^\ell_n}_{\mathrm{up}}\rangle \in \mathbf{val}[s^\ell_n] \setminus \mathbf{val}[s^{\ell+1}_n]\ \text{”}.$$

Claim 1.4.4.3. (α) $p \Vdash_{\mathbb{Q}^*_\infty(K,\Sigma,\Sigma^\perp)} (\forall i \in \omega)(\exists^\infty n \in \omega)(\dot{g}(n) < f_i(n))$,
(β) *If $h <^* f_i$ for all $i \in \omega$, then $p \Vdash_{\mathbb{Q}^*_\infty(K,\Sigma,\Sigma^\perp)}$ “ $h <^* \dot{g}$ ”.*

Proof of the claim. (α) Suppose $q \geq p$, and $i, N < \omega$. Passing to a stronger condition (using "gluing and cutting") we may assume that

- $\mathbf{nor}[t^q_0] > 2^{i+4}$ and
- $m^{t^q_0}_{\mathrm{dn}} = m^{t^p_{m_n}}_{\mathrm{dn}}$, $m^{t^q_0}_{\mathrm{up}} = m^{t^p_{m_n}}_{\mathrm{up}}$ for some $n > N + i + 1$.

Choose $w \in \mathrm{pos}(w^q, t^q_0)$ such that $\langle w^q, w\rangle \notin s^{n-i}_n$ and look at the condition $q' = (w, t^q_1, t^q_2, \dots)$.

(β) Let $q \geq p$. Passing to a stronger condition if necessary, we may assume that for some increasing sequence $\langle N_k : k < \omega\rangle \subseteq \omega$ we have:

- $m^{t^q_k}_{\mathrm{dn}} = m^{t^p_{m_{N_k}}}_{\mathrm{dn}}$, $m^{t^q_k}_{\mathrm{up}} = m^{t^p_{m_{N_{k+1}}}}_{\mathrm{dn}}$, $\mathbf{nor}[t^q_k] > 5$ for all $k < \omega$,
- if $n \geq N_0$ then $h(n) < n - 5$.

Using "super–gluing" choose creatures $s_k \in K$ (for $k \in \omega$) such that

- $m^{s_k}_{\mathrm{dn}} = m^{t^p_{m_{N_k}}}_{\mathrm{dn}}$, $m^{s_k}_{\mathrm{up}} = m^{t^p_{m_{N_{k+1}}}}_{\mathrm{dn}}$, and
- $\mathbf{nor}[s_k] \geq \min\{\mathbf{nor}[s^{h(n)+1}_n] : N_k \leq n < N_{k+1}\} - 1$, and
- $(\forall \langle u, v\rangle \in \mathbf{val}[s_k])(\forall n \in [N_k, N_{k+1}))(\langle v{\restriction}m^{s^0_n}_{\mathrm{dn}}, v{\restriction}m^{s^0_n}_{\mathrm{up}}\rangle \in \mathbf{val}[s^{h(n)+1}_n])$.

Apply "linked" to choose creatures $t_k \in \Sigma(s_k) \cap \Sigma(t^q_k)$ such that

$$1 + \mathbf{nor}[t_k] \geq \min\{\mathbf{nor}[s_k], \mathbf{nor}[t^q_k]\} \geq \min\{\mathbf{nor}[t^q_k], 2^{n-h(n)} : N_k \leq n < N_{k+1}\}.$$

Then $q^* = (w^q, s_0, s_1, s_2, \dots)$ is a condition in $\mathbb{Q}^*_\infty(K,\Sigma,\Sigma^\perp)$, $q^* \geq q$ and it forces that $(\forall n \geq N_0)(h(n) < \dot{g}(n))$. $\square$

$\square$

Remark 1.4.5. Note that 1.4.4(1) applies to forcing notions $\mathbb{Q}^*_\infty(K,\Sigma)$ too, see 1.1.7.

Definition 1.4.6. *A ccc forcing notion $\mathbb{P}$ is* nice *if there is a partition $\langle P_m : m<\omega\rangle$ of $\mathbb{P}$ such that*

($\bigstar$) *if $\langle p_n : n<\omega\rangle \subseteq \mathbb{P}$ is a maximal antichain in $\mathbb{P}$, $m\in\omega$, then there is $N<\omega$ such that*

$$(\forall p\in P_m)(\exists n<N)(p, p_n \textit{ are compatible }).$$

Theorem 1.4.7 (Miller [12], Brendle and Judah [2]; see [1, Thm. 6.5.11]). *If $\mathcal{F}\subseteq\omega^\omega$ is an unbounded family and $\mathbb{P}$ is a nice ccc forcing notion, then*

$$\Vdash_{\mathbb{P}} \text{“ the family } \mathcal{F} \text{ is unbounded ”.}$$

Remark 1.4.8. Since no dominating reals can be added at limit stages of FS iterations of ccc forcing notions (see [27, Con. VI.3.17]), it follows from 1.4.7 that FS iterations of nice forcing notions do not add dominating reals.

Definition 1.4.9. *(1) Let (K,Σ) be a local creating pair (or a local tree–creating pair) for $\mathbf{H}$. We say that (K,Σ) is* Cohen–producing *if for each $n\in\omega$ there is a set $A_n\subseteq\mathbf{H}(n)$ such that*

if $t\in K$, $\mathbf{nor}[t]>1$, $u\in\mathrm{dom}(\mathbf{val}[t])$, $\mathrm{lh}(u)=n$,
then there are v_0, v_1 such that $\langle u,v_0\rangle,\langle u,v_1\rangle\in\mathbf{val}[t]$ and $v_1(n)\in A_n$ and $v_0(n)\notin A_n$.

(2) A creating pair (K,Σ) is of the BCB–type *if it is local, forgetful and satisfies the following condition:*

($\circledast^{\mathrm{BCB}}$) *for every sequence $\langle s_n : n<\omega\rangle$ of creatures from K with $m^{s_n}_{\mathrm{dn}}=i$, $\mathbf{nor}[s_n]\geq 2$ (for all n), there are $a_0,\ldots,a_m\in\mathbf{H}(i)$ and an increasing sequence $\langle n_k : k<\omega\rangle\subseteq\omega$ such that*

$$(\forall a\in\mathbf{H}(i)\setminus\{a_0,\ldots,a_m\})(\forall^\infty k\in\omega)$$
$$(\forall u\in\mathrm{dom}(\mathbf{val}[s_{n_k}]))(\langle u, u^\frown\langle a\rangle\rangle\in\mathbf{val}[s_{n_k}]).$$

(3) A local tree–creating pair (K,Σ) is of the BCB$^{\mathrm{tree}}$–type *if*

($\circledast^{\mathrm{BCB}}_{\mathrm{tree}}$) *for every $\eta\in\bigcup\limits_{n<\omega}\prod\limits_{i<n}\mathbf{H}(i)$ and a sequence $\langle s_n : n<\omega\rangle\subseteq\mathrm{LTCR}_\eta\cap K$ such that $\mathbf{nor}[s_n]\geq 2$, there are $a_0,\ldots,a_m\in\mathbf{H}(\mathrm{lh}(\eta))$ and an increasing sequence $\langle n_k : k<\omega\rangle\subseteq\omega$ such that*

$$(\forall a\in\mathbf{H}(\mathrm{lh}(\eta))\setminus\{a_0,\ldots,a_m\})(\forall^\infty k\in\omega)(\eta^\frown\langle a\rangle\in\mathrm{pos}(s_{n_k})).$$

Remark 1.4.10. (1) Note that if $\mathbf{H}(i)$ is finite for each $i\in\omega$ then any local forgetful creating pair (local tree creating pair) is of the BCB–type (BCB$^{\mathrm{tree}}$–type, respectively).

(2) The difference between BCB and BCB$^{\mathrm{tree}}$ is not serious, the two notions are just fitted to their contexts.

(3) "BCB" stands for "bounded – co-bounded". The "bounded" part reflects what is stated in (1) above, and the "co-bounded" is supposed to point out the analogy to the co-bounded topology on ω in the case when each $\mathbf{H}(i)$ is infinite; compare Miller [12].

Theorem 1.4.11. *Assume that $h : \omega \times \omega \longrightarrow \omega$ is a regressive function and $\mathcal{F} \subseteq (\omega \setminus 2)^{\omega}$ is a countable h–closed family which is $\geq^*$–directed.*

(1) If (K, Σ) is a local Cohen–producing h–linked creating pair (tree–creating pair, respectively), then the forcing notion $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ ($\mathbb{Q}^{\rm tree}_{\mathcal{F}}(K, \Sigma)$, resp.) adds a Cohen real.

(2) If (K, Σ) is an h–linked tree–creating pair of the $\mathrm{BCB}^{\rm tree}$*–type, then the forcing notion $\mathbb{Q}^{\rm tree}_{\mathcal{F}}(K, \Sigma)$ is nice.*

(3) If a creating pair (K, Σ) is h–linked and of the BCB*–type, then the forcing notion $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ is nice.*

Proof. (1) Let (K, Σ) be a creating pair for $\mathbf{H}$ and let sets $A_n \subseteq \mathbf{H}(n)$ (for $n \in \omega$) witness that it is Cohen–producing. Let $\dot{c}$ be a $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$–name for a real in 2^{ω} defined by

$$\Vdash_{\mathbb{Q}^*_{\mathcal{F}}(K,\Sigma)} (\forall n \in \omega)(\dot{c}(n) = 1 \ \Leftrightarrow\ \dot{W}(n) \in A_n)$$

(where $\dot{W}$ is the name for the $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$–generic real). Suppose that a condition $p \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ is such that $\mathbf{nor}[t^p_n] > 1$ for all $n \in \omega$. Let $\sigma : [\mathrm{lh}(w^p), N] \longrightarrow 2$, $\mathrm{lh}(w^p) \leq N < \omega$. It should be clear that there is $w \in \mathrm{pos}(w^p, t^p_0, \ldots, t^p_{N-\mathrm{lh}(w^p)})$ such that

$$(\forall n \in [\mathrm{lh}(w^p), N])(w(n) \in A_n \ \Leftrightarrow\ \sigma(n) = 1).$$

Hence easily $\dot{c}$ is a name for a Cohen real.

(2) Let $\nu \in \bigcup\limits_{n\in\omega} \prod\limits_{i<n} \mathbf{H}(i)$, $f \in \mathcal{F}$ and let

$$P_{\nu,f} \stackrel{\rm def}{=} \{p \in \mathbb{Q}^{\rm tree}_{\mathcal{F}}(K, \Sigma) : \mathrm{root}(p) = \nu \ \&\ (\forall \eta \in T^p)(\mathbf{nor}[t^p_\eta] \geq f(\mathrm{lh}(\eta)))\}.$$

Suppose that $\langle p_n : n < \omega\rangle \subseteq \mathbb{Q}^{\rm tree}_{\mathcal{F}}(K, \Sigma)$ is a maximal antichain such that, for each $n \in \omega$ and $\eta \in T^{p_n}$, we have $\mathbf{nor}[t^{p_n}_\eta] \geq 2$.

Claim 1.4.11.1. *There is $N < \omega$ such that*

$$(\forall q \in P_{\nu,f})(\exists n < N)(q, p_n \textit{ are compatible }).$$

Proof of the claim. Assume not. Then we may choose a sequence $\langle q_k : k < \omega\rangle \subseteq P_{\nu,f}$ such that for each $n < k < \omega$ the conditions q_k and p_n are incompatible. We inductively build a tree T and a system $\langle s_\eta : \eta \in T\rangle$ together with a sequence $\langle X_n, Y_n : n < \omega\rangle$ so that

(α) $X_n \subseteq \bigcup\limits_{m\in\omega} \prod\limits_{i<m} \mathbf{H}(i)$, $Y_{n+1} \subseteq Y_n \in [\omega]^{\omega}$,

(β) $(\forall\eta\in X_n)(\forall^\infty k\in Y_n)(\eta\in T^{q_k})$,
(γ) $\mathbf{nor}[s_\eta]\geq f(\mathrm{lh}(\eta))$, $T=\bigcup\limits_{n\in\omega}X_n$.

Fix a bijection $\#:\bigcup\limits_{m\in\omega}\prod\limits_{i<m}\mathbf{H}(i)\longrightarrow\omega$ such that $\eta_0\lhd\eta_1$ implies $\#(\eta_0)<\#(\eta_1)$.

We declare that $\nu=\mathrm{root}(T)$, $X_0=\{\nu\}$, $Y_0=\omega$.

Suppose we have arrived to the $(n+1)^{\mathrm{th}}$ stage of the construction and X_n, Y_n have been already defined so that the clauses (α), (β) above are satisfied. Let $\eta\in X_n$ be such that

$$\#(\eta)=\min\{\#(\eta'):\eta'\in X_n\}.$$

Let $Y_n'\in[\omega]^\omega$ consist of these $k\in Y_n$ that $\eta\in T^{q_k}$ (remember (β)). Apply $(\circledast^{\mathrm{BCB}}_{\mathrm{tree}})$ of 1.4.9(3) to the sequence $\langle t^{q_k}_\eta:k\in Y_n'\rangle$ to choose an infinite set $Y_{n+1}\subseteq Y_n'$ such that, letting $k^*=\min(Y_{n+1})$ and $s_\eta=t^{q_{k^*}}_\eta$ we have

$$(\forall\eta'\in\mathrm{pos}(s_\eta))(\forall^\infty k\in Y_{n+1})(\eta'\in\mathrm{pos}(t^{q_k}_\eta)).$$

Finally, we let $X_{n+1}=(X_n\setminus\{\eta\})\cup\mathrm{pos}(s_\eta)$. This finishes the description of the inductive step.

After the construction is carried out we let $q^*=\langle s_\eta:\eta\in T\rangle$. It should be clear that $q^*\in\mathbb{Q}^{\mathrm{tree}}_{\mathcal{F}}(K,\Sigma)$ (and even $q^*\in P_{\nu,f}$). Consequently we find $n<\omega$ such that the conditions p_n and q^* are compatible.

Suppose that $\nu\trianglelefteq\mathrm{root}(p_n)$. Then necessarily $\mathrm{root}(p_n)\in T$. It follows from our construction (remember clause (β)) that we may find $k>n$ such that $\mathrm{root}(p_n)\in T^{q_k}$. But then, using the assumption that (K,Σ) is h–linked and $\mathcal{F}$ is h–closed and $\geq^*$–directed, we immediately get that the conditions p_n,q_k are compatible, contradicting the choice of the q_k. Similarly, if $\mathrm{root}(p_n)\lhd\nu$ then taking any $k>n$ we get that the conditions q_k, p_n are compatible, again a contradiction. □

Since the conditions of the form used above are dense in $\mathbb{Q}^{\mathrm{tree}}_{\mathcal{F}}(K,\Sigma)$ one easily concludes that the forcing notion $\mathbb{Q}^{\mathrm{tree}}_{\mathcal{F}}(K,\Sigma)$ is nice.

(3) Similarly.

□

1.5. Examples

Our first example recalls the forcing notion of [16, §3]. Let us start with presenting the main tool for this type of constructions – norms determined by Hall's Marriage Theorem.

Definition 1.5.1. *Let* $\mathbf{H}:\omega\longrightarrow\mathcal{H}(\omega_1)$.

(1) Let $\mathcal{K}^{\mathbf{H}}$ *be the collection of all finite non-empty families* Δ *of finite partial functions* f *such that* $\emptyset\neq\mathrm{dom}(f)\subseteq\omega$ *and* $f(n)\in\mathbf{H}(n)$ *for all* $n\in\mathrm{dom}(f)$. *For integers* $m_0<m_1$ *let*

$$\mathcal{K}^{\mathbf{H}}_{m_0,m_1}\stackrel{\mathrm{def}}{=}\{\Delta\in\mathcal{K}^{\mathbf{H}}:(\forall f\in\Delta)(\mathrm{dom}(f)\subseteq[m_0,m_1))\}.$$

(2) Let $\Delta \in \mathcal{K}^{\mathbf{H}}$, $k \in \omega$. *A function* $F : \Delta \longrightarrow [\omega]^k$ *is* a k–selector for Δ *if*

$$(\forall f, f' \in \Delta)\big(F(f) \subseteq \mathrm{dom}(f) \ \textit{and} \ f \neq f' \ \Rightarrow F(f) \cap F(f') = \emptyset\big).$$

(3) For $\Delta_0, \Delta_1 \in \mathcal{K}^{\mathbf{H}}$ *we write* $\Delta_0 \preceq \Delta_1$ *whenever*

$$(\forall f \in \Delta_0)(\exists g \in \Delta_1)(g \subseteq f).$$

(4) We define the Hall norms of a set $\Delta \in \mathcal{K}^{\mathbf{H}}$ *as follows:*

$$\begin{aligned}
\mathbf{hn}^+(\Delta) &\stackrel{\mathrm{def}}{=} \max\{k+1 : k \in \omega \textit{ and there is an } k\textit{–selector for } \Delta\},\\
\mathbf{hn}(\Delta) &\stackrel{\mathrm{def}}{=} \max\{k+1 : k \in \omega \textit{ and for every } \Delta' \subseteq \Delta \textit{ there is } \Delta'' \subseteq \Delta'\\
&\qquad\qquad \textit{such that elements of } \Delta'' \textit{ have pairwise disjoint}\\
&\qquad\qquad \textit{domains and } |\bigcup_{f\in\Delta''} \mathrm{dom}(f)| \geq k \cdot |\Delta'|\},\\
\mathbf{HN}(\Delta) &\stackrel{\mathrm{def}}{=} \max\{\mathbf{hn}(\Delta') : \Delta \preceq \Delta'\}.
\end{aligned}$$

Lemma 1.5.2. *(1) If* $\Delta \in \mathcal{K}^{\mathbf{H}}$ *and* $k_0 \in \omega$ *then*

$$\mathbf{hn}^+(\Delta) > k_0 \ \textit{ if and only if } \ (\forall \Delta' \subseteq \Delta)(|\bigcup\{\mathrm{dom}(f) : f \in \Delta'\}| \geq k_0 \cdot |\Delta'|)$$

and $1 \leq \mathbf{hn}(\Delta) \leq \mathbf{hn}^+(\Delta) \leq \mathbf{HN}(\Delta)$.
(2) If $\Delta_0, \Delta_1 \in \mathcal{K}^{\mathbf{H}}$ *and* $\mathbf{xx} \in \{\mathbf{hn}, \mathbf{hn}^+, \mathbf{HN}\}$ *then*

$$\mathbf{xx}(\Delta_0 \cup \Delta_1) \geq \lfloor \min\{\frac{\mathbf{xx}(\Delta_0)}{2}, \frac{\mathbf{xx}(\Delta_1)}{2}\}\rfloor.$$

(3) If $m^0_0 < m^0_1 \leq m^1_0 < m^1_1 \leq \cdots \leq m^k_0 < m^k_1 < \omega$, $\Delta_i \in \mathcal{K}^{\mathbf{H}}_{m^i_0, m^i_1}$ *(for* $i \leq k$*) and* $\mathbf{xx} \in \{\mathbf{hn}, \mathbf{hn}^+, \mathbf{HN}\}$ *then*

$$\mathbf{xx}(\bigcup_{i\leq k} \Delta_i) = \min\{\mathbf{xx}(\Delta_i) : i \leq k\}.$$

(4) Suppose that $m_0 < m < m_1 < \omega$ *and* $\Delta \in \mathcal{K}^{\mathbf{H}}_{m_0,m_1}$. *Let*

$$\begin{aligned}
\Delta_0 &= \{f{\restriction}[m_0, m) : f \in \Delta \ \&\ |\mathrm{dom}(f) \cap [m_0, m)| \geq \tfrac{1}{2}|\mathrm{dom}(f)|\},\\
\Delta_1 &= \{f{\restriction}[m, m_1) : f \in \Delta \ \&\ |\mathrm{dom}(f) \cap [m, m_1)| \geq \tfrac{1}{2}|\mathrm{dom}(f)|\}.
\end{aligned}$$

Then, for $i < 2$, *either* $\Delta_i = \emptyset$ *or* $\mathbf{hn}(\Delta_i) \geq \frac{1}{2}\mathbf{hn}(\Delta)$.

Proof. (1) It follows from Hall's Theorem (see Hall [4]) and the definitions of the norms.
(2)–(4) Straightforward (compare [16, Claim 3.1.2]). □

Example 1.5.3. Let $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$, $|\mathbf{H}(n)| \geq 2$ for all $n \in \omega$.
We construct a $\otimes$–creating triple $(K_{\mathbf{H}}, \Sigma_{\mathbf{H}}, \Sigma^{\perp}_{\mathbf{H}})$ for $\mathbf{H}$ which:

(1) is semi–normal (see 1.4.2(3)), forgetful (see 1.1.2(4)) and super–gluing (see 1.4.2(4)),
(2) has the cutting property (see 1.1.12(3)),
(3) is really finitary (see 1.3.3) provided $\mathbf{H}(n)$ is finite for each $n \in \omega$.

Construction. Let $K_{\mathbf{H}}$ consist of all creatures $t \in \mathrm{CR}[\mathbf{H}]$ such that

- $\mathbf{dis}[t] = (m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}, \Delta_t)$ for some $\Delta_t \in \mathcal{K}^{\mathbf{H}}_{m_0,m_1} \cup \{\emptyset\}$ such that, if $\Delta_t \neq \emptyset$, $\mathbf{hn}^+(\Delta_t) > 1$,
- $\mathbf{val}[t] = \{\langle u, v\rangle \in \prod\limits_{i<m^t_{\mathrm{dn}}} \mathbf{H}(i) \times \prod\limits_{i<m^t_{\mathrm{up}}} \mathbf{H}(i) : u \vartriangleleft v \ \&\ (\forall f \in \Delta_t)(f \not\subseteq v)\}$,
- if $\Delta_t = \emptyset$ then $\mathbf{nor}[t] = m^t_{\mathrm{dn}} + 1$, otherwise $\mathbf{nor}[t] = \log_8(\mathbf{hn}(\Delta_t))$.

(Note that $\mathbf{hn}^+(\Delta_t) > 1$ implies $\mathbf{val}[t] \neq \emptyset$.) For $t_0, \dots, t_n \in K_{\mathbf{H}}$ such that $m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}}$ (for $i < n$) let

$$\Sigma_{\mathbf{H}}(t_0, \dots, t_n) = \{t \in K_{\mathbf{H}} : m^t_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}} \ \&\ m^t_{\mathrm{up}} = m^{t_n}_{\mathrm{up}} \ \&\ \bigcup_{i\le n} \Delta_{t_i} \subseteq \Delta_t\}.$$

It should be clear that $\Sigma_{\mathbf{H}}$ is a composition operation on $K_{\mathbf{H}}$, $K_{\mathbf{H}}$ is countable and forgetful, and if each $\mathbf{H}(n)$ is finite then $K_{\mathbf{H}}$ is really finitary.

For a creature $t \in K_{\mathbf{H}}$ we define $\Sigma^{\perp}_{\mathbf{H}}(t)$ as follows. It consists of all finite sets $\{s_0, \dots, s_n\} \subseteq K_{\mathbf{H}}$ (a suitable enumeration) such that

- $m^t_{\mathrm{dn}} = m^{s_0}_{\mathrm{dn}} < m^{s_0}_{\mathrm{up}} = m^{s_1}_{\mathrm{dn}} < \cdots < m^{s_{n-1}}_{\mathrm{up}} = m^{s_n}_{\mathrm{dn}} < m^{s_n}_{\mathrm{up}} = m^t_{\mathrm{up}}$, and
- $(\forall f \in \Delta_t)(\exists \ell \le n)(f{\restriction}[m^{s_\ell}_{\mathrm{dn}}, m^{s_\ell}_{\mathrm{up}}) \in \Delta_{s_\ell})$.

It is clear that $\Sigma^{\perp}_{\mathbf{H}}$ is a decomposition operation on $K_{\mathbf{H}}$, so $(K_{\mathbf{H}}, \Sigma_{\mathbf{H}}, \Sigma^{\perp}_{\mathbf{H}})$ is a $\otimes$–creating triple.

It follows from 1.5.2(2) that $(K_{\mathbf{H}}, \Sigma_{\mathbf{H}})$ is linked, and using 1.5.2(3) one easily shows that it is super–gluing. Similarly, $(K_{\mathbf{H}}, \Sigma_{\mathbf{H}}, \Sigma^{\perp}_{\mathbf{H}})$ has the cutting property by 1.5.2(4).

Note that

$(*)$ if $f \in \prod\limits_{i=m_0}^{m_1-1} \mathbf{H}(i)$, then $\mathbf{hn}(\{f\}) = \mathbf{HN}(\{f\}) = m_1 - m_0 + 1$.

Hence, using 1.5.2(2), we may easily conclude that $(K_{\mathbf{H}}, \Sigma_{\mathbf{H}})$ is reducible. However, it is not normal – one can build $s, t \in K_{\mathbf{H}}$ such that $\mathbf{nor}[s] < \mathbf{nor}[t]$ but $\mathbf{val}[t] \subseteq \mathbf{val}[s]$ (which is in some sense paradoxical, and this is why we modify this example in 1.5.5).

Claim 1.5.3.1. $(K_{\mathbf{H}}, \Sigma_{\mathbf{H}}, \Sigma^{\perp}_{\mathbf{H}})$ *is semi–normal.*

Proof of the claim. Let $n \in \omega$, $t \in K_{\mathbf{H}}$ be such that $\mathbf{nor}[t] > 2^{n+2}$, $m^t_{\mathrm{up}} - m^t_{\mathrm{dn}} > 2^{2^{n+4}}$. We may assume that $\Delta_t \neq \emptyset$ (remember $(*)$). We choose inductively a sequence $\langle \Delta_\ell, A_\ell : \ell \le n\rangle$ such that

(1) $\Delta_\ell \in \mathcal{K}^{\mathbf{H}}_{m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}}$, $A_\ell \subseteq \prod\limits_{i=m^t_{\mathrm{dn}}}^{m^t_{\mathrm{up}}-1} \mathbf{H}(i)$, $A_\ell \preceq \Delta_\ell$,

(2) $\Delta_0 = \Delta_t$,

(3) $8^{2^{n+2-\ell}-1} \le \mathbf{hn}(\Delta_{\ell+1}) = \mathbf{HN}(\Delta_{\ell+1}) = \mathbf{HN}(A_{\ell+1}) < 8^{2^{n+2-\ell}}$ (for $\ell < n$),

(4) $(\forall f \in A_\ell)(\forall g \in \bigcup\limits_{k<\ell} \Delta_k)(g \not\subseteq f)$.

There are no problems for $\ell = 0$ (note that there are practically no restrictions on A_0, and Δ_0 is determined). So suppose that we have arrived to a stage $\ell + 1 \le n$ of the construction. It follows from 1.5.2(1,2) that $\mathbf{hn}(\bigcup\limits_{k\le\ell} \Delta_k) > 8^{2^{n+3-\ell}-2}$. Let

$$A^* = \{f \in \prod_{i=m^t_{\rm dn}}^{m^t_{\rm up}-1} \mathbf{H}(i) : (\forall g \in \bigcup_{k\le\ell} \Delta_k)(g \not\subseteq f)\}.$$

Necessarily $\mathbf{HN}(A^*) \le 2$. Now, using $(*)$ and 1.5.2(2), we may choose $A_{\ell+1} \subseteq A^*$ such that $8^{2^{n+2-\ell}-1} \le \mathbf{HN}(A_{\ell+1}) < 8^{2^{n+2-\ell}}$. Next, we pick $\Delta_{\ell+1} \in \mathcal{K}^{\mathbf{H}}_{m^t_{\rm dn},m^t_{\rm up}}$ satisfying $A_{\ell+1} \preceq \Delta_{\ell+1}$ and $\mathbf{HN}(A_{\ell+1}) = \mathbf{hn}(\Delta_{\ell+1}) = \mathbf{HN}(\Delta_{\ell+1})$. This finishes the construction.

For $\ell \le n$ let $s_\ell \in \Sigma_{\mathbf{H}}(t)$ be such that $\Delta_{s_\ell} = \bigcup\limits_{k\le\ell} \Delta_k$. It should be clear that

(α) $s_0 = t$, $s_{\ell+1} \in \Sigma(s_\ell)$, $2^{n+1-\ell} < \mathbf{nor}[s_{\ell+1}] \le 2^{n-\ell+2}$ (for $\ell < n$).

We claim that additionally

(β) if $s \in K_{\mathbf{H}}$, $m^s_{\rm dn} = m^t_{\rm dn}$, $m^s_{\rm up} = m^t_{\rm up}$ and $\mathbf{nor}[s] > 2^{n-\ell+3}$, $\ell < n$, then

$$(\forall u \in {\rm dom}(\mathbf{val}[s]))(\exists v)(\langle u, v\rangle \in \mathbf{val}[s] \setminus \mathbf{val}[s_{\ell+1}])$$

(what will finish the proof of the claim). So suppose $s \in K_{\mathbf{H}}$, $m^s_{\rm dn} = m^t_{\rm dn}$, $m^s_{\rm up} = m^t_{\rm up}$ and $\mathbf{nor}[s] > 2^{n-\ell+3}$, $\ell < n$. Let $u \in \prod\limits_{i<m^t_{\rm dn}} \mathbf{H}(i)$. If $\Delta_s = \emptyset$, then we may take $v \in \prod\limits_{i<m^t_{\rm up}} \mathbf{H}(i)$ such that $u \vartriangleleft v$ and $(\exists f \in \Delta_t)(f \subseteq v)$, so clearly $\langle u, v\rangle \in \mathbf{val}[s] \setminus \mathbf{val}[s_{\ell+1}]$. Assume now that $\Delta_s \neq \emptyset$, so $\mathbf{hn}(\Delta_s) > 8^{2^{n-\ell+3}}$. Since $\mathbf{HN}(A_{\ell+1}) < 8^{2^{n+2-\ell}} < \mathbf{hn}(\Delta_s)$, there is $f \in A_{\ell+1}$ such that $(\forall g \in \Delta_s)(g \not\subseteq f)$. Clearly $\langle u, u^\frown f\rangle \in \mathbf{val}[s] \setminus \mathbf{val}[s_{\ell+1}]$. $\square$

Finally note that the forcing notion $\mathbb{Q}^*_\infty(K_{\mathbf{H}}, \Sigma_{\mathbf{H}}, \Sigma^\perp_{\mathbf{H}})$ is not trivial. $\square$

Conclusion 1.5.4. Let $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$ be such that $|\mathbf{H}(n)| \ge 2$ for all $n < \omega$. Then the forcing notion $\mathbb{Q}^*_\infty(K_{\mathbf{H}}, \Sigma_{\mathbf{H}}, \Sigma^\perp_{\mathbf{H}})$ (where $(K_{\mathbf{H}}, \Sigma_{\mathbf{H}}, \Sigma^\perp_{\mathbf{H}})$ is as defined in 1.5.3) is σ-$*$–linked and it adds a dominating real. Consequently it is not ω–nw–nep (by [21]).

If one looks at $\mathbf{val}[t]$ for $t \in K_{\mathbf{H}}$ in 1.5.3, then it is clear that $\mathbf{HN}$ is more appropriate to determine the norms of creatures. We presented 1.5.3 as it is a direct relative of the forcing notion of [16, §3]. However, it seems that the following example presents a nicer member of this family.

Example 1.5.5. Let $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$, $|\mathbf{H}(n)| \ge 2$ for all $n \in \omega$.
We construct a $\otimes$–creating triple $(K_{1.5.5}, \Sigma_{1.5.5}, \Sigma^\perp_{1.5.5})$ for $\mathbf{H}$ which:

(1) is almost normal (see the construction), regular (see 1.3.2), forgetful and super–gluing,

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(2) has the cutting property,
(3) is really finitary provided $\mathbf{H}(n)$ is finite for each $n \in \omega$.

Construction. It is similar to 1.5.3, but instead of **hn** we use **HN**. So $K_{1.5.5}$ consists of $t \in \mathrm{CR}[\mathbf{H}]$ such that

- $\mathbf{dis}[t] = (m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}, \Delta_t)$ for some $\Delta_t \in \mathcal{K}^{\mathbf{H}}_{m_0,m_1} \cup \{\emptyset\}$ such that, if $\Delta_t \neq \emptyset$, $\mathbf{hn}^+(\Delta_t) > 1$,
- $\mathbf{val}[t] = \{\langle u, v\rangle \in \prod\limits_{i<m^t_{\mathrm{dn}}} \mathbf{H}(i) \times \prod\limits_{i<m^t_{\mathrm{up}}} \mathbf{H}(i) : u \lhd v \ \&\ (\forall f \in \Delta_t)(f \not\subseteq v)\}$,
- if $\Delta_t = \emptyset$ then $\mathbf{nor}[t] = \log_8(m^t_{\mathrm{up}} - m^t_{\mathrm{dn}}) + 2m^t_{\mathrm{dn}} + 1$, otherwise $\mathbf{nor}[t] = \log_8(\mathbf{HN}(\Delta_t))$.

For $t_0, \dots, t_n \in K_{1.5.5}$ such that $m^{t_i}_{\mathrm{up}} = m^{t_{i+1}}_{\mathrm{dn}}$ (for $i < n$) let

$$\Sigma_{1.5.5}(t_0, \dots, t_n) = \{t \in K_{1.5.5} : m^t_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}} \ \&\ m^t_{\mathrm{up}} = m^{t_n}_{\mathrm{up}} \ \&\ \bigcup_{i\le n} \Delta_{t_i} \subseteq \Delta_t\}.$$

For $t \in K_{1.5.5}$, $\Sigma^{\perp}_{1.5.5}(t)$ consists of all finite sets $\{s_0, \dots, s_n\} \subseteq K_{1.5.5}$ such that

- $m^t_{\mathrm{dn}} = m^{s_0}_{\mathrm{dn}} < m^{s_0}_{\mathrm{up}} = m^{s_1}_{\mathrm{dn}} < \cdots < m^{s_{n-1}}_{\mathrm{up}} = m^{s_n}_{\mathrm{dn}} < m^{s_n}_{\mathrm{up}} = m^t_{\mathrm{up}}$, and
- $(\forall f \in \Delta_t)(\exists \ell \le n)(f{\restriction}[m^{s_\ell}_{\mathrm{dn}}, m^{s_\ell}_{\mathrm{up}}) \in \Delta_{s_\ell})$.

Clearly $(K_{1.5.5}, \Sigma_{1.5.5}, \Sigma^{\perp}_{1.5.5})$ is a forgetful super–gluing $\otimes$–creating triple. It is linked, and it is really finitary provided $\mathbf{H}(n)$ is finite for each $n \in \omega$.

Claim 1.5.5.1. *Suppose $s, t \in K_{1.5.5}$ are such that* $\mathbf{nor}[s] < \mathbf{nor}[t]$. *Then*

$$(\forall u \in \mathrm{dom}(\mathbf{val}[t]))(\exists v)(\langle u, v\rangle \in \mathbf{val}[t] \setminus \mathbf{val}[s]).$$

Proof of the claim. We may assume that $m^s_{\mathrm{dn}} = m^t_{\mathrm{dn}}$, $m^s_{\mathrm{up}} = m^t_{\mathrm{up}}$ (otherwise trivial). It follows from the assumptions (and the definition of $\mathbf{nor}[s]$) that $\Delta_s \neq \emptyset$. If $\Delta_t = \emptyset$, then the conclusion is immediate, so assume $\Delta_t \neq \emptyset$. Thus $\Delta_s, \Delta_t \in \mathcal{K}^{\mathbf{H}}$ and $\mathbf{HN}(\Delta_s) < \mathbf{HN}(\Delta_t)$. Choose $\Delta \in \mathcal{K}^{\mathbf{H}}$ such that $\Delta_t \preceq \Delta$ and $\mathbf{hn}(\Delta) = \mathbf{HN}(\Delta_t)$. Note that, by 1.5.2(1), we have $\mathbf{hn}(\Delta) = \mathbf{HN}(\Delta) = \mathbf{hn}^+(\Delta)$. Consequently, we may choose $\Delta' \in \mathcal{K}^{\mathbf{H}}$ such that elements of Δ' have pairwise disjoint domains, $\Delta \preceq \Delta'$ and $\mathbf{hn}(\Delta') = \mathbf{hn}(\Delta)$. Now, for some $f \in \Delta_s$ we have $(\forall g \in \Delta')(g \not\subseteq f)$. By the choice of Δ' we may build $f^* \in \prod\limits_{i=m^s_{\mathrm{dn}}}^{m^s_{\mathrm{up}}-1} \mathbf{H}(i)$ such that $f \subseteq f^*$ and $(\forall g \in \Delta')(g \not\subseteq f^*)$. Since $\Delta_t \preceq \Delta'$ we are done. □

Using 1.5.5.1 we see that $(K_{1.5.5}, \Sigma_{1.5.5})$ is almost normal in the following sense: the reducibility demand (see 1.4.2(1)) holds for those $t \in K_{1.5.5}$ for which $\Delta_t \neq \emptyset$. However, this is enough to carry out, e.g., the proof of 1.4.4(3) with almost no changes.

