

TORSION MODULES, LATTICES AND P-POINTS

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ABSTRACT

Answering a long-standing question in the theory of torsion modules, we show that weakly productively bounded domains are necessarily productively bounded. (See the Introduction for definitions.) Moreover, we prove a twin result for the ideal lattice L of a domain equating weak and strong global intersection conditions for families $(X_i)_{i \in I}$ of subsets of L with the property that $\bigcap_{i \in I} A_i \neq 0$ whenever $A_i \in X_i$. Finally, we show that for domains with Krull dimension (and countably generated extensions thereof), these lattice-theoretic conditions are equivalent to productive boundedness.

0. Introduction

This paper continues a series of articles (see [3, 1, 2, 8]) dealing with the following problem about torsion modules.

For which (right) Ore domains R is it true that, given any family $(M_i)_{i \in I}$ of torsion (right) R -modules, the intersection of annihilators, $\bigcap_{i \in J} \text{ann}(M_i)$, is non-zero for some cofinite subset $J \subseteq I$ provided that the direct product $\prod_{i \in I} M_i$ is torsion?

In other words, over which Ore domains is the obvious sufficient condition for a direct product of torsion modules to be torsion also necessary? We call such Ore domains (right) *productively bounded*. The fact that Dedekind domains are productively bounded plays a pivotal role in the theory of direct-sum decompositions of direct products of modules over such domains [7], which triggered interest in the property for more general rings. The known classes of productively bounded domains include all Ore domains of countable Krull dimension [3] and all commutative noetherian domains [1].

As a consequence of our main results, these classes of positive examples can be enlarged: namely, each commutative domain which is a countably generated extension of a productively bounded domain inherits this property (Section 4).

Already in the first investigation of the topic [3], it turned out that the following weakened condition on R , accordingly labelled (right) *weak productive boundedness*, is far more accessible in concrete situations: whenever $(M_i)_{i \in I}$ is a family of R -modules such that $\prod_{i \in I} M_i$ is torsion, the annihilators $\text{ann}_R(M_i)$ are non-zero for all but finitely many $i \in I$. This naturally raised the question of whether weak productive boundedness implies the full boundedness condition in general. Our main theorem answers this question affirmatively (Section 1). The proof rests on the notion of a p-point, a certain type of ultrafilter on ω (defined below).

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There is an immediately neighbouring pair of properties for the dual ideal lattice of R which, at least on the face of it, is somewhat stronger than the two boundedness conditions for R discussed so far. Consider any lattice L which is complete and finitely join-irreducible, the latter meaning that its largest element 1 is not the join of two strictly smaller elements. (Observe that the dual ideal lattice of an Ore domain satisfies these conditions.) If $(X_i)_{i \in I}$ is a family of non-empty subsets of L , an element $x = (x_i)_{i \in I}$ of $\prod_{i \in I} X_i$ is referred to as a *transversal* of $(X_i)_{i \in I}$, and such a transversal is said to be *bounded* if $\bigvee \{x_i : i \in I\} < 1$. We say that L is *uniformly transversally bounded* if for each family $(X_i)_{i \in I}$ of non-empty subsets of L all of whose transversals are bounded, there is a cofinite subset J of I such that $\bigvee (\bigcup_{i \in J} X_i) < 1$; moreover, we call L *transversally bounded* if for each family $(X_i)_{i \in I}$ of non-empty subsets of L , all of whose transversals are bounded, $\bigvee X_i < 1$ for all but finitely many $i \in I$.

Given a domain R , we denote by L_R the dual of its right ideal lattice. It is easy to see that transversal boundedness of L_R (respectively uniform transversal boundedness of L_R) entails weak productive boundedness (respectively productive boundedness) of R . It remains open whether the reverse implications hold in general. However, we show in Section 3 that for domains with Krull dimension in the sense of Gordon and Robson [4], all of these boundedness conditions are equivalent. This completes a round-trip from torsion modules to ultrafilters through lattices which began with the papers we listed at the outset.