Claim 1.5.5.2. $(K_{1.5.5}, \Sigma_{1.5.5}, \Sigma^{\perp}_{1.5.5})$ *is regular and has the cutting property.*

Proof of the claim. First we show the regularity. So suppose that

$$(w, t_0, \dots, t_n), (u, s_0, \dots, s_m) \in \mathrm{FC}(K_{1.5.5}, \Sigma_{1.5.5}, \Sigma^{\perp}_{1.5.5})$$

are such that $(w, t_0, \dots, t_n) \le (u, s_0, \dots, s_m)$, $m^{s_m}_{\mathrm{up}} \le m^{t_n}_{\mathrm{up}}$, $\mathbf{nor}[s_0] \ge 3$, and $\mathbf{nor}[t_\ell] \ge 3$, where $\ell \le n$ is such that $m^{t_\ell}_{\mathrm{dn}} < m^{s_0}_{\mathrm{dn}} < m^{t_\ell}_{\mathrm{up}} \le m^{s_0}_{\mathrm{up}}$. It follows from the definition of $\le$ (see 1.1.3) that

$$(\forall f \in \Delta_{t_\ell})\big(f{\restriction}m^{s_0}_{\mathrm{dn}} \not\subseteq u \ \text{ or } \ (\exists g \in \Delta_{s_0})(g \subseteq f)\big).$$

Let $t', t'' \in K_{1.5.5}$ be such that $m^{t'}_{\mathrm{dn}} = m^{t_\ell}_{\mathrm{dn}}$, $m^{t'}_{\mathrm{up}} = m^{t''}_{\mathrm{dn}} = m^{s_0}_{\mathrm{dn}}$, $m^{t''}_{\mathrm{up}} = m^{t_\ell}_{\mathrm{up}}$ and

$$\begin{aligned}\Delta_{t'} &= \{f{\restriction}[m^{t_\ell}_{\mathrm{dn}}, m^{s_0}_{\mathrm{dn}}) : f \in \Delta_{t_\ell} \ \&\ f{\restriction}m^{s_0}_{\mathrm{dn}} \not\subseteq u\},\\ \Delta_{t''} &= \{f{\restriction}[m^{s_0}_{\mathrm{dn}}, m^{t_\ell}_{\mathrm{up}}) : f \in \Delta_{t_\ell} \ \&\ f{\restriction}m^{s_0}_{\mathrm{dn}} \subseteq u\}.\end{aligned}$$

Then $\{t', t''\} \in \Sigma^{\perp}_{1.5.5}(t_\ell)$ and clearly $u \in \mathrm{pos}(w, t_0, \dots, t_{\ell-1}, t')$. The only thing left is to show that the norm of t'' is at least 2. If $\Delta_{t''} = \emptyset$, then it is clearly true as $m^{t''}_{\mathrm{dn}} \ge 1$. So suppose that $\Delta_{t''} \ne \emptyset$ and we have to argue that $\mathbf{HN}(\Delta_{t''}) \ge 64$. But this is clear as $\Delta_{t''} \preceq \Delta_{s_0}$.

Now let us show that $(K_{1.5.5}, \Sigma_{1.5.5}, \Sigma^{\perp}_{1.5.5})$ has the cutting property. Let $t \in K_{1.5.5}$, $\mathbf{nor}[t] > 1$, $m^t_{\mathrm{dn}} < m < m^t_{\mathrm{up}}$. Choose $\Delta \in \mathcal{K}^{\mathbf{H}}_{m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}}$ such that elements of Δ have pairwise disjoint domains, $\Delta_t \preceq \Delta$ and $\mathbf{HN}(\Delta_t) = \mathbf{hn}(\Delta)$ (like in the proof of 1.5.5.1). Put

$$\begin{aligned}\Delta^0 &= \{f{\restriction}[m^t_{\mathrm{dn}}, m) : f \in \Delta \ \&\ |\mathrm{dom}(f) \cap [m^t_{\mathrm{dn}}, m)| \ge \tfrac{1}{2}|\mathrm{dom}(f)|\},\\ \Delta^1 &= \{f{\restriction}[m, m^t_{\mathrm{up}}) : f \in \Delta \ \&\ |\mathrm{dom}(f) \cap [m, m^t_{\mathrm{up}})| \ge \tfrac{1}{2}|\mathrm{dom}(f)|\}.\end{aligned}$$

Let $s_0, s_1 \in K_{1.5.5}$ be such that $m^{s_0}_{\mathrm{dn}} = m^t_{\mathrm{dn}}$, $m^{s_0}_{\mathrm{up}} = m = m^{s_1}_{\mathrm{dn}}$, $m^{s_1}_{\mathrm{up}} = m^t_{\mathrm{up}}$ and

$$\begin{aligned}\Delta_{s_0} &= \{g{\restriction}[m^t_{\mathrm{dn}}, m) : g \in \Delta_t \ \&\ (\exists f \in \Delta^0)(f \subseteq g)\},\\ \Delta_{s_1} &= \{g{\restriction}[m, m^t_{\mathrm{up}}) : g \in \Delta_t \ \&\ (\exists f \in \Delta^1)(f \subseteq g)\}.\end{aligned}$$

Now check. □

□

Conclusion 1.5.6. The forcing notion $\mathbb{Q}^*_\infty(K_{1.5.5}, \Sigma_{1.5.5}, \Sigma^{\perp}_{1.5.5})$ is σ-$*$–linked Souslin ccc and it adds a dominating real. Consequently it adds a Cohen real (by [25]) and it is not ω–nw–nep (by [21]).

Hall's norms are of special interest because of "gluing and cutting", but we may use them to build local creating pairs to.

Example 1.5.7. Let $\mathbf{H}^* : \omega \longrightarrow \mathcal{H}(\omega_1)$, $2 \le |\mathbf{H}^*(i)| < \omega$ for all $i \in \omega$. Suppose that $\bar{n} = \langle n_k : k < \omega\rangle \subseteq \omega$ is an increasing sequence such that $\lim\limits_{k\to\infty} n_{k+1} - n_k = \infty$ and let $\mathbf{H} = \mathbf{H}^*[\bar{n}] : \omega \longrightarrow \mathcal{H}(\omega_1)$ be defined by $\mathbf{H}(k) = \prod\limits_{i=n_k}^{n_{k+1}-1} \mathbf{H}^*(i)$.

We construct a really finitary creating pair $(K_{1.5.7}, \Sigma_{1.5.7})$ for $\mathbf{H}$ which is local, forgetful, normal and Cohen–producing (see 1.4.9(1)).

Construction. Let $K_{1.5.7}$ consist of creatures $t \in \mathrm{CR}[\mathbf{H}]$ such that

- $\mathbf{dis}[t] = (m_t, \Delta_t)$ for some $m_t < \omega$ and $\Delta_t \in \mathcal{K}^{\mathbf{H}^*}_{n_{m_t}, n_{m_t+1}} \cup \{\emptyset\}$ such that, if $\Delta_t \neq \emptyset$, $\mathbf{hn}^+(\Delta_t) > 1$,
- $\mathbf{val}[t] = \{\langle u, v\rangle \in \prod\limits_{i<m_t} \mathbf{H}(i) \times \prod\limits_{i \le m_t} \mathbf{H}(i) : u \lhd v \ \& \ (\forall f \in \Delta_t)(f \not\subseteq v(m_t))\}$,
- if $\Delta_t \neq \emptyset$, then $\mathbf{nor}[t] = \log_8(\mathbf{HN}(\Delta_t))$, otherwise $\mathbf{nor}[t] = \log_8(n_{m_t+1} - n_{m_t} + 1)$.

The operation $\Sigma_{1.5.7}$ gives non-empty results for singletons only and

$$\Sigma_{1.5.7}(t) = \{s \in K_{1.5.7} : m^t_{\mathrm{dn}} = m^s_{\mathrm{dn}} \ \& \ \Delta_t \subseteq \Delta_s\}.$$

Clearly, $(K_{1.5.7}, \Sigma_{1.5.7})$ is a really finitary creating pair which is local, forgetful and linked (remember 1.5.2(2)).

Claim 1.5.7.1. $(K_{1.5.7}, \Sigma_{1.5.7})$ *is normal and Cohen–producing.*

Proof of the claim. First note that $(K_{1.5.7}, \Sigma_{1.5.7})$ is reducible (remember $(*)$ of the construction for 1.5.3; note that here there are no problems caused by $\Delta_t = \emptyset$). Next note that (the proof of) 1.5.5.1 applies here too.

To show that $(K_{1.5.7}, \Sigma_{1.5.7})$ is Cohen–producing fix (for $k \in \omega$) an $a_k \in \mathbf{H}^*(n_k)$ and let

$$A_k = \{f \in \prod_{i=n_k}^{n_{k+1}-1} \mathbf{H}^*(i) : f(n_k) = a_k\} \subseteq \mathbf{H}(k).$$

Suppose that $t \in K_{1.5.7}$, $\mathbf{nor}[t] > 1$, $m^t_{\mathrm{dn}} = k$ and let $u \in \mathrm{dom}(\mathbf{val}[t])$. If $\Delta_t = \emptyset$, then we easily choose $v_0, v_1 \in \mathrm{rng}(\mathbf{val}[t])$ extending u and such that $v_0(k) \notin A_k$, $v_1(k) \in A_k$ (remember $|\mathbf{H}^*(n_k)| \ge 2$). So suppose that $\Delta_t \neq \emptyset$ and thus $\mathbf{HN}(\Delta_t) > 8$. Then we may find $\Delta' \in \mathcal{K}^{\mathbf{H}^*}_{n_k, n_{k+1}}$ such that $\Delta_t \preceq \Delta'$, elements of Δ' have pairwise disjoint domains all of size > 2. Now we easily build $v_0, v_1 \in \mathrm{dom}(\mathrm{rng}[t])$, both extending u and such that $v_0(k) \notin A_k$, $v_1(k) \in A_k$. □

□

Conclusion 1.5.8. Assume $\mathbf{H}^*, \bar{n}, \mathbf{H}$ and $(K_{1.5.7}, \Sigma_{1.5.7})$ are as in 1.5.7.

(1) Suppose $f : \omega \times \omega \longrightarrow \omega$ is a fast function such that

$$(\forall k \in \omega)(\forall^\infty \ell \in \omega)(f(k,\ell) < \log_8(n_{\ell+1} - n_\ell))$$

(e.g. $f(k,\ell) = 2^{2k}$). Then $\mathbb{Q}^*_f(K_{1.5.7}, \Sigma_{1.5.7})$ is a non-trivial σ-$*$–linked Borel ccc forcing notion which adds a dominating real (so it adds a Cohen real and it is not ω–nw–nep).

(2) Let $h(n,m) = \max\{0, m-1\}$ (so $h : \omega \times \omega \longrightarrow \omega$ is a regressive function). Suppose that $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ is a countable h–closed $\ge^*$–directed family such that $(\forall f \in \mathcal{F})(\forall^\infty \ell \in \omega)(f(\ell) < \log_8(n_{\ell+1} - n_\ell))$ (e.g. $\mathcal{F} = \{f_k : k < \omega\}$, $f_k(\ell) = \max\{2, \lfloor \frac{1}{k} \log_8(n_{\ell+1} - n_\ell)\rfloor\}$). Then $\mathbb{Q}^*_{\mathcal{F}}(K_{1.5.7}, \Sigma_{1.5.7})$ is a non-trivial σ–$*$–linked very Borel ccc forcing notion, it adds a Cohen real and it is nice (so it preserves unbounded families).

Our next examples generalize (in some sense) the Eventually Different Real Forcing of Miller [12].

Example 1.5.9. Let $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$, $|\mathbf{H}(k)| \geq 4$. For $k < \omega$, let N_k be $|\mathbf{H}(k)|$ if $\mathbf{H}(k)$ is finite, and 2^{k+2} otherwise. Assume $\lim\limits_{k \mapsto \infty} N_k = \infty$. Let $h : \omega \times \omega \longrightarrow \omega$ be given by

$$h(k,n) = \begin{cases} n-1 & \text{if } n \geq N_k, \\ 2n - N_k & \text{if } \frac{7}{8}N_k < n < N_k, \\ 1 & \text{otherwise.} \end{cases}$$

(Note that h is regressive.) We construct an h–linked creating pair $(K_{1.5.9}, \Sigma_{1.5.9})$ for $\mathbf{H}$ which is local, forgetful, Cohen–producing and of the BCB–type (see 1.4.9).

Construction. Let $K_{1.5.9}$ be the collection of all $t \in \mathrm{CR}[\mathbf{H}]$ such that

- $\mathbf{dis}[t] = (k_t, E_t)$ for some $k_t < \omega$ and $E_t \subseteq \mathbf{H}(k_t)$ such that $0 < |E_t| < N_{k_t}$,
- $\mathbf{val}[t] = \{\langle u, v\rangle \in \prod\limits_{i<k_t} \mathbf{H}(i) \times \prod\limits_{i \leq k_t} \mathbf{H}(i) : u \lhd v \ \&\ v(k_t) \notin E_t\}$,
- if $|E_t| \geq \frac{1}{4}N_{k_t}$ then $\mathbf{nor}[t] = 1$; otherwise $\mathbf{nor}[t] = N_{k_t} - |E_t|$.

The operation $\Sigma_{1.5.9}$ is natural: it gives non-empty results for singletons only and

$$\Sigma_{1.5.9}(t) = \{s \in K_{1.5.9} : k_s = k_t \ \&\ E_t \subseteq E_s\}.$$

It should be clear that $(K_{1.5.9}, \Sigma_{1.5.9})$ is a local forgetful creating pair for $\mathbf{H}$.

To show that it is h–linked suppose that $k > 1, t_0, t_1 \in K_{1.5.9}, \mathbf{nor}[t_0], \mathbf{nor}[t_1] \geq k$ and $\ell = m^{t_0}_{\mathrm{dn}} = m^{t_1}_{\mathrm{dn}}$. Then $|E_{t_0}|, |E_{t_1}| < \frac{1}{4}N_\ell$ and thus $0 < |E_{t_0} \cup E_{t_1}| < N_\ell$. Let $s \in K_{1.5.9}$ be such that $k_s = \ell$, $E_s = E_{t_0} \cup E_{t_1}$. Clearly $s \in \Sigma_{1.5.9}(t_0) \cap \Sigma_{1.5.9}(t_1)$. If $h(\ell, k) = 1$ then clearly $\mathbf{nor}[s] \geq h(\ell, k)$, so suppose $h(\ell, k) > 1$. Necessarily $\frac{7}{8}N_\ell < k < N_\ell$, so $|E_{t_0}|, |E_{t_1}| < \frac{1}{8}N_\ell$ and therefore $|E_s| < \frac{1}{4}N_\ell$. Hence

$$\mathbf{nor}[s] = N_\ell - |E_s| \geq N_\ell - |E_{t_0}| - |E_{t_1}| \geq 2k - N_\ell = h(\ell, k).$$

Let us show now that $(K_{1.5.9}, \Sigma_{1.5.9})$ is Cohen–producing. For each $n \in \omega$ choose a set $A_n \subseteq \mathbf{H}(n)$ such that $|A_n| = \lfloor \frac{1}{2}|\mathbf{H}(n)| \rfloor$ if $\mathbf{H}(n)$ is finite, and A_n is infinite co-infinite if $\mathbf{H}(n)$ is infinite. Suppose that $t \in K_{1.5.9}$, $\mathbf{nor}[t] > 1$. Then $E_t \subseteq \mathbf{H}(k_t)$, $|E_t| < \frac{1}{4}N_{k_t}$, so we may choose $a_1 \in A_{k_t} \setminus E_{k_t}$ and $a_0 \in \mathbf{H}(k_t) \setminus (A_{k_t} \cup E_{k_t})$, and we easily finish.

Finally, let us argue that $(K_{1.5.9}, \Sigma_{1.5.9})$ is of the BCB–type, To this end suppose that $\langle s_n : n < \omega\rangle \subseteq K_{1.5.9}$, $m^{s_n}_{\mathrm{dn}} = k_{s_n} = \ell$, $\mathbf{nor}[s_n] \geq 2$. Then $|E_{s_n}| < \frac{1}{4}N_\ell$ for each n.

If $\mathbf{H}(\ell)$ is finite, then the demand in $(\circledast^{\mathrm{BCB}})$ of 1.4.9(2) is trivially satisfied (just take $\{a_0, \ldots, a_m\} = \mathbf{H}(\ell)$).

So suppose that $\mathbf{H}(\ell)$ is infinite and to simplify notation let $\mathbf{H}(\ell) = \omega$. Let $E_{s_n} = \{b^n_0, \ldots, b^n_{k_n-1}\}$ be the increasing enumeration; $k_n = |E_{s_n}|$. We may find an infinite set $Y \subseteq \omega$ and $k^* \leq k < \omega$ such that

- $k_n = k$ for each $n \in Y$,
- $\langle b^n_i : n \in Y\rangle$ is constant for each $i < k^*$,

- $\langle b^n_i : n \in Y\rangle$ is strictly increasing for each $i \in [k^*, k)$.

Suppose $a \in \mathbf{H}(\ell) \setminus \{b^n_i : i < k^*\}$ for some (equivalently: all) $n \in Y$. Then, for sufficiently large $n \in Y$, for every $i \in [k^*, k)$ we have $b^n_i > a$. Consequently $(\forall^\infty n \in Y)(a \notin E_{s_n})$, so we may easily finish. □

Conclusion 1.5.10. Let $\mathbf{H}$, $\langle N_k : k < \omega\rangle$, h and $(K_{1.5.9}, \Sigma_{1.5.9})$ be as in 1.5.9. Suppose that $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ is a countable h–closed $\geq^*$–directed family such that for some $f \in \mathcal{F}$ we have $(\forall^\infty k \in \omega)(f(k) < N_k)$ (e.g. $\mathcal{F} = \{f_\ell : \ell < \omega\}$, $f_\ell(k) = N_k - 2^\ell$ if $N_k > 2^{\ell+1}$, $f_\ell(k) = 2$ otherwise). Then $\mathbb{Q}^*_{\mathcal{F}}(K_{1.5.9}, \Sigma_{1.5.9})$ is a non-trivial σ–$*$–linked Borel ccc forcing notion which adds a Cohen real and is nice (so it preserves unbounded families).

If $\mathbf{H}$ and $\mathcal{F}$ are as in 1.5.10, and $\mathbf{H}(k)$ is finite for each k, then we may use 1.3.4(3) to get that the forcing notion $\mathbb{Q}^*_{\mathcal{F}}(K_{1.5.9}, \Sigma_{1.5.9})$ is very Borel ccc. We may prove the same conclusion without the additional assumption on $\mathbf{H}$ (see 1.5.11 below). Unfortunately, this proof is very specific for $\mathbb{Q}^*_{\mathcal{F}}(K_{1.5.9}, \Sigma_{1.5.9})$ and it does not generalize to cover more forcing notions of the form $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$

Proposition 1.5.11. *Assume that* $\mathbf{H}$, $\langle N_k : k < \omega\rangle$, h, $K = K_{1.5.9}$ *and* $\Sigma = \Sigma_{1.5.9}$ *are as in 1.5.9, and* $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ *is a countable h–closed* $\leq^*$*–directed family such that*

$$(\exists f \in \mathcal{F})(\forall^\infty k \in \omega)(f(k) < N_k).$$

Then the forcing notion $\mathbb{Q}^*_{\mathcal{F}}(K_{1.5.9}, \Sigma_{1.5.9})$ *is very Borel ccc.*

Proof. The only thing that should be shown is that being a maximal antichain is a Borel relation (remember 1.5.10 and so 1.3.4(1c)). Put $Z = \bigcup\limits_{i<\omega} \prod\limits_{j<i} \mathbf{H}(i) \setminus \{\langle\rangle\}$, $\mathcal{X} = (Z \cup \{\langle\rangle\}) \times K^\omega$, and $\mathcal{Y} = \mathcal{X}^\omega$ and $\mathcal{Z} = \omega^Z \times \omega^Z$.

Then $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are Polish spaces (each equipped with the respective product topology), $\mathbb{Q}^*_\emptyset(K, \Sigma)$ and $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ are Π^0_1 and Σ^0_2 subsets of $\mathcal{X}$, respectively, and for our conclusion it is enough to show that

$$P \stackrel{\text{def}}{=} \Big\{\langle p_n : n < \omega\rangle \in \mathcal{Y} : (\forall n < \omega)(p_n \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)) \text{ and } \{p_n : n < \omega\} \text{ is pre-dense in } \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)\Big\}$$

is a Borel subset of $\mathcal{Y}$.

Plainly we may assume that

$$(\forall f \in \mathcal{F})(\forall k < \omega)(2 \leq f(k) < N_k)$$

(as we may modify suitable the family $\mathcal{F}$ without changing the forcing notion). Now, for $f \in \mathcal{F}$ we define

$$C_f = \{(h_0, h_1) \in \mathcal{Z} : (\forall \eta \in Z)(h_0(\eta) < \mathrm{lh}(\eta) \,\&\, h_1(\eta) < N_{h_0(\eta)} - f(h_0(\eta)))\}.$$

Clearly each C_f is a compact subset of $\mathcal{Z}$.

Claim 1.5.11.1. *Suppose that $\bar{p} = \langle p_n : n < \omega\rangle \in \mathcal{Y}$, where $p_n \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ are such that $\mathbf{nor}[t^{p_n}_k] > 1$ (for all $k, n < \omega$). Then the following are equivalent:*

(A)$_{\bar{p}}$ There is a condition $p \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ incompatible with every p_n (for $n < \omega$).

(B)$_{\bar{p}}$ There are $\eta \in Z$ and $f \in \mathcal{F}$ and $(h_0, h_1) \in C_f$ such that

(i) for every $\nu \in Z$, if $\eta \lhd \nu$ then $h_0(\nu) \geq \mathrm{lh}(\eta)$, and

(ii) for every $\nu_0, \nu_1 \in Z$, if $\eta \lhd \nu_0$, $\eta \lhd \nu_1$ and $\nu_0, \nu_1 \in \bigcup\limits_{n<\omega} \mathrm{POS}(p_n)$ and $h_0(\nu_0) = h_0(\nu_1)$ and $h_1(\nu_0) = h_1(\nu_1)$, then $\nu_0(h_0(\nu_0)) = \nu_1(h_0(\nu_1))$.

Proof of the claim. Assume (A)$_{\bar{p}}$ and pick $p \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ and $f \in \mathcal{F}$ such that

$$(\forall k < \omega)(\mathbf{nor}[t^p_k] = N_{\mathrm{lh}(w^p)+k} - |E_{t^p_k}| = f(\mathrm{lh}(w^p) + k)),$$

and p is incompatible with all p_n (for $n < \omega$). For $\ell = \mathrm{lh}(w^p)+k$, $k < \omega$, let $\langle x^\ell_m : m < N_\ell - f(\ell)\rangle$ be an enumeration of $E_{t^p_k}$. Note that, as $p \perp p_n$, if $\nu \in \mathrm{POS}(p_n)$ and $w^p \lhd \nu$, then for some $h_0(\nu) \in [\mathrm{lh}(w^p), \mathrm{lh}(\nu))$ and $h_1(\nu) < N_{h_0(\nu)} - f(h_0(\nu))$ we have $\nu(h_0(\nu)) = x^{h_0(\nu)}_{h_1(\nu)}$. Letting $\eta = w^p$ we may now easily define (h_0, h_1) so that η, f, (h_0, h_1) witness (B)$_{\bar{p}}$.

Suppose now that η, f and (h_0, h_1) witness (B)$_{\bar{p}}$. Let $w^p = \eta$ and for $\ell = \mathrm{lh}(w^p) + k$, $k < \omega$ and $m < N_\ell - f(\ell)$ let $x^\ell_m \in \mathbf{H}(\ell)$ be such that

(∗) if $\nu \in \bigcup\limits_{n<\omega} \mathrm{POS}(p_n)$, $w^p \lhd \nu$ and $h_0(\nu) = \ell$ and $h_1(\nu) = m$, then $x^\ell_m = \nu(\ell)$.

(The choice of the x^ℓ_m's is possible by (B)$_{\bar{p}}$(ii).) Now, for each $k < \omega$ pick $t^p_k \in K$ so that

$$m^{t^p_k}_{\mathrm{dn}} = \mathrm{lh}(w^p) + k \quad \text{and} \quad E_{t^p_k} = \{x^{\mathrm{lh}(w^p)+k}_m : m < N_{\mathrm{lh}(w^p)+k} - f(\mathrm{lh}(w^p) + k)\}.$$

Notice that for sufficiently large k we have $f(\mathrm{lh}(w^p) + k) > \frac{7}{8} N_{\mathrm{lh}(w^p)+k}$, so for those k we will also have $\mathbf{nor}[t^p_k] = f(\mathrm{lh}(w^p) + k)$. Hence $p \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ and easily it is a condition incompatible with all p_n's. □

For $\eta \in Z$ and $f \in \mathcal{F}$, let $B^{\eta,f}$ consist of all $(h_0, h_1, \bar{p}) \in \mathcal{Z} \times \mathcal{Y}$ such that

- $(h_0, h_1) \in C_f$, $\bar{p} = \langle p_n : n < \omega\rangle \in \mathcal{Y}$, where $p_n \in \mathbb{Q}^*_{\emptyset}(K, \Sigma)$ are such that $\mathbf{nor}[t^{p_n}_k] > 1$ for all $k, n < \omega$, and
- $(\forall \nu \in Z)(\eta \lhd \nu \Rightarrow \mathrm{lh}(\eta) \leq h_0(\nu))$, and
- for all $\nu_0, \nu_1 \in Z \cap \bigcup\limits_{n<\omega} \mathrm{POS}(p_n)$ we have

$$h_0(\nu_0) = h_0(\nu_1)\ \&\ h_1(\nu_0) = h_1(\nu_1) \quad \Rightarrow \quad \nu_0(h_0(\nu_0)) = \nu_1(h_1(\nu_1)).$$

It should be clear that $B^{\eta,f}$ is a closed subset of $C_f \times \mathcal{Y}$ and hence (as C_f is compact) the set

$$A^{\eta,f} = \{\bar{p} \in \mathcal{Y} : (\exists (h_0, h_1) \in C_f)((h_0, h_1, \bar{p}) \in B^{\eta,f})\}$$

is a closed subset of $\mathcal{Y}$, and $A^{\eta,f} \cap \left(\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)\right)^\omega$ is Borel. Now, pick a Borel function $\pi : \mathcal{Y} \longrightarrow \mathcal{Y}$ such that

if $\bar{p} = \langle p_n : n < \omega\rangle \in \left(\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)\right)^\omega$ and $\bar{q} = \langle q_n : n < \omega\rangle = \pi(\bar{p})$, then

- for each $n < \omega$, $q_n \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ and $(\forall k < \omega)(\mathbf{nor}[t^{q_n}_k] > 1)$,
- for each $n < \omega$, for some $m < \omega$ and $k < \omega$ we have

$$w^{q_n} \in \mathrm{pos}(w^{p_m}, t^{p_m}_0, \dots, t^{p_m}_k) \quad \text{and} \quad q_n = (w^{q_n}, t^{p_m}_{k+1}, t^{p_m}_{k+2}, \dots),$$

- if $m < \omega$, $k < \omega$, $w \in \mathrm{pos}(w^{p_m}, t^{p_m}_0, \dots, t^{p_m}_k)$, and $(\forall \ell > k)(\mathbf{nor}[t^{p_m}_\ell] > 1)$, then $(w, t^{p_m}_{k+1}, t^{p_m}_{k+2}, \dots) \in \{q_n : n < \omega\}$.

Clearly, $\bar{p} \in \big(\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)\big)^\omega$ is pre-dense if and only if so is $\pi(\bar{p})$. Hence, for $\bar{p} \in \big(\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)\big)^\omega$ we have
$\bar{p}$ not is pre-dense if and only if
there are $\eta \in Z$ and $f \in \mathcal{F}$ such that $\pi(\bar{p}) \in A^{\eta, f}$
(remember 1.5.11.1). Since both Z and $\mathcal{F}$ are countable, the proof of the proposition is completed. □

The construction presented in 1.5.9 is a particular case of a more general method of building linked creating pairs from some of the examples presented in [15]. First let us recall the following definition.

Definition 1.5.12 (See [15, Def. 5.2.5]). *Let (K, Σ) be a creating pair. We say that a creature $t \in K$ is* (n, m)–additive *if for all $t_0, \dots, t_{n-1} \in \Sigma(t)$ such that* $\mathbf{nor}[t_i] \le m$ *(for $i < n$) there is $s \in \Sigma(t)$ such that*

$$t_0, \dots, t_{n-1} \in \Sigma(s) \quad \textit{and} \quad \mathbf{nor}[s] \le \max\{\mathbf{nor}[t_\ell] : \ell < n\} + 1.$$

Example 1.5.13. Suppose that (K, Σ) is a local and forgetful creating pair for $\mathbf{H}$, and it satisfies the demand ($\boxplus$) of 1.4.2(2). Let $\bar{t}^* = \langle t^*_0, t^*_1, t^*_2, \dots \rangle \in \mathrm{PC}_\infty(K, \Sigma)$ be such that each t^*_n is $(2, \mathbf{nor}[t^*_n])$–additive and $m^{t_0}_{\mathrm{dn}} = 0$. We construct a local linked creating pair $(K^c_{\bar{t}^*}, \Sigma^c_{\bar{t}^*})$ (*the $\bar{t}^*$–dual of* (K, Σ)).

Construction. For $n < \omega$ and a creature $t \in \Sigma(t^*_n)$ let a creature t^c be such that

- $\mathbf{nor}[t^c] = \max\{0, \mathbf{nor}[t^*_n] - \mathbf{nor}[t]\}$,
- $\mathbf{val}[t^c] = \big(\prod\limits_{i<n} \mathbf{H}(i) \times \prod\limits_{i \le n} \mathbf{H}(i)\big) \setminus \mathbf{val}[t]$,
- $\mathbf{dis}[t^c] = (\mathbf{dis}[t], c)$.

[The creature t^c is defined only if $\mathbf{val}[t^c] \ne \emptyset$.] Let $K^c_{\bar{t}^*}$ be the collection of all (correctly defined) t^c (for $t \in \Sigma(t^*_n)$, $n < \omega$). For $t^c \in K^c_{\bar{t}^*}$ (defined as above for $t \in \Sigma(t^*_n)$) we let

$$\Sigma^c_{\bar{t}^*}(t^c) = \{s^c : t \in \Sigma(s) \;\&\; s \in \Sigma(t^*_n)\}.$$

□

The examples of local creating pairs have their (local) tree–creating variants too. They can be constructed like the following example.

Example 1.5.14. Let $\mathbf{H} \in \omega^\omega$ be a strictly increasing function such that $\mathbf{H}(0) > 2$. We construct a really finitary, normal (local) tree–creating pair $(K_{1.5.14}, \Sigma_{1.5.14})$ for $\mathbf{H}$ which is Cohen–producing.

Construction. The family $K_{1.5.14}$ consists of tree–creatures $t \in \mathrm{LTCR}[\mathbf{H}]$ such that

- $\mathbf{dis}[t] = (m_t, \eta_t, E_t)$ such that $m_t < \omega$, $\eta_t \in \prod\limits_{i<m_t} \mathbf{H}(i)$ and $\emptyset \neq E_t \subseteq \mathbf{H}(m_t)$, $E_t \neq \mathbf{H}(m_t)$,
- $\mathbf{val}[t] = \{\langle \eta_t, \nu\rangle : \eta_t \lhd \nu \in \prod\limits_{i\le m_t} \mathbf{H}(i) \ \&\ \nu(m_t) \notin E_t\}$,
- $\mathbf{nor}[t] = \log_4(\frac{\mathbf{H}(m_t)}{|E_t|})$.

The tree composition $\Sigma_{1.5.14}$ is natural: it gives non empty results for singletons only and then

$$\Sigma_{1.5.14}(t) = \{s \in K_{1.5.14} : \eta_s = \eta_t \ \&\ E_t \subseteq E_s\}.$$

Now check. $\square$

Conclusion 1.5.15. Suppose $\mathbf{H} \in \omega^\omega$ is strictly increasing, $\mathbf{H}(0) > 4$.

(1) The forcing notion $\mathbb{Q}_1^{\mathrm{tree}}(K_{1.5.14}, \Sigma_{1.5.14})$ is non-trivial, σ–$*$–linked and it adds a dominating real.

(2) Assume that $f : \omega \times \omega \longrightarrow \omega$ is a fast function such that

$$(\forall n, m < \omega)(f(n,m) < \log_4(\mathbf{H}(m))).$$

Then the forcing notion $\mathbb{Q}_f^{\mathrm{tree}}(K_{1.5.14}, \Sigma_{1.5.14})$ is non-trivial, σ–$*$–linked, Borel ccc, and it adds a dominating real.

(3) Suppose that $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ is a countable $\geq^*$–directed family such that

$$(\exists f \in \mathcal{F})(\forall^\infty n \in \omega)(f(n) < \log_4(\mathbf{H}(n))) \quad \text{and}$$
$$(\forall f \in \mathcal{F})(\exists g \in \mathcal{F})(\forall^\infty n \in \omega)(g(n) < f(n) - 1).$$

Then the forcing notion $\mathbb{Q}_{\mathcal{F}}^{\mathrm{tree}}(K_{1.5.14}, \Sigma_{1.5.14})$ is non-trivial, σ–$*$–linked, very Borel ccc, and it adds a Cohen real and is nice.

2. More constructions

In this section we introduce more schemes for building ccc forcing notions as well as more norm conditions that can be used in conjunctions with the methods presented in the previous section.

2.1. Mixtures with random

Definition 2.1.1. *Let* $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$. *We say that* $(K, \Sigma, \mathbf{F})$ *is* a mixing triple for $\mathbf{H}$ *if*

(a) (K, Σ) *is a (local) tree–creating pair for* $\mathbf{H}$*,*

(b) for each $\eta \in \bigcup\limits_{m\in\omega} \prod\limits_{i<m} \mathbf{H}(i)$ *there is* $t^*_\eta \in K$ *such that*

$$(\forall t \in \mathrm{LTCR}_\eta[\mathbf{H}] \cap K)(t \in \Sigma(t^*_\eta)),$$

(c) $\mathbf{F} = \langle F_\eta : \eta \in \bigcup\limits_{m\in\omega} \prod\limits_{i<m} \mathbf{H}(i)\rangle$*, where for each* $\eta \in \bigcup\limits_{m\in\omega} \prod\limits_{i<m} \mathbf{H}(i)$*:*

(d) $F_\eta : [0,1]^{\mathrm{pos}(t^*_\eta)} \longrightarrow [0,1]$*,*

(e) if $\langle r_\nu : \nu \in \mathrm{pos}(t^*_\eta)\rangle, \langle r'_\nu : \nu \in \mathrm{pos}(t^*_\eta)\rangle \in [0,1]^{\mathrm{pos}(t^*_\eta)}$*,* $r_\nu \le r'_\nu$ *for all* $\nu \in \mathrm{pos}(t^*_\eta)$*, then* $F_\eta(r_\nu : \nu \in \mathrm{pos}(t^*_\eta)) \le F_\eta(r'_\nu : \nu \in \mathrm{pos}(t^*_\eta))$*,*

(f) if $\langle r_\nu : \nu \in \mathrm{pos}(t^*_\eta)\rangle \in [0,1]^{\mathrm{pos}(t^*_\eta)}$*,* $\varepsilon > 0$*, then there are* $r'_\nu < r_\nu$ *(for* $\nu \in \mathrm{pos}(t^*_\eta)$*) such that for each* $\langle r''_\nu : \nu \in \mathrm{pos}(t^*_\eta)\rangle \in [0,1]^{\mathrm{pos}(t^*_\eta)}$ *satisfying* $r'_\nu < r''_\nu \le r_\nu$ *(for* $\nu \in \mathrm{pos}(t^*_\eta)$*) we have*

$$F_\eta(r_\nu : \nu \in \mathrm{pos}(t^*_\eta)) - \varepsilon < F_\eta(r''_\nu : \nu \in \mathrm{pos}(t^*_\eta)),$$

(g) if $r_\nu \ge \varepsilon > 0$ *for* $\nu \in \mathrm{pos}(t^*_\eta)$ *then* $F_\eta(r_\nu : \nu \in \mathrm{pos}(t^*_\eta)) \ge \varepsilon$*.*

Definition 2.1.2. *Let* $(K, \Sigma, \mathbf{F})$ *be a mixing triple for* $\mathbf{H}$*.*

(1) Let $T^* = T^*_{K,\Sigma} \subseteq \bigcup\limits_{m\in\omega} \prod\limits_{i<m} \mathbf{H}(i)$ *be a tree such that*

$$\mathrm{root}(T^*) = \langle\rangle \quad \textit{and} \quad (\forall \eta \in T^*)(\mathrm{succ}_{T^*}(\eta) = \mathrm{pos}(t^*_\eta)).$$

(2) If $X \subseteq \mathrm{pos}(t^*_\eta)$*,* $\eta \in T^*$ *and* $\langle r_\nu : \nu \in X\rangle \subseteq [0,1]$*, then we define* $F_\eta(r_\nu : \nu \in X)$ *as* $F_\eta(r^*_\nu : \nu \in \mathrm{pos}(t^*_\eta))$*, where*

$$r^*_\nu = \begin{cases} r_\nu & \textit{if } \nu \in X, \\ 0 & \textit{if } \nu \in \mathrm{pos}(t^*_\eta) \setminus X. \end{cases}$$

(3) Suppose that $p = \langle t^p_\eta : \eta \in T^p\rangle \in \mathbb{Q}^{\mathrm{tree}}_\emptyset(K, \Sigma)$ *and* $A \subseteq T^p$ *is a front of* T^p*. We let* $T[p, A] = \{\eta \in T^p : (\exists \rho \in A)(\eta \trianglelefteq \rho)\}$*, and we define* $\mu^{\mathbf{F}}_{p,A} = \mu_{p,A} : T[p, A] \longrightarrow [0,1]$ *by downward induction as follows:*
- *if* $\eta \in A$ *then* $\mu_{p,A}(\eta) = 1$*,*
- *if* $\mu_{p,A}(\nu)$ *has been defined for all* $\nu \in \mathrm{pos}(t^p_\eta)$*,* $\eta \in T[p, A] \setminus A$*, then we put* $\mu_{p,A}(\eta) = F_\eta(\mu_{p,A}(\nu) : \nu \in \mathrm{pos}(t^p_\eta))$*.*

(4) For $p = \langle t^p_\eta : \eta \in T^p\rangle \in \mathbb{Q}^{\mathrm{tree}}_\emptyset(K, \Sigma)$ *we define*

$$\mu^{\mathbf{F}}(p) = \inf\{\mu_{p,A}(\mathrm{root}(p)) : A \textit{ is a front of } T^p\}.$$

(5) Let $\mathbb{Q}^{\mathrm{mt}}_\emptyset(K, \Sigma, \mathbf{F}) = \{p \in \mathbb{Q}^{\mathrm{tree}}_\emptyset(K, \Sigma) : \mu^{\mathbf{F}}(p) > 0\}$ *be equipped with the partial order inherited from* $\mathbb{Q}^{\mathrm{tree}}_\emptyset(K, \Sigma)$*. Similarly we define forcing notions* $\mathbb{Q}^{\mathrm{mt}}_1(K, \Sigma, \mathbf{F})$*,* $\mathbb{Q}^{\mathrm{mt}}_f(K, \Sigma, \mathbf{F})$*,* $\mathbb{Q}^{\mathrm{mt}}_{\mathcal{F}}(K, \Sigma, \mathbf{F})$ *(for suitable* f *and* $\mathcal{F}$*).*

Definition 2.1.3. *A mixing triple* $(K, \Sigma, \mathbf{F})$ *is* ccc–complete *if*

(a) for each $\eta \in T^*_{K,\Sigma}$ *and* $A \subseteq \mathrm{pos}(t^*_\eta)$*, there is a unique tree–creature* $t_A \in \Sigma(t^*_\eta)$ *such that* $\mathrm{pos}(t_A) = A$*,*

(b) if $\eta \in T^*_{K,\Sigma}$*,* $A \subseteq B \subseteq \mathrm{pos}(t^*_\eta)$*, then* $t_A \in \Sigma(t_B)$ *and* $\mathbf{nor}[t_A] \le \mathbf{nor}[t_B]$*,*

(c) if $r_\nu = r'_\nu + r''_\nu$*,* $r_\nu, r'_\nu, r''_\nu \in [0,1]$ *(for* $\nu \in \mathrm{pos}(t^*_\eta)$*,* $\eta \in T^*_{K,\Sigma}$*), then*

$$F_\eta(r_\nu : \nu \in \mathrm{pos}(t^*_\eta)) = F_\eta(r'_\nu : \nu \in \mathrm{pos}(t^*_\eta)) + F_\eta(r''_\nu : \nu \in \mathrm{pos}(t^*_\eta)).$$

Lemma 2.1.4. *Let $(K, \Sigma, \mathbf{F})$ be a ccc–complete mixing triple for* $\mathbf{H}$. *Suppose that $p_0, \dots p_m \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ are such that $\sum\limits_{\ell\leq m} \mu^{\mathbf{F}}(p_\ell) > 1$ and* $\mathrm{root}(p_0) = \dots = \mathrm{root}(p_m)$. *Then for some $\ell < n \leq m$ the conditions p_ℓ, p_n are compatible in $\mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$.*

Proof. Let $\nu = \mathrm{root}(p_0) = \cdots = \mathrm{root}(p_m)$. For each $\ell < n \leq m$ such that $[T^{p_\ell}] \cap [T^{p_n}] \neq \emptyset$ choose a tree $T_{\ell,n} \subseteq T^*_{K,\Sigma}$ satisfying

$$\max(T_{\ell,n}) = \emptyset, \quad \mathrm{root}(T_{\ell,n}) = \nu \quad \text{and} \quad [T_{\ell,n}] = [T^{p_\ell}] \cap [T^{p_n}].$$

Let $p_{\ell,n} \in \mathbb{Q}^{\mathrm{tree}}_{\emptyset}(K, \Sigma)$ be such that $T^{p_{\ell,n}} = T_{\ell,n}$ (defined if $[T^{p_\ell}] \cap [T^{p_n}] \neq \emptyset$, $\ell < n \leq m$). Our aim is to show that for some $\ell < n \leq m$, $p_{\ell,n}$ is defined and belongs to $\mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ (i.e., $\mu^{\mathbf{F}}(p_{\ell,n}) > 0$). So suppose that for each $\ell < n \leq m$, either $[T^{p_\ell}] \cap [T^{p_n}] = \emptyset$ or $\mu^{\mathbf{F}}(p_{\ell,n}) = 0$. Let $\varepsilon = 2^{-(m+1)}(\sum\limits_{\ell\leq m} \mu^{\mathbf{F}}(p_\ell) - 1) > 0$ and let $p \in \mathbb{Q}^{\mathrm{tree}}_{\emptyset}(K, \Sigma)$ be such that $\mathrm{root}(p) = \nu$ and $T^p = T^{p_0} \cup \dots \cup T^{p_m}$ (clearly $\min\{\mu^{\mathbf{F}}(p_\ell) : \ell \leq m\} \leq \mu^{\mathbf{F}}(p) \leq 1$). Choose a front A of T^p such that for each $\ell < n \leq m$, if $p_{\ell,n}$ is defined and $A_{\ell,n} = A \cap T^{p_{\ell,n}}$, then $\mu_{p_{\ell,n},A_{\ell,n}}(\nu) < \varepsilon$, and if $[T^{p_\ell}] \cap [T^{p_n}] = \emptyset$ then $T^{p_\ell} \cap T^{p_n} \subseteq T[p, A] \setminus A$).

Claim 2.1.4.1. *For each $\eta \in T[p, A]$ we have*

$$(\otimes) \quad \mu_{p,A}(\eta) \geq \sum_{\ell\leq m} \mu_{p_\ell,A^\ell}(\eta) - \sum_{\ell<n\leq m} \mu_{p_{\ell,n},A_{\ell,n}}(\eta),$$

where $A^\ell = A \cap T^{p_\ell}$ (for $\ell \leq m$), and if $\ell < n \leq m$ and $p_{\ell,n}$ is not defined or $\eta \notin T[p_{\ell,n}, A_{\ell,n}]$ then we stipulate $\mu_{p_{\ell,n},A_{\ell,n}}(\eta) = 0$ (and similarly $\mu_{p_\ell,A^\ell}(\eta) = 0$ if $\eta \notin T[p_\ell, A^\ell]$).

Proof of the claim. We show this by downward induction on $\mathrm{lh}(\eta)$.

First suppose that $\eta \in A$. Let $k = |\{\ell \leq m : \eta \in A^\ell\}| = \sum\limits_{\ell\leq m} \mu_{p_\ell,A^\ell}(\eta)$. Then $\sum\limits_{\ell<n\leq m} \mu_{p_{\ell,n},A_{\ell,n}}(\eta) = \binom{k}{2}$ and

$$\sum_{\ell\leq m} \mu_{p_\ell,A^\ell}(\eta) - \sum_{\ell<n\leq m} \mu_{p_{\ell,n},A_{\ell,n}}(\eta) = k - \frac{k(k-1)}{2} \leq 1 = \mu_{p,A}(\eta).$$

Suppose now that $(\otimes)$ has been shown for all $\rho \in \mathrm{pos}(t^p_\eta)$, $\eta \in T[p, A] \setminus A$. Let

$$X = \{\rho \in \mathrm{pos}(t^p_\eta) : \sum_{\ell\leq m} \mu_{p_\ell,A^\ell}(\rho) \geq \sum_{\ell<n\leq m} \mu_{p_{\ell,n},A_{\ell,n}}(\rho)\}$$

and $Y = \mathrm{pos}(t^p_\eta) \setminus X$. It follows from the inductive hypothesis and 2.1.3(c) that

$$F_\eta(\mu_{p,A}(\rho) : \rho \in X) \geq \sum_{\ell\leq m} F_\eta(\mu_{p_\ell,A^\ell}(\rho) : \rho \in X) - \sum_{\ell<n\leq m} F_\eta(\mu_{p_{\ell,n},A_{\ell,n}}(\rho) : \rho \in X).$$

[Note that though 2.1.3(c) guarantees the additivity of F_η only when $r_\nu = r'_\nu + r''_\nu$, $r_\nu, r'_\nu, r''_\nu \in [0,1]$, we can first prove that

$$F_\eta(\frac{1}{M} \cdot r_\nu : \nu \in \mathrm{pos}(t^*_\eta)) = \frac{1}{M} F_\eta(r_\nu : \nu \in \mathrm{pos}(t^*_\nu)).$$

Next, we may reduce the needed additivity to the one postulated in 2.1.3(c) by dividing all terms by suitably large M.] Now, by 2.1.1(e),

$$\sum_{\ell \le m} F_\eta(\mu_{p_\ell, A^\ell}(\rho) : \rho \in Y) \le \sum_{\ell < n \le m} F_\eta(\mu_{p_{\ell,n}, A_{\ell,n}}(\rho) : \rho \in Y),$$

and hence

$$\begin{aligned} F_\eta(\mu_{p,A}(\rho) : \rho \in Y) \ge & \sum_{\ell \le m} F_\eta(\mu_{p_\ell, A^\ell}(\rho) : \rho \in Y) \\ & - \sum_{\ell < n \le m} F_\eta(\mu_{p_{\ell,n}, A_{\ell,n}}(\rho) : \rho \in Y). \end{aligned}$$

Since

$$\begin{aligned} \mu_{p,A}(\eta) &= F_\eta(\mu_{p,A}(\rho) : \rho \in X) + F_\eta(\mu_{p,A}(\rho) : \rho \in Y), \\ \mu_{p_\ell, A^\ell}(\eta) &= F_\eta(\mu_{p_\ell, A^\ell}(\rho) : \rho \in X) + F_\eta(\mu_{p_\ell, A^\ell}(\rho) : \rho \in Y), \\ \mu_{p_{\ell,n}, A_{\ell,n}}(\eta) &= F_\eta(\mu_{p_{\ell,n}, A_{\ell,n}}(\rho) : \rho \in X) + F_\eta(\mu_{p_{\ell,n}, A_{\ell,n}}(\rho) : \rho \in Y), \end{aligned}$$

we may easily finish. $\square$

Now we apply 2.1.4.1 to $\nu = \mathrm{root}(p_0)$. We get then

$$\begin{aligned} 1 \ge \mu_{p,A}(\nu) \ge & \sum_{\ell \le m} \mu_{p_\ell, A^\ell}(\nu) - \sum_{\ell < n \le m} \mu_{p_{\ell,n}, A_{\ell,n}}(\nu) \\ > & \sum_{\ell \le m} \mu^{\mathbf{F}}(p_\ell) - 2^{m+1} \cdot \varepsilon = 1, \end{aligned}$$

a contradiction. $\square$

Corollary 2.1.5. *Let $(K, \Sigma, \mathbf{F})$ be a ccc–complete mixing triple for* $\mathbf{H}$.

(1) The forcing notion $\mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ satisfies the ccc.
(2) If $f : \omega \times \omega \longrightarrow \omega$ is a fast function and (K, Σ) is linked, then the forcing notions $\mathbb{Q}^{\mathrm{mt}}_{f}(K, \Sigma, \mathbf{F})$ and $\mathbb{Q}^{\mathrm{mt}}_{1}(K, \Sigma, \mathbf{F})$ are ccc.
(3) If $h : \omega \times \omega \longrightarrow \omega$ is regressive, (K, Σ) is h–linked and $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ is either countable or $\ge^*$–directed, then $\mathbb{Q}^{\mathrm{mt}}_{\mathcal{F}}(K, \Sigma, \mathbf{F})$ is ccc.

Remark 2.1.6. (1) Forcing notions determined by mixing triples are in some sense mixtures of the random real forcing with forcings determined by tree–creating pairs. The "mt" in $\mathbb{Q}^{\mathrm{mt}}_{*}(K, \Sigma, \mathbf{F})$ stands for "**m**easured **t**ree".

(2) Because of 2.1.5(1) (and the proof of 2.1.4) we can be very generous as far as the demands on the norms are concerned, and still we may easily ensure that the resulting forcing notion satisfies the ccc. For example, if $(K, \Sigma, \mathbf{F})$ is a ccc–complete mixing triple, (K, Σ) is semi–linked in the sense that the demand of 1.2.3(1) is satisfied whenever $\mathrm{lh}(\eta)$ is even, and

$$\mathbb{Q}^{\mathrm{mt}}_{1/2}(K, \Sigma, \mathbf{F}) = \{p \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F}) : (\forall \eta \in [T^p])(\lim_{k\to\infty} \mathbf{nor}[t^p_{\eta\restriction 2k}] = \infty)\},$$

then $\mathbb{Q}^{\mathrm{mt}}_{1/2}(K, \Sigma, \mathbf{F})$ is ccc too.

(3) This type of constructions (i.e., mixture–like) for not-ccc case will be presented in [14] and [17, §2].

Let us finish this subsection with showing that the forcing notions $\mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ tend to have many features of the random real forcing.

Definition 2.1.7. *Let $(K, \Sigma, \mathbf{F})$ be a mixing triple, $p \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$.*

(1) A function $\mu : T^p \longrightarrow [0, 1]$ is a semi–$\mathbf{F}$–measure on p *if*

$$(\forall \eta \in T^p)\big(\mu(\eta) \leq F_\eta(\mu(\nu) : \nu \in \mathrm{pos}(t^p_\eta))\big).$$

(2) If above the equality holds (for each $\eta \in T^p$), then μ is called an $\mathbf{F}$–measure.

Proposition 2.1.8. *Assume $(K, \Sigma, \mathbf{F})$ is a mixing triple, $p \in \mathbb{Q}^{\mathrm{tree}}_{\emptyset}(K, \Sigma)$.*

(1) If $\mu : T^p \longrightarrow [0, 1]$ is semi–$\mathbf{F}$–measure on p, then for each $\eta \in T^p$ we have $\mu(\eta) \leq \mu^{\mathbf{F}}(p^{[\eta]})$.

(2) The mapping $\eta \mapsto \mu^{\mathbf{F}}(p^{[\eta]}) : T^p \longrightarrow [0, 1]$ is an $\mathbf{F}$–measure on p.

(3) If there is a semi–$\mathbf{F}$–measure μ on p such that $\mu(\mathrm{root}(p)) > 0$, then $p \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$.

Proof. Straightforward. □

Proposition 2.1.9. *Suppose that $(K, \Sigma, \mathbf{F})$ is a ccc–complete mixing triple, and $p_0, \ldots, p_m \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ are such that* $\mathrm{root}(p_0) = \cdots = \mathrm{root}(p_m)$. *Let $p \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ be such that $T^p = T^{p_0} \cup \cdots \cup T^{p_m}$. Then:*

(1) $\mu^{\mathbf{F}}(p) \leq \sum\limits_{\ell \leq m} \mu^{\mathbf{F}}(p_\ell)$,

(2) if $[T^{p_\ell}] \cap [T^{p_n}] = \emptyset$ for $\ell < n \leq m$ (or just $p_0, \ldots, p_m$ are pairwise incompatible in $\mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$), then $\mu^{\mathbf{F}}(p) = \sum\limits_{\ell \leq m} \mu^{\mathbf{F}}(p_\ell)$,

(3) $\{p_0, \ldots, p_m\}$ is pre-dense above p.

Proof. Like 2.1.4. □

Lemma 2.1.10. *Let $(K, \Sigma, \mathbf{F})$ be a ccc–complete mixing triple.*

(1) Suppose that conditions $p, q \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ are such that $\mathrm{root}(p) = \mathrm{root}(q)$, *$p \leq q$, $\mu^{\mathbf{F}}(q) < \mu^{\mathbf{F}}(p)$, and let $0 < \varepsilon < 1$. Then there is a condition $r \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ stronger than p and incompatible with q and such that* $\mathrm{root}(r) = \mathrm{root}(p)$, *$\mu^{\mathbf{F}}(r) \geq (1 - \varepsilon) \cdot (\mu^{\mathbf{F}}(p) - \mu^{\mathbf{F}}(q))$.*

(2) If $p \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$, $\varepsilon > 0$ then there is $\eta \in T^p$ such that $\mu^{\mathbf{F}}(p^{[\eta]}) > (1-\varepsilon)$.

Proof. 1) Let $\nu = \mathrm{root}(p)$. Choose a front A of T^p such that

$$\mu_{q,A\cap T^q}(\nu) < \mu^{\mathbf{F}}(q) + \varepsilon\cdot(\mu^{\mathbf{F}}(p) - \mu^{\mathbf{F}}(q)) < \mu^{\mathbf{F}}(p)$$

(so necessarily $A \setminus T^q \neq \emptyset$). Take $r_0, r_1 \in \mathbb{Q}^{\rm tree}_\emptyset(K,\Sigma)$ such that

$$\begin{aligned} T^{r_0} &= \{\eta \in T^p : (\exists \rho \in A\setminus T^q)(\eta \trianglelefteq \rho \text{ or } \rho \vartriangleleft \eta)\}\\ T^{r_1} &= \{\eta \in T^p : (\exists \rho \in A\cap T^q)(\eta \trianglelefteq \rho \text{ or } \rho \vartriangleleft \eta)\}. \end{aligned}$$

Clearly $\mathrm{root}(r_i) = \nu$, $A\cap T^{r_i}$ is a front of T^{r_i}, $A = (A\cap T^{r_0}) \cup (A\cap T^{r_1})$ and $T^q \subseteq T^{r_1}$. Hence, $r_1 \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$ and $\mu^{\mathbf{F}}(q) \le \mu^{\mathbf{F}}(r_1) < \mu^{\mathbf{F}}(q) + \varepsilon\cdot(\mu^{\mathbf{F}}(p) - \mu^{\mathbf{F}}(q))$. Now, using 2.1.3(c), we may conclude that $\mu^{\mathbf{F}}(r_0) \ge (1-\varepsilon)\cdot(\mu^{\mathbf{F}}(p) - \mu^{\mathbf{F}}(q))$, finishing the proof.
2) Straightforward. $\square$

Proposition 2.1.11. *Assume that $(K,\Sigma,\mathbf{F})$ is a ccc–complete mixing triple, $p \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$. Let $m < \omega$, $\varepsilon > 0$ and let $\dot\tau$ be a $\mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$–name such that $p \Vdash \dot\tau < m$. Then there are $X \subseteq m$ and conditions $q_\ell \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$ (for $\ell \in X$) such that*

(α) $q_\ell \Vdash \dot\tau = \ell$,
(β) $\mathrm{root}(q_\ell) = \mathrm{root}(p)$,
(γ) $\sum\limits_{\ell\in X} \mu^{\mathbf{F}}(q_\ell) \ge (1-\varepsilon)\mu^{\mathbf{F}}(p)$.

Proof. Let $\nu = \mathrm{root}(p)$. For each $\ell < m$ define $\mu_\ell : T^p \longrightarrow [0,1]$ by:

$$\mu_\ell(\eta) = \sup\{\mu^{\mathbf{F}}(q) : q \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F}) \;\&\; q \ge p^{[\eta]} \;\&\; \mathrm{root}(q) = \eta \;\&\; q \Vdash \dot\tau = \ell\}$$

(with the convention that $\sup\emptyset = 0$). It follows from 2.1.1(f) that each μ_ℓ is an $\mathbf{F}$–measure on p.

Claim 2.1.11.1. $\mu^{\mathbf{F}}(p) = \sum\limits_{\ell<m} \mu_\ell(\nu)$.

Proof of the claim. First note that, by a suitable modification of 2.1.4, we have $\mu^{\mathbf{F}}(p) \ge \sum\limits_{\ell<m}\mu_\ell(\nu)$. So suppose that $\mu^{\mathbf{F}}(p) > \sum\limits_{\ell<m}\mu_\ell(\nu)$, and let $n\in\omega$ be such that $(1-\frac{2}{n})\cdot\mu^{\mathbf{F}}(p) > \sum\limits_{\ell<m}\mu_\ell(\nu)$. Let

$$S = \{\eta \in T^p : (1-\frac{1}{n})\cdot\mu^{\mathbf{F}}(p^{[\eta]}) > \sum_{\ell<m}\mu_\ell(\eta)\},$$

and let $T \subseteq S$ be a tree such that $\mathrm{root}(T) = \nu$ and $\eta \in T$, $\rho \in \mathrm{succ}_{T^p}(\eta)\cap S$ imply $\rho \in T$. Clearly $\max(T) = \emptyset$, so we may choose $r \in \mathbb{Q}^{\rm tree}_\emptyset(K,\Sigma)$ so that $T = T^r$. If $\mu^{\mathbf{F}}(r) > 0$, then ($r \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$ and) we may choose a condition $q \in \mathbb{Q}^{\rm mt}_\emptyset(K,\Sigma,\mathbf{F})$ stronger than r which decides the value of $\dot\tau$. By 2.1.10(2) we find $\eta \in T^q$ such that $\mu^{\mathbf{F}}(q^{[\eta]}) > (1-\frac{1}{n}) \ge (1-\frac{1}{n})\mu^{\mathbf{F}}(p^{[\eta]})$, what contradicts $T^q \subseteq T^r \subseteq S$. Therefore $\mu^{\mathbf{F}}(r) = 0$, and like in 2.1.10(1) we may build a condition $q \ge p$ and a front A of T^q such that

- $\mathrm{root}(q) = \nu$, $A \cap T^r = \emptyset$,
- $T^q = \{\eta \in T^p : (\exists \rho \in A)(\eta \trianglelefteq \rho \text{ or } \rho \vartriangleleft \eta)\}$,
- $(1 - \frac{1}{n})\mu^{\mathbf{F}}(p^{[\eta]}) \leq \sum\limits_{\ell<m} \mu_\ell(\eta) \leq \mu^{\mathbf{F}}(p^{[\eta]})$ for each $\eta \in A$, and
- $\mu^{\mathbf{F}}(q) > (1 - \frac{1}{n})\mu^{\mathbf{F}}(p)$.

Now, we may easily conclude that $\sum\limits_{\ell<m} \mu_\ell(\nu) \geq (1 - \frac{2}{n}) \cdot \mu^{\mathbf{F}}(p)$, getting a contradiction. $\square$

The conclusion of the proposition follows immediately from 2.1.11.1 and the definition of μ_ℓ's (so we take $X = \{\ell < m : \mu_\ell(\nu) > 0\}$ and suitable q_ℓ's for $\ell \in X$). $\square$

Proposition 2.1.12. *Suppose that* $(K, \Sigma, \mathbf{F})$ *is a ccc–complete mixing triple and* $\dot{\tau}$ *is a* $\mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$*–name for an ordinal. Let* $p \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$, $0 < \varepsilon < 1$. *Then there is a condition* $q \geq p$ *and a front* A *of* T^q *such that*

(α) $\mathrm{root}(q) = \mathrm{root}(p)$, $\mu^{\mathbf{F}}(q) > (1 - \varepsilon)\mu^{\mathbf{F}}(p)$,
(β) *for each* $\eta \in A$ *the condition* $q^{[\eta]}$ *decides the value of* $\dot{\tau}$.

Proof. Let

$$B = \{\eta \in T^p : \text{for some } p^* \geq p \text{ we have: } \mathrm{root}(p^*) = \eta,\ \mu^{\mathbf{F}}(p^*) \geq (1 - \tfrac{\varepsilon}{2})\mu^{\mathbf{F}}(p^{[\eta]}), \text{ and } p^* \text{ decides } \dot{\tau} \text{ on a front}\}.$$

It follows from 2.1.10 that, if $q \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ is a condition stronger than p, then $T^q \cap B \neq \emptyset$. If $\mathrm{root}(p) \in B$, then we are clearly done, so suppose $\mathrm{root}(p) \notin B$. Note that if $\eta \in T^p \setminus B$ then $\mathrm{succ}_{T^p}(\eta) \setminus B \neq \emptyset$. Thus

$$T \stackrel{\mathrm{def}}{=} \{\eta \in T^p : (\forall \nu \trianglelefteq \eta)(\nu \notin B)\}$$

is a tree with $\max(T) = \emptyset$, $T \cap B = \emptyset$, $\mathrm{root}(T) = \mathrm{root}(p)$. This T determines a condition $r \in \mathbb{Q}^{\mathrm{tree}}_{\emptyset}(K, \Sigma)$, $\mathrm{root}(r) = \mathrm{root}(p)$. It follows from the previous remark that $\mu^{\mathbf{F}}(r) = 0$. Take a front A of T^p such that $\mu_{r, A\cap T^r}(\mathrm{root}(r)) < \frac{1}{4} \cdot \varepsilon \cdot \mu^{\mathbf{F}}(p)$ and $A \subseteq T^r \cup B$. For each $\nu \in A \setminus T^r$ fix a condition q_ν such that $p^{[\nu]} \leq q_\nu$, $\mu^{\mathbf{F}}(q_\nu) \geq (1 - \frac{\varepsilon}{2})\mu^{\mathbf{F}}(p^{[\nu]})$, and q_ν decides $\dot{\tau}$ on a front. Let q be such that

$$T^q = \{\eta \in T^p : (\exists \nu \in A \setminus T^r)(\eta \in T^{q_\nu} \text{ or } \nu \trianglelefteq \eta)\}.$$

It should be clear that q is a condition as required $\square$

2.2. *Exotic norm conditions*

The norm conditions introduced in the first section have their counterparts in the non-ccc case (as presented in [15]). Here we formulate more norm conditions which may be used to build ccc forcing notions from linked creating pairs (or tree–creating pairs), and which seem to be very ccc–specific. Let us start with a norm condition that allows us to include into our framework the "Mathias with ultrafilter" forcing notion.

Definition 2.2.1. *(1) A local creating pair (K, Σ) for* **H** *is* strongly linked *if it is full (see 1.1.2(5)), linked and*

$(\otimes)^{\rm sl}$ *there are $t^{\min}_\ell \in K$ (for $\ell < \omega$) such that $m^{t^{\min}_\ell}_{\rm dn} = \ell$ and if $t \in K$, $m^t_{\rm dn} = \ell$, then $t^{\min}_\ell \in \Sigma(t)$.*

If additionally for each $\ell < \omega$ we have

$$(\forall u \in {\rm dom}(\mathbf{val}[t^{\min}_\ell]))(\exists!\, v \in {\rm rng}(\mathbf{val}[t^{\min}_\ell]))(u \lhd v),$$

then we say that (K, Σ) is strongly$^+$ linked.

(2) A local tree–creating pair (K, Σ) for **H** *is* (tree-) strongly linked *if it is linked and*

$(\otimes)^{\rm sl}_{\rm tree}$ *there are $t^{\min}_\eta \in K \cap {\rm LTCR}_\eta[\mathbf{H}]$ (for $\eta \in \bigcup\limits_{m\in\omega}\prod\limits_{i<m}\mathbf{H}(i)$) such that if*

$$t \in K \cap {\rm LTCR}_\eta[\mathbf{H}],\ \eta \in \bigcup_{m\in\omega}\prod_{i<m}\mathbf{H}(i),\ \textit{then } t^{\min}_\eta \in \Sigma(t).$$

If, additionally, $|{\rm pos}(t^{\min}_\eta)| = 1$ for each η then we say that (K, Σ) is strongly$^+$ linked.

(3) Let D be a non-principal ultrafilter on ω. We define norm conditions $\mathcal{C}(D)$ and $\mathcal{C}^{\rm tree}(D)$ (for the contexts of creating pairs and tree–creating pairs, respectively) and the corresponding forcing notions $\mathbb{Q}^*_D(K, \Sigma)$, $\mathbb{Q}^{\rm tree}_D(K, \Sigma)$ as follows.

- *A sequence $\langle t_i : i < \omega\rangle$ satisfies $\mathcal{C}(D)$ if for some $\ell < \omega$ we have:*

$$(\forall i < \omega)(m^{t_i}_{\rm dn} = \ell + i) \quad \textit{and} \quad \lim_D \langle \mathbf{nor}[t_{j-\ell}] : \ell \le j < \omega\rangle = \infty.$$

For a local creating pair (K, Σ), $\mathbb{Q}^*_D(K, \Sigma)$ is the forcing notion

$$\mathbb{Q}^*_{\mathcal{C}(D)}(K, \Sigma) = \{p \in \mathbb{Q}^*_\emptyset(K, \Sigma) : \langle t^p_i : i < \omega\rangle \textit{ satisfies } \mathcal{C}(D)\}.$$

- *A system $\langle t_\eta : \eta \in T\rangle \subseteq {\rm LTCR}[\mathbf{H}]$ satisfies $\mathcal{C}^{\rm tree}(D)$ if T is a tree, $t_\eta \in {\rm LTCR}_\eta[\mathbf{H}]$ and ${\rm pos}(t_\eta) = {\rm succ}_T(\eta) \subseteq \prod\limits_{i\le {\rm lh}(\eta)} \mathbf{H}(i)$ for each $\eta \in T$, and $(\forall \eta \in [T])(\lim_D \langle \mathbf{nor}[t_{\eta\restriction k}] : {\rm lh}({\rm root}(T)) \le k < \omega\rangle = \infty)$.*
  For a local tree creating pair (K, Σ), $\mathbb{Q}^{\rm tree}_D(K, \Sigma)$ is the forcing notion

$$\mathbb{Q}^{\rm tree}_{\mathcal{C}(D)}(K, \Sigma) = \{p \in \mathbb{Q}^{\rm tree}_\emptyset(K, \Sigma) : \langle t^p_\eta : \eta \in T^p\rangle \textit{ satisfies } \mathcal{C}^{\rm tree}(D)\}.$$

Remark 2.2.2. Strongly linked (and especially strongly$^+$ linked) creating pairs resemble omittory creating pairs of [15, Def. 2.1.1] – in both cases the practical examples are such that we may "omit" some of the creatures from a condition p. Here the "omitting" is done by replacing t^p_i by the suitable $t^{\min}_\ell$ (see 2.4.5).

Proposition 2.2.3. *Let D be a non-principal ultrafilter on ω. If (K, Σ) is a local strongly linked creating pair (tree–creating pair, respectively), then the forcing notion $\mathbb{Q}^*_D(K, \Sigma)$ ($\mathbb{Q}^{\rm tree}_D(K, \Sigma)$, respectively) is σ–centered.*

Proof. Straightforward. □

Forcing notions $\mathbb{Q}^*_D(K, \Sigma)$, though similar to the Mathias forcing notion, do not have (in general) as many nice properties as this one. For example "deciding formulas without changing the root" may easily fail, even though we may have some kind of continuous reading of names.

Definition 2.2.4 (See [15, Def. 1.2.9]). *Let D be a non-principal ultrafilter on ω, (K,Σ) be a local strongly linked creating pair and $\dot\tau$ be a $\mathbb{Q}^*_D(K,\Sigma)$–name for an ordinal. We say that a condition $p\in\mathbb{Q}^*_D(K,\Sigma)$* approximates $\dot\tau$ at t^p_n *(or at n) whenever the following demand is satisfied:*

$(*)$ *for each $w_1\in\mathrm{pos}(w^p,t^p_0,\ldots,t^p_{n-1})$, if there is a condition $r\in\mathbb{Q}^*_D(K,\Sigma)$ stronger than p and such that $w^r=w_1$ and r decides the value of $\dot\tau$, then the condition $(w_1,t^p_n,t^p_{n+1},\ldots)$ decides the value of $\dot\tau$*

Proposition 2.2.5. *Assume that D is a Ramsey ultrafilter on ω and (K,Σ) is a local, really finitary and strongly$^+$ linked creating pair. Then for each $p\in\mathbb{Q}^*_D(K,\Sigma)$ and a name $\dot\tau$ for an ordinal, there is a condition $q\geq p$ which approximates $\dot\tau$ at every n and such that $w^p=w^q$.*

Proof. The proof follows the lines of the appropriate proof for the Mathias forcing notion (see e.g. [1, §7.4]). Let $\langle t^{\min}_\ell:\ell<\omega\rangle$ witness that (K,Σ) satisfies $(\otimes)^{\rm sl}$ of 2.2.1(1), $(\forall u\in\mathrm{dom}(\mathbf{val}[t^{\min}_\ell]))(\exists! v\in\mathrm{rng}(\mathbf{val}[t^{\min}_\ell]))(u\lhd v)$. For simplicity, we assume that $\mathbf{nor}[t^{\min}_\ell]\leq 1$ (for $\ell<\omega$).

For a condition $p\in\mathbb{Q}^*_D(K,\Sigma)$ and $n\in\omega$ let

$$\mathrm{supp}^n(p)\stackrel{\rm def}{=}\{m^{t^p_i}_{\rm dn}:i<\omega\ \&\ \mathbf{nor}[t^p_i]>n+1\}\in D.$$

Choose inductively conditions $p_n\in\mathbb{Q}^*_D(K,\Sigma)$ such that for each $n<\omega$:

(1) $p_0=p$, $p_n\leq p_{n+1}$, $w^{p_n}=w^p$, and $t^{p_{n+1}}_i=t^{p_n}_i$ for $i<n$,

(2) if $w\in\mathrm{pos}(w^p,t^p_0,\ldots,t^p_{n-1})$ and there is $p^*\geq p$ such that $w^{p^*}=w$ and p^* decides $\dot\tau$, then $(w,t^{p_{n+1}}_n,t^{p_{n+1}}_{n+1},\ldots)$ decides $\dot\tau$.

(Note that "strongly linked" implies that if $w^{q_0}=w^{q_1}$, then q_0,q_1 are compatible; also remember that (K,Σ) is full.) Since D is Ramsey, we may choose an increasing sequence $\langle i_n:n<\omega\rangle\subseteq\omega\setminus\mathrm{lh}(w^p)$ such that

$$\{i_n:n<\omega\}\in D\quad\text{ and }\quad(\forall n\in\omega)(i_n+2<i_{n+1}\in\mathrm{supp}^{n+1}(p_{i_n-\mathrm{lh}(w^p)+2})).$$

For $j<\omega$ let

$$t^q_j=\begin{cases} t^{p_{i_n-\mathrm{lh}(w^p)+2}}_j & \text{if } j=i_{n+1}-\mathrm{lh}(w^p),\ n\in\omega,\\ t^{\min}_j & \text{if } j+\mathrm{lh}(w^p)\notin\{i_{n+1}:n<\omega\}.\end{cases}$$

It should be clear that $q\stackrel{\rm def}{=}(w^p,t^q_0,t^q_1,\ldots)\in\mathbb{Q}^*_D(K,\Sigma)$ is a condition stronger than p and for every $w\in\mathrm{pos}(w^p,t^p_0,\ldots,t^p_{i_n-\mathrm{lh}(w^p)})$, $n\in\omega$ we have

$$(w,t^q_{i_n-\mathrm{lh}(w^p)+1},t^q_{i_n-\mathrm{lh}(w^p)+2},\ldots)\geq(w,t^{p_{i_n-\mathrm{lh}(w^p)+2}}_{i_n-\mathrm{lh}(w^p)+1},t^{p_{i_n-\mathrm{lh}(w^p)+2}}_{i_n-\mathrm{lh}(w^p)+2},\ldots).$$

Hence easily q approximates $\dot\tau$ at all points of the form $i_{n+1}-\mathrm{lh}(w^p)+1$ (for $n<\omega$), and by the additional demand on $t^{\min}_\ell$ (in "strongly$^+$") we conclude that q approximates $\dot\tau$ at all $n<\omega$. $\square$

Proposition 2.2.6. *Suppose that* (K, Σ) *is a strongly*$^+$ *linked local creating pair (with* $t_n^{\min}$ *witnessing this). Assume that* $\mathbf{nor}[t_n^{\min}] \le 1$ *(for* $n < \omega$*) and*

$(*)$ *for each* $n \in \omega$ *there are disjoint sets* $A_n, B_n \subseteq \mathbf{H}(n)$ *such that*

- *if* $\langle u, v\rangle \in \mathbf{val}[t_n^{\min}]$*, then* $v(n) \notin A_n \cup B_n$*,*
- *if* $t \in K$*,* $\mathbf{nor}[t] > 1$*,* $u \in \mathrm{dom}(\mathbf{val}[t])$*,* $\mathrm{lh}(u) = n$ *then there are* v_0, v_1 *such that* $\langle u, v_0\rangle, \langle u, v_1\rangle \in \mathbf{val}[t]$ *and* $v_1(n) \in A_n$ *and* $v_0(n) \in B_n$*.*

Let D *be an ultrafilter on* ω*. Then the forcing notion* $\mathbb{Q}^*_D(K, \Sigma)$ *adds a Cohen real.*

Proof. Let $\dot{W}$ be the name for $\mathbb{Q}^*_D(K, \Sigma)$–generic real and let $\dot{K} = \{\dot{k}_n : n < \omega\}$ be a name for a subset of ω such that

$$\dot{K} = \{k \in \omega : \dot{W}(k) \in A_k \cup B_k\}.$$

(Clearly $\dot{K}$ is infinite.) Let $\dot{c} \in 2^\omega$ be given by $\dot{c}(n) = 0$ if and only if $\dot{W}(\dot{k}_n) \in A_n$. It should be clear that $\dot{c}$ is a name for a Cohen real over $\mathbf{V}$. $\square$

Now we will give some norm conditions that can be used in the context of local and forgetful creating pairs. Note that if (K, Σ) is of that type, then for each $t \in K$ we have (unique) set $P_t \subseteq \mathbf{H}(m^t_{\mathrm{dn}})$ such that

$$\langle u, v\rangle \in \mathbf{val}[t] \quad \text{if and only if} \quad v(m^t_{\mathrm{dn}}) \in P_t,\ u \lhd v \ \text{ and } \ \mathrm{lh}(v) = \mathrm{lh}(u) + 1$$

(for some, equivalently all, $u \in \mathrm{dom}(\mathbf{val}[t])$). The set P_t corresponds to $\mathrm{pos}(t)$ in the tree–creatures context, and below we will use the notation $\mathrm{pos}(t)$ for it (hoping that this does not cause any confusion). Our next definition is a variant of 2.1.3(a,b) for the case of local forgetful creating pairs.