For the dual ideal lattices of *arbitrary* domains, we prove transversal boundedness to be equivalent to uniform transversal boundedness (Section 2). Interestingly, this equivalence distinguishes dual ideal lattices of domains from abstract lattices. In fact, the second author showed that the continuum hypothesis guarantees the existence of lattices which are transversally bounded, but not uniformly so (see [8]). In Section 2 of the present paper, we observe that this conclusion is actually independent of ZFC, even independent of ZFC plus the negation of the continuum hypothesis.

Throughout, we shall assume that R is a commutative integral domain. However, all of our results in Sections 1–3 actually carry over to arbitrary Ore domains in a quite obvious manner (for module read ‘right module’ and for ideal read ‘right ideal’). This is not true for the theorems of Section 4, where we warn the reader about this point.

That R is productively bounded (respectively weakly productively bounded) will be abbreviated by ‘ R is PB’ (respectively ‘ R is wPB’). Moreover, transversal boundedness and uniform transversal boundedness will be denoted by ‘TB’ and ‘UTB’, respectively, for short.

We recall the definitions of some special kinds of ultrafilters on ω (see [5, pp. 257–9], for example). An ultrafilter U on ω is called a *p-point* (respectively a *Ramsey ultrafilter*) if, whenever ω is expressed as a disjoint union $\coprod_{k \in \omega} N_k$ of subsets N_k , none of which belongs to U , there is a set $Y \in U$ such that $Y \cap N_k$ is finite (respectively $|Y \cap N_k| \leq 1$) for all $k \in \omega$. It is known that either of CH or $(\text{MA} + \neg \text{CH})$ implies the existence of Ramsey ultrafilters. On the other hand, by a result due to the third author [6], the non-existence of p-points is consistent with $\text{ZFC} + \neg \text{CH}$.

1. Products of torsion modules

Our first aim is to prove that every weakly productively bounded domain is actually productively bounded. We start by recording a lemma from [3] to which we shall refer repeatedly.

LEMMA 1 (see [3, Lemma 1.1]). *If R is weakly productively bounded, then for every uncountable family \mathcal{A} of ideals such that $\bigcap \mathcal{A} = 0$, there is a countable subfamily \mathcal{A}' of \mathcal{A} such that $\bigcap \mathcal{A}' = 0$.*

Next we list some consequences of the assumption that there is a counterexample to our claim. Call a domain R *ultrafiltral* if there is a non-principal ultrafilter U on ω , together with a family $(A_n)_{n \in \omega}$ of ideals of R such that, for every subset Y of ω , the intersection $\bigcap_{n \in Y} A_n$ is zero if and only if $Y \in U$. It is essentially proved in [3] and [2] that R is ultrafiltral in case R is weakly productively bounded without being productively bounded; we shall sketch the proof and strengthen the conclusion.

LEMMA 2. *Suppose that R is weakly productively bounded but not productively bounded. Then there is a p -point U , together with a family $(A_n)_{n \in \omega}$ of ideals of R such that, for every subset Y of ω , the intersection $\bigcap_{n \in Y} A_n$ is zero if and only if $Y \in U$.*

Proof. It follows from the assumptions on R and Lemma 1 that there is a countable family $(M_i)_{i \in I}$ of torsion modules such that $\prod_{i \in I} M_i$ is torsion and $\bigcap_{i \in I} \text{ann}(M_i) = 0$. Let $A_i = \text{ann}(M_i)$. As in [3, Theorem 6.1, proof of Step II], there is a subset L of I such that $\bigcap_{i \in L} A_i = 0$, and for any pair of disjoint subsets J and K of L , either $\bigcap_{i \in J} A_i \neq 0$ or $\bigcap_{i \in K} A_i \neq 0$. Without loss of generality, we can assume that $I = L = \omega$. If U is defined to be $\{Y \subseteq \omega : \bigcap_{i \in Y} A_i = 0\}$, U is a non-principal ultrafilter on ω . We claim that U is a p -point.

Suppose that $\omega = \coprod_{k \in \omega} N_k$ such that $N_k \notin U$ for every $k \in \omega$. For each $m \in \omega$, let

$$T_m = \bigoplus \{M_n : n \in \bigcup_{k \geq m} N_k\}. \quad (1.1)$$

Then, for all m , the annihilator $\text{ann}(T_m) = \bigcap \{A_n : n \in \bigcup_{k \geq m} N_k\}$ is zero since $\bigcup_{k \geq m} N_k \in U$ (note that the finite union $\bigcup_{k < m} N_k$ does not belong to U). Consequently, since R is wPB, the product $\prod_{m \in \omega} T_m$ is not torsion. Let $y = (y(m))_{m \in \omega}$ be an element of $\prod_{m \in \omega} T_m$ which is not torsion, that is, $\bigcap_{m \in \omega} \text{ann}(y(m)) = 0$. Moreover, let Y be the union of the supports of the $y(m)$; more precisely,

$$Y = \{n \in \omega : \exists m \text{ s.t. } y(m) \text{ has a non-zero projection on } M_n\}.$$

(Here we refer to the canonical projections associated with the definition of T_m in (1.1).) Then Y belongs to U since $\bigcap_{n \in Y} A_n \subseteq \text{ann}(y) = 0$. On the other hand, for all $k \in \omega$, the intersection $Y \cap N_k$ is clearly finite by construction.

The contradiction we seek now follows from the following observation.

LEMMA 3. *Suppose that there is a p -point U , together with a family $(A_n)_{n \in \omega}$ of ideals of R , such that for every subset Y of ω , the intersection $\bigcap_{n \in Y} A_n$ is zero if and only if Y belongs to U . Then there is an uncountable family \mathcal{B} of ideals of R such that $\bigcap \mathcal{B} = 0$, while $\bigcap \mathcal{B}' \neq 0$ for every countable subfamily \mathcal{B}' of \mathcal{B} .*

Proof. Let $\bar{U} = \{S \subseteq \omega : S \text{ is infinite and } S \notin U\}$. For each $S \in \bar{U}$ and $m \in \omega$, we moreover define

$$B_{S,m} = \bigcap \{A_n : n \in S \setminus \{0, 1, \dots, m\}\}.$$

Note that the $B_{S,m}$ form an ascending chain of ideals each of which is non-zero because S (and hence $S \setminus \{0, 1, \dots, m\}$) does not belong to U .

Let $B_S = \bigcup_{m \in \omega} B_{S,m}$, and set $\mathcal{B} = \{B_S : S \in \bar{U}\}$. First we claim that $\bigcap \mathcal{B} = 0$. Suppose, to the contrary, that $\bigcap \mathcal{B}$ contains a non-zero element r . If $X = \{n \in \omega : r \in A_n\}$, then clearly X does not belong to U since $r \in \bigcap_{n \in X} A_n$. Therefore $\omega \setminus X \in U$, and in particular, $\omega \setminus X$ is infinite. Whenever we write $\omega \setminus X$ as the disjoint union of two infinite subsets, these cannot both belong to U . Hence there is a subset $S_1 \subseteq \omega \setminus X$ which belongs to \bar{U} . But then $r \in B_{S_1}$, and consequently $r \in B_{S_1,m}$ for some m . This means that $S_1 \setminus \{0, 1, \dots, m\}$ is contained in X , a contradiction to the choice of S_1 .

It remains to be proved that $\bigcap \mathcal{B}' \neq 0$ for every countable subset \mathcal{B}' of \mathcal{B} . Observe that so far we have used only the ultrafilter properties of U . Now we shall use the fact that U is a p-point to show that, whenever $\{S_n : n \in \omega\} \subseteq \bar{U}$, there exists $S \in \bar{U}$ such that $B_S \subseteq \bigcap \{B_{S_n} : n \in \omega\}$. This will clearly imply our claim concerning countable subfamilies of \mathcal{B} .

If $\bigcup_{n \in \omega} S_n \in \bar{U}$, we can take $S = \bigcup_{n \in \omega} S_n$. So assume $\bigcup_{n \in \omega} S_n \in U$. Let $N_0 = \omega \setminus \bigcup_{n \in \omega} S_n$, $N_1 = S_0$, and for $k > 1$, let $N_k = S_{k-1} \setminus \bigcup_{n < k-1} S_n$. Then $N_k \notin U$ for all $k \in \omega$, and ω is the disjoint union of the N_k . Hence there exists $Y \in U$ such that for all $k \in \omega$, the intersection $Y \cap N_k$ is finite. Let $S = \omega \setminus Y$. Clearly S is infinite, since $Y \cap S_0$ is finite, and so $S \in \bar{U}$.