Definition 2.2.7. *A local forgetful creating pair* (K, Σ) *for* $\mathbf{H}$ *is* complete *if*

(a) for each $i \in \omega$ *and a nonempty set* $A \subseteq \mathbf{H}(i)$*, there is a unique creature* $t^i_A \in K$ *such that* $m^{t^i_A}_{\mathrm{dn}} = i$ *and* $\mathrm{pos}(t^i_A) = A$*,*

(b) if $i \in \omega$*,* $A \subseteq B \subseteq \mathbf{H}(i)$*, then* $t^i_A \in \Sigma(t^i_B)$ *and* $\mathbf{nor}[t^i_A] \le \mathbf{nor}[t^i_B]$*.*

Definition 2.2.8. *Let* $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$ *be such that* $(\forall n \in \omega)(|\mathbf{H}(n)| > 2^n)$ *and let* (K, Σ) *be a complete creating pair for* $\mathbf{H}$*.*

(1) A 1–norming system *(for* $\mathbf{H}$*) is a pair* $(\bar{K}, \bar{g})$ *such that*

(α) $\bar{K} = \langle K_\ell : \ell \in \omega\rangle$ *is a sequence of infinite pairwise disjoint subsets of* ω*,* $\min(K_\ell) \ge \ell$*,*

(β) $\bar{g} = \langle g_\rho : \rho \in 2^{<\omega}\rangle$*, where for each* $\ell < \omega$*:*

(γ) *if* $\rho \in 2^\ell$ *then* $g_\rho \in \prod\limits_{m \in K_\ell} \mathbf{H}(m)$*, and*

(δ) *for every* $m \in K_\ell$*, there are no repetitions in* $\langle g_\rho(m) : \rho \in 2^\ell\rangle$*.*

(2) Let $\mathcal{C}(\mathbf{nor})$ *be a norm condition for* K *and* $(\bar{K}, \bar{g})$ *be a* 1*–norming system. We define* $(\bar{K}, \bar{g})$–modified version $\mathcal{C}(\mathbf{nor})^{\bar{K},\bar{g}}$ of $\mathcal{C}(\mathbf{nor})$ *by*

a sequence $\langle t_i : i < \omega\rangle$ *satisfies* $\mathcal{C}(\mathbf{nor})^{\bar{K},\bar{g}}$ *if and only if* *it satisfies* $\mathcal{C}(\mathbf{nor})$ *and for some* $\rho_0, \ldots, \rho_k \in 2^\omega$, $k < \omega$ *we have*

$$(\forall i, \ell < \omega)(\forall \rho \in 2^\ell)([m^{t_i}_{\rm dn} \in K_\ell \ \&\ g_\rho(m^{t_i}_{\rm dn}) \notin {\rm pos}(t_i)] \Rightarrow \\ [\rho \vartriangleleft \rho_0 \vee \ldots \vee \rho \vartriangleleft \rho_k]).$$

(3) If $\mathcal{C}(\mathbf{nor})$ *is one of* $\mathcal{C}^\infty(\mathbf{nor})$, $\mathcal{C}^{\mathcal{F}}(\mathbf{nor})$ *or* $\mathcal{C}^f(\mathbf{nor})$ *(for suitable* $f, \mathcal{F}$*; see 1.1.5), then the forcing notions corresponding to* $(\bar{K}, \bar{g})$*–modified versions of* $\mathcal{C}(\mathbf{nor})$ *will be denoted by* $\mathbb{Q}^{\bar{K},\bar{g}}_\infty(K, \Sigma)$, $\mathbb{Q}^{\bar{K},\bar{g}}_{\mathcal{F}}(K, \Sigma)$, $\mathbb{Q}^{\bar{K},\bar{g}}_f(K, \Sigma)$*, respectively.*

Proposition 2.2.9. *Let* $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$ *be such that* $(\forall n \in \omega)(|\mathbf{H}(n)| > 2^n)$*. Assume that* (K, Σ) *is a complete creating pair, and* $(\bar{K}, \bar{g})$ *is a* 1*–norming system (for* $\mathbf{H}$*).*

(1) If $f : \omega \times \omega \longrightarrow \omega$ *is fast and* (K, Σ) *is linked, then* $\mathbb{Q}^{\bar{K},\bar{g}}_\infty(K, \Sigma)$ *and* $\mathbb{Q}^{\bar{K},\bar{g}}_f(K, \Sigma)$ *are* σ*-*$*$*–linked Souslin forcing notions.*

(2) Assume that $h : \omega \times \omega \longrightarrow \omega$ *is regressive and* $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ *is an* h*–closed family which is either countable or* $\geq^*$*–directed. Suppose* (K, Σ) *is local and* h*–linked. Then the forcing notion* $\mathbb{Q}^{\bar{K},\bar{g}}_{\mathcal{F}}(K, \Sigma)$ *is* σ*-*$*$*–linked, and if* $\mathcal{F}$ *is countable and* $\geq^*$*–directed, then* $\mathbb{Q}^{\bar{K},\bar{g}}_{\mathcal{F}}(K, \Sigma)$ *is also Souslin.*

Proof. Straightforward. □

Definition 2.2.10. *Let* (K, Σ) *be a local forgetful creating pair for* $\mathbf{H}$.

(1) A 2–norming system *is a sequence* $\bar{U} = \langle U_{\rho,k} : \rho \in 2^{<\omega} \ \&\ k < \omega\rangle$ *of pairwise disjoint infinite subsets of* ω *such that* ${\rm lh}(\rho) \leq \min(U_{\rho,k})$.

(2) For a norm condition $\mathcal{C}(\mathbf{nor})$ *and a* 2*–norming system* $\bar{U}$ *we define* $\bar{U}$–modified version $\mathcal{C}(\mathbf{nor})^{\bar{U}}$ of $\mathcal{C}(\mathbf{nor})$ *by*
a sequence $\langle t_i : i < \omega\rangle$ *satisfies* $\mathcal{C}(\mathbf{nor})^{\bar{U}}$ *if and only if*
it satisfies $\mathcal{C}(\mathbf{nor})$ *and for some* $\rho_0, \ldots, \rho_\ell \in 2^\omega$ *and* $k_0, \ldots, k_\ell < \omega$, $\ell < \omega$*, for every* $i, k < \omega$ *and* $\rho \in 2^{<\omega}$ *we have:*

$$[m^{t_i}_{\rm dn} \in U_{\rho,k} \ \&\ {\rm pos}(t_i) \neq \mathbf{H}(m^{t_i}_{\rm dn})] \\ \Rightarrow [\rho \vartriangleleft \rho_0 \vee \ldots \vee \rho \vartriangleleft \rho_\ell \text{ and } k \in \{k_0, \ldots, k_\ell\}].$$

We will use notation $\mathbb{Q}^{\bar{U}}_\infty(K, \Sigma)$, $\mathbb{Q}^{\bar{U}}_{\mathcal{F}}(K, \Sigma)$, $\mathbb{Q}^{\bar{U}}_f(K, \Sigma)$ *for the respective forcing notions (and suitable* $f, \mathcal{F}$*).*

Proposition 2.2.11. *Let* (K, Σ) *be a complete creating pair, and* $\bar{U}$ *be a* 2*–norming system.*

(1) If $f : \omega\times\omega \longrightarrow \omega$ *is fast and* (K, Σ) *is linked, then* $\mathbb{Q}^{\bar{U}}_\infty(K, \Sigma)$ *and* $\mathbb{Q}^{\bar{U}}_f(K, \Sigma)$ *are* σ*-*$*$*–linked Souslin forcing notions.*

(2) Assume that $h : \omega \times \omega \longrightarrow \omega$ *is regressive and* $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ *is an* h*–closed family which is either countable or* $\geq^*$*–directed. Suppose* (K, Σ) *is local and* h*–linked. Then the forcing notion* $\mathbb{Q}^{\bar{U}}_{\mathcal{F}}(K, \Sigma)$ *is* σ*-*$*$*–linked, and if* $\mathcal{F}$ *is countable and* $\geq^*$*–directed, then* $\mathbb{Q}^{\bar{U}}_{\mathcal{F}}(K, \Sigma)$ *is also Souslin.*

Proof. Straightforward. □

2.3. Universal forcing notions

Here we introduce constructions involving very peculiar norm conditions. As a matter of fact, norms are not important in that type of constructions, but they still provide examples. Prototypes for the method described here are the Universal Meager forcing notion $\mathbb{UM}$ and forcing notions related to variants of the PP–property (see [27, Ch.VI, §2.12], [15, Ch.7]).

Definition 2.3.1. *Let* (K, Σ) *be a tree–creating pair for* $\mathbf{H}$. A finite candidate for (K, Σ) *is a system* $\langle t_\eta : \eta \in \hat{S}\rangle$ *such that*

(i) $S \subseteq \bigcup\limits_{k \le \mathrm{lev}(S)} \prod\limits_{i<k} \mathbf{H}(i)$ *is a tree of height* $\mathrm{lev}(S) < \omega$*, each node in* S *has a successor at the last level* $\mathrm{lev}(S)$*,*
(ii) $\hat{S} = S \setminus \max(S)$ *(i.e., non-maximal nodes of* S*),*
(iii) $t_\eta \in \mathrm{LTCR}_\eta[\mathbf{H}] \cap K$ *and* $\mathrm{pos}(t_\eta) = \mathrm{succ}_S(\eta)$ *(for* $\eta \in \hat{S}$*).*

The collection of finite candidates for (K, Σ) *is denoted* $\mathrm{FC}(K, \Sigma)$*. It is equipped with the following order (similar to that of* $\mathbb{Q}^{\mathrm{tree}}_\emptyset(K, \Sigma)$*):*

$\langle t^0_\eta : \eta \in \hat{S}^0\rangle \le \langle t^1_\eta : \eta \in \hat{S}^1\rangle$ *if and only if* $\mathrm{lev}(S^0) \le \mathrm{lev}(S^1)$ *and*

$$(\forall \eta \in S^1)(\mathrm{lh}(\eta) < \mathrm{lev}(S^0) \quad\Rightarrow\quad \eta \in S^0 \ \&\ t^1_\eta \in \Sigma(t^0_\eta)).$$

Remark 2.3.2. Finite candidates for tree–creating pairs correspond to that for creating pairs (see 1.1.9). In general, finite candidates do not have to be finite (just the respective tree is of finite height), but if (K, Σ) is finitary then they are.

Definition 2.3.3. *Let* $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$. A universality parameter $\mathfrak{p}$ for $\mathbf{H}$ *is a tuple* $(K^{\mathfrak{p}}, \Sigma^{\mathfrak{p}}, \mathcal{F}^{\mathfrak{p}}, \mathcal{G}^{\mathfrak{p}}) = (K, \Sigma, \mathcal{F}, \mathcal{G})$ *such that*

(α) (K, Σ) *is a really finitary local tree–creating pair for* $\mathbf{H}$*,*
(β) $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ *is either countable or* $\le^*$*–directed [note the direction of the inequality!],*
(γ) *elements of* $\mathcal{G}$ *are quadruples* $(\langle t_\eta : \eta \in \hat{S}\rangle, n_{\mathrm{dn}}, n_{\mathrm{up}}, \bar{r})$ *such that*
- $\langle t_\eta : \eta \in \hat{S}\rangle \in \mathrm{FC}(K, \Sigma)$*,* $\mathrm{root}(S) = \langle\rangle$*,*
- $n_{\mathrm{dn}} \le n_{\mathrm{up}} \le \mathrm{lev}(S)$*,*
- $\bar{r} = \langle r_i : i \in \mathrm{dom}(\bar{r})\rangle$*,* $r_i \in \omega$*,* $\mathrm{dom}(\bar{r}) \subseteq [n_{\mathrm{dn}}, n_{\mathrm{up}}]$*,*

(δ) **if:**
- $(\langle t^0_\eta : \eta \in \hat{S}^0\rangle, n^0_{\mathrm{dn}}, n^0_{\mathrm{up}}, \bar{r}^0) \in \mathcal{G}$*,*
- $\langle t^1_\eta : \eta \in \hat{S}^1\rangle \in \mathrm{FC}(K, \Sigma)$*,* $\langle t^0_\eta : \eta \in \hat{S}^0\rangle \le \langle t^1_\eta : \eta \in \hat{S}^1\rangle$ *and*
- $\bar{r}^1 = \langle r^1_i : i \in \mathrm{dom}(\bar{r}^1)\rangle$*,* $r^1_i \in \omega$*,* $\mathrm{dom}(\bar{r}^0) \subseteq \mathrm{dom}(\bar{r}^1)$*, and* $r^0_i \le r^1_i$ *for* $i \in \mathrm{dom}(\bar{r}^0)$*, and*
- $n^1_{\mathrm{dn}} \le n^0_{\mathrm{dn}}$*,* $n^0_{\mathrm{up}} \le n^1_{\mathrm{up}} \le \mathrm{lev}(S^1)$ *and* $\mathrm{dom}(\bar{r}^1) \subseteq [n^1_{\mathrm{dn}}, n^1_{\mathrm{up}}]$*,*

then: $(\langle t^1_\eta : \eta \in \hat{S}^1\rangle, n^1_{\mathrm{dn}}, n^1_{\mathrm{up}}, \bar{r}^1) \in \mathcal{G}$*,*

(ε) *for some increasing function* $F = F^{\mathcal{G}} \in \omega^\omega$, **if:**
- $(\langle t^\ell_\eta : \eta \in \hat{S}^\ell\rangle, n^\ell_{\mathrm{dn}}, n^\ell_{\mathrm{up}}, \bar{r}^\ell) \in \mathcal{G}$ *(for* $\ell < 2$*),* $\mathrm{lev}(S^0) = \mathrm{lev}(S^1)$*,*
- $\langle t_\eta : \eta \in \hat{S}\rangle \in \mathrm{FC}(K, \Sigma)$*,* $\langle t_\eta : \eta \in \hat{S}\rangle \le \langle t^\ell_\eta : \eta \in \hat{S}^\ell\rangle$ *(for* $\ell < 2$*),*
- $\mathrm{lev}(S) < n^0_{\mathrm{dn}}$*,* $n^0_{\mathrm{up}} < n^1_{\mathrm{dn}}$*,* $F(n^1_{\mathrm{up}}) < \mathrm{lev}(S^1)$*,*

then: *there is* $(\langle t^*_\eta : \eta \in \hat{S}^*\rangle, n^*_{\rm dn}, n^*_{\rm up}, \bar{r}^*) \in \mathcal{G}$ *such that*

- $n^*_{\rm dn} = n^0_{\rm dn}$, $n^*_{\rm up} = F(n^1_{\rm up})$, $\mathrm{dom}(\bar{r}^*) = [n^*_{\rm dn}, n^*_{\rm up}]$, $\bar{r}^* \supseteq \bar{r}^0 \cup \bar{r}^1$, *and* $r^*_i \leq 1$ *for all* $i \in [n^*_{\rm dn}, n^*_{\rm up}] \setminus (\mathrm{dom}(\bar{r}^0) \cup \mathrm{dom}(\bar{r}^1))$,
- $\mathrm{lev}(S^*) = \mathrm{lev}(S^0) = \mathrm{lev}(S^1)$, $S \subseteq S^*$ *and* $t^*_\eta = t_\eta$ *for* $\eta \in \hat{S}$,
- $\langle t^*_\eta : \eta \in \hat{S}^*\rangle \leq \langle t^\ell_\eta : \eta \in \hat{S}^\ell\rangle$ *(for* $\ell < 2$*).*

Remark 2.3.4. In Definition 2.3.5 below, we may think about the forcing notion $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ in the following way. We have a criterion for "small trees" provided by $\mathcal{G}$ and $\mathcal{F}$ (these are $(\mathcal{G}, f)$–narrow trees, see 2.3.5(c)). We try to add a small tree that will almost cover all small trees from the ground model. So, naturally, a condition p consists of a small tree (it is the system $\langle t^p_\eta : \eta \in T^p\rangle$), in which some finite part ("below N^p") is declared to be fixed. Now, when we extend the condition p, we cannot change the tree below the level N^p, but above that level we may *increase* it. The function f^p controls in some sense the "smallness" of the tree T^p. See more later.

Definition 2.3.5. *Let* $\mathfrak{p} = (K, \Sigma, \mathcal{F}, \mathcal{G})$ *be a universality parameter for* $\mathbf{H}$. *We define a forcing notion* $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ *as follows.*
A condition *in* $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ *is a triple* $p = (N^p, \langle t^p_\eta : \eta \in T^p\rangle, f^p)$ *such that*

(a) $\langle t^p_\eta : \eta \in T^p\rangle \in \mathbb{Q}^{\rm tree}_\emptyset(K, \Sigma)$, $\mathrm{root}(T^p) = \langle\rangle$,
(b) $N^p \in \omega$, $f^p \in \mathcal{F}$,
(c) the system $\langle t^p_\eta : \eta \in T^p\rangle$ *is* $(\mathcal{G}, f^p)$–narrow, *which means:*

for infinitely many $n < \omega$, *for some* $(\langle t_\eta : \eta \in \hat{S}\rangle, n_{\rm dn}, n_{\rm up}, \bar{r}) \in \mathcal{G}$ *we have*

- $n_{\rm dn} = n$, *and* $(\forall i \in \mathrm{dom}(\bar{r}))(r_i \leq f^p(i))$ *and*
- *if* $\eta \in T^p$, $\mathrm{lh}(\eta) < \mathrm{lev}(S)$, *then* $\eta \in S$ *and* $t^p_\eta \in \Sigma(t_\eta)$.

The relation $\leq$ *on* $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ *is given by:*
$(N^0, \langle t^0_\eta : \eta \in T^0\rangle, f^0) \leq (N^1, \langle t^1_\eta : \eta \in T^1\rangle, f^1)$ *if and only if*

- $N^0 \leq N^1$, $\langle t^0_\eta : \eta \in T^0\rangle \geq \langle t^1_\eta : \eta \in T^1\rangle$ *(in* $\mathbb{Q}^{\rm tree}_\emptyset(K, \Sigma)$*), and*
- *if* $\eta \in T^1$, $\mathrm{lh}(\eta) < N^0$ *then* $\eta \in T^0$ *and* $t^0_\eta = t^1_\eta$, *and*
- $(\forall^\infty i \in \omega)(f^0(i) \leq f^1(i))$.

Proposition 2.3.6. *If* $\mathfrak{p} = (K, \Sigma, \mathcal{F}, \mathcal{G})$ *is a universality parameter, then* $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ *is a* σ*–centered forcing notion. If additionally* $\mathcal{F}$ *is countable* and $\leq^*$*–directed then* $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ *is Borel ccc.*

Proof. It is easy to check that the relation $\leq$ of $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ is transitive (so $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ is a forcing notion). Let us argue that it is σ–centered when $\mathcal{F}$ is countable (the case of $\leq^*$–directed $\mathcal{F}$ can be handled similarly).

For $\langle t_\eta : \eta \in S\rangle \in \mathrm{FC}(K, \Sigma)$, $f \in \mathcal{F}$ let

$$Q_f^{\langle t_\eta:\eta\in S\rangle} = \{p \in \mathbb{Q}^{\rm tree}(\mathfrak{p}) : N^p = \mathrm{lev}(S) \text{ and } S \subseteq T^p \text{ and } f^p = f \text{ and } (\forall \eta \in T^p)(\mathrm{lh}(\eta) < N^p \Rightarrow \eta \in S \;\&\; t^p_\eta = t_\eta)\}.$$

Claim 2.3.6.1. *Each $Q_f^{\langle t_\eta:\eta\in S\rangle}$ is a directed subset of* $\mathbb{Q}^{\rm tree}(\mathfrak{p})$.

Proof of the claim. Let $(N^\ell, \langle t^\ell_\eta : \eta \in T^\ell\rangle, f^\ell) \in Q_f^{\langle t_\eta:\eta\in S\rangle}$ (for $\ell < 2$). (Thus $N^\ell = \mathrm{lev}(S)$, $f^\ell = f$.)

Let $F^{\mathcal{G}} \in \omega^\omega$ be the increasing function given by 2.3.3(ε). Pick a sequence

$$\mathrm{lev}(S)+1 = n^{0,0}_{\rm dn} < n^{0,0}_{\rm up} < n^{1,0}_{\rm dn} < n^{1,0}_{\rm up} < \cdots < n^{0,k}_{\rm dn} < n^{0,k}_{\rm up} < n^{1,k}_{\rm dn} < n^{1,k}_{\rm up} < \cdots$$

such that $F^{\mathcal{G}}(n^{1,k}_{\rm up}) + 1 < n^{0,k+1}_{\rm dn}$ and (for $\ell < 2$ and $k \in \omega$)

$$(\langle t^\ell_\eta : \eta \in T^\ell \;\&\; \mathrm{lh}(\eta) < n^{\ell,k}_{\rm up}\rangle, n^{\ell,k}_{\rm dn}, n^{\ell,k}_{\rm up}, f{\restriction}[n^{\ell,k}_{\rm dn}, n^{\ell,k}_{\rm up}]) \in \mathcal{G}$$

(possible by the definition of the forcing $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ and 2.3.3(δ)). Now build inductively a system $\langle t^*_\eta : \eta \in T^*\rangle \in \mathbb{Q}^{\rm tree}_\emptyset(K, \Sigma)$ as follows.

We declare that $\mathrm{root}(T^*) = \mathrm{root}(S) = \langle\rangle$, and if $\eta \in T^*$, $\mathrm{lh}(\eta) < \mathrm{lev}(S)$, then $t^*_\eta = t_\eta$ and $\mathrm{succ}_{T^*}(\eta) = \mathrm{pos}(t^*_\eta)$.

Suppose we have defined T^* up to the level $n^{0,k}_{\rm dn} - 1$, so we know t^*_η for $\mathrm{lh}(\eta) < n^{0,k}_{\rm dn} - 1$. Let S_k be the tree of height $n^{0,k}_{\rm dn} - 1$ built from these t^*_η (so it is the respective "initial part" of our future T^*), and assume that $\langle t^*_\eta : \eta \in \hat{S}_k\rangle \le \langle t^\ell_\eta : \eta \in T^\ell\rangle$ (for $\ell = 0, 1$). Apply 2.3.3(ε) to get $\langle t^*_\eta : \eta \in \hat{S}_{k+1}\rangle$ such that

$$(\langle t^*_\eta : \eta \in \hat{S}_{k+1}\rangle, n^{0,k}_{\rm dn}, n^{1,k}_{\rm up}, f{\restriction}[n^{0,k}_{\rm dn}, n^{1,k}_{\rm up}]) \in \mathcal{G},$$

and $S_k \subseteq S_{k+1}$, $\mathrm{lev}(S_{k+1}) = n^{0,k+1}_{\rm dn} - 1$ and $\langle t^*_\eta : \eta \in \hat{S}_{k+1}\rangle \le \langle t^\ell_\eta : \eta \in T^\ell\rangle$ (for $\ell < 2$). We declare that T^* up to the level $n^{0,k+1}_{\rm dn} - 1$ is S_{k+1} (and the respective t^*_η are as chosen above).

Plainly, $(\mathrm{lev}(S), \langle t^*_\eta : \eta \in T^*\rangle, f) \in Q_f^{\langle t_\eta:\eta\in S\rangle}$ is a condition stronger than both $(N^0, \langle t^0_\eta : \eta \in T^0\rangle, f^0)$ and $(N^1, \langle t^1_\eta : \eta \in T^1\rangle, f^1)$. □

The rest should be clear. □

2.4. Examples

Example 2.4.1. Let $\mathbf{H}(i) = \omega$. Suppose that $\mathbf{D}, \mathbf{B}, \mathbf{S}$ are functions such that

- $\mathrm{dom}(\mathbf{D}) \subseteq \omega^{<\omega}$, $\mathbf{D}(\eta)$ is a non-principal ultrafilter on ω (for $\eta \in \mathrm{dom}(\mathbf{D})$),
- $\mathrm{dom}(\mathbf{S}) = \mathrm{dom}(\mathbf{B}) = \omega^{<\omega} \setminus \mathbf{D}(\eta)$ and for each $\eta \in \mathrm{dom}(\mathbf{B})$:

$$2 \le \mathbf{B}(\eta) \in \omega \cup \{\omega\}, \quad \mathbf{S}(\eta) = \langle s^\eta_k : k \in \mathbf{B}(\eta)\rangle \subseteq (0,1), \text{ and } \sum_{k\in\mathbf{B}(\eta)} s^\eta_k = 1.$$

We build a ccc–complete (see 2.1.3) mixing triple $(K_{2.4.1}, \Sigma_{2.4.1}, \mathbf{F}_{2.4.1})$ for $\mathbf{H}$ (for the parameters $\mathbf{B}, \mathbf{D}, \mathbf{S}$).

Construction. Let $K_{2.4.1}$ consist of all tree creatures $t \in \mathrm{LTCR}[\mathbf{H}]$ such that

- $\mathbf{dis}[t] = (n_t, \eta_t, A_t)$ for some $n_t \in \omega$, $\eta_t \in \prod_{i<n_t} \mathbf{H}(i)$ and $A_t \subseteq \omega$ such that

$$A_t \in \mathbf{D}(\eta_t) \quad \text{if } \eta_t \in \mathrm{dom}(\mathbf{D}), \qquad \text{and} \qquad \emptyset \ne A_t \subseteq \mathbf{B}(\eta_t) \quad \text{if } \eta_t \in \mathrm{dom}(\mathbf{B}),$$

- $\mathbf{nor}[t] = n_t$,
- $\mathbf{val}[t] = \{\langle \eta_t, \nu\rangle : \eta_t \vartriangleleft \nu \in \prod\limits_{i\leq n_t} \mathbf{H}(i) \ \&\ \nu(n_t) \in A_t\}$.

The operation $\Sigma_{2.4.1}$ is natural:

$$\Sigma_{2.4.1}(t) = \{s \in K_{2.4.1} : n_s = n_t \ \&\ \eta_s = \eta_t \ \&\ A_s \subseteq A_t\}.$$

For $\eta \in \prod\limits_{i<n} \mathbf{H}(i)$ let $t^*_\eta \in K_{2.4.1} \cap \mathrm{LTCR}_\eta[\mathbf{H}]$ be such that

$$A_{t^*_\eta} = \begin{cases} \omega & \text{if } \eta \in \mathrm{dom}(\mathbf{D}),\\ \mathbf{B}(\eta) & \text{if } \eta \in \mathrm{dom}(\mathbf{B}),\end{cases}$$

and for $\langle r_\nu : \nu \in \mathrm{pos}(t^*_\eta)\rangle \in [0,1]^{\mathrm{pos}(t^*_\eta)}$ let

$$F^{2.4.1}_\eta(r_\nu : \nu \in \mathrm{pos}(t^*_\eta)) = \begin{cases} \sum\limits_{k\in\mathbf{B}(\eta)} s^\eta_k \cdot r_{\eta^\frown\langle k\rangle} & \text{if } \eta \in \mathrm{dom}(\mathbf{B}),\\ \lim_{\mathbf{D}(\eta)} \langle r_{\eta^\frown\langle k\rangle} : k \in \omega\rangle & \text{if } \eta \in \mathrm{dom}(\mathbf{D}).\end{cases}$$

Let $\mathbf{F}_{2.4.1} = \langle F^{2.4.1}_\eta : \eta \in \bigcup\limits_{n\in\omega} \prod\limits_{i<n} \mathbf{H}(i)\rangle$. Check that $(K_{2.4.1}, \Sigma_{2.4.1}, \mathbf{F}_{2.4.1})$ is a ccc–complete mixing triple for $\mathbf{H}$. $\square$

Conclusion 2.4.2. Let $\mathbf{H}, \mathbf{D}, \mathbf{B}$ and $\mathbf{S}$ be as in 2.4.1. Then the forcing notion $\mathbb{Q}^{\mathrm{mt}}_\emptyset(K_{2.4.1}, \Sigma_{2.4.1}, \mathbf{F}_{2.4.1})$ (for the parameters $\mathbf{D}, \mathbf{B}, \mathbf{S}$) is ccc (and non-trivial).

Remark 2.4.3. If $\mathrm{dom}(\mathbf{D}) = \omega^{<\omega}$ then $\mathbb{Q}^{\mathrm{mt}}_\emptyset(K_{2.4.1}, \Sigma_{2.4.1}, \mathbf{F}_{2.4.1})$ is equivalent to the "Laver with ultrafilters" forcing notion.

If $\mathrm{dom}(\mathbf{D}) = \emptyset$ and $\mathbf{B}(\eta)$ is finite (for each η) then $\mathbb{Q}^{\mathrm{mt}}_\emptyset(K_{2.4.1}, \Sigma_{2.4.1}, \mathbf{F}_{2.4.1})$ is the random real forcing (with weights determined by $\mathbf{S}$ in an obvious way).

Between these two extremes we have cases of "mixtures of random with ultrafilters" and our next observation applies to most of them. It could be formulated with a larger generality (e.g. regarding $\mathrm{dom}(\mathbf{D})$), what should be clear after reading the proof.

Proposition 2.4.4. *Let $\mathbf{H}(n) = \omega$ for $n \in \omega$, and let $X \subseteq \omega$ be an infinite co-infinite set. Suppose that*

(a) $\mathbf{D}$ is a function such that $\mathrm{dom}(\mathbf{D}) = \{\eta \in \omega^{<\omega} : \mathrm{lh}(\eta) \in X\}$ and $\mathbf{D}(\eta)$ is a non-principal Ramsey ultrafilter on ω (for $\eta \in \mathrm{dom}(\mathbf{D})$),
(b) $\bar{n} = \langle n_\ell : \ell \in \omega \setminus X\rangle$ is a sequence of integers, $n_\ell \geq 1$,
(c) $\mathbf{B}, \mathbf{S}$ are functions such that $\mathrm{dom}(\mathbf{B}) = \mathrm{dom}(\mathbf{S}) = \{\eta \in \omega^{<\omega} : \mathrm{lh}(\eta) \notin X\}$, $\mathbf{B}(\eta) = n_{\mathrm{lh}(\eta)}$ and $\mathbf{S}(\eta) = \langle \frac{1}{n_{\mathrm{lh}(\eta)}} : k < n_{\mathrm{lh}(\eta)}\rangle$.

Let $(K_{2.4.1}, \Sigma_{2.4.1}, \mathbf{F}_{2.4.1})$ be the mixing triple built for the parameters $\mathbf{D}, \mathbf{B}, \mathbf{S}$ as in 2.4.1, and let $f \in \omega^\omega$ be such that $f(n) \geq 2$ (for $n \in \omega$).

Then for every $\mathbb{Q}^{\mathrm{mt}}_\emptyset(K_{2.4.1}, \Sigma_{2.4.1}, \mathbf{F}_{2.4.1})$–name $\dot\tau$ for a real in $\prod\limits_{i\in\omega} f(i)$, there are an increasing sequence $\langle m_j : j \in \omega\rangle \subseteq \omega$ and a function $g \in \prod\limits_{i\in\omega} f(i)$ such that

$$\Vdash_{\mathbb{Q}^{\mathrm{mt}}_\emptyset(K_{2.4.1},\Sigma_{2.4.1},\mathbf{F}_{2.4.1})} \text{“ } (\forall^\infty j \in \omega)(\dot\tau{\restriction}[m_j, m_{j+1}) \neq g{\restriction}[m_j, m_{j+1})) \text{ ”}.$$

Hence, in particular, forcing with $\mathbb{Q}^{\rm mt}_{\emptyset}(K_{2.4.1}, \Sigma_{2.4.1}, \mathbf{F}_{2.4.1})$ *does not add Cohen reals (but it clearly adds a dominating real).*

Proof. For notational convenience, let $(K_{2.4.1}, \Sigma_{2.4.1}, \mathbf{F}_{2.4.1}) = (K, \Sigma, \mathbf{F})$.

Note that we may assume that $f(n) > 2^{2^{n+2}}$ (as we may work with the mapping $n \mapsto f{\restriction}[k_n, k_{n+1})$ for some increasing $\langle k_n : n < \omega\rangle$ instead). Let $X = \{x_m : m < \omega\}$ be the increasing enumeration, and let m_k be defined by: $m_0 = 0$ and $m_{k+1} = m_k + 2^k \cdot (1 + \prod\limits_{\ell \in x_k \setminus X} n_\ell)^k$ (for $k \in \omega$). [Here we keep the convention that if $x_k \setminus X = \emptyset$, then $\prod\limits_{\ell \in x_k \setminus X} n_\ell = 1$; or just assume that $x_0 > 0$.]

Let $\dot{\tau}$ be a $\mathbb{Q}^{\rm mt}_{\emptyset}(K, \Sigma, \mathbf{F})$–name for an element of $\prod\limits_{i \in \omega} f(i)$, and let p be a condition in $\mathbb{Q}^{\rm mt}_{\emptyset}(K, \Sigma, \mathbf{F})$. We may assume that $\mathrm{lh}(\mathrm{root}(p)) \in X$, and just for simplicity let $\mathrm{lh}(\mathrm{root}(p)) = x_0$.

We define inductively a tree $T \subseteq T^p$, mappings $Y : T \cap \bigcup\limits_{i<\omega} \omega^{x_i+1} \longrightarrow [\omega]^\omega$, $\pi : T \cap \bigcup\limits_{i<\omega} \omega^{x_i+1} \longrightarrow \omega$, a function $g \in \prod\limits_{i \in \omega} f(i)$, and a system $\langle q_\eta : \eta \in T \cap \bigcup\limits_{i<\omega} \omega^{x_i+1}\rangle$ of conditions from $\mathbb{Q}^{\rm mt}_{\emptyset}(K, \Sigma, \mathbf{F})$.

We declare $\mathrm{root}(T) = \mathrm{root}(p)$. Using 2.1.11 and 2.1.9 we choose an increasing sequence of conditions $\langle q^k_{\mathrm{root}(T)} : k < \omega\rangle$ and values for $g(m_k) < f(m_k)$ (thus defining $g{\restriction}\{m_k : k \in \omega\}$) such that

- $q^0_{\mathrm{root}(T)} \geq p$, $\mathrm{root}(q^k_{\mathrm{root}(T)}) = \mathrm{root}(T)$ and $\mu^{\mathbf{F}}(q^k_{\mathrm{root}(T)}) > \frac{3}{4}\mu^{\mathbf{F}}(p)$,
- $q^k_{\mathrm{root}(p)} \Vdash \dot{\tau}(m_k) \neq g(m_k)$.

Since $\mathbf{D}(\mathrm{root}(T))$ is a Ramsey ultrafilter we may choose a set $\{a_k : k < \omega\} \in \mathbf{D}(\mathrm{root}(T))$ (the increasing enumeration) such that

$$\mathrm{root}(T)^\frown\langle a_k\rangle \in T^{q^k_{\mathrm{root}(T)}} \quad \text{and} \quad \mu^{\mathbf{F}}((q^k_{\mathrm{root}(T)})^{[\mathrm{root}(T)^\frown\langle a_k\rangle]}) > \frac{3}{4}\mu^{\mathbf{F}}(p).$$

We declare that $\mathrm{root}(T)^\frown a_k \in T$ for all $k < \omega$ and we let

$$\pi(\mathrm{root}(T)^\frown\langle a_k\rangle) = k, \quad q_{\mathrm{root}(T)^\frown\langle a_k\rangle} = (q^k_{\mathrm{root}(T)})^{[\mathrm{root}(T)^\frown\langle a_k\rangle]}.$$

Next we choose pairwise disjoint sets $Y(\mathrm{root}(T)^\frown\langle a_k\rangle) \subseteq \omega \setminus \{m_\ell : \ell \in \omega\}$ such that $m_{k+1} \cap Y(\mathrm{root}(T)^\frown\langle a_k\rangle) = \emptyset$ and

$$(\forall \ell < \omega)(|Y(\mathrm{root}(T)^\frown\langle a_k\rangle) \cap (m_{\ell+k+1}, m_{\ell+k+2})| = 1).$$

Suppose now that we have already defined $T \cap \omega^{x_i+1}$, $i < \omega$, together with $Y(\eta) \subseteq \omega \setminus \{m_k : k \in \omega\}$, $\pi(\eta) \in \omega$, $q_\eta \in \mathbb{Q}^{\rm mt}_{\emptyset}(K, \Sigma, \mathbf{F})$ (for $\eta \in T \cap \omega^{x_j+1}$, $j \leq i$) and

$$g_i \stackrel{\rm def}{=} g{\restriction}(\{m_k : k < \omega\} \cup \bigcup\{Y(\eta) : \eta \in T \cap \omega^{x_j+1}, j < i\})$$

so that the following conditions are satisfied:

(α) if $\nu \vartriangleleft \eta$, and $\pi(\nu), \pi(\eta)$ are defined, then $\pi(\nu) < \pi(\eta)$,
(β) if $\nu \in T \cap \omega^{x_j}$, $j \leq i$, then there are no repetitions in the sequence $\langle \pi(\eta) : \nu \vartriangleleft \eta \in T \cap \omega^{x_j+1} \rangle$,
(γ) if $\eta \in T \cap \omega^{x_j+1}$, $j \leq i$, then

$$Y_\eta \cap m_{\pi(\eta)+1} = \emptyset \quad \text{and} \quad (\forall \ell < \omega)(|Y_\eta \cap [m_{\pi(\eta)+\ell+1}, m_{\pi(\eta)+\ell+2})| = 1),$$

(δ) the (defined) $Y(\eta)$'s are pairwise disjoint,
(ε) for each $\eta \in T \cap \omega^{x_i+1}$ we have: $\mathrm{root}(q_\eta) = \eta$ and

$$q_\eta \Vdash (\forall k \leq \pi(\eta))(g_i{\restriction}[m_k, m_{k+1}) \nsubseteq \dot{\tau}),$$

(ζ) if $q \in \mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ is a condition such that $\mathrm{root}(q) = \mathrm{root}(T)$, $T^q \cap \omega^{x_i+1} = T \cap \omega^{x_i+1}$ and $q^{[\eta]} = q_\eta$ for all $\eta \in \omega^{x_i+1}$, then $\mu^{\mathbf{F}}(q) \geq (\frac{1}{2} + \frac{1}{2^{i+2}})\mu^{\mathbf{F}}(p)$.

[Check that these conditions are satisfied at the first stage of the construction.]
Note that it follows from clauses (α) and (β) that for each k we have

$$|\{\eta \in T \cap \omega^{\leq x_i+1} : \pi(\eta) \leq k\}| \leq 2^k(1 + \prod_{\ell \in x_i \setminus X} n_\ell)^k,$$

and hence (by clause (γ))

$$(\boxtimes_k) \qquad |[m_{k+1}, m_{k+2}) \cap \bigcup\{Y(\eta) : \eta \in T \cap \omega^{\leq x_i+1}\}| \leq 2^k(1 + \prod_{\ell \in x_i \setminus X} n_\ell)^k.$$

Fix $\eta \in T \cap \omega^{x_i+1}$ (and note that $\pi(\eta) \geq i$). Choose an increasing sequence $\langle q^k_\eta : \pi(\eta) < k < \omega \rangle$ of conditions in $\mathbb{Q}^{\mathrm{mt}}_{\emptyset}(K, \Sigma, \mathbf{F})$ and values for $g{\restriction}Y_\eta$ such that

- $q^k_\eta \geq q_\eta$, $\mathrm{root}(q^k_\eta) = \eta$ and $\mu^{\mathbf{F}}(q^k_\eta) > (1 - 2^{-(i+4)})\mu^{\mathbf{F}}(q_\eta)$,
- if $j \in Y_\eta \cap [m_k, m_{k+1})$, $\pi(\eta) < k < \omega$, then $q^k_\eta \Vdash \dot{\tau}(j) \neq g(j)$,
- the sequence $\langle T^{q^k_\eta} \cap \omega^{x_{i+1}} : \pi(\eta) < k < \omega \rangle$ is constant (and let $\{\nu_\ell : \ell < \ell^*\}$ be the enumeration of $T^{q^k_\eta} \cap \omega^{x_{i+1}}$; necessarily $\ell^* \leq \prod\limits_{k=x_i+1}^{x_{i+1}-1} n_k$),
- $r_\ell \stackrel{\mathrm{def}}{=} \lim\limits_{k \to \infty} \mu^{\mathbf{F}}((q^k_\eta)^{[\nu_\ell]}) > 0$ for each $\ell < \ell^*$ (note that the limit exists as the sequence is non-increasing).

[Why possible? Use 2.1.11 and 2.1.9, remember our additional assumption on f.]
Choose $a^\ell_k \in \omega$ (for $\ell < \ell^*$ and $k > \pi(\eta)$) so that:

- $\{a^\ell_k : \pi(\eta) < k < \omega\} \in \mathbf{D}(\nu_\ell)$,
- $\nu_\ell{}^\frown\langle a^\ell_k \rangle \in T^{q^k_\eta}$ and $\mu^{\mathbf{F}}((q^k_\eta)^{[\eta^\frown\langle a^\ell_k \rangle]}) > (1 - 2^{-(i+4)})r_\ell$

(remember that each $\mathbf{D}(\nu_\ell)$ is a Ramsey ultrafilter). Now we declare that $\nu_\ell{}^\frown a^\ell_k \in T$ for all $\ell < \ell^*$ and $\pi(\eta) < k < \omega$ and we let

$$\pi(\nu_\ell{}^\frown\langle a^\ell_k \rangle) = k, \quad q_{\nu_\ell{}^\frown\langle a^\ell_k \rangle} = (q^k_\eta)^{[\nu_\ell{}^\frown\langle a^\ell_k \rangle]}.$$

This finishes the definitions of $T \cap \omega^{\leq x_{i+1}+1}$ and of $\pi(\nu), q_\nu$ for $\nu \in T \cap \omega^{x_{i+1}+1}$. It should be clear that (the respective variants of) clauses (α), (β), (ε) and (ζ) are satisfied. Using $(\boxtimes_k)$ we may easily choose sets $Y(\nu)$ (for $\nu \in T \cap \omega^{x_{i+1}+1}$) so that the demands (γ), (δ) hold. The construction is finished.

The tree T is perfect and it determines a condition $q \in \mathbb{Q}^{\rm tree}_\emptyset(K, \Sigma)$.

Claim 2.4.4.1. $\mu^{\mathbf{F}}(q) > 0$ *(so* $q \in \mathbb{Q}^{\rm mt}_\emptyset(K, \Sigma, \mathbf{F})$*).*

Proof of the claim. First note that the clause (ζ) is not enough to show this, as there are fronts in T^q which are not included in any $T^q \cap \omega^{\leq x_i+1}$. However, we may use the specific way the construction was carried out to build a semi–**F**–measure $\mu : T^q \longrightarrow [0, 1]$ such that $\mu(\mathrm{root}(q)) = \frac{3}{8}\mu^{\mathbf{F}}(p) > 0$ (what is enough by 2.1.8). So, if $\eta \in T \cap \omega^{x_i+1}$, $i < \omega$, then we let $\mu(\eta) = (1 - \frac{1}{2^{i+1}})\mu^{\mathbf{F}}(q_\eta)$; if $\eta \in T$, $x_i + 1 < \mathrm{lh}(\eta) \leq x_{i+1}$ then $\mu(\eta) = F_\eta(\mu(\nu) : \nu \in \mathrm{succ}_T(\eta))$. Now check. $\square$

Thus q is a condition stronger than p and it forces that $(\forall k \in \omega)(\dot\tau{\restriction}[m_k, m_{k+1}) \neq g{\restriction}[m_k, m_{k+1}))$. Since $\mathbb{Q}^{\rm mt}_\emptyset(K, \Sigma, \mathbf{F})$ satisfies the ccc, we may easily finish. $\square$

Our next example is a small modification of 1.5.14. In a similar way we may modify other examples from the previous section to produce more strongly linked creating pairs.

Example 2.4.5. Let $\mathbf{H} \in \omega^\omega$ be a strictly increasing function such that $\mathbf{H}(0) > 2$. We construct a really finitary, strongly$^+$ linked creating pair $(K_{2.4.5}, \Sigma_{2.4.5})$ for $\mathbf{H}$ which satisfies the demands of 2.2.6 (in particular $(*)$ there).

Construction. The family $K_{2.4.5}$ consists of creatures $t \in \mathrm{CR}[\mathbf{H}]$ such that

- $\mathbf{dis}[t] = (m_t, E_t)$ such that $m_t < \omega$ and $\emptyset \neq E_t \subseteq \mathbf{H}(m_t) \setminus \{0\}$,
- $\mathbf{val}[t] = \{\langle u, v\rangle \in \prod\limits_{i<m_t} \mathbf{H}(i) \times \prod\limits_{i\leq m_t} \mathbf{H}(i) : u \lhd v \;\&\; v(m_t) \notin E_t\}$,
- $\mathbf{nor}[t] = \log_4(\frac{\mathbf{H}(m_t)}{|E_t|})$.

The operation $\Sigma_{2.4.5}$ is natural:

$$\Sigma_{2.4.5}(t) = \{s \in K_{2.4.5} : m_s = m_t \;\&\; E_t \subseteq E_s\}.$$

Now check. $\square$

Conclusion 2.4.6. Suppose that $\mathbf{H} \in \omega^\omega$ is strictly increasing, $\mathbf{H}(0) > 2$ and $(K_{2.4.5}, \Sigma_{2.4.5})$ is built as in 2.4.5 for $\mathbf{H}$. Let D be a Ramsey ultrafilter on ω. Then the forcing notion $\mathbb{Q}^*_D(K_{2.4.5}, \Sigma_{2.4.5})$ is σ–centered, adds a Cohen real and adds a dominating real.

Now we turn to universality parameters. As said before, one of the prototypes here is the Universal Meager forcing notion $\mathbb{UM}$. Let us represent it as $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ (for a suitable $\mathfrak{p}$).

Example 2.4.7. We construct a universality parameter $\mathfrak{p}$ such that $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ is equivalent to $\mathbb{UM}$.

Construction. Let $\mathbf{H} : \omega \longrightarrow \omega \setminus 2$ and let K consists of tree creatures t for $\mathbf{H}$ such that

- $\mathbf{dis}[t] = (m_t, \eta_t, A_t)$ for some $m_t < \omega$, $\eta_t \in \bigcup\limits_{i<m_t} \mathbf{H}(i)$ and $\emptyset \neq A_t \subseteq \mathbf{H}(m_t)$,
- $\mathbf{nor}[t] = |A_t|$,
- $\mathbf{val}[t] = \{\langle \eta_t, \eta_t{}^\frown\langle a\rangle\rangle : a \in A_t\}$.

The operation Σ is natural, so $s \in \Sigma(t)$ if and only if $\eta_s = \eta_t$ and $A_s \subseteq A_t$. Let $\mathcal{F} = \{f\}$, $f(i) = 2$.

$\mathcal{G}$ consists of quadruples $(\langle t_\eta : \eta \in \hat{S}\rangle, n_{\rm dn}, n_{\rm up}, \bar{r})$ such that

- $\langle t_\eta : \eta \in \hat{S}\rangle \in \mathrm{FC}(K, \Sigma)$,
- $n_{\rm dn} \leq n_{\rm up} \leq \mathrm{lev}(S)$,
- $\bar{r} = \langle r_i : i \in \mathrm{dom}(\bar{r})\rangle$, $r_i < \omega$, $\mathrm{dom}(\bar{r}) \subseteq [n_{\rm dn}, n_{\rm up}]$,
- if $\eta \in S$, $\mathrm{lh}(\eta) = n_{\rm dn}$ then for some $\nu \in S$ we have $\eta \trianglelefteq \nu$, $\mathrm{lh}(\nu) < n_{\rm up}$ and $\mathbf{nor}[t_\nu] < \mathbf{H}(\mathrm{lh}(\nu))$.

Easily $\mathfrak{p} = (K, \Sigma, \mathcal{F}, \mathcal{G})$ is a universality parameter and $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ is the Universal Meager forcing notion, $\square$

Remark 2.4.8. Our next example 2.4.9 captures a number of constructions related to the PP–property. Under the assumptions on (K, Σ) as there, we may think that we have a way to measure how large splittings are, and this fully determines what are the tree-creatures in K (and what are the norms). The function $\mathbf{F}$ is used to define (possibly totally not related) norms of sets of nodes of the same length. Thus $\mathbf{F}$ may just count how many elements are in $\max(S)$ (in this case the universality parameter given by 2.4.9 is related to the PP–property). Other possibilities for $\mathbf{F}$ include taking the maximum value of $\mathbf{nor}[t_\eta]$, or taking the product of all relevant $\mathbf{nor}[t_\eta]$'s.

Example 2.4.9. Assume that $\mathbf{H} : \omega \longrightarrow \omega \setminus 2$ is strictly increasing and a family $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ is either countable or $\leq^*$–directed [note the direction of the inequality]. Let (K, Σ) be a local, really finitary, tree–creating pair for $\mathbf{H}$ such that

- for each $\eta \in \prod\limits_{i<n} \mathbf{H}(i)$, $n < \omega$ and a nonempty $A \subseteq \mathbf{H}(n)$ there is a unique $t_{\eta,A} \in \mathrm{LTCR}_\eta[\mathbf{H}] \cap K$ with $\mathrm{pos}(t_{\eta,A}) = \{\eta{}^\frown\langle a\rangle : a \in A\}$, and
- if $|A| = 1$, then $\mathbf{nor}[t_{\eta,A}] \leq 1$, and
- if $\emptyset \neq B \subseteq A \subseteq \mathbf{H}(n)$, then $t_{\eta,B} \in \Sigma(t_{\eta,A})$ and $\mathbf{nor}[t_{\eta,B}] \leq \mathbf{nor}[t_{\eta,A}]$.

Furthermore, let $\mathbf{F} : \mathrm{FC}(K, \Sigma) \longrightarrow \mathbb{R}^{\geq 0}$ be such that

if $\langle t_\eta : \eta \in \hat{S}\rangle \in \mathrm{FC}(K, \Sigma)$, $\mathbf{nor}[t_\eta] \leq 1$ (for $\eta \in \hat{S}$), then $\mathbf{F}(\langle t_\eta : \eta \in \hat{S}\rangle) \leq 1$.

We construct $\mathcal{G} = \mathcal{G}^{K,\Sigma}_{\mathbf{F}}$ such that $(K, \Sigma, \mathcal{F}, \mathcal{G})$ is a universality parameter.

Construction. For a nonempty set $Y \subseteq \prod\limits_{j<i} \mathbf{H}(j)$, $i < \omega$, we define $\mathrm{NOR}(Y)$ as the value of $\mathbf{F}(\langle t_\eta : \eta \in \hat{S}\rangle)$ for S such that $\max(S) = Y$ and $\mathrm{root}(S) = \langle\rangle$. (Note that under our assumptions on (K, Σ) there is exactly one such $\langle t_\eta : \eta \in \hat{S}\rangle \in \mathrm{FC}(K, \Sigma)$.)

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Let $\mathcal{G}$ consists of all quadruples $(\langle t_\eta : \eta \in \hat{S}\rangle, n_{\rm dn}, n_{\rm up}, \bar{r})$ satisfying the demands of 2.3.3(γ) and such that

for some sequence $\langle Y_i : i \in \mathrm{dom}(\bar{r})\rangle$ we have

- $Y_i \subseteq \prod\limits_{j<i} \mathbf{H}(j)$, $\mathrm{NOR}(Y_i) \leq r_i$,
- $(\forall \eta \in S \cap \prod\limits_{j<n_{\rm up}} \mathbf{H}(j))(\exists i \in \mathrm{dom}(\bar{r}))(\eta{\restriction}i \in Y_i)$.

Now check. $\square$

Example 2.4.10. A universality parameter $\mathfrak{p}$ such that $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ is the "universal closed measure zero" forcing notion.

Construction. Let $\mathbf{H}$, (K, Σ) and $\mathcal{F}$ be as defined in the construction for 2.4.7.

Let $\mathcal{G}$ consists of all quadruples $(\langle t_\eta : \eta \in \hat{S}\rangle, n_{\rm dn}, n_{\rm up}, \bar{r})$ satisfying the demands of 2.3.3(γ) and such that

$$\frac{|S \cap \prod\limits_{i<n_{\rm up}} \mathbf{H}(i)|}{|\prod\limits_{i<n_{\rm up}} \mathbf{H}(i)|} \leq \sum_{i \in \mathrm{dom}(\bar{r})} \frac{1}{(i+1)^2}.$$

Let $\mathfrak{p} = (K, \Sigma, \mathcal{F}, \mathcal{G})$. Note that the forcing notion $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ is equivalent to $\mathbb{Q}$ defined as follows.

A condition in $\mathbb{Q}$ is a pair (N, T) such that $N < \omega$ and $T \subseteq \bigcup\limits_{n<\omega} \prod\limits_{i<n} \mathbf{H}(i)$ is a tree such that $[T]$ is a measure zero subset of $\prod\limits_{i<\omega} \mathbf{H}(i)$;

the order of $\mathbb{Q}$ is the natural one: $(N_0, T_0) \leq (N_1, T_1)$ if and only if $N_0 \leq N_1$, $T_0 \subseteq T_1$ and $T_1 \cap \prod\limits_{i<N_0} \mathbf{H}(i) \subseteq T_0$. $\square$

3. Interlude: ideals

Here we introduce σ–ideals determined by forcing notions discussed in this paper. Most of the content of this part is well known and belongs to folklore (some of this material is presented in Judah and Rosłanowski [7], [6]).

3.1. Generic ideals

We will show how a Souslin ccc forcing notion adding one real produces a ccc Borel σ–ideal on some Polish space. While we could do this in a larger generality (e.g., considering any name for a real, not only the ones of the form specified in 3.1.1(3) below, compare [22, §4] and [19, §6, §7]), we have decided to use the specific form of the forcing notions we want to deal with and simplify the notation and arguments (loosing slightly on generality, but it will be clear how possible generalizations go).

Context 3.1.1. (1) $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$, $|\mathbf{H}(n)| \geq 2$ for all $n \in \omega$; $\mathcal{T}_n = \prod\limits_{i<n} \mathbf{H}(i)$ (for $n \in \omega$) and $\mathcal{T} = \bigcup\limits_{n<\omega} \mathcal{T}_n$. Let $\mathcal{X} = \prod\limits_{n\in\omega} \mathbf{H}(n) = [\mathcal{T}]$ be equipped with the natural product (Polish) topology.

(2) $\mathbb{P}$ is a Souslin ccc forcing notion with a parameter $r \in 2^\omega$ (which also encodes $\mathbf{H}$), so we have $\mathbf{\Sigma}^1_1$–formulas $\varphi_0(\cdot, r)$, $\varphi_1(\cdot,\cdot,r)$, $\varphi_2(\cdot,\cdot,r)$ defining $\mathbb{P}$, $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$, respectively.
(3) $\dot{W} = \langle p_\eta : \eta \in \mathcal{T}\rangle \subseteq \mathbb{P}$ is such that
  (α) $p_{\langle\rangle} = \emptyset_{\mathbb{P}}$, and if $\eta \lhd \nu \in \mathcal{T}$ then $p_\eta \leq_{\mathbb{P}} p_\nu$,
  (β) $\langle p_{\eta^\frown\langle a\rangle} : a \in \mathbf{H}(\mathrm{lh}(\eta))\rangle$ is a maximal antichain above p_η,
  (γ) for each $p \in \mathbb{P}$ there is $n < \omega$ such that

$$|\{\eta \in \mathcal{T}_n : p, p_\eta \text{ are compatible }\}| \geq 2,$$

  (δ) if $p, q \in \mathbb{P}$ are incompatible, then there is $\eta \in \mathcal{T}$ such that p, p_η are compatible but q, p_η are incompatible.
  [We will treat $\dot{W}$ as a $\mathbb{P}$–name for a real in $\mathcal{X}$ such that $p_\eta \Vdash \eta \lhd \dot{W}$, and so $\Vdash \dot{W} \notin \mathbf{V}$.]

Definition 3.1.2. *For $\mathbb{P}$, $\dot{W}$, $\mathbf{H}$, $\mathcal{X}$ as in 3.1.1, let $\mathcal{I}_{\mathbb{P},\dot{W}}$ be the collection of all Borel subsets B of $\mathcal{X}$ such that*

$$\Vdash_{\mathbb{P}} \text{“ } \dot{W} \notin B \text{ ”}.$$

Proposition 3.1.3. *(1) $\mathcal{I}_{\mathbb{P},\dot{W}}$ is a ccc σ–ideal of Borel subsets of $\mathcal{X}$.*
(2) $\mathcal{I}_{\mathbb{P},\dot{W}}$ contains all singletons.
(3) Let $\psi(p,c)$ be the formula

$$\text{“ } p \in \mathbb{P} \text{ and } c \in 2^\omega \text{ is a Borel code (for a set } \sharp c \subseteq \mathcal{X}) \text{ and } p \Vdash \dot{W} \in \sharp c \text{ ”}.$$

  (a) If M is a transitive model of ZFC*, *$p, r, c, \dot{W} \in M$, then*

$$\psi(p,c) \quad \Leftrightarrow \quad M \models \psi(p,c).$$

  (b) There are a $\mathbf{\Sigma}^1_2$–formula ψ_0 and a $\mathbf{\Pi}^1_2$–formula ψ_1 (both with the parameter r) such that

$$\psi(p,c) \equiv \psi_0(p,c) \equiv \psi_1(p,c)$$

  (i.e., the equivalences are provable in ZFC*).*
(4) The formula “$c \in 2^\omega$ is a Borel code (for a set $\sharp c \subseteq \mathcal{X}$) and $\sharp c \in \mathcal{I}_{\mathbb{P},\dot{W}}$” is absolute between transitive models of ZFC* *(containing r, $\dot{W}$, c).*

Proof. (1), (2) Straightforward.
(3) See (the proof of) [1, Lemma 3.6.12].
(4) Follows from (3) and the definition of $\mathcal{I}_{\mathbb{P},\dot{W}}$ (remember that “being a maximal antichain of $\mathbb{P}$” is absolute; see [1, Lemma 3.6.4]). □

Definition 3.1.4. *Let $\mathbb{P}$, $\dot{W}$, $\mathbf{H}$, $\mathcal{X}$ be as in 3.1.1.*

(1) Let $x \in \mathcal{X}$ and M be a transitive model of ZFC*, *$r, \dot{W} \in M$. We say that x is $\mathcal{I}_{\mathbb{P},\dot{W}}$–generic over M, if $x \notin B$ for every Borel set B from $\mathcal{I}_{\mathbb{P},\dot{W}}$ coded in M.*
(2) For a condition $p \in \mathbb{P}$ let

$$S(p) = S_{\dot{W}}(p) \stackrel{\text{def}}{=} \{\eta \in \mathcal{T} : p_\eta, p \text{ are compatible }\}.$$

(3) For a maximal antichain $\mathcal{A} \subseteq \mathbb{P}$ let

$$B_{\mathcal{A}} = B_{\dot{W},\mathcal{A}} \stackrel{\text{def}}{=} \{x \in \mathcal{X} : (\forall p \in \mathcal{A})(\exists n < \omega)(x{\restriction}n \notin S(p))\}.$$

Let $\mathcal{I}^0_{\mathbb{P},\dot{W}}$ be the family of all subsets of $\mathcal{X}$ that can be covered by a set of the form $B_{\mathcal{A}}$ (for a maximal antichain $\mathcal{A} \subseteq \mathbb{P}$).

Proposition 3.1.5. *(a) For each $p \in \mathbb{P}$, $S(p)$ is a perfect subtree of $\mathcal{T}$. If $p \le q$, then $S(q) \subseteq S(p)$. If $p, q \in \mathbb{P}$ and $S(q) \subseteq S(p)$, then $q \Vdash p \in \Gamma_{\mathbb{P}}$.*
(b) If $G \subseteq \mathbb{P}$ is a generic filter over $\mathbf{V}$, *then* $\mathbf{V}[G] = \mathbf{V}[\dot{W}^G]$.
(c) $\mathcal{I}^0_{\mathbb{P},\dot{W}}$ is an ideal of subsets of $\mathcal{X}$; sets $B_{\mathcal{A}}$ (for a maximal antichain $\mathcal{A} \subseteq \mathbb{P}$) are $\mathbf{\Pi}^0_2$.
(d) Let $x \in \mathcal{X}$ and M be a transitive model of ZFC*, *r, $\dot{W} \in M$. Then x is $\mathcal{I}_{\mathbb{P},\dot{W}}$–generic over M if and only if there is a $\mathbb{P}^M$–generic filter $G \subseteq \mathbb{P}^M$ over M such that $\dot{W}^G = x$.*
(e) $\mathcal{I}_{\mathbb{P},\dot{W}}$ is the σ–ideal generated by $\mathcal{I}^0_{\mathbb{P},\dot{W}}$. Every set from $\mathcal{I}_{\mathbb{P},\dot{W}}$ can be covered by a $\mathbf{\Sigma}^0_3$ set from $\mathcal{I}_{\mathbb{P},\dot{W}}$.

Proof. Straightforward (or see [7, §2]). □

Conclusion 3.1.6. The quotient algebra $\mathbf{Borel}(\mathcal{X})/\mathcal{I}_{\mathbb{P},\dot{W}}$ is a ccc complete Boolean algebra. The mapping

$$\pi : \mathbb{P} \longrightarrow \mathbf{Borel}(\mathcal{X})/\mathcal{I}_{\mathbb{P},\dot{W}} : p \mapsto \big[[S(p)]\big]_{\mathcal{I}_{\mathbb{P},\dot{W}}}$$

satisfies:

(1) $\mathrm{rng}(\pi)$ is a dense subset of the algebra $\mathbf{Borel}(\mathcal{X})/\mathcal{I}_{\mathbb{P},\dot{W}}$,
(2) $(\forall p, q \in \mathbb{P})(p \perp q \;\Leftrightarrow\; \pi(p) \cap \pi(q) = \mathbf{0})$,
(3) $(\forall p, q \in \mathbb{P})(q \Vdash p \in \Gamma_{\mathbb{P}} \;\Leftrightarrow \pi(q) \subseteq \pi(p))$.

Consequently, the complete Boolean algebra $\mathrm{RO}(\mathbb{P})$ (of regular open subsets of $\mathbb{P}$) determined by $\mathbb{P}$ is isomorphic to $\mathbf{Borel}(\mathcal{X})/\mathcal{I}_{\mathbb{P},\dot{W}}$. Moreover, π maps $\dot{W}$ onto the canonical name for the generic real in $\mathbf{Borel}(\mathcal{X})/\mathcal{I}_{\mathbb{P},\dot{W}}$, so for a Borel code c (for a Borel subset $\sharp c$ of $\mathcal{X}$) we have $[\![\dot{W} \in \sharp c]\!]_{\mathrm{RO}(\mathbb{P})} = [\sharp c]_{\mathcal{I}_{\mathbb{P},\dot{W}}}$.

Remark 3.1.7. It follows from 3.1.6 that we have nice description of names for reals in the extensions via $\mathbb{P}$.

(1) If $\dot{\tau}$ is a $\mathbb{P}$–name for an element of $\mathcal{X}$, then there is a Borel function $f : \mathcal{X} \longrightarrow \mathcal{X}$ such that $\Vdash_{\mathbb{P}} f(\dot{W}) = \dot{\tau}$.
(2) If $\dot{B}$ is a $\mathbb{P}$–name for a Borel subset of $\mathcal{X}$, then there is a Borel set $A \subseteq \mathcal{X} \times \mathcal{X}$ such that $\Vdash_{\mathbb{P}} \dot{B} = (A)_{\dot{W}}$, where $(A)_x = \{y : (x, y) \in A\}$.

(See [1, Lemma 3.7.1].)

3.2. Universality ideals

For a forcing notion $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ (where $\mathfrak{p}$ is a universality parameter) we may consider the ccc ideal defined as in 3.1.2, however there is another Borel σ–ideal related to $\mathbb{Q}^{\rm tree}(\mathfrak{p})$ (justifying the term "universality forcing notion").

Definition 3.2.1. *Let $\mathfrak{p} = (K, \Sigma, \mathcal{F}, \mathcal{G})$ be a universality parameter for* **H**.

(1) We say that $\mathfrak{p}$ is suitable *whenever:*

(a) for every $f \in \mathcal{F}$ and $n < \omega$ there is $N > n$ such that **if** $(\langle t_\eta : \eta \in \hat{S}\rangle, n_{\rm dn}, n_{\rm up}, f{\restriction}[n_{\rm dn}, n_{\rm up}]) \in \mathcal{G}$, $N \le n_{\rm dn}$ *and* $\eta \in \prod\limits_{i<n} \mathbf{H}(i)$, **then** $(\exists \nu \in \prod\limits_{i<{\rm lev}(S)} \mathbf{H}(i))(\eta \lhd \nu \ \&\ \nu \notin S)$,

(b) for every $f \in \mathcal{F}$ and $n < \omega$ there is $N > n$ such that **if** $\langle t_\eta : \eta \in \hat{S}\rangle \in {\rm FC}(K, \Sigma)$, ${\rm root}(S) = \langle\rangle$, ${\rm lev}(S) = n$, $\eta \in \prod\limits_{i<N} \mathbf{H}(i)$ *and* $\eta{\restriction}n \in S$, **then** *there is* $(\langle t^*_\nu : \nu \in \hat{S}^*\rangle, n_{\rm dn}, n_{\rm up}, f{\restriction}[n_{\rm dn}, n_{\rm up}]) \in \mathcal{G}$ *such that* $n < n_{\rm dn} \le n_{\rm up} < N$, $S \subseteq S^*$ *and* $t_\nu = t^*_\nu$ *for* $\nu \in \hat{S}$ *and* $\eta \in S^*$.

(2) $\mathcal{I}^0_\mathfrak{p}$ is the collection of subsets A of $\prod\limits_{i<\omega} \mathbf{H}(i)$ such that for some $f \in \mathcal{F}$ and a $(\mathcal{G}, f)$–narrow system $\langle t_\eta : \eta \in T\rangle \in \mathbb{Q}^{\rm tree}_\emptyset(K, \Sigma)$ (see 2.3.5(c)) with ${\rm root}(T) = \langle\rangle$ *we have $A \subseteq [T]$.*

Trees T as above will be called $(\mathcal{G}, f)$–narrow.

(3) $\mathcal{I}_\mathfrak{p}$ is the σ–ideal of subsets of $\prod\limits_{i<\omega} \mathbf{H}(i)$ generated by $\mathcal{I}^0_\mathfrak{p}$.

(4) $\dot{T}_\mathfrak{p}$ is a $\mathbb{Q}^{\rm tree}(\mathfrak{p})$–name such that

$$\Vdash_{\mathbb{Q}^{\rm tree}(\mathfrak{p})} \dot{T}_\mathfrak{p} = \bigcup\{T^p \cap \prod_{i<N^p} \mathbf{H}(i) : p \in \Gamma_{\mathbb{Q}^{\rm tree}(\mathfrak{p})}\}.$$

Remark 3.2.2. The universality parameters constructed in Examples 2.4.7, 2.4.10 are suitable. Also the parameters given by Example 2.4.9 can be easily made suitable: condition 3.2.1(1)(b) easily follows. To ensure 3.2.1(1)(a) we may require that **F**, $\mathcal{F}$ satisfy

$(*)$ if $n < \omega$, $f \in \mathcal{F}$, $\eta \in \prod\limits_{i<n} \mathbf{H}(i)$ and $Y \subseteq \prod\limits_{i\le n} \mathbf{H}(i)$ satisfy ${\rm NOR}(Y) \le f(n)$, then $(\exists y \in \mathbf{H}(n))(\eta^\frown\langle y\rangle \notin Y)$.

(But clearly much less is enough.)

Proposition 3.2.3. *Let $\mathfrak{p} = (K, \Sigma, \mathcal{F}, \mathcal{G})$ be a suitable universality parameter for* **H**.

(1) Every set in $\mathcal{I}^0_\mathfrak{p}$ is nowhere dense (in the product topology of $\prod\limits_{i<\omega} \mathbf{H}(i)$); all singletons belong to $\mathcal{I}^0_\mathfrak{p}$.

(2) $\mathcal{I}_\mathfrak{p}$ is a proper Borel σ–ideal of subsets of $\prod\limits_{i<\omega} \mathbf{H}(i)$.

(3) If $\mathcal{F}$ is $\le^*$–directed then $\mathcal{I}^0_\mathfrak{p}$ is an ideal.

(4) In $\mathbf{V}^{\mathbb{Q}^{\rm tree}(\mathfrak{p})}$, $\dot{T}_{\mathfrak{p}}$ *is a tree with no maximal branches which is* $(\mathcal{G}, h)$*–narrow for some function* h *(possibly* $h \notin \mathcal{F}$*). If* $\mathcal{F}$ *is a singleton, then* $h \in \mathcal{F}$*.*

(5) Suppose that $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ *is finite support iteration of ccc forcing notions such that for some increasing sequence* $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \delta$, $\dot{\mathbb{Q}}_{\alpha_n}$ *is (forced to be)* $\mathbb{Q}^{\rm tree}(\mathfrak{p})$*. Let* $\dot{T}_n$ *be the name for the tree* $\dot{T}_{\mathfrak{p}}$ *added at stage* α_n*. Then, in* $\mathbf{V}^{\mathbb{P}_\delta}$*, if* $T \in \mathbf{V}$ *is* $(\mathcal{G}, f)$*–narrow for some* $f \in \mathcal{F}$ *then* $T \subseteq \dot{T}_n$ *for some* $n < \omega$*.*

(6) If in 5 above we additionally assume that $\mathcal{F} = \{f\}$ *then, in* $\mathbf{V}^{\mathbb{P}_\delta}$*, there is a set from* $\mathcal{I}_{\mathfrak{p}}$ *which contains all Borel sets from* $\mathcal{I}_{\mathfrak{p}}$ *coded in* $\mathbf{V}$*.*

4. Sweet and Sour

4.1. On amalgamation

One of the major problems in set theory of the 20th century was the following: suppose $(\mathbb{P}_1, \leq_{\mathbb{P}_1})$, $(\mathbb{P}_2, \leq_{\mathbb{P}_2})$ are ccc partial orders and $\mathbb{P}_1 \times \mathbb{P}_2$ is equipped with the coordinate-wise order (so $(p_1, p_2) \leq (q_1, q_2)$ iff $p_1 \leq_{\mathbb{P}_1} q_1$ & $p_2 \leq_{\mathbb{P}_2} q_2$). Does $(\mathbb{P}_1 \times \mathbb{P}_2, \leq)$ satisfy the ccc? The interest in this question came from topology as well as from its close relation to the Souslin Problem. The intensive investigations of the productivity of ccc for forcing notions (or Boolean algebras, or topologies) resulted in many interesting concepts and discoveries. Several *strong ccc* properties implying that the product of respective partial orders satisfies the ccc were introduced. Examples of those properties include

- **The Knaster Property:** A partial order $(\mathbb{P}, \leq_{\mathbb{P}})$ has *the Knaster property* if for every uncountable family $\mathcal{A} \subseteq \mathbb{P}$ there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that members of $\mathcal{B}$ are pairwise compatible.
- **The pre-caliber ω_1:** $(\mathbb{P}, \leq_{\mathbb{P}})$ has *pre-caliber* ω_1 if for every uncountable $\mathcal{A} \subseteq \mathbb{P}$ there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that every finite subfamily of $\mathcal{B}$ has a common upper bound in $\mathbb{P}$.

(Note that pre-caliber $\omega_1 \Rightarrow$ Knaster Property $\Rightarrow$ productive ccc.) Lastly, the Souslin Problem was completed by Solovay and Tennenbaum [30] who proved that it is consistent that every ccc forcing notion has the Knaster Property (and thus, consistently, ccc is productive).

Since the productivity question occurred to be so productive, one can ask if we may generalize the problem by replacing *the product* by *the amalgamation* of forcing notions. Let us recall the definition of this operation.