To verify the claimed inclusion, we need to show that $B_{S,m} \subseteq B_{S_n}$ for all $m, n \in \omega$. Our construction permits us to write $S_n = \tilde{S}_n \cup F_n$ for $n \in \omega$, where $\tilde{S}_n \subseteq S$ and $F_n \subseteq Y \cap \bigcup_{k \leq n+1} N_k$ is finite. This clearly implies $\tilde{S}_n \setminus \{0, 1, \dots, m\} \subseteq S \setminus \{0, 1, \dots, m\}$ for all $m \in \omega$. Therefore

$$\begin{aligned} \bigcap \{A_\ell : \ell \in S \setminus \{0, 1, \dots, m\}\} &\subseteq \bigcap \{A_\ell : \ell \in \tilde{S}_n \setminus \{0, 1, \dots, m\}\} \\ &= \bigcap \{A_\ell : \ell \in S_n \setminus (F_n \cup \{0, 1, \dots, m\})\} \subseteq B_{S_n}, \end{aligned}$$

which completes the proof of the lemma.

THEOREM 4. *Every weakly productively bounded domain is productively bounded.*

Proof. Suppose, to the contrary, that R is wPB but not PB. By Lemma 2, the hypothesis of Lemma 3 is satisfied. But then R is not wPB by Lemma 1, a contradiction.

2. Lattices

Primarily, this section is devoted to proving that for every domain R , transversal boundedness of the dual ideal lattice L_R entails uniform transversal boundedness of that lattice. This does not follow from Theorem 4, because it is unresolved whether every productively bounded domain R gives rise to a uniformly transversally bounded lattice L_R ; but it does follow from Lemma 3 and previous results of the second author.

As we mentioned earlier, this implication cannot be extended to general lattices in the presence of the continuum hypothesis. We shall subsequently explore what happens when the continuum hypothesis fails.

THEOREM 5. *For every domain R , transversal boundedness of L_R is equivalent to uniform transversal boundedness of L_R .*

We need to review some of the terminology and results of [8] before being able to apply the insights of the previous section. Given any ultrafilter U on ω , we consider an associated complete lattice $L(U) = \{A \subseteq \omega : A \notin U\} \cup \{\omega\}$, ordered by inclusion, in which meets coincide with set-theoretic intersections, and joins are given by

$$\bigvee \{A_i : i \in I\} = \begin{cases} \bigcup_{i \in I} A_i, & \text{if } \bigcup_{i \in I} A_i \notin U, \\ \omega, & \text{otherwise,} \end{cases}$$

The following lemma combines two results of [8].

LEMMA 6. *Let L be a complete, finitely join-irreducible lattice which is transversally bounded, but not uniformly so. Then there exists a p-point U , together with a complete upper subsemilattice L' of L which (as a complete upper semilattice) is isomorphic to $L(U)$.*

In case $L = L_R$ for a domain R , there is a family $(X_n)_{n \in \omega}$ of subsets of L such that a semilattice L' as above and an isomorphism $\phi : L(U) \rightarrow L'$ can be constructed as follows: if $A_n = \bigcap X_n$, then L' consists of all intersections of subfamilies of the family $(A_n)_{n \in \omega}$, and $\phi(\{n\}) = A_n$.

Proof. By Corollary C of [8], there exists a complete upper subsemilattice L' of L which is isomorphic to an upper semilattice of the form $L(U)$ for some ultrafilter U on ω . The lattice L' , being closed under suprema in L , clearly inherits the property of being TB, and hence $L(U)$ is TB. But, by Theorem E(I) of [8], this guarantees that U is a p-point.

The claim concerning a realization of L' in case L is the dual ideal lattice of a domain, is an immediate consequence of part (c) of Corollary C of [8].