Definition 4.1.1. *(1) Suppose* $\mathbb{P}, \mathbb{Q}$ *are forcing notions such that* $\mathbb{P} \lessdot {\rm RO}(\mathbb{Q})$*. Then* $(\mathbb{Q} : \mathbb{P})$ *is a* $\mathbb{P}$*–name for a forcing notion which is a suborder of* $\mathbb{Q}$*,*

$$p \Vdash_{\mathbb{P}} \text{“ } q \in (\mathbb{Q} : \mathbb{P}) \text{ ”} \qquad \textit{if and only if}$$
$$\textit{every } p' \in \mathbb{P} \textit{ stronger than } p \textit{ is compatible with } q \textit{ in } {\rm RO}(\mathbb{Q}).$$

(2) Suppose that $\mathbb{P}, \mathbb{Q}_0, \mathbb{Q}_1$ *are forcing notions and* $f_\ell : \mathbb{P} \longrightarrow {\rm RO}(\mathbb{Q}_\ell)$ *(for* $\ell < 2$*) are complete embeddings.* The amalgamation of $\mathbb{Q}_0, \mathbb{Q}_1$ over f_0, f_1 *is*

$$\mathbb{Q}_0 \times_{f_0, f_1} \mathbb{Q}_1 = \{(q_1, q_2) \in \mathbb{Q}_0 \times \mathbb{Q}_1 : (\exists p \in \mathbb{P})\, (p \Vdash \text{“ } q_0 \in (\mathbb{Q}_0 : f_0[\mathbb{P}]) \text{ \& } q_1 \in (\mathbb{Q}_1 : f_1[\mathbb{P}]) \text{”})\}$$

ordered in the natural way (so $(q_0, q_1) \leq (q_0', q_1')$ *if and only if* $q_0 \leq q_0'$ *and* $q_1 \leq q_1'$*).*

It can be easily seen from the definition of the amalgamation that $\mathbb{Q}_0 \times_{f_0,f_1} \mathbb{Q}_1$ is equivalent to $\mathbb{P} * \big((\mathbb{Q}_0 : f_0[\mathbb{P}]) \times (\mathbb{Q}_1 : f_1[\mathbb{P}])\big)$. So, if $\mathbb{P} \lessdot \mathbb{Q}_0$, $\mathbb{P} \lessdot \mathbb{Q}_1$ (and f_0, f_1 are the identity mappings) then

$$\mathbb{Q}_0 \times_{\mathbb{P}} \mathbb{Q}_1 \stackrel{\text{def}}{=} \mathbb{Q}_0 \times_{f_0,f_1} \mathbb{Q}_1 = \mathbb{P} * \big((\mathbb{Q}_0 : \mathbb{P}) \times (\mathbb{Q}_1 : \mathbb{P})\big)$$

and the amalgamation can be thought of as a generalization of the product. Thus the question when amalgamation of ccc forcing notions satisfies the ccc can be thought of as a generalization of the productivity problem. However, there is no nice answer here. By Shelah [24, §1], a Cohen real adds a Souslin tree, so let $\dot{S}$ be a $\mathbb{C}$–name for a Souslin tree (ordered naturally). The forcing notion $\mathbb{Q} = \mathbb{C} * \dot{S}$ satisfies the ccc, but the amalgamation $\mathbb{Q} \times_{\mathbb{C}} \mathbb{Q} = \mathbb{C} * (\dot{S} \times \dot{S})$ does not satisfy the ccc (this example exists in any universe of ZFC, of course). Even demanding stronger variants of ccc (which clearly worked for products) may not help for amalgamations: Roitman [13] gave an example of $\mathbb{P}$, $\dot{\mathbb{Q}}_1$, $\dot{\mathbb{Q}}_2$ such that

(a) $\mathbb{P} * \dot{\mathbb{Q}}_1$, $\mathbb{P} * \dot{\mathbb{Q}}_2$ have pre-caliber ω_1, but
(b) $\Vdash_{\mathbb{P}}$" $\dot{\mathbb{Q}}_1 \times \dot{\mathbb{Q}}_2$ does not satisfy the ccc ", so
(c) $\big(\mathbb{P} * \dot{\mathbb{Q}}_1\big) \times_{\mathbb{P}} \big(\mathbb{P} * \dot{\mathbb{Q}}_2\big)$ does not satisfy the ccc.

However, there are *sweet* (=strong ccc) properties of forcing notions which are preserved by amalgamations.

Definition 4.1.2 (Shelah [24, Def. 7.2]).
A triple $(\mathbb{P}, \mathcal{D}, \bar{E})$ *is* model of sweetness *(on* $\mathbb{P}$*) whenever:*

(i) $\mathbb{P}$ *is a forcing notion,* $\mathcal{D}$ *is a dense subset of* $\mathbb{P}$*,*
(ii) $\bar{E} = \langle E_n : n < \omega\rangle$*, each* E_n *is an equivalence relation on* $\mathcal{D}$ *such that* $\mathcal{D}/E_n$ *is countable,*
(iii) equivalence classes of each E_n *are* $\leq_{\mathbb{P}}$*–directed,* $E_{n+1} \subseteq E_n$*,*
(iv) if $\{p_i : i \leq \omega\} \subseteq \mathcal{D}$*,* $p_i\ E_i\ p_\omega$ *(for* $i \in \omega$*) ,then*

$$(\forall n \in \omega)(\exists q \geq p_\omega)(q\ E_n\ p_\omega\ \&\ (\forall i \geq n)(p_i \leq q)),$$

(v) if $p, q \in \mathcal{D}$*,* $p \leq q$ *and* $n \in \omega$ *then there is* $k \in \omega$ *such that*

$$(\forall p' \in [p]_{E_k})(\exists q' \in [q]_{E_n})(p' \leq q').$$

If there is a model of sweetness on $\mathbb{P}$*, then we say that* $\mathbb{P}$ *is* sweet.

Definition 4.1.3 (Stern [31, Def. 1.2]). *Let* $\mathbb{P}$ *be a forcing notion and* τ *be a topology on* $\mathbb{P}$*. We say that* $(\mathbb{P}, \tau)$ *is* a model of topological sweetness *whenever the following conditions are satisfied:*

(i) the topology τ *has a countable basis,*
(ii) $\emptyset_{\mathbb{P}}$ *is an isolated point in* τ*,*
(iii) if a sequence $\langle p_n : n < \omega\rangle \subseteq \mathbb{P}$ *is* τ*–converging to* $p \in \mathbb{P}$*,* $q \geq p$ *and* W *is a* τ*–neighbourhood of* q*, then there is a condition* $r \in \mathbb{P}$ *such that*

(a) $r \in W$, $r \geq q$,
(b) the set $\{n \in \omega : p_n \leq r\}$ *is infinite.*

A forcing notion $\mathbb{P}$ *is* topologically sweet, *if there is a topology* τ *on* $\mathbb{P}$ *such that* $(\mathbb{P}, \tau)$ *is a model of topological sweetness.*

Definition 4.1.4. *(1) Let* $(\mathbb{P}_\ell, \mathcal{D}_\ell, \bar{E}_\ell)$ *(for* $\ell < 2$*) be models of sweetness. We say that* $(\mathbb{P}_1, \mathcal{D}_1, \bar{E}_1)$ extends the model $(\mathbb{P}_0, \mathcal{D}_0, \bar{E}_0)$ *if*

- $\mathbb{P}_0 \lessdot \mathbb{P}_1$, $\mathcal{D}_0 \subseteq \mathcal{D}_1$ *and* $E^0_n = E^1_n \restriction \mathcal{D}_0$ *for each* $n \in \omega$,
- *if* $p \in \mathcal{D}_0$, $n \in \omega$, *then* $[p]_{E^1_n} \subseteq \mathcal{D}_0$,
- *if* $p \leq q$, $p \in \mathcal{D}_1$, $q \in \mathcal{D}_0$, *then* $p \in \mathcal{D}_0$.

(2) Let $(\mathbb{P}_\ell, \tau_\ell)$ *(for* $\ell < 2$*) be models of topological sweetness. We say that* $(\mathbb{P}_1, \tau_1)$ extends the model $(\mathbb{P}_0, \tau_0)$ *if* $\mathbb{P}_0 \lessdot \mathbb{P}_1$, $\mathbb{P}_0$ *is a* τ_1*–open subset of* $\mathbb{P}_1$ *and* $\tau_1 \restriction \mathbb{P}_0 = \tau_0$.

Theorem 4.1.5. *(1)* (Shelah [24, 7.5]) *Suppose that* $(\mathbb{P}_\ell, \mathcal{D}_\ell, \bar{E}_\ell)$ *(for* $\ell < 2$*) are models of sweetness, and* $f_\ell : \mathbb{P} \longrightarrow \mathrm{RO}(\mathbb{P}_\ell)$ *(for* $\ell < 2$*) are complete embeddings. Then there is a model of sweetness* $(\mathbb{P}_0 \times_{f_0, f_1} \mathbb{P}_1, \mathcal{D}^*, \bar{E}^*)$ *based on the amalgamation* $\mathbb{P}_0 \times_{f_0, f_1} \mathbb{P}_1$ *and extending each* $(\mathbb{P}_\ell, \mathcal{D}_\ell, \bar{E}_\ell)$ *for* $\ell < 2$.

(2) (Stern [31, §2.2]) *Suppose that* $(\mathbb{P}_\ell, \tau_\ell)$ *(for* $\ell < 2$*) are models of topological sweetness, and* $f_\ell : \mathbb{P} \longrightarrow \mathrm{RO}(\mathbb{P}_\ell)$ *(for* $\ell < 2$*) are complete embeddings. Then there is a model of topological sweetness* $(\mathbb{P}_0 \times_{f_0, f_1} \mathbb{P}_1, \tau^*)$ *based on the amalgamation* $\mathbb{P}_0 \times_{f_0, f_1} \mathbb{P}_1$ *and extending each* $(\mathbb{P}_\ell, \tau_\ell)$ *for* $\ell < 2$.

The real interest in preserving ccc in amalgamations comes from the independence results related to regularity properties of definable sets. Assuming the existence of an inaccessible cardinal, Solovay [29] showed the consistency of the following two statements

(M) every projective set of reals is Lebesgue measurable,
(B) every projective set of reals has the Baire property.

The key ingredient of Solovay's proof was the observation that the Collapse Algebra is very homogeneous. Shelah [24] proved that

- (M) implies ω_1 is inaccessible in **L**, but
- the consistency of (B) can be established without inaccessible cardinals.

For the latter result one builds a homogeneous ccc forcing notion adding a lot of Cohen reals. Homogeneity is obtained by multiple use of amalgamation (see [6] for a full explanation of how this works), the Cohen reals come from compositions with the Universal Meager forcing notion $\mathbb{UM}$ or with the Hechler forcing notion $\mathbb{D}$. An important feature of sweetness is that it is preserved in compositions with $\mathbb{UM}$, $\mathbb{D}$.

Proposition 4.1.6. *(1)* (Shelah [24, §7]) *If* $\mathbb{P}$ *is sweet then both* $\mathbb{P} * \dot{\mathbb{UM}}$ *and* $\mathbb{P} * \dot{\mathbb{D}}$ *are sweet.*

(2) (Stern [31, 3.3]) *If* $\mathbb{P}$ *is topologically sweet then both* $\mathbb{P} * \dot{\mathbb{UM}}$ *and* $\mathbb{P} * \dot{\mathbb{D}}$ *are topologically sweet.*

4.2. More on sweetness

The original model of Shelah [24] for (B) (i.e., "all projective sets of reals have the Baire property") satisfies also CH. Can we have (B)+¬CH? Judah and Shelah [9] aimed at exactly this, but the proof there has an irreparable gap. However, a recent work in progress [18] gives a totally different way of building models for (B)+¬CH. Then we may ask for models of ZFC in which (B) holds and some cardinal characteristics of the continuum are large. Thus we would like to use (in addition to $\mathbb{UM}$ or $\mathbb{D}$) other forcings in the constructions as in [24], [18]. However, the proofs for 4.1.6 were very specific and it has not been clear if there are more forcing notions which could be composed with sweet ones. In this subsection we show that sweetness of 4.1.2 is essentially everything we need. We also show that our methods may lead to sweet and/or topologically sweet forcing notions.

Definition 4.2.1. *Let $\mathcal{B}$ be a countable basis of a topology on a forcing notion $\mathbb{P}$. We say that $(\mathbb{P}, \mathcal{B})$ is* a model of iterable sweetness *if*

(i) $\mathcal{B}$ is closed under finite intersections,
(ii) each $U \in \mathcal{B}$ is $\leq_{\mathbb{P}}$–directed and $p \leq q \in U \;\Rightarrow\; p \in U$,
(iii) if $\langle p_n : n \leq \omega\rangle \subseteq U \in \mathcal{B}$ and the sequence $\langle p_n : n < \omega\rangle$ converges to p_ω (in the topology generated by $\mathcal{B}$), then there is a condition $p \in U$ such that $(\forall n \leq \omega)(p_n \leq p)$.

Proposition 4.2.2. *Assume that $(\mathbb{P}, \mathcal{D}, \bar{E})$ is a sweetness model on $\mathbb{P}$, $\bar{E} = \langle E_n : n < \omega\rangle$, $\mathcal{D} = \mathbb{P}$. Furthermore, suppose that any two compatible elements of $\mathbb{P}$ have a least upper bound, i.e., if $p_0, p_1 \in \mathbb{P}$ are compatible, then there is $q \geq p_0, p_1$ such that $(\forall r \in \mathbb{P})(r \geq p_0 \;\&\; r \geq p_1 \;\Rightarrow\; r \geq q)$. For $\bar{p} = \langle p_\ell : \ell \leq k\rangle \subseteq \mathbb{P}$ and $\bar{n} = \langle n_\ell : \ell \leq k\rangle \subseteq \omega$ let*

$$U(\bar{p}, \bar{n}) \stackrel{\text{def}}{=} \{q \in \mathbb{P} : (\forall \ell \leq k)(\exists q' \in [p_\ell]_{E_{n_\ell}})(q \leq q')\},$$

and let $\mathcal{B}$ be the collection of all sets $U(\bar{p}, \bar{n})$ (for $\bar{p} \subseteq \mathbb{P}$, $\bar{n} \subseteq \omega$, $\mathrm{lh}(\bar{p}) = \mathrm{lh}(\bar{n})$). Then $(\mathbb{P}, \mathcal{B})$ is a model of iterable sweetness.

Proof. Plainly, $\mathcal{B}$ is closed under finite intersections. Since the equivalence classes of each E_n are directed and any two compatible members of $\mathbb{P}$ have a least upper bound, we may conclude that the elements of $\mathcal{B}$ are directed and downward closed.

Before we verify the demand 4.2.1(iii), let us first note that if $p \in U(\bar{p}, \bar{n})$, $\bar{p} = \langle p_\ell : \ell \leq k\rangle$, $\bar{n} = \langle n_\ell : \ell \leq k\rangle$, then for some N we have $U(\langle p\rangle, \langle N\rangle) \subseteq U(\bar{p}, \bar{n})$. [Why? Use 4.1.2(v) to choose N such that

$$(\forall \ell \leq k)(\forall q \in [p]_{E_N})(\exists q' \in [p_\ell]_{E_{n_\ell}})(q \leq q').$$

Clearly this N is as required.]

Now suppose that a sequence $\langle p_n : n < \omega\rangle$ converges to p_ω in the topology generated by $\mathcal{B}$, and $p_\omega, p_n \in U(\bar{q}, \bar{n}) \in \mathcal{B}$ for all $n < \omega$. Take N such that $U(\langle p_\omega\rangle, \langle N\rangle) \subseteq U(\bar{q}, \bar{n})$. Choose an increasing sequence $\langle m_i : i < \omega\rangle$ such that

$$(\forall i < \omega)(\forall n \geq m_i)(p_n \in U(\langle p_\omega\rangle, \langle N + 1 + i\rangle)).$$

Next pick conditions $p_i^* \in [p_\omega]_{E_{N+1+i}}$ such that

$$(\forall i < \omega)(\forall n \in [m_i, m_{i+1}))(p_n \leq p_i^*)$$

(remember that each $[p_\omega]_{E_{N+1+i}}$ is directed). It follows from 4.1.2(iv) that we may find a condition $q' \geq p_\omega$ such that $q' \in [p_\omega]_{E_N}$ and $(\forall i < \omega)(p_i^* \leq q')$. Then

$$(\forall n \geq m_0)(p_n \leq q') \quad \text{and} \quad q' \in U(\bar{q}, \bar{n}).$$

Since $U(\bar{q}, \bar{n})$ is directed and $q', p_0, \ldots, p_{m_0} \in U(\bar{q}, \bar{n})$, the conditions $q', p_0, \ldots, p_{m_0}$ have an upper bound in $U(\bar{q}, \bar{n})$ – let q be such an upper bound. Then q is as needed to justify 4.2.1(iii). □

Lemma 4.2.3. *Assume that* $(\mathbb{P}, \tau)$ *is a model of topological sweetness.*

(1) If $p, q \in \mathbb{P}$, $p \leq q$ *and* $q \in U \in \tau$, *then there is an open neighbourhood* V *of* p *such that*

$$(\forall r \in V)(\exists r' \in U)(r \leq r').$$

(2) If $m \in \omega$, $p \in U \in \tau$, *then there is an open neighbourhood* V *of* p *such that any* $p_0, \ldots, p_m \in V$ *have a common upper bound in* U.

Proof. Straightforward. □

Theorem 4.2.4. *Suppose that* $(\mathbb{P}, \tau)$ *is a model of topological sweetness and* $\dot{\mathcal{B}}, \dot{\mathbb{Q}}$ *are* $\mathbb{P}$*–names such that*

$$\Vdash_{\mathbb{P}} \text{“ } (\dot{\mathbb{Q}}, \dot{\mathcal{B}}) \textit{ is a model of iterable sweetness ”}.$$

Then there is dense subset $\mathbb{R}$ *of the iteration* $\mathbb{P} * \dot{\mathbb{Q}}$ *and a topology* τ^* *on* $\mathbb{R}$ *such that* $\mathbb{P} \subseteq \mathbb{R}$ *and* $(\mathbb{R}, \tau^*)$ *is a model of topological sweetness extending the model* $(\mathbb{P}, \tau)$.

Proof. Let $\dot{V}_n$ be $\mathbb{P}$–names such that

$$\Vdash_{\mathbb{P}} \text{“ } \dot{V}_0 = \{\emptyset_{\dot{\mathbb{Q}}}\} \text{ and } \{\dot{V}_n : 0 < n < \omega\} \text{ enumerates } \dot{\mathcal{B}} \setminus \{\emptyset\} \text{ ”}$$

(note that $\Vdash$ “ $(\dot{\mathbb{Q}}, \{\dot{V}_n : n < \omega\})$ is a model of iterable sweetness ” and also $\Vdash$ “ $\emptyset_{\dot{\mathbb{Q}}} \in \dot{V}_n$ for all n ”, remember 4.2.1(ii)). Let $\mathcal{U}$ be a countable basis of the topology τ and let

$$\mathbb{R} \stackrel{\text{def}}{=} \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} : p \neq \emptyset_{\mathbb{P}} \text{ and } (\exists n < \omega)(p \Vdash \dot{q} \in \dot{V}_n)\} \cup \{(\emptyset_{\mathbb{P}}, \dot{\emptyset}_{\dot{\mathbb{Q}}})\}.$$

For $U \in \mathcal{U}$, $\bar{U} = \langle U_0, \ldots, U_{m-1} \rangle \subseteq \mathcal{U}$ and $\bar{n} = \langle n_0, \ldots, n_{m-1} \rangle \subseteq \omega$ and $\bar{k} = \langle k_0, \ldots, k_M \rangle \subseteq \omega$ $(m, M < \omega)$ we put

$$U^*(U, \bar{U}, \bar{n}, \bar{k}) \stackrel{\text{def}}{=} \{(p, \dot{q}) \in \mathbb{R} : p \in U \text{ and } (\forall \ell \leq M)(p \Vdash \dot{q} \in \dot{V}_{k_\ell}) \text{ and } (\forall \ell < m)(\exists p' \in U_\ell)(p \leq p' \,\&\, p' \Vdash \dot{q} \in \dot{V}_{n_\ell})\}.$$

Let $\mathcal{C}$ be the collection of all sets of the form $U^*(U, \bar{U}, \bar{n}, \bar{k})$ (for suitable parameters $U, \bar{U}, \bar{n}, \bar{k}$).

Claim 4.2.4.1. *(1) The family $\mathcal{C}$ forms a countable basis of a topology on $\mathbb{R}$; we will denote this topology by τ^*. $\emptyset_{\mathbb{R}}$ is an isolated point in τ^*.*
(2) If $(p, \dot{q}), (p', \dot{q}') \in \mathbb{R}$, $(p, \dot{q}) \leq (p', \dot{q}')$ and $(p', \dot{q}') \in U^* \in \mathcal{C}$, then there is $V^* \in \mathcal{C}$ such that $(p, \dot{q}) \in V^*$ and $(\forall r \in V^*)(\exists r' \in U^*)(r \leq r')$.

Proof of the claim. 1) Should be clear.
2) Let $U^* = U^*(U, \langle U_0, \ldots, U_{m-1}\rangle, \langle n_0, \ldots, n_{m-1}\rangle, \langle k_0, \ldots, k_M\rangle)$ and let $p'_\ell \in U_\ell$ (for $\ell < m$) be such that $p' \leq p'_\ell$ and $p'_\ell \Vdash \dot{q}' \in \dot{V}_{n_\ell}$. Choose $U'_\ell \in \mathcal{U}$ such that $p'_\ell \in U'_\ell \subseteq U_\ell$ and every two members of U'_ℓ have a common upper bound in U_ℓ (possible by 4.2.3(2)). Next pick $U', U'' \in \mathcal{U}$ such that $p' \in U'' \subseteq U' \subseteq U$ and

- every member of U' has an upper bound in each of the sets $U'_0, \ldots, U'_{m-1}$,
- every $M + 1$ elements of U'' have a common upper bound in U'.

Let $U^+ \in \mathcal{U}$ be such that $p \in U^+$ and each element of U^+ has an upper bound in U'', and let k be such that $p \Vdash \dot{q} \in \dot{V}_k$. Put

$$V^* = U^*(U^+, \langle U'_0, \ldots, U'_{m-1}, U'', \ldots, U''\rangle, \langle n_0, \ldots, n_{m-1}, k_0, \ldots, k_M\rangle, \langle k\rangle) \in \mathcal{C}.$$

First we show that $(p, \dot{q}) \in V^*$. By our choices, $p \in U^+$ and $p \Vdash \dot{q} \in \dot{V}_k$. For $\ell < m$, $p'_\ell \in U'_\ell$ is a condition stronger than $p' \geq p$ and $p'_\ell \Vdash \dot{q} \leq \dot{q}' \in \dot{V}_{n_\ell}$, so $p'_\ell \Vdash \dot{q} \in \dot{V}_{n_\ell}$ (remember 4.2.1(ii)). Next, for $i \leq M$, we have $p' \in U''$ and $p' \Vdash \dot{q} \leq \dot{q}' \in \dot{V}_{k_i}$, so $p' \Vdash \dot{q} \in \dot{V}_{k_i}$.

Now, suppose that $(p^*, \dot{q}^*) \in V^*$. Then $p^* \in U^+$ and we have conditions $p^*_\ell \in U'_\ell$ (for $\ell < m$) and conditions $p^{**}_i \in U''$ (for $i \leq M$) such that

- $p^*_\ell \geq p^*$ and $p^{**}_i \geq p^*$,
- $p^*_\ell \Vdash \dot{q}^* \in \dot{V}_{n_\ell}$ and $p^{**}_i \Vdash \dot{q}^* \in \dot{V}_{k_i}$.

Pick a condition $p^+ \in U'$ stronger than all p^{**}_i (for $i \leq M$). We claim that $(p^+, \dot{q}^*) \in U^*$. Clearly $p^+ \in U$ and $p^+ \Vdash \dot{q}^* \in \dot{V}_{k_i}$ (for $i \leq M$). Fix $\ell < m$. By the choice of U', we find $p^+_\ell \in U'_\ell$ stronger than p^+. By the choice of U'_ℓ, we find a condition $p_\ell \in U_\ell$ stronger than both p^+_ℓ and p^*_ℓ. Then $p_\ell \Vdash \dot{q}^* \in \dot{V}_{n_\ell}$ and we are done. □

Claim 4.2.4.2. *Suppose that a sequence $\langle (p_k, \dot{q}_k) : k < \omega\rangle \subseteq \mathbb{R}$ is τ^*–converging to $(p_\omega, \dot{q}_\omega)$, and $(p_\omega, \dot{q}_\omega) \in U^*(U, \langle U_0, \ldots, U_{m-1}\rangle, \langle n_0, \ldots, n_{m-1}\rangle, \bar{k}) \in \mathcal{C}$. Then there are an infinite set $X \subseteq \omega$ and conditions $p^*_\ell \geq p_\omega$ (for $\ell < m$) such that for each $\ell < m$ and $k \in X$:*

(i) $p^*_\ell \in U_\ell$, $p^*_\ell \geq p_k$,
(ii) $p^*_\ell \Vdash \dot{q}_k \in \dot{V}_{n_\ell}$.

Proof of the claim. Pick $p^\ell_\omega \in U_\ell$, $p^\ell_\omega \geq p_\omega$ such that $p^\ell_\omega \Vdash \dot{q}_\omega \in \dot{V}_{n_\ell}$ (for $\ell < m$). Fix sequences $\langle W^\ell_j : j < \omega\rangle \subseteq \mathcal{U}$ (for $\ell < m$) such that

- $p^\ell_\omega \in W^\ell_{j+1} \subseteq W^\ell_j$,
- $\{W^\ell_j : j < \omega\}$ forms a basis of neighbourhoods of p^ℓ_ω.

Clearly $(p_\omega,\dot q_\omega)\in U^*(U,\langle W^0_j,\ldots,W^{m-1}_j\rangle,\langle n_0,\ldots,n_{m-1}\rangle,\bar k)$ for every $j<\omega$, so we may pick an increasing sequence $\langle k_j:j<\omega\rangle\subseteq\omega$ such that

$$(\forall j<\omega)((p_{k_j},\dot q_{k_j})\in U^*(U,\langle W^0_j,\ldots,W^{m-1}_j\rangle,\langle n_0,\ldots,n_{m-1}\rangle,\bar k)).$$

Let $p^\ell_{k_j}\in W^\ell_j$ be such that $p_{k_j}\le p^\ell_{k_j}$, $p^\ell_{k_j}\Vdash \dot q_{k_j}\in\dot V_{n_\ell}$ (for $\ell<m$, $j<\omega$). Each sequence $\langle p^\ell_{k_j}:j<\omega\rangle$ τ–converges to p^ℓ_ω so we may find an infinite set $A\subseteq\omega$ and conditions $p^*_\ell\in U_\ell$ (for $\ell<m$) such that

$$p^*_\ell\ge p^\ell_\omega\ge p_\omega\quad\text{and}\quad(\forall j\in A)(\forall\ell<m)(p^*_\ell\ge p^\ell_{k_j}).$$

Let $X=\{k_j:j\in A\}$. □

Claim 4.2.4.3. *Suppose $\langle(p_n,\dot q_n):n<\omega\rangle\subseteq\mathbb{R}$ is τ^*–converging to $(p_\omega,\dot q_\omega)$. Then there is $X\in[\omega]^\omega$ such that:*

($\boxtimes$) if $(p_\omega,\dot q_\omega)\in U^*(U,\langle U_0,\ldots,U_{m-1}\rangle,\langle n_0,\ldots,n_{m-1}\rangle,\bar k)\in\mathcal{C}$,
then *for some $N\in\omega$ and $p^*_\ell\in U_\ell$ (for $\ell<m$) we have:*
(i) $p^*_\ell\ge p_\omega$, $(\forall n\in X\setminus N)(p^*_\ell\ge p_n)$,
(ii) $(\forall n\in X\setminus N)(\forall\ell<m)(p^*_\ell\Vdash\dot q_n\in\dot V_{n_\ell})$.

Proof of the claim. Let $\langle U^*_i:i<\omega\rangle$ enumerate all sets $U^*\in\mathcal{C}$ to which $(p_\omega,\dot q_\omega)$ belongs. Apply 4.2.4.2 to choose inductively a decreasing sequence $\langle X_i:i<\omega\rangle\subseteq[\omega]^\omega$ such that for each $i<\omega$:

> if $U^*_i=U^*(U,\langle U_0,\ldots,U_{m-1}\rangle,\langle n_0,\ldots,n_{m-1}\rangle,\bar k)$, then there are conditions $p^i_\ell\ge p_\omega$ (for $\ell<m$) satisfying
>
> $$(\forall\ell<m)(\forall n\in X_i)(p^i_\ell\in U_\ell\ \&\ p^i_\ell\ge p_n\ \&\ p^i_\ell\Vdash\dot q_n\in\dot V_{n_\ell}).$$

Next pick an infinite set $X\subseteq\omega$ almost included in all X_n's. □

Claim 4.2.4.4. *Suppose that $\langle(p_n,\dot q_n):n<\omega\rangle\subseteq\mathbb{R}$ τ^*–converges to $(p_\omega,\dot q_\omega)\in U^*\in\mathcal{C}$. Then there is a condition $(p^*,\dot q^*)\in U^*$ stronger than $(p_\omega,\dot q_\omega)$ and such that*

$$(\exists^\infty n\in\omega)((p_n,\dot q_n)\le(p^*,\dot q^*)).$$

Proof of the claim. Let $U^*=U^*(U,\langle U_0,\ldots,U_{m-1}\rangle,\langle n_0,\ldots,n_{m-1}\rangle,\langle k_0,\ldots,k_M\rangle)$, and let $p^\ell_\omega\in U_\ell$ (for $\ell<m$) be such that $p^\ell_\omega\ge p_\omega$ and $p^\ell_\omega\Vdash\dot q_\omega\in\dot V_{n_\ell}$. Pick $U',U'_0,\ldots,U'_{m-1}\in\mathcal{U}$ such that

- $p_\omega\in U'\subseteq U$, $p^\ell_\omega\in U'_\ell\subseteq U_\ell$,
- any 3 elements of U'_ℓ have a common upper bound in U_ℓ,
- every element of U' has an upper bound in each of $U'_0,\ldots,U'_{m-1}$.

Apply 4.2.4.3 to choose $X \in [\omega]^\omega$ such that the condition ($\boxtimes$) of 4.2.4.3 holds. Note that then

$$p_\omega \Vdash \text{“ the sequence } \langle \dot{q}_n : n \in X\rangle \text{ is } \dot{\mathcal{B}}\text{–convergent to } \dot{q}_\omega \text{ ”.}$$

[Why? If not, then we may pick a condition $r \geq p_\omega$ and an integer N such that

$$r \Vdash \text{“ } \dot{q}_\omega \in \dot{V}_N \ \& \ (\exists^\infty n \in X)(\dot{q}_n \notin \dot{V}_N) \text{ ”.}$$

Pick $W \in \mathcal{U}$ such that $r \in W$ and any 2 members of W are compatible, and apply ($\boxtimes$) of 4.2.4.3 to $U^*(U, \langle W\rangle, \langle N\rangle, \langle k_0, \ldots, k_M\rangle)$. We get a condition $r^* \in W$ such that

$$(\forall^\infty n \in X)(r^* \geq p_n \ \& \ r^* \Vdash \dot{q}_n \in \dot{V}_N).$$

Since r^*, r are compatible, we get a contradiction.]

Since $\langle p_n : n \in X\rangle$ τ–converges to p_ω, we may find an infinite $X' \subseteq X$ and a condition $p^* \in U'$ such that

$$p^* \geq p_\omega \ \& \ (\forall n \in X')(p^* \geq p_n).$$

Next use ($\boxtimes$) of 4.2.4.3 to pick conditions $p'_\ell \in U'_\ell$ and $N \in \omega$ such that

$$(\forall n \in X' \setminus N)\big(p'_\ell \geq p_n \ \text{ and } \ p'_\ell \Vdash \dot{q}_n \in \dot{V}_{n_\ell} \quad \text{and} \quad (\forall i \leq M)(p_n \Vdash \dot{q}_n \in \dot{V}_{k_i})\big).$$

By the choice of $U', U'_0, \ldots, U'_{m-1}$ we get conditions $p^*_\ell \in U_\ell$ such that $p^*_\ell \geq p^*$, $p^*_\ell \geq p'_\ell$, $p^*_\ell \geq p^\ell_\omega$ (for $\ell < m$).

Now we are going to define a $\mathbb{P}$–name $\dot{q}^*$ for a condition in $\dot{\mathbb{Q}}$. Let $\mathcal{A}$ be a maximal antichain of $\mathbb{P}$ such that for each $\ell < m$ and $r \in \mathcal{A}$:

- either $r \geq p^*_\ell$ or r, p^*_ℓ are incompatible, and
- either $r \geq p^*$ or r, p^* are incompatible.

Fix $r \in \mathcal{A}$. If r, p^* are incompatible, then let $\dot{q}_r$ be $\dot{\emptyset}_{\dot{\mathbb{Q}}}$. Assume $r \geq p^*$ and let

$$I = \{n_\ell : \ell < m \ \& \ r \geq p^*_\ell\} \cup \{k_i : i \leq M\} \quad (\neq \emptyset).$$

The condition r forces that the sequence $\langle \dot{q}_n : n \in X' \setminus N\rangle$ converges to $\dot{q}_\omega$, and $\dot{q}_\omega \in \bigcap\limits_{j\in I} \dot{V}_j$, and $\dot{q}_n \in \bigcap\limits_{j\in I} \dot{V}_j$ (for all $n \in X' \setminus N$). Applying 4.2.1(i+iii) we find a $\mathbb{P}$–name $\dot{q}_r$ for an element of $\dot{\mathbb{Q}}$ such that

$$r \Vdash \text{“ } (\forall n \in X' \setminus N)(\dot{q}_n \leq \dot{q}_r \ \& \ \dot{q}_\omega \leq \dot{q}_r \ \& \ \dot{q}_r \in \bigcap_{j\in I} \dot{V}_j) \text{ ”.}$$

Now, let $\dot{q}^*$ be a $\mathbb{P}$–name such that $r \Vdash \dot{q}^* = \dot{q}_r$ (for $r \in \mathcal{A}$).

Look at the condition $(p^*, \dot{q}^*) \in \mathbb{R}$. Clearly $p^* \Vdash \dot{q}^* \in \dot{V}_{k_i}$ (for all $i \leq M$) and $p^*_\ell \geq p^*$, $p^*_\ell \Vdash \dot{q}^* \in \dot{V}_{n_\ell}$ (for $\ell < m$), so $(p^*, \dot{q}^*) \in U^*$. Moreover, if $n \in (X' \setminus N) \cup \{\omega\}$, then $p^* \geq p_n$ and $p^* \Vdash \dot{q}_n \leq \dot{q}$, so $(p^*, \dot{q}^*) \geq (p_n, \dot{q}_n)$ and we are done. $\square$

Claim 4.2.4.5. *If* $(p, \dot{q}) \in U^* \in \mathcal{C}$*, then there is* $V^* \in \mathcal{C}$ *such that* $(p, \dot{q}) \in V^*$ *and any two conditions* $(p_0, \dot{q}_0), (p_1, \dot{q}_1) \in V^*$ *have a common upper bound in* U^*.

Proof of the claim. Let $U^* = U^*(U, \langle U_0, \ldots, U_{m-1}\rangle, \langle n_0, \ldots, n_{m-1}\rangle, \langle k_0, \ldots, k_M\rangle)$ and let $p_\ell \in U_\ell$ (for $\ell < m$) be such that $p \leq p_\ell$ and $p_\ell \Vdash \dot{q} \in \dot{V}_{n_\ell}$. Pick $U'_\ell \in \mathcal{U}$ such that $p_\ell \in U'_\ell \subseteq U_\ell$ and every three members of U'_ℓ have a common upper bound in U_ℓ. Also choose $U', U'' \in \mathcal{U}$ such that $p \in U'' \subseteq U' \subseteq U$, and each member of U' has an upper bound in every U'_ℓ (for $\ell < m$) and every two members of U'' have a common upper bound in U'. Put

$$V^* = U^*(U'', \langle U'_0, \ldots, U'_{m-1}\rangle, \langle n_0, \ldots, n_{m-1}\rangle, \langle k_0, \ldots, k_M\rangle) \in \mathcal{C}.$$

Clearly $(p, \dot{q}) \in V^*$. Suppose now that $(p_0, \dot{q}_0), (p_1, \dot{q}_1) \in V^*$ and let $p^i_\ell \in U'_\ell$ be such that $p^i_\ell \Vdash \dot{q}_i \in \dot{V}_{n_\ell}$ and $p_i \leq p^i_\ell$ (for $i = 0, 1$ and $\ell < m$). Also let $p^* \in U'$ be stronger than both p_0 and p_1, and for each $\ell < m$ let $p^*_\ell \in U_\ell$ be stronger than both p^* and p^0_ℓ and p^1_ℓ. Now, like in the proof of 4.2.4.4, choose a $\mathbb{P}$–name $\dot{q}^*$ for a condition in $\dot{\mathbb{Q}}$ such that

$$\begin{aligned} p^* &\Vdash \text{“ } \dot{q}^* \geq \dot{q}_0 \ \& \ \dot{q}^* \geq \dot{q}_1 \text{ ”, and} \\ p^*_\ell &\Vdash \text{“ } \dot{q}^* \in \dot{V}_{n_\ell} \text{ ” for } \ell < m \text{, and} \\ p^* &\Vdash \text{“ } \dot{q}^* \in \dot{V}_{k_\ell} \text{ ” for } \ell \leq M. \end{aligned}$$

(Remember that the sets $\bigcap\limits_{j \in I} \dot{V}_j$ are forced to be directed by 4.2.1(i+ii).) Then $(p^*, \dot{q}^*) \in U^*$ is a condition stronger than both $(p_0, \dot{q}_0)$ and $(p_1, \dot{q}_1)$. □

Now we may put together 4.2.4.4, 4.2.4.1(2) and 4.2.4.5 to conclude that the topology τ^* satisfies the demand 4.1.3(iii), finishing the proof of the theorem.

Note that $(p, \dot{\emptyset}_{\dot{\mathbb{Q}}}) \in \mathbb{R}$ for each $p \in \mathbb{P}$ and the mapping $p \mapsto (p, \dot{\emptyset}_{\dot{\mathbb{Q}}})$ is a homeomorphic embedding of $(\mathbb{P}, \tau)$ into $(\mathbb{R}, \tau^*)$, so we may think that τ is the restriction of τ^* to $\mathbb{P} \subseteq \mathbb{R}$. Moreover, under this interpretation, $\mathbb{P}$ is an open subset of $\mathbb{R}$. □

Proposition 4.2.5. *(1) In the cases discussed in 1.1.14(1,2), the forcing notion under considerations is topologically sweet, provided* K *is countable.*

(2) If (K, Σ) *is a really finitary (see 1.3.3) linked tree–creating pair, and* f *is a fast function, then the forcing notion* $\mathbb{Q}^{\text{tree}}_f(K, \Sigma)$ *is topologically sweet.*

(3) Let $\mathfrak{p} = (K, \Sigma, \mathcal{F}, \mathcal{G})$ *be a universality parameter for* $\mathbf{H}$*. Assume that*

(a) $\mathcal{F}$ *is either countable or* $<\omega_1 - \leq^*$*–directed, and*

(b) for each $\eta \in \bigcup\limits_{n<\omega} \prod\limits_{i<n} \mathbf{H}(i)$ *there is* $t^{\max}_\eta \in \mathrm{LTCR}_\eta[\mathbf{H}] \cap K$ *such that*

$$(\forall t \in \mathrm{LTCR}_\eta[\mathbf{H}] \cap K)(t \in \Sigma(t^{\max}_\eta)).$$

Then the forcing notion $\mathbb{Q}^{\text{tree}}(\mathfrak{p})$ *is sweet (and thus iterably sweet, provided elements compatible in* $\mathbb{Q}^{\text{tree}}(\mathfrak{p})$ *have the least upper bound).*

Proof. 1) Let $(K, \Sigma, \Sigma^\perp)$ be a $\otimes$–creating triple for $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$. Suppose that $(K, \Sigma, \Sigma^\perp)$ is linked, gluing and has the cutting property.

For $\mathbf{c} = (w, t_0, \dots, t_n) \in \mathrm{FC}(K, \Sigma, \Sigma^\perp)$ and $N < \omega$ let

$$U(\mathbf{c}, N) = \{p \in \mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp) : w^p = w \quad \& \quad (\forall k \le n)(t^p_k = t_k) \quad \& \quad (\forall k > n)(\mathbf{nor}[t^p_k] \ge N)\}.$$

Let τ be the topology generated by the sets $U(\mathbf{c}, N)$ (for $\mathbf{c} \in \mathrm{FC}(K, \Sigma, \Sigma^\perp)$ and $N < \omega$) and $\{\emptyset_{\mathbb{Q}^*_\infty(K,\Sigma,\Sigma^\perp)}\}$. It is straightforward to check that $(\mathbb{Q}^*_\infty(K, \Sigma, \Sigma^\perp), \tau)$ is a model of topological sweetness.
Other instances of 1) and 2) can be handled similarly.
3) We consider the case when $\mathcal{F}$ is countable only (if $\mathcal{F}$ is $<\omega_1 - \le^*$–directed the proof is similar). We put $\mathcal{D} = \mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$ and we define relations E_n (for $n < \omega$) on $\mathcal{D}$ as follows:

$p_0\ E_n\ p_1$ if and only if
$N^{p_0} = N^{p_1}$, $f^{p_0} = f^{p_1}$ and

$$(\forall \eta \in T^{p_0})(\mathrm{lh}(\eta) < N^{p_0} + n \quad \Rightarrow \quad \eta \in T^{p_1}\ \&\ t^{p_0} = t^{p_1}).$$

We claim that $(\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p}), \mathcal{D}, \langle E_n : n < \omega\rangle)$ is a model of sweetness. Plainly, each E_n is an equivalence relation with countably many equivalence classes, $E_{n+1} \subseteq E_n$. Similarly as in 2.3.6.1 one can show that the equivalence classes of each E_n are directed and that the clause 4.1.2(v) is satisfied. Let us show that the demand 4.1.2(iv) holds.

So suppose $p_i\ E_i\ p_\omega$ (for $i \ge n$, $n < \omega$). Thus $f^{p_i} = f^{p_\omega} = f$, $N^{p_i} = N^{p_\omega} = N$ (for $i \ge n$). We build inductively a system $\langle t_\eta : \eta \in T\rangle \in \mathbb{Q}^{\mathrm{tree}}_\emptyset(K, \Sigma)$ as follows.

First we let $M_0 = N + n$ and we declare

$$\eta \in T^{p_\omega}\ \&\ \mathrm{lh}(\eta) < M_0 \quad \Rightarrow \quad \eta \in T\ \&\ t_\eta = t^{p_\omega}_\eta.$$

Suppose that we have defined $\langle t_\eta : \eta \in T\ \&\ \mathrm{lh}(\eta) < M_k\rangle \in \mathrm{FC}(K, \Sigma)$ already. Pick $M'_k > M_k$ such that for some $n^k_{\mathrm{dn}}, n^k_{\mathrm{up}}$ we have $M_k < n^k_{\mathrm{dn}} \le n^k_{\mathrm{up}} \le F^{\mathcal{G}}(n^k_{\mathrm{up}}) < M'_k$ and

$$(\langle t^{p_\omega}_\eta : \eta \in T^{p_\omega}\ \&\ \mathrm{lh}(\eta) < M'_k\rangle, n^k_{\mathrm{dn}}, n^k_{\mathrm{up}}, f{\restriction}[n^k_{\mathrm{dn}}, n^k_{\mathrm{up}}]) \in \mathcal{G}.$$

Next, choose M_{k+1} and $n^{k,i}_{\mathrm{dn}}, n^{k,i}_{\mathrm{up}}$ (for $i \in [n, M'_k]$) such that $M'_k < n^{k,n}_{\mathrm{dn}}$, $n^{k,i}_{\mathrm{dn}} \le n^{k,i}_{\mathrm{up}} \le F^{\mathcal{G}}(n^{k,i}_{\mathrm{up}}) < n^{k,i+1}_{\mathrm{dn}}$-2, $F^{\mathcal{G}}(n^{k,M'_k}_{\mathrm{up}}) < M_{k+1}$ and

$$(\langle t^{p_i}_\eta : \eta \in T^{p_i}\ \&\ \mathrm{lh}(\eta) < M_{k+1}\rangle, n^{k,i}_{\mathrm{dn}}, n^{k,i}_{\mathrm{up}}, f{\restriction}[n^{k,i}_{\mathrm{dn}}, n^{k,i}_{\mathrm{up}}]) \in \mathcal{G}.$$

Let $\langle t^k_\eta : \eta \in \hat{S}_k\rangle \in \mathrm{FC}(K, \Sigma)$ be such that $\mathrm{root}(S_k) = \langle\rangle$, $\mathrm{lev}(S_k) = M_{k+1}$, and $t^k_\eta = t^{p_\omega}_\eta$ when $\mathrm{lh}(\eta) < M'_k$ and $t^k_\eta = t^{\max}_\eta$ when $M'_k \le \mathrm{lh}(\eta) < \mathrm{lev}(S_k)$. Apply repeatedly 2.3.3(ε) to get $\langle t_\eta : \eta \in T\ \&\ \mathrm{lh}(\eta) < M_{k+1}\rangle \in \mathrm{FC}(K, \Sigma)$ such that

$$(\langle t_\eta : \eta \in T\ \&\ \mathrm{lh}(\eta) < M_{k+1}\rangle, n^k_{\mathrm{dn}}, n^{k,M'_k}_{\mathrm{up}}, f{\restriction}[n^k_{\mathrm{dn}}, n^{k,M'_k}_{\mathrm{up}}]) \in \mathcal{G},$$

and $\langle t_\eta : \eta \in T\ \&\ \mathrm{lh}(\eta) < M_{k+1}\rangle \le \langle t^k_\eta : \eta \in \hat{S}_k\rangle$, and

$$\langle t_\eta : \eta \in T\ \&\ \mathrm{lh}(\eta) < M_{k+1}\rangle \le \langle t^{p_i}_\eta : \eta \in T^{p_i}\rangle \qquad \text{for all } i \in [n, M'_k].$$

Note that then $\langle t_\eta : \eta \in T\ \&\ \mathrm{lh}(\eta) < M_{k+1}\rangle \le \langle t^{p_i}_\eta : \eta \in T^{p_i}\rangle$ for all $i \in [n, \omega]$.

After the construction is carried out one easily checks that $q = (N, \langle t_\eta : \eta \in T\rangle, f) \in \mathbb{Q}^{\rm tree}(\mathfrak{p})$ is a condition stronger than all p_i's (for $i \in [n, \omega]$) and $q\ E_n\ p_\omega$. $\square$

Definition 4.2.6. *Suppose that a function $h : \omega \times \omega \longrightarrow \omega$ is regressive, $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$, and (K, Σ) is a local creating pair for $\mathbf{H}$. We define $\mathbb{Q}^h_{\mathcal{F}}(K, \Sigma)$ as the suborder of $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ consisting of conditions $p \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ such that*

$$(\exists f \in \mathcal{F})(\forall k < \omega)(\forall^\infty n < \omega)(h^{(k)}(m^{t^p_n}, \lfloor \mathbf{nor}[t^p_n]\rfloor) \geq f(m^{t^p_n})),$$

where $h^{(k+1)}(i, j) = h^{(k)}(i, h(i, j))$, $h^{(0)}(i, j) = j$.

Remark 4.2.7. Note that the norm condition introduced in 4.2.6 is in many cases nothing new. If for each $f \in \mathcal{F}$ there is $f^+ \in \mathcal{F}$ such that

$$(\forall k < \omega)(\forall^\infty n < \omega)(h^{(k)}(n, f(n)) \geq f^+(n)),$$

then clearly $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma) = \mathbb{Q}^h_{\mathcal{F}}(K, \Sigma)$.

Proposition 4.2.8. *Let $h : \omega \times \omega \longrightarrow \omega$ be regressive and $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ be a countable family. Assume that (K, Σ) is a local h–linked creating pair for $\mathbf{H}$ such that K is countable and*

$(*)$ *if $s, t \in K$ and $s \in \Sigma(t)$, then $\mathbf{nor}[s] \leq \mathbf{nor}[t]$.*

Then the forcing notion $\mathbb{Q}^h_{\mathcal{F}}(K, \Sigma)$ is topologically sweet.

Proof. For a finite candidate $\mathbf{c} = (w, t_0, \ldots, t_m) \in \mathrm{FC}(K, \Sigma)$ and sequences $\bar{f} = \langle f_\ell : \ell \leq \ell^*\rangle \subseteq \mathcal{F}$ and $\bar{k} = \langle k_\ell : \ell \leq \ell^*\rangle \subseteq \omega$ we let

$$\begin{aligned} U(\mathbf{c}, \bar{f}, \bar{k}) = \{p \in \mathbb{Q}^h_{\mathcal{F}}(K, \Sigma) : w^p = w \quad \& \quad (\forall n \leq m)(t^p_n = t_n) \quad \& \\ (\forall \ell \leq \ell^*)(\forall n > m)(h^{(k_\ell)}(m^{t^p_n}_{\rm dn}, \lfloor \mathbf{nor}[t^p_n]\rfloor) \geq f_\ell(m^{t^p_n})) \quad \& \\ (\forall \ell \leq \ell^*)(\forall k < \omega)(\forall^\infty n < \omega)(h^{(k)}(m^{t^p_n}_{\rm dn}, \lfloor \mathbf{nor}[t^p_n]\rfloor) \geq f_\ell(m^{t^p_n}))\ \}. \end{aligned}$$

Plainly the sets $U(\mathbf{c}, \bar{f}, \bar{k})$ (for suitable $\mathbf{c}, \bar{f}, \bar{k}$) and $\{\emptyset_{\mathbb{Q}^h_{\mathcal{F}}(K,\Sigma)}\}$ constitute a basis of a topology τ on $\mathbb{Q}^h_{\mathcal{F}}(K, \Sigma)$. It is not difficult to check that $(\mathbb{Q}^h_{\mathcal{F}}(K, \Sigma), \tau)$ is a model of topological sweetness. $\square$

4.3. The sour part of the spectrum

The main point of the sweet properties is that amalgamations of sweet forcing notions are ccc, see 4.1.5. A kind of opposite behaviour is when amalgamating results in a forcing notion that collapses ω_1. This effect will be called sourness, and we have a number of variants of it (see 4.3.2 below).

In this part, whenever we use $\mathbb{Q}$–names for elements of a space $\mathcal{X} = \prod\limits_{i<\omega} \mathbf{H}(i)$, we assume that they are in *a standard form*. Thus, a $\mathbb{Q}$–name $\dot{\tau}$ for a real in $\mathcal{X}$ is a system

$$\langle q^n_{\eta,k} : n < \omega\ \&\ \eta \in \prod_{i<n} \mathbf{H}(i)\ \&\ k < N_\eta\rangle,$$

where $q^n_{\eta,k} \in \mathbb{Q}$, $N_\eta \leq \omega$ and for each $n < \omega$

$$\langle q^n_{\eta,k} : \eta \in \prod_{i<n} \mathbf{H}(i) \ \&\ k < N_\eta \rangle$$

is a maximal antichain of $\mathbb{Q}$, and

$$\nu \vartriangleleft \eta \quad \Rightarrow \quad (\forall k < N_\eta)(\exists m < N_\nu)(q^{\mathrm{lh}(\nu)}_{\nu,m} \leq q^{\mathrm{lh}(\eta)}_{\eta,k}).$$

(The intension is that $q^n_{\eta,k} \Vdash \eta \vartriangleleft \dot{\tau}$.) If ($\mathbb{Q}$ is ccc and) $\dot{\tau}_0$ is a $\mathbb{Q}$–name for an element of $\mathcal{X}$ then there is a standard name $\dot{\tau}$ such that $\Vdash \dot{\tau}_0 = \dot{\tau}$. The point of using standard names is that their specific forms allows us to bound the complexity of some formulas; see, e.g., [1, 3.6.12].

Context 4.3.1. Let $\mathbb{P}$, $\dot{W}$, $\mathcal{X}$ be as in 3.1.1, and let $\mathbb{Q}_0$, $\mathbb{Q}_1$ be Souslin ccc forcing notions. Suppose that, for $\ell < 2$, $\dot{\tau}_\ell$ is a standard $\mathbb{Q}_\ell$–name for an element of $\mathcal{X}$. Furthermore, suppose that there are isomorphisms $f_\ell : \mathrm{RO}(\mathbb{P}) \xrightarrow{\text{onto}} (\mathrm{RO}(\mathbb{Q}_\ell))_{\dot{\tau}_\ell}$ mapping $\dot{W}$ onto $\dot{\tau}_\ell$ (i.e., $f_\ell([\![\eta \vartriangleleft \dot{W}]\!]_{\mathrm{RO}(\mathbb{P})}) = [\![\eta \vartriangleleft \dot{\tau}_\ell]\!]_{\mathrm{RO}(\mathbb{Q}_\ell)}$).
Let r be a real encoding all parameters required for the definitions of the above objects (including partial orders and the respective incompatibility relations).

Definition 4.3.2. *Let $\mathbb{P}$, $\dot{W}$, $\mathcal{X}$, $\mathbb{Q}_i$, $\dot{\tau}_\ell$, f_ℓ, r be as in 4.3.1.*

(a) The amalgamation $\mathbb{Q}_0 \times_{f_0,f_1} \mathbb{Q}_1$ will be also denoted $\mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1} \mathbb{Q}_1$.

We say that

(b) $(\mathbb{Q}_0, \dot{\tau}_0)$ is weakly sour to $(\mathbb{Q}_1, \dot{\tau}_1)$ *if the amalgamation $\mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1} \mathbb{Q}_1$ fails the ccc;*

(c) $(\mathbb{Q}_0, \dot{\tau}_0)$ is sour to $(\mathbb{Q}_1, \dot{\tau}_1)$ *whenever the following condition holds:*
  ($\boxplus$) *if $\mathbf{V} \subseteq \mathbf{V}'$ are universes of* ZFC, *$r \in \mathbf{V}$, $G_0, G_1 \in \mathbf{V}'$, $G_\ell \subseteq \mathbb{Q}^{\mathbf{V}}_\ell$ is generic over $\mathbf{V}$ and $\dot{\tau}^{G_0}_0 = \dot{\tau}^{G_1}_1$, then $\mathbf{V}' \models$ " $\omega^{\mathbf{V}}_1$ is countable ";*

(d) $(\mathbb{Q}_0, \dot{\tau}_0)$ is explicitly sour to $(\mathbb{Q}_1, \dot{\tau}_1)$ *if there are sequences $\langle E_m : m < \omega \rangle$ and $\langle q^\ell_{\alpha,n} : \alpha < \omega_1 \ \&\ n < \omega \rangle$ (for $\ell < 2$) such that*
  (i) each E_m is an equivalence relation on ω_1 with at most countably many equivalence classes,
  (ii) $\{q^\ell_{\alpha,n} : n < \omega\} \subseteq \mathbb{Q}_\ell$ is predense in $\mathbb{Q}_\ell$ (for each $\alpha < \omega_1$, $\ell < 2$),
  (iii) if $\alpha < \beta < \omega_1$, $m < \omega$, $\alpha\ E_m\ \beta$ and $n_0, n_1 < m$ and both $(q^0_{\alpha,n_0}, q^1_{\alpha,n_1})$ and $(q^0_{\beta,n_0}, q^1_{\beta,n_1})$ are in the amalgamation $\mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1} \mathbb{Q}_1$, then the conditions $(q^0_{\alpha,n_0}, q^1_{\alpha,n_1})$, $(q^0_{\beta,n_0}, q^1_{\beta,n_1})$ are incompatible (in $\mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1} \mathbb{Q}_1$);

(e) $(\mathbb{Q}_0, \dot{\tau}_0)$ is very explicitly sour to $(\mathbb{Q}_1, \dot{\tau}_1)$ *if there are a closed perfect subset C of ω^ω and Borel functions $g_\ell : C \times \omega \longrightarrow \mathbb{Q}_\ell$ such that*
  (i) $\{g_\ell(x, n) : n < \omega\}$ is predense in $\mathbb{Q}_\ell$ (for each $x \in C$, $\ell < 2$),
  (ii) if $x_0, x_1 \in C$ are distinct, $x_0{\restriction}m = x_1{\restriction}m$, $k_0, k_1 < m$, then there are $n < \omega$ and disjoint sets $A_0, A_1 \subseteq \prod_{i<n} \mathbf{H}(i)$ such that for $\ell < 2$:

$$(\forall q \in \mathbb{Q}_\ell)([q \geq g_\ell(x_0, k_\ell) \ \&\ q \geq g_\ell(x_1, k_\ell)] \quad \Rightarrow \quad q \Vdash \dot{\tau}_\ell{\restriction}n \in A_\ell).$$

We say that $\mathbb{Q}_0$ is sour to $\mathbb{Q}_1$ over $(\mathbb{P}, \dot{W})$ *if there are names* $\dot{\tau}_0, \dot{\tau}_1$ *(as above) such that* $(\mathbb{Q}_0, \dot{\tau}_0)$ *is sour to* $(\mathbb{Q}_1, \dot{\tau}_1)$ *(and similarly for the variants).*
A forcing notion $\mathbb{Q}$ *is* sour over $(\mathbb{P}, \dot{W})$ *if it is sour to itself over* $(\mathbb{P}, \dot{W})$ *(and similarly for the other notions).*
We may skip $\dot{W}$ *and say "over* $\mathbb{P}$*" if it is clear what* $\dot{W}$ *we consider.*

Remark 4.3.3. Sourness (4.3.2(c)) is a strong way to say that the amalgamation $\mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1} \mathbb{Q}_1$ collapses ω_1. Explicit sourness (4.3.2(d)) guarantees that we have a nice witness for the collapse, see 4.3.4 below. What is the point of "very explicitly sour"? On one hand that condition implies that the amalgamation collapses the continuum, and on the other hand the name for the collapsing function is encoded in a nice way by a real. Note that the properties of the functions g_0, g_1 required in 4.3.2(e)(i,ii) are $\mathbf{\Pi}^1_1$ (remember that $\dot{\tau}_\ell$ are standard names), and thus we have suitable absoluteness.

Proposition 4.3.4. *Let* $(\mathbb{Q}_0, \dot{\tau}_0)$, $(\mathbb{Q}_1, \dot{\tau}_1)$, $(\mathbb{P}, \dot{W})$ *and r be as in 4.3.1.*

(1) If $(\mathbb{Q}_0, \dot{\tau}_0)$ *is explicitly sour to* $(\mathbb{Q}_1, \dot{\tau}_1)$ *in every universe* $\mathbf{V}$ *of* ZFC *containing r, then* $(\mathbb{Q}_0, \dot{\tau}_0)$ *is sour to* $(\mathbb{Q}_1, \dot{\tau}_1)$.
(2) If $(\mathbb{Q}_0, \dot{\tau}_0)$ *is very explicitly sour to* $(\mathbb{Q}_1, \dot{\tau}_1)$ *with functions* g_0, g_1 *witnessing this, then* $(\mathbb{Q}_0, \dot{\tau}_0)$ *is explicitly sour to* $(\mathbb{Q}_1, \dot{\tau}_1)$ *in every universe of* ZFC *containing r and (the Borel codes for)* g_0, g_1.

Proof. 1) Suppose that $\mathbf{V} \subseteq \mathbf{V}'$ are universes of ZFC, $r \in \mathbf{V}$, $G_0, G_1 \in \mathbf{V}'$, $G_\ell \subseteq \mathbb{Q}^{\mathbf{V}}_\ell$ is generic over $\mathbf{V}$ and $\dot{\tau}_0^{G_0} = \dot{\tau}_1^{G_1}$.
Assume $\mathbf{V}' \models \omega_1^{\mathbf{V}} = \omega_1$.

Claim 4.3.4.1. *(1) If* $q_\ell \in \mathbb{Q}_\ell$ *and* $(q_0, q_1) \notin \mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1} \mathbb{Q}_1$*, then for some disjoint Borel sets* $B_0, B_1 \subseteq \mathcal{X}$ *we have* $q_0 \Vdash \dot{\tau}_0 \in B_0$ *and* $q_1 \Vdash \dot{\tau}_1 \in B_1$*. Consequently, if* $q_0 \in G_0$ *and* $q_1 \in G_1$*, then* $(q_0, q_1) \in \mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1} \mathbb{Q}_1$.
(2) If $(q_0, q_1), (q_0', q_1') \in \mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1} \mathbb{Q}_1$ *are incompatible in* $\mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1} \mathbb{Q}_1$*, then for some disjoint Borel sets* $B_0, B_1 \subseteq \mathcal{X}$ *we have (for* $\ell = 0, 1$*):*

$$(\forall p \in \mathbb{Q}_\ell)(p \geq q_\ell \;\&\; p \geq q_\ell' \quad \Rightarrow \quad p \Vdash \dot{\tau}_\ell \in B_\ell).$$

Proof of the claim. Straightforward if you remember 3.1.6. □

Now, let $\langle E_m : m < \omega \rangle$, $\langle q^\ell_{\alpha,k} : \alpha < \omega_1^{\mathbf{V}}, k < \omega \rangle \in \mathbf{V}$ witness that $(\mathbb{Q}_0, \dot{\tau}_0)$ is explicitly sour to $(\mathbb{Q}_1, \dot{\tau}_1)$ (in $\mathbf{V}$). By 4.3.2(d)(ii) we know that, in $\mathbf{V}'$,

$$(\forall \alpha < \omega_1)(\exists k^0_\alpha, k^1_\alpha < \omega)(q^0_{\alpha,k^0_\alpha} \in G_0 \;\&\; q^1_{\alpha,k^1_\alpha} \in G_1).$$

For some $k^0, k^1 < \omega$ the set

$$Y = \{\alpha < \omega_1 : k^0_\alpha = k^0 \;\&\; k^1_\alpha = k^1\}$$

is uncountable. Let $m = k^0 + k^1 + 1$. It follows from 4.3.2(d)(i) that there are distinct $\alpha, \beta \in Y$ such that $\alpha\ E_m\ \beta$. By 4.3.4.1(1) we know $(q^0_{\alpha,k^0}, q^1_{\alpha,k^1}), (q^0_{\beta,k^0}, q^1_{\beta,k^1}) \in \mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1} \mathbb{Q}_1$, so by 4.3.2(d)(iii) these two conditions are incompatible in $\mathbb{Q}_0 \times_{\dot{\tau}_0=\dot{\tau}_1}$

$\mathbb{Q}_1$. But then, using 4.3.4.1(2), we find disjoint Borel sets $B_0, B_1 \subseteq \mathcal{X}$ such that $\dot{\tau}_0^{G_0} \in B_0$ and $\dot{\tau}_1^{G_1} \in B_1$, a contradiction to $\dot{\tau}_0^{G_0} = \dot{\tau}_1^{G_1}$.
2) Let $\mathbf{V}$ contain r and g_0, g_1. Working in $\mathbf{V}$, pick a sequence $\langle x_\alpha : \alpha < \omega_1\rangle$ of pairwise distinct members of C. Put:

- $q^0_{\alpha,n} = g_0(x_\alpha, n)$, $q^1_{\alpha,n} = g_1(x_\alpha, n)$,
- $\alpha\ E_m\ \beta$ if and only if $x_\alpha{\restriction}m = x_\beta{\restriction}m$.

Now check. □

Proposition 4.3.5. *Let $r, \mathbb{P}, \dot{W}, \mathbb{Q}_\ell, \dot{\tau}_\ell$ (for $\ell < 2$) be as in 4.3.1. Moreover, let $\mathbb{Q}_\ell, \dot{W}_\ell, \mathcal{X}_\ell, \mathbf{H}_\ell$ (for $\ell < 2$) be as in 3.1.1 (with the real r encoding all the needed parameters). Assume that*

(a) ω_1 is not an inaccessible cardinal in $\mathbf{L}$,
(b) $(\mathbb{Q}_0, \dot{\tau}_0)$ is sour to $(\mathbb{Q}_1, \dot{\tau}_1)$,
(c) for every real s and a Borel set $B \subseteq \mathcal{X}_\ell$ coded in $\mathbf{L}[r, s]$, if $B \notin \mathcal{I}_{\mathbb{Q}_\ell, \dot{W}_\ell}$ then there is an $\mathcal{I}_{\mathbb{Q}_\ell, \dot{W}_\ell}$–generic real over $\mathbf{L}[r, s]$ belonging to the set B.

Then there is a $\mathbf{\Sigma}^1_3$ set which does not have $\mathcal{I}_{\mathbb{P}, \dot{W}}$–Baire property.

Proof. Since $\mathbb{Q}_\ell, \dot{W}_\ell$ are as in 3.1.1, we have a nice description of $\mathbb{Q}_\ell$–names for reals, see 3.1.7. So we have Borel functions $h_\ell : \mathcal{X}_\ell \longrightarrow \mathcal{X}$ such that $\Vdash_{\mathbb{Q}_\ell} \dot{\tau}_\ell = h_\ell(\dot{W}_\ell)$. It follows from the assumed properties of $\dot{\tau}_\ell$ that

$(*)_1$ for every Borel set $A \subseteq \mathcal{X}$, if $A \notin \mathcal{I}_{\mathbb{P}, \dot{W}}$ then $h_\ell^{-1}[A] \notin \mathcal{I}_{\mathbb{Q}_\ell, \dot{W}_\ell}$.

(Note that h_ℓ is coded by the real r as well.) Since ω_1 is not inaccessible in $\mathbf{L}$, for some real a we have $\omega_1^{\mathbf{L}[a]} = \omega_1$. Let

$$X_\ell = \{x \in \mathcal{X} : \text{for some } \mathcal{I}_{\mathbb{Q}_\ell, \dot{W}_\ell}\text{–generic real } y \in \mathcal{X}_\ell \text{ over } \mathbf{L}[a, r] \text{ we have } h_\ell(y) = x\}.$$

(Note that if y is $\mathcal{I}_{\mathbb{Q}_\ell, \dot{W}_\ell}$–generic over $\mathbf{L}[a, r]$, then it determines a $\mathbb{Q}_\ell^{\mathbf{L}[a,r]}$–generic filter $G \subseteq \mathbb{Q}_\ell^{\mathbf{L}[a,r]}$ over $\mathbf{L}[a, r]$ and $h_\ell(y) = \dot{\tau}_\ell^G$; remember 3.1.5 and the choice of h_ℓ.)

$(*_2)$ X_ℓ is a $\mathbf{\Sigma}^1_3$ subset of $\mathcal{X}_\ell$.

[Why? Note that, by 3.1.3(4), the formula "c is a Borel code for a set $\sharp c \subseteq \mathcal{X}_\ell$ and $\sharp c \in \mathcal{I}_{\mathbb{Q}_\ell, \dot{W}_\ell}$" is (equivalent in ZFC to) a $\mathbf{\Pi}^1_2$ formula. Hence easily the formula "y is $\mathcal{I}_{\mathbb{Q}_\ell, \dot{W}_\ell}$–generic over $\mathbf{L}[a, r]$" is $\mathbf{\Pi}^1_2$.]

$(*_3)$ For every Borel set $B \subseteq \mathcal{X}$, if $B \notin \mathcal{I}_{\mathbb{P}, \dot{W}}$ then $X_\ell \cap B \neq \emptyset$.

[Immediate by $(*_1)$ and the assumption (c).]

$(*_4)$ $X_0 \cap X_1 = \emptyset$.

[It follows from the sourness and the assumption that $\omega_1^{\mathbf{L}[a]} = \omega_1$.]
Finally note that $(*_3) + (*_4)$ implies that both sets X_0, X_1 do not have the $\mathcal{I}_{\mathbb{P}, \dot{W}}$–Baire property. □

Corollary 4.3.6. *Suppose that* r, $\mathbb{P}$, $\dot{W}$, $\mathbb{Q}_\ell$, $\dot{\tau}_\ell$, $\dot{W}_\ell$ *are as in 4.3.5, and clauses (a)–(c) there hold, and* $\mathbb{Q}_0 = \mathbb{Q}_1$, $\dot{W}_0 = \dot{W}_1$. *Assume additionally that, for some* $k < \omega$, *there are* $\mathbb{Q}_0^{(k)}$*–names* $\dot{\rho}_\ell^0, \dot{\rho}_\ell^1$ *(for* $\ell < 2$*) and a* $\mathbb{Q}_0$*–name* $\dot{\tau}$ *such that*

(d) $\Vdash_{\mathbb{Q}_0}$ " $\dot{\tau} \in \mathcal{X}$ *is* $\mathcal{I}_{\mathbb{P},\dot{W}}$*–generic over* $\mathbf{V}$ ",

(e) $\Vdash_{\mathbb{Q}_0^{(k)}}$ " $\dot{\rho}_\ell^0, \dot{\rho}_\ell^1 \in \mathcal{X}_0$ *are* $\mathcal{I}_{\mathbb{Q}_0,\dot{W}_0}$*–generic over* $\mathbf{V}$ ", *and for* $\ell < 2$ *and every* $\mathcal{I}_{\mathbb{Q}_0,\dot{W}_0}$*–positive Borel set* $B \subseteq \mathcal{X}_0$ *there is* $q \in \mathbb{Q}_0^{(k)}$ *such that* $q \Vdash$ " $\dot{\rho}_\ell^0 \in B$ ",

(f) $\Vdash_{\mathbb{Q}_0^{(k)}}$ " $\dot{\tau}[\dot{\rho}_\ell^0] = \dot{\tau}_\ell[\dot{\rho}_\ell^1]$ ".

(Above, $\mathbb{Q}_0^{(k)}$ *stands for the iteration of length* k *of the forcing notion* $\mathbb{Q}_0$.*)*
Then there is a projective subset of $\mathcal{X}_0$ *that does not have the* $\mathcal{I}_{\mathbb{Q}_0,\dot{W}_0}$*–Baire property.*

Let us turn to getting sourness for some of the forcing notions discussed in this paper. Of course, because of 4.1.5 the forcing notions covered by 4.2.5 are not sour (they are in the sweet kingdom, after all). However, there are sour examples around. Let us introduce them starting with exotic norm conditions which were chosen specially with the sourness in mind.

Proposition 4.3.7. *Let* (K, Σ) *be a local forgetful and complete (see 2.2.7) creating pair for* $\mathbf{H}$.

(1) Assume (K, Σ) *is linked and*

(α) $(\forall n < \omega)(|\mathbf{H}(n)| > 2^n)$,

(β) $(\bar{K}, \bar{g})$ *is a 1–norming system for* $\mathbf{H}$ *(see 2.2.8),* $\bar{K} = \langle K_\ell : \ell < \omega \rangle$, $\bar{g} = \langle g_\rho : \rho \in 2^{<\omega} \rangle$,

(γ) *if* $A \subseteq \mathbf{H}(n)$, $a \in A$ *and* $\mathbf{nor}[t_A^n] > 1$ *then* $\mathbf{nor}[t_{A\setminus\{a\}}^n] \geq \mathbf{nor}[t_A^n] - 1$,

(δ) *letting* $A_n = \mathbf{H}(n) \setminus \{g_\rho(n) : \rho \in 2^\ell\}$ *for* $n \in K_\ell$, *and* $A_n = \mathbf{H}(n)$ *for* $n \notin \bigcup_{\ell<\omega} K_\ell$, *we have* $\lim_{n\to\infty} \mathbf{nor}[t_{A_n}^n] = \infty$.

(Above, for $n \in \omega$ *and* $A \subseteq \mathbf{H}(n)$, t_A^n *is the unique creature* $t \in K$ *with* $m_{\rm dn}^t = n$ *and* $\mathrm{pos}(t) = A$*; see 2.2.7.)*

Then the forcing notion $\mathbb{Q}_\infty^{\bar{K},\bar{g}}(K, \Sigma)$ *(see 2.2.8) is very explicitly sour over Cohen.*

(2) If (K, Σ), $(\bar{K}, \bar{g})$ *and* $\mathbf{H}$ *are as in (1), and* $f : \omega \times \omega \longrightarrow \omega$ *is fast and*

$$(\forall n < \omega)(\forall i \geq n)(f(n, i) \leq \mathbf{nor}[t_{\mathbf{H}(i)}^i]),$$

then $\mathbb{Q}_f^{\bar{K},\bar{g}}(K, \Sigma)$ *is very explicitly sour over Cohen.*

(3) Assume that $h : \omega \times \omega \longrightarrow \omega$ *is regressive,* $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ *is* h*–closed, countable and* $\geq^*$*–directed. Suppose that* (K, Σ) *is* h*–linked and clauses* (α) – (γ) *hold true, and*

(δ^+) *for* A_n *as in* (δ)*, for each* $f \in \mathcal{F}$ *we have* $(\forall^\infty n \in \omega)(f(n) \leq \mathbf{nor}[t_{A_n}^n])$.

Then the forcing notion $\mathbb{Q}_{\mathcal{F}}^{\bar{K},\bar{g}}(K, \Sigma)$ *is very explicitly sour over Cohen.*

(4) Assume that (K, Σ) *is linked, satisfies the demand 1(*γ*), and* $\lim_{n\to\infty} \mathbf{nor}[t_{\mathbf{H}(n)}^n] = \infty$. *Let* $\bar{U} = \langle U_{\rho,k} : \rho \in 2^{<\omega}\ \&\ k < \omega \rangle$ *be a 2–norming system (see 2.2.10). Then the forcing notion* $\mathbb{Q}_\infty^{\bar{U}}(K, \Sigma)$ *(see 2.2.10) is very explicitly sour over Cohen.*

(5) Similarly for forcing notions $\mathbb{Q}^{\bar U}_f(K,\Sigma)$, $\mathbb{Q}^{\bar U}_{\mathcal{F}}(K,\Sigma)$ (under assumptions parallel to that in 2,3 above, with clause (δ^+) replaced by just $f(n) \le \mathbf{nor}[t^n_{\mathbf{H}(n)}]$ for $n<\omega$, $f\in\mathcal{F}$).

Proof. 1) Let us think about the Cohen forcing $\mathbb{C}$ as the set of partial functions from $2^{<\omega}$ to 2 with the relation of extension; $\mathbf{H}_{\mathbb{C}}(\rho)=2$ for $\rho\in 2^{<\omega}$ (so we interpret ω as $2^{<\omega}$), and $\dot W_{\mathbb{C}}$ is the natural name for the $\mathbb{C}$–generic real in $2^{2^{<\omega}}$.

Let $\dot W$ be the name for $\mathbb{Q}^{\bar K,\bar g}_\infty(K,\Sigma)$–generic real, i.e., $\Vdash \dot W=\bigcup\{w^q : q\in \Gamma_{\mathbb{Q}^{\bar K,\bar g}_\infty(K,\Sigma)}\}$. Let $\dot\tau_0,\dot\tau_1$ be standard $\mathbb{Q}^{\bar K,\bar g}_\infty(K,\Sigma)$–names for functions from $2^{<\omega}$ to 2 such that

- $\dot\tau_0(\rho)=1$ if and only if $(\exists m\in K_{\mathrm{lh}(\rho)+1})(\dot W(m)\in\{g_{\rho^\frown\langle 0\rangle}(m), g_{\rho^\frown\langle 1\rangle}(m)\})$,
- $\dot\tau_1(\rho)=1-\dot\tau_0(\rho)$.

Claim 4.3.7.1.

$$\Vdash_{\mathbb{Q}^{\bar K,\bar g}_\infty(K,\Sigma)} \text{“ } \dot\tau_0,\dot\tau_1 \textit{ are } \mathbb{C}\textit{–generic over } \mathbf{V} \text{ ”;}$$

moreover $[\![\dot\tau_\ell\in B]\!]_{\mathbb{Q}^{\bar K,\bar g}_\infty(K,\Sigma)}\neq\mathbf{0}$ *for any non-meager Borel set* $B\subseteq 2^{2^{<\omega}}$*. Hence* $(\mathbb{C},\dot W_{\mathbb{C}})$, $(\mathbb{Q}^{\bar K,\bar g}_\infty(K,\Sigma),\dot\tau_0)$, $(\mathbb{Q}^{\bar K,\bar g}_\infty(K,\Sigma),\dot\tau_1)$ *are as in 4.3.1.*

Proof of the claim. Let $q\in\mathbb{Q}^{\bar K,\bar g}_\infty(K,\Sigma)$ and let $\rho_0,\ldots,\rho_k\in 2^\omega$ be such that

$$\rho\in 2^\ell \;\&\; m^{t^q_i}_{\mathrm{dn}}\in K_\ell \;\&\; g_\rho(m^{t^q_i}_{\mathrm{dn}})\notin\mathrm{pos}(t^q_i) \quad\Rightarrow\quad \rho\vartriangleleft\rho_0\vee\ldots\vee\rho\vartriangleleft\rho_k.$$

Pick $N\ge\mathrm{lh}(w^q)$ such that $\rho_0{\restriction}N,\ldots,\rho_k{\restriction}N$ are pairwise distinct and

(a) $m^{t^q_i}_{\mathrm{dn}}\ge N \quad\Rightarrow\quad \mathbf{nor}[t^q_i]\ge 2$, and
(b) $n\ge N \quad\Rightarrow\quad \mathbf{nor}[t^n_{A_n}]\ge 2$, where A_n is as in the assumption (δ).

Suppose $M>N$ and $h:\{\rho\in 2^{<\omega}: N<\mathrm{lh}(\rho)\le M\}\longrightarrow 2$. Build a condition $p=(w^p,t^p_0,t^p_1,\ldots)$ such that

(c) $w^q\vartriangleleft w^p$, $w^p(m)\in\mathrm{pos}(t^q_j)$ if $m=m^{t^q_j}_{\mathrm{dn}}<\mathrm{lh}(w^p)$, and $t^p_i\in\Sigma(t^q_j)$ whenever $m^{t^q_j}_{\mathrm{dn}}=m^{t^p_i}_{\mathrm{dn}}$,
(d) if $N<\mathrm{lh}(\rho)\le M$, $h(\rho)=1$, then for some $m=m(\rho)\in K_{\mathrm{lh}(\rho)+1}$ we have $m<\mathrm{lh}(w^p)$ and $w^p(m)\in\{g_{\rho^\frown\langle 0\rangle}(m),g_{\rho^\frown\langle 1\rangle}(m)\}\cap\mathrm{pos}(t^q_i)$, where $m^{t^q_i}_{\mathrm{dn}}=m$,
(e) if $m^{t^q_i}_{\mathrm{dn}}\ge\mathrm{lh}(w^p)$ then $\mathbf{nor}[t^q_i]\ge 2^{M+3}$,
(f) if $m<\mathrm{lh}(w^p)$ is not any of the $m(\rho)$'s from clause (d) above (for $h(\rho)=1$), $m\in K_\ell$, $\ell>N$ (so $m>N$), then $w^p(m)\in\mathrm{pos}(t^q_i)\setminus\{g_\rho(m):\rho\in 2^\ell\}$, where $m^{t^q_i}_{\mathrm{dn}}=m$ (remember: (K,Σ) is linked and (a)+(b)),
(g) if $m=\mathrm{lh}(w^p)+i\in K_\ell$, $N<\ell\le M+1$ and $m^{t^q_j}_{\mathrm{dn}}=m$, then $t^p_i\in\Sigma(t^q_j)$ is such that

$$\mathbf{nor}[t_i^p] \geq \mathbf{nor}[t_j^q] - 2^{M+1} \quad \text{and} \quad \mathrm{pos}(t_i^p) \cap \{g_\rho(m) : \rho \in 2^\ell\} = \emptyset$$

(remember assumption (γ) and clause (e)),

(h) if $m = \mathrm{lh}(w^p) + i \notin \bigcup\limits_{\ell=N+1}^{M} K_\ell$, $m = m_{\mathrm{dn}}^{t_j^q}$, then $t_i^p = t_j^q$.

It should be clear that we can build $p \in \mathbb{Q}_\infty^{\bar{K},\bar{g}}(K, \Sigma)$ satisfying the demands (c)–(h) and that then $q \leq p$ and $p \Vdash h \subseteq \dot{\tau}_0$.

Now we easily conclude that $\dot{\tau}_0$ is Cohen over $\mathbf{V}$; the rest should be clear too. $\square$

For $\rho \in 2^\omega$ and $k < \omega$ let $p_{\rho,k} \in \mathbb{Q}_\infty^{\bar{K},\bar{g}}(K, \Sigma)$ be such that $w^{p_{\rho,k}} = \langle\rangle$, if $i \in K_\ell$, $i > k$, then $\mathrm{pos}(t_i^{p_{\rho,k}}) = \mathbf{H}(i) \setminus \{g_{\rho\restriction\ell}(i)\}$, and if either $i \leq k$ or $i \notin \bigcup\limits_{\ell\in\omega} K_\ell$ then $\mathrm{pos}(t_i^{p_{\rho,k}}) = \mathbf{H}(i)$. Note that the function $g^* : 2^\omega \times \omega \longrightarrow \mathbb{Q}_\infty^{\bar{K},\bar{g}}(K, \Sigma) : (\rho, k) \mapsto p_{\rho,k}$ is Borel.

Claim 4.3.7.2. *(1) For each $\rho \in 2^\omega$ the set $\{p_{\rho,k} : k < \omega\}$ is predense in $\mathbb{Q}_\infty^{\bar{K},\bar{g}}(K, \Sigma)$.*

(2) If $\rho_0, \rho_1 \in 2^\omega$, $\rho_0\restriction m = \rho_1\restriction m = \sigma$, $\rho_0(m) = 0$, $\rho_1(m) = 1$ and $k < m$, then

$$(\forall q \in \mathbb{Q}_\infty^{\bar{K},\bar{g}}(K, \Sigma))(q \geq p_{\rho_0,k} \;\&\; q \geq p_{\rho_1,k} \quad \Rightarrow \quad q \Vdash \dot{\tau}_0(\sigma) = 0).$$

Proof of the claim. (1) Straightforward.

(2) Note that if $q \geq p_{\rho_0,k}$, $q \geq p_{\rho_1,k}$ then for each i, $n = m_{\mathrm{dn}}^{t_i^q} \in K_{m+1}$ implies ($n \geq m+1 > k$ and) $g_{\rho_0\restriction(m+1)}(n), g_{\rho_1\restriction(m+1)}(n) \notin \mathrm{pos}(t_i^q)$ (and, of course, $\sigma^\frown\langle 0\rangle = \rho_0\restriction(m+1)$, $\sigma^\frown\langle 1\rangle = \rho_1\restriction(m+1)$). Also, if $n < \mathrm{lh}(w^q)$, $n \in K_{m+1}$, then ($n > k$ and) $w^q(n) \notin \{g_{\rho_0\restriction(m+1)}(n), g_{\rho_1\restriction(m+1)}\}$. $\square$

Now one easily shows that $(\mathbb{Q}_\infty^{\bar{K},\bar{g}}(K, \Sigma), \dot{\tau}_0)$ is very explicitly sour to $(\mathbb{Q}_\infty^{\bar{K},\bar{g}}(K, \Sigma), \dot{\tau}_1)$.

2), 3) Similarly.

4) Here we think about the Cohen forcing $\mathbb{C}$ as the set of finite partial functions from $2^{<\omega} \times \omega$ to 2 ordered by the extension; $\mathbf{H}_\mathbb{C}$, $\dot{W}_\mathbb{C}$ are interpreted suitably.

For each $n < \omega$ pick $a_n \in \mathbf{H}(n)$. Let $\dot{W}$ be the name for $\mathbb{Q}_\infty^{\bar{U}}(K, \Sigma)$–generic real, and take standard $\mathbb{Q}_\infty^{\bar{U}}(K, \Sigma)$–names $\dot{\tau}_0, \dot{\tau}_1$ for functions from $2^{<\omega} \times \omega$ from 2 such that

- $\dot{\tau}_0(\sigma, k) = 0$ if and only if $(\forall n \in U_{\sigma^\frown\langle 0\rangle,k} \cup U_{\sigma^\frown\langle 1\rangle,k})(\dot{W}(n) \neq a_n)$,
- $\dot{\tau}_1(\sigma, k) = 1 - \dot{\tau}_0(\sigma, k)$.

Claim 4.3.7.3.

$$\Vdash_{\mathbb{Q}_\infty^{\bar{U}}(K,\Sigma)} \text{“ } \dot{\tau}_0, \dot{\tau}_1 \textit{ are } \mathbb{C}\textit{–generic over } \mathbf{V} \text{ ”;}$$

moreover $[\![\dot{\tau}_\ell \in B]\!]_{\mathbb{Q}_\infty^{\bar{U}}(K,\Sigma)} \neq \mathbf{0}$ for any non-meager Borel set $B \subseteq 2^{2^{<\omega}}$. Hence $(\mathbb{C}, \dot{W}_\mathbb{C})$, $(\mathbb{Q}_\infty^{\bar{U}}(K, \Sigma), \dot{\tau}_0)$, $(\mathbb{Q}_\infty^{\bar{U}}(K, \Sigma), \dot{\tau}_1)$ are as in 4.3.1.

Proof of the claim. Quite similar to 4.3.7.1. $\square$

Now, for $\rho \in 2^\omega$ and $n, k < \omega$ let $p^n_{\rho,k} \in \mathbb{Q}^{\bar{U}}_\infty(K, \Sigma)$ be such that $w^{p^n_{\rho,k}} = \langle\rangle$, and if $i \in U_{\rho\restriction m,k}$, $m < \omega$, $n \leq i$, then $\mathrm{pos}(t_i^{p^n_{\rho,k}}) = \mathbf{H}(i) \setminus \{a_i\}$, and $\mathrm{pos}(t_i^{p^n_{\rho},k}) = \mathbf{H}(i)$ is all other cases.

Claim 4.3.7.4. *(1) For each $\rho \in 2^\omega$ and $k \in \omega$, the set $\{p^n_{\rho,k} : n < \omega\}$ is predense in $\mathbb{Q}^{\bar{U}}_\infty(K, \Sigma)$.*
(2) If $k < \omega$, $\rho_0, \rho_1 \in 2^\omega$, $\rho_0\restriction m = \rho_1\restriction m = \sigma$, $\rho_0(m) \neq \rho_1(m)$ and $n \leq m$, then

$$(\forall q \in \mathbb{Q}^{\bar{U}}_\infty(K, \Sigma))(q \geq p^n_{\rho_0,k} \;\&\; q \geq p^n_{\rho_1,k} \quad\Rightarrow\quad q \Vdash \dot{\tau}_0(\sigma, k) = 0).$$

Proof of the claim. Straightforward. $\square$

Now we easily finish.
5) Similarly. $\square$

Definition 4.3.8. *Let (K, Σ) be a local creating pair for $\mathbf{H}$ and let $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$.* A sourness system for $(K, \Sigma, \mathcal{F})$ *is a pair $(\bar{g}, \bar{\ell})$ such that*

(α) $\bar{\ell} = \langle \ell_k : k < \omega\rangle \subseteq \omega$ is increasing,
(β) $\bar{g} = \langle g_\rho : \rho \in 2^{<\omega}\rangle$, if $\rho \in 2^k$ then $g_\rho \in \prod\limits_{i<\ell_k} \mathcal{P}(\mathbf{H}(i))$, and

$$\rho \lhd \rho' \;\Rightarrow\; g_\rho \lhd g_{\rho'},$$

(γ) for each $n \in [\ell_k, \ell_{k+1})$, $k < \omega$, the sets $\{g_\rho(n) : \rho \in 2^{k+1}\}$ are pairwise disjoint and non-empty, and $\mathbf{H}(n) \setminus \bigcup\{g_\rho(n) : \rho \in 2^{k+1}\} \neq \emptyset$,
(δ) if $f \in \mathcal{F}$, $\langle t_0, t_1, \dots\rangle \in \mathrm{PC}(K, \Sigma)$ (see 1.1.9), $m^{t_0}_{\mathrm{dn}} = \ell_{k_0}$, $\mathbf{nor}[t_n] \geq f(n+\ell_{k_0})$ for $n < \omega$ then:
(i) for some $N < \omega$, for each $k \geq k_0$, we have

$$|\{\rho \in 2^{k+1} : |\{n \in [\ell_k, \ell_{k+1}) : g_\rho(n) \cap \mathrm{pos}(t_{n-\ell_{k_0}}) \neq \emptyset\}| < 2^{k+1}\}| \leq N,$$

(ii) for some $k_1 \geq k_0$ we have

$$(\forall k \geq k_1)(\forall n \in [\ell_k, \ell_{k+1}))(\mathrm{pos}(t_{n-\ell_{k_0}}) \setminus \bigcup\{g_\rho(n) : \rho \in 2^{k+1}\} \neq \emptyset)$$

Theorem 4.3.9. *Suppose that $h : \omega \times \omega \longrightarrow \omega$ is regressive, and $\mathcal{F} \subseteq (\omega \setminus 2)^\omega$ is a countable h–closed and $\geq^*$–directed family. Let (K, Σ) be a local, h–linked and complete creating pair for $\mathbf{H}$ such that*

(a) for some $f^* \in \mathcal{F}$ we have that $(\forall n < \omega)(\mathbf{nor}[t^n_{\mathbf{H}(n)}] \geq f^*(n))$ (see 2.2.7),
(b) there is a sourness system $(\bar{g}, \bar{\ell})$ for $(K, \Sigma, \mathcal{F})$; let $\bar{g} = \langle g_\rho : \rho \in 2^{<\omega}\rangle$, $\bar{\ell} = \langle \ell_k : k < \omega\rangle$,
(c) if $A \subseteq \mathbf{H}(n)$, $\ell_k \leq n < \ell_{k+1}$, $\rho \in 2^{k+1}$ and $A \setminus g_\rho(n) \neq \emptyset$, then

$$\mathbf{nor}[t^n_{A\setminus g_\rho(n)}] \geq h(n, \mathbf{nor}[t^n_A]).$$

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Then the forcing notion $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ *is very explicitly sour over Cohen.*

Proof. Here we interpret the Cohen forcing notion $\mathbb{C}$ in the standard way, i.e., it is $(2^{<\omega}, \lhd)$ (and $\mathbf{H}_{\mathbb{C}}$, $\dot{W}_{\mathbb{C}}$ are natural).

For $\rho \in 2^\omega$ let $g_\rho = \bigcup\limits_{i\in\omega} g_{\rho\restriction i} \in \prod\limits_{i\in\omega} \mathcal{P}(\mathbf{H}(i))$ (remember 4.3.8(β)). Let $\dot{W}$ be the canonical name for the $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$–generic real and let $\dot{\tau}_0, \dot{\tau}_1$ be standard $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$–names for reals in 2^ω such that

- $\dot{\tau}_0(k) = 0$ if and only if there are $\rho_0, \rho_1 \in 2^\omega$ such that $\rho_0\restriction k = \rho_1\restriction k$, $\rho_0(k) \neq \rho_1(k)$ and $(\forall n \geq \ell_k)(\dot{W}(n) \notin g_{\rho_0}(n) \cup g_{\rho_1}(n))$;
- $\dot{\tau}_1(n) = 1 - \dot{\tau}_0(n)$.

Claim 4.3.9.1. $(\mathbb{C}, \dot{W}_{\mathbb{C}})$, $(\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma), \dot{\tau}_0)$, $(\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma), \dot{\tau}_1)$ *are as in 4.3.1.*

Proof of the claim. We will show that $\dot{\tau}_0$ is (a name for) a Cohen real over $\mathbf{V}$; then the rest should be clear.

So suppose $p = (w^p, t_0^p, t_1^p, \dots) \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ and we may assume that $\mathrm{lh}(w^p) = \ell_{k_0}$, $k_0 < \omega$. For $k \geq k_0$ let

$$\Upsilon_k^p = \Upsilon_k \stackrel{\text{def}}{=} \{\rho \in 2^{k+1} : |\{n \in [\ell_k, \ell_{k+1}) : g_\rho(n) \cap \mathrm{pos}(t^p_{n-\ell_{k_0}}) \neq \emptyset\}| < 2^{k+1}\},$$

and let N be such that $(\forall k \geq k_0)(|\Upsilon_k| < N)$ (remember 4.3.8(δ(i)). Let

$$u_k = \{i \leq k : (\exists \rho_0, \rho_1 \in \Upsilon_k)(\rho_0\restriction i = \rho_1\restriction i \ \&\ \rho_0(i) \neq \rho_1(i))\}.$$

Clearly, for all $k \geq k_0$, $|u_k| \leq N$, and thus we may choose an infinite set $A \subseteq \omega$ such that $\{u_k : k \in A\}$ forms a Δ–system with the heart, say, u^*. Take $k_1 > \max(u^*) + k_0 + N$ such that

$(*_1)$ $(\forall k \geq k_1)(\forall n \in [\ell_k, \ell_{k+1}))(\mathrm{pos}(t^p_{n-\ell_{k_0}}) \setminus \bigcup\{g_\rho(n) : \rho \in 2^{k+1}\} \neq \emptyset)$

(remember 4.3.8(δ(ii)) and suppose that $v \subseteq [k_1, k_2)$, $k_2 > k_1$. We are going to build a condition $q \geq p$ such that

$$q \Vdash (\forall j \in [k_1, k_2))(\dot{\tau}_0(j) = 0 \ \Leftrightarrow\ j \in v).$$

To this end, pick $i > k_2$ such that

$(*_2)$ $u_i \cap [k_1, k_2) = \emptyset$, and

$(*_3)$ for some $f \in \mathcal{F}$ we have

$$(\forall n \geq \ell_{i+1})(f(n) < h^{(k_2)}(n, \mathbf{nor}[t^p_{n-\ell_{k_0}}]),$$

where $h^{(k+1)}(n, m) = h(n, h^{(k)}(n, m))$.

(Possible by the choice of k_1 and the assumption that $\mathcal{F}$ is h–closed.) Since $|\Upsilon_i| < N < 2^{k_1}$, we may find $\rho^* \in 2^\omega$ such that $\rho^*\restriction k_1 \notin \{\sigma\restriction k_1 : \sigma \in \Upsilon_i\}$. For $k \in v$ fix $\rho_k \in 2^\omega$ such that $\rho_k\restriction k = \rho^*\restriction k$, $\rho_k(k) = 1 - \rho^*(k)$.

Let $\langle \sigma_j : j < 2^{i+1} - |v| - 1\rangle$ enumerate $2^{i+1} \setminus \{\rho^*\restriction(i+1), \rho_k\restriction(i+1) : k \in v\}$. By induction on $j < 2^{i+1} - |v| - 1$ define $n^*_j \in [\ell_i, \ell_{i+1}) \cup \{*\}$ as follows:

if there is $n \in [\ell_i, \ell_{i+1}) \setminus \{n^*_{j'} : j' < j\}$ such that $g_{\sigma_j}(n) \cap \mathrm{pos}(t^p_{n-\ell_{k_0}}) \neq \emptyset$,
then n^*_j is the first such number,
otherwise n^*_j is $*$.

Now we choose $w^q, t^q_0, t^q_1, \ldots$ so that:

(i) $\mathrm{lh}(w^q) = \ell_{i+1}$, $w^p \vartriangleleft w^q$;

(ii) if $n \in [\ell_i, \ell_{i+1})$, $n = n^*_j$, $j < 2^{i+1} - |v| - 1$, then $w^q(n) \in g_{\sigma_j}(n) \cap \mathrm{pos}(t^p_{n-\ell_{k_0}})$;

(iii) if $\ell_{k_0} \leq n < \ell_{k_1}$, then $w^q(n) \in \mathrm{pos}(t^p_{n-\ell_{k_0}})$;

(iv) if $n \in [\ell_{k_1}, \ell_i) \cup ([\ell_i, \ell_{i+1}) \setminus \{n^*_j : j < 2^{i+1} - |v| - 1\})$, then

$$w^q(n) \in \mathrm{pos}(t^p_{n-\ell_{k_0}}) \setminus \bigcup \{g_\rho(n) : \rho \in 2^{i+1}\};$$

(v) $t^q_n \in \Sigma(t^p_{n+\ell_{i+1}-\ell_{k_0}})$ is such that $\mathbf{nor}[t^q_n] \geq f(n + \ell_{i+1})$ and

$$\big(g_{\rho^*}(n + \ell_{i+1}) \cup \bigcup_{k \in v} g_{\rho_k}(n + \ell_{i+1})\big) \cap \mathrm{pos}(t^q_n) = \emptyset$$

(where f is given by $(*_3)$).

[Why is the choice possible? Demands (i)–(iii) are easy; (iv) can be satisfied by $(*_1)$, remember 4.3.8(β); (v) is possible by the assumption (c) of the theorem and $(*_3)$.] One easily checks that the demands (i)–(v) imply $q = (w^q, t^q_0, t^q_1, \ldots)$ is a condition in $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ stronger than p. Also, by (ii)+(iv)+(v),

$$q \Vdash (\forall k \in v)(\forall n \geq \ell_k)(\dot{W}(n) \notin g_{\rho^*}(n) \cup g_{\rho_k}(n))$$

(remember 4.3.8(γ); thus $g_{\sigma_j}(n) \cap g_{\rho^*}(n) = g_{\sigma_j}(n) \cap g_{\rho_k}(n) = \emptyset$ in clause (ii)). Hence $q \Vdash (\forall k \in v)(\dot{\tau}_0(k) = 0)$. Now we argue that $q \Vdash (\forall k \in [k_1, k_2) \setminus v)(\dot{\tau}_0(k) = 1)$. If not, then for some $k \in [k_1, k_2) \setminus v$ we find $\rho^+_0, \rho^+_1 \in 2^{i+1}$ such that

$$\rho^+_0 {\restriction} k = \rho^+_1 {\restriction} k, \quad \rho^+_0(k) \neq \rho^+_1(k) \quad \text{and}$$
$$(\forall n \in [\ell_k, \ell_{i+1}))(w^q(n) \notin g_{\rho^+_0}(n) \cup g_{\rho^+_1}(n)).$$

Necessarily, $\{\rho^+_0, \rho^+_1\} \not\subseteq \{\rho^* {\restriction}(i+1), \rho_k {\restriction}(i+1) : k \in v\}$. Moreover, if $\rho^+_\ell \in \{\sigma_j : j < 2^{i+1} - |v| - 1\}$, then $\rho^+_\ell \in \Upsilon_i$ (as if $\rho^+_\ell = \sigma_j \notin \Upsilon_i$ then $n^*_j \in [\ell_i, \ell_{i+1})$ and $w^q(n^*_j) \in g_{\rho^+_\ell}(n^*_j)$). Since $\rho^* {\restriction} k_1 \notin \{\sigma {\restriction} k_1 : \sigma \in \Upsilon_i\}$, we may conclude that $\rho^+_0, \rho^+_1 \in \{\sigma_j : j < 2^{i+1} - |v| - 1\}$ (and thus both are in Υ_i). However, then we get $k \in u_i$, what contradicts $(*_2)$. □

For $\rho \in 2^\omega$ and $n \in \omega$ let $p_{\rho,n} \in \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ be such that $w^{p_{\rho,n}} = \langle\rangle$, $t^{p_{\rho,n}}_k = t^k_{\mathbf{H}(k)}$ for $k < n$, and $t^{p_{\rho,n}}_k = t^k_{\mathbf{H}(k) \setminus g_\rho(k)}$ for $k \geq n$. Let

$$g_0, g_1 : 2^\omega \times \omega \longrightarrow \mathbb{Q}^*_{\mathcal{F}}(K, \Sigma) : (\rho, n) \mapsto p_{\rho,n}.$$

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It is straightforward to check that g_0, g_1 witness $(\mathbb{Q}^*_{\mathcal{F}}(K,\Sigma), \dot{\tau}_0)$ is very explicitly sour to $(\mathbb{Q}^*_{\mathcal{F}}(K,\Sigma), \dot{\tau}_1)$ (note that if $k, k' < m$, $\rho_0, \rho_1 \in 2^\omega$, $\rho_0{\restriction}m = \rho_1{\restriction}m$, $\rho_0(m) \neq \rho_1(m)$ and $q \geq g_0(\rho_0,k), g_1(\rho_1,k')$, then $q \Vdash$" $\dot{\tau}_0(m) = 0$ & $\dot{\tau}_1(m) = 1$ "). $\square$

Remark 4.3.10. In 4.3.9, in the assumptions on the family $\mathcal{F}$, instead of demanding that "$\mathcal{F}$ is countable h–closed and $\geq^*$–directed", we may require that "$\mathcal{F}$ is h–closed and either countable or $\geq^*$–directed", and then conclude that $\mathbb{Q}^*_{\mathcal{F}}(K,\Sigma)$ is explicitly sour. The "countable and $\geq^*$–directed" assumption is needed only to be in the context of 4.3.1 (i.e., to make sure that $\mathbb{Q}^*_{\mathcal{F}}(K,\Sigma)$ is Souslin).

4.4. Conclusions

Conclusion 4.4.1. (1) Let $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\omega_1)$. The following forcing notions are topologically sweet:
(a) $\mathbb{Q}^*_\infty(K_{\mathbf{H}}, \Sigma_{\mathbf{H}}, \Sigma^\perp_{\mathbf{H}})$ of 1.5.3 and $\mathbb{Q}^*_\infty(K_{1.5.5}, \Sigma_{1.5.5}, \Sigma^\perp_{1.5.5})$,
(b) $\mathbb{Q}^*_f(K_{1.5.7}, \Sigma_{1.5.7})$ for f as in 1.5.8(1), and $\mathbb{Q}^*_{\mathcal{F}}(K_{1.5.7}, \Sigma_{1.5.7})$ for $\mathcal{F}$ as defined in 1.5.8(2) (the "e.g." part).

Conclusion 4.4.2. The forcing notions $\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$ for the universality parameters $\mathfrak{p}$ defined in 2.4.7, 2.4.9 and 2.4.10 are iterably sweet.

Conclusion 4.4.3. Let $\mathbf{H} : \omega \longrightarrow \omega$, $\mathbf{H}(i) \geq 2^{2^{i+1}}$, and let h be as in 1.5.9.

(1) Let $f^0_k(n) = \max\{2, \mathbf{H}(n) - 2^{kn}\}$ and $\mathcal{F}_0 = \{f^0_k : k < \omega\}$. Then the forcing notion $\mathbb{Q}^*_{\mathcal{F}_0}(K_{1.5.9}, \Sigma_{1.5.9})$ (constructed for $\mathbf{H}$ as in 1.5.9) is topologically sweet.
(2) Let $f^1_k(n) = \max\{2, \mathbf{H}(n) - 2^k\}$, $\mathcal{F}_1 = \{f^1_k : k < \omega\}$. Then the forcing notion $\mathbb{Q}^*_{\mathcal{F}_1}(K_{1.5.9}, \Sigma_{1.5.9})$ is very explicitly sour over Cohen.
Similarly if $f^2_k(n) = \max\{2, \mathbf{H}(n) - (k2^n)\}$, $\mathcal{F}_2 = \{f^2_k : k < \omega\}$.

Proof. (2) We are going to apply 4.3.9. First note that even though $(K_{1.5.9}, \Sigma_{1.5.9})$ as defined in 1.5.9 is not complete we can easily make it so, or restrict our attention to the forcing notion below some condition (the problems with completeness come from the technical requirement in the definition of $t \in K_{1.5.9}$ that $E_t \neq \emptyset$).

We are going to build a sourness system $(\bar{g}, \bar{\ell})$ for $(K_{1.5.9}, \Sigma_{1.5.9}, \mathcal{F}_1)$ such that the demand 4.3.9(c) holds.

Let $\ell_0 = 0$, $\ell_{k+1} = \ell_k + 2^{2^k}$. For $\rho \in 2^{<\omega}$ pick g_ρ such that

($\oplus_1$) if $\rho \in 2^k$, then $g_\rho \in \prod\limits_{i<\ell_k} \mathbf{H}(i)$, and $\rho \lhd \rho' \Rightarrow g_\rho \lhd g_{\rho'}$,

($\oplus_2$) if $n \in [\ell_k, \ell_{k+1})$, $k < \omega$, then there are no repetitions in the sequence $\langle g_\rho(n) : \rho \in 2^{k+1}\rangle$.

We claim that, letting $\bar{\ell} = \langle \ell_k : k < \omega\rangle$ and $\bar{g} = \langle g_\rho : \rho \in 2^{<\omega}\rangle$, $(\bar{g}, \bar{\ell})$ is as required (we identify $\mathbf{H}(i)$ with $[\mathbf{H}(i)]^1$, of course). Clauses 4.3.8(α)–(γ) are clear.

Suppose that $\langle t_0, t_1, \dots\rangle \in \mathrm{PC}(K_{1.5.9}, \Sigma_{1.5.9})$, $m^{t_n} = n$, $\mathbf{dis}[t_n] = (n, E_n)$, $\mathbf{nor}[t_n] \geq f^1_N(n)$. Then, for large enough n, $|E_n| \leq 2^N$, so let $M = \max\{|E_n| :$

$n < \omega\}$. Assume that $\rho_i \in 2^{k+1}$ (for $i \leq M$) are pairwise distinct. By $(\oplus_2)$, for each $n \in [\ell_k, \ell_{k+1})$ there is $i \leq M$ such that $g_{\rho_i}(n) \notin E_n$. Hence for some $i \leq M$

$$|\{n \in [\ell_k, \ell_{k+1}) : g_{\rho_i}(n) \notin E_n\}| \geq \frac{\ell_{k+1} - \ell_k}{M+1} = \frac{2^{2^k}}{M+1}.$$

Hence we easily conclude that 4.3.8(δ(i)) holds. The demands 4.3.8(δ(ii)) and 4.3.9(c) are even easier.

For $\mathcal{F}_2$ we proceed similarly, but we choose g_ρ so that $g_\rho(n) \in [\mathbf{H}(n)]^{2^n}$. $\square$

5. Epilogue

A general problem that we have in mind in this paper is classifying "nice" ccc forcing notions, in particular finding dividing lines in this family, or at least natural properties. We should explain what we mean. A forcing notion is "nice" if it has a quite absolute definition, so Borel is natural, but Souslin is more central (see 1.3.1 and also [19]), but we may be happy with just "one of the form presented in this paper". A dividing line is a property of such definitions, so that both it and its negation is meaningful (that is we can prove theorems from both). Thus a dividing line may serve as a division to cases in solving problems. (On parallel in Model Theory see [26] and [23].)

The first (possible) dividing line we considered here is determined by "being ω–nw–nep" (or just "being very Souslin ccc"). In some sense, one can consider very Souslin forcing notion as those which are really close to random and Cohen. We have examples of very Borel ccc forcing notions (see 1.5.8(2), 1.5.11, 1.5.15(3)), and forcing notions which are not ω–nw–nep (see 1.5.4, 1.5.6, 1.5.8(1), 1.5.15(1,2)). The argument for "not being ω–nw–nep" was in all cases the same: adding a dominating real. So we arrive to the following question.

Problem 5.1. Suppose $\mathbb{P}$ is a Borel ccc forcing notion which is not equivalent to a ω–nw–nep forcing. Does $\mathbb{P}$ add a dominating real?

If one looks at 1.3.4(3) and 1.5.11, then the following (perhaps less central but still intriguing) question related to ω–nw–nep forcing notions arises.

Problem 5.2. Assume $\mathbf{H}$, K, Σ, $\mathcal{F}$ and h are as in 1.3.4(1c) or as in 1.3.4(2b). Is the forcing notion $\mathbb{Q}^*_{\mathcal{F}}(K, \Sigma)$ (or $\mathbb{Q}^{\rm tree}_{\mathcal{F}}(K, \Sigma)$, respectively) very Borel ccc? (Of course, we are interested in non-finitary (K, Σ).)

The second dividing line originates in [24] and studies of the Baire property (and measurability) of projective sets. To get a model in which all projective sets have the Baire property, [24] uses sweetness while [31] applies topological sweetness. However, the use of the two variants of sweetness might be slightly confusing. What we really need for this type of construction are two properties, say, (a)–sweetness and (b)–sweetness such that

(i) if $\mathbb{P}$ is (a)–sweet and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$–name for a (b)–sweet forcing notion, then $\mathbb{P} * \dot{\mathbb{Q}}$ is (a)–sweet,

(ii) if $\mathbb{P}_0, \mathbb{P}_1$ are (a)–sweet then for sufficiently many forcing notions $\mathbb{Q}$ and their two complete embeddings $f_\ell : \mathbb{Q} \longrightarrow \mathrm{RO}(\mathbb{P}_\ell)$, the amalgamation $\mathbb{P}_0 \times_{f_0, f_1} \mathbb{P}_1$ is (a)–sweet,
(iii) the Universal Meager forcing notion $\mathbb{UM}$ is (b)–sweet,
(iv) (a)–sweetness implies the ccc.

(Note that in (ii) we do not require that all amalgamations are (a)–sweet, we just need to cover the amalgamations needed to ensure suitable homogeneity of the Boolean algebra we construct; see [6]) To some extend this approach was materialized in 4.2.4: the topological sweetness may serve as (a)–sweetness and iterable sweetness is a good candidate for (b)–sweetness. It should be remarked here, that it is quite surprising that compositions of (topologically) sweet forcing notions with the Universal Meager $\mathbb{UM}$ (or the Hechler forcing notion $\mathbb{D}$) are topologically sweet because *the second iterand is sweet*. (The respective proofs in [24], [31] were somewhat less general.) Still, it is very reasonable to ask

Problem 5.3. Can 4.2.4 be improved by weakening the demands on $\dot{\mathbb{Q}}$? Can you find (a)–sweetness and (b)–sweetness satisfying (i)–(iv) and weaker than topological sweetness and iterable sweetness, respectively?

The sweet/sour division is sometimes very surprising – compare 4.4.3(1) and 4.4.3(2). The forcing notions $\mathbb{Q}^*_{\mathcal{F}_0}(K_{1.5.9}, \Sigma_{1.5.9})$ and $\mathbb{Q}^*_{\mathcal{F}_2}(K_{1.5.9}, \Sigma_{1.5.9})$ (of 4.4.3) look very similar and one could expect that both are like the Cohen forcing. However, the first is topologically sweet (so not so far from Cohen) while the other is very explicitly sour (so one could even say that worse than random).

Topological sweetness occurs to be not so seldom (see 4.4.1), however it does not imply that we could make real use of these forcing notions in constructions like [18]. These forcings seem to be quite far from the iterable sweetness, so we conjecture that the following has an affirmative answer.

Problem 5.4. Let $\mathbb{Q}$ be one of the forcing notion covered by 4.2.5(1,2) and 4.2.8. Is there $k < \omega$ such that the (k–step) iteration $\mathbb{Q}^{(k)}$ is sour over Cohen? Over $\mathbb{Q}$?

Proposition 4.2.5(3) gives sweet forcing notions, so they could be of some use in constructions like that in [18]. However, do we really need to force additionally with these forcings? In particular:

Problem 5.5. Is there a universality parameter $\mathfrak{p}$ satisfying the requirements of 4.2.5(3) such that no finite iteration of the Universal Meager forcing notion adds a $\mathbb{Q}^{\mathrm{tree}}(\mathfrak{p})$–generic real? Does the Universal Meager forcing add generic reals for the forcing of 2.4.10? Of 2.4.9?

An intriguing thing is that in the cases we proved sourness over the Cohen, we actually got that the considered forcing notion is very explicitly sour over Cohen (so in particular the amalgamation collapses $\mathfrak{c}$).

Problem 5.6. Let $\mathbb{Q}$ be a Souslin ccc (or just nep ccc) forcing notion, and $(\mathbb{P}, \dot{W})$ is as in 3.1.1.

(1) Assume that $\mathbb{Q}$ is sour over $(\mathbb{P}, \dot{W})$. Is it explicitly sour over $(\mathbb{P}, \dot{W})$? Very explicitly?

(2) Suppose that $\mathbb{Q}$ is not topologically sweet and adds a Cohen real. Is it (weakly) sour over Cohen?

Since sweet/sour division is related to the Baire property of projective sets, let us finish with the following general problem.

Problem 5.7. (1) Let $(\mathbb{P}, \dot{W})$ be as in 3.1.1, $\mathbb{P}$ be of the type studied in this paper. For a cardinal κ, let $\mathcal{I}^{\kappa}_{\mathbb{P},\dot{W}}$ be a $<\kappa$–complete ideal generated by $\mathcal{I}_{\mathbb{P},\dot{W}}$.
What is the consistency strength of the statement "every projective subset of $\mathcal{X}$ has the $\mathcal{I}^{\kappa}_{\mathbb{P},\dot{W}}$–Baire property" ?
(We conjecture that it is always either ZFC or "ZFC + there exists an inaccessible cardinal ", and we would like to characterize and/or describe this dividing line.)

(2) Similarly for (typically non-ccc) ideals $\mathcal{I}_{\mathfrak{p}}$ determined by suitable universality parameters $\mathfrak{p}$ (see 3.2.1).

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