Proof of Theorem 5. Suppose, to the contrary, that L_R is TB without being UTB, and let U , L' , $(X_n)_{n \in \omega}$ and A_n be as in Lemma 6. In particular, U is a p-point. Moreover, the definition of $L(U)$ immediately yields the following string of equivalences for any $Y \subseteq \omega$: $Y \in U$ if and only if $\bigvee Y = \omega$ in $L(U)$ if and only if $\phi(\bigvee Y) = 0$. But $\phi(\bigvee Y) = \bigcap_{n \in Y} A_n$, and so the p-point U and the family $(A_n)_{n \in \omega}$ of ideals satisfy the hypothesis of Lemma 3. The conclusion of Lemma 3, when combined with Lemma 1, shows that R fails to be wPB. On the other hand, transversal boundedness of L_R clearly forces R to be wPB. This contradiction completes the proof.

The second author has shown that CH implies the existence of a complete finitely join-irreducible lattice which is transversally bounded without having the uniform boundedness property ([8, p. 204]). Here we observe that this conclusion is independent of \neg CH.

THEOREM 7. *It is undecidable in $ZFC + \neg$ CH whether there is a complete finitely join-irreducible lattice which is transversally bounded, but not uniformly so.*

Proof. The axioms $MA + \neg$ CH imply that there is a Ramsey ultrafilter (see, for example, [5, p. 259]). Hence, by Example J and Theorem G of [8], any model of

ZFC + MA + \neg CH admits a complete finitely join-irreducible lattice which is TB but not UTB.

On the other hand, in a model of ZFC + \neg CH without p-points, there is no such lattice by Lemma 6.

3. Domains with Krull dimension

For domains with Krull dimension, we can show the equivalence of the ring-theoretic and lattice-theoretic properties considered in the previous sections. Moreover, in this case, all of these boundedness conditions follow from the (on the face of it comparatively weak) condition on uncountable families of ideals which arises as a consequence of weak productive boundedness in Lemma 1. More precisely, we have the following.

THEOREM 8. *Suppose that R is a domain with Krull dimension and L_R is its dual ideal lattice. Then the following statements are equivalent.*

- (1) R is weakly productively bounded.
- (2) R is productively bounded.
- (3) L_R is transversally bounded.
- (4) L_R is uniformly transversally bounded.
- (5) For every uncountable family \mathcal{A} of ideals of R such that $\bigcap \mathcal{A} = 0$, there is a countable subfamily \mathcal{A}' of \mathcal{A} such that $\bigcap \mathcal{A}' = 0$.

REMARK. Our argument for the crucial implication (5) \Rightarrow (4) was inspired by the proof of Theorem 8 in [1]. We recall the pivotal definition introduced there. Given a complete lattice L , we start by fixing a family $(X_i)_{i \in I}$ of subsets of L . An element $y \in L$ is said to be *tame* relative to an element $x \in L$ if there exists an infinite subset $K \subseteq I$, together with a family of elements $x_k \in X_k$ for $k \in K$, such that $x \vee \bigvee_{k \in S} x_k \geq y$ for every infinite subset S of K . For the convenience of the reader, we include Lemma 7 of [1].

LEMMA 9. *Let L and $(X_i)_{i \in I}$ be as above, and suppose that L does not contain a complete upper subsemilattice isomorphic to 2^ω . Then there exists a finite subset $F \subseteq I$, together with a family $(x_i)_{i \in F}$ of elements $x_i \in X_i$, such that every element of $\bigcup_{i \in I \setminus F} X_i$ is tame relative to $\bigvee_{i \in F} x_i$.*

Proof of Theorem 8. The implications (4) \Rightarrow (2) and (3) \Rightarrow (1) are known, and (2) \Rightarrow (1) and (4) \Rightarrow (3) are trivial. Moreover, (1) \Rightarrow (5) follows from Lemma 1. Hence it suffices to prove (5) \Rightarrow (4). We assume (5), and let $(X_i)_{i \in I}$ be a family of non-empty subsets of the dual ideal lattice L_R such that for each cofinite subset $J \subseteq I$, the intersection of the ideals in $\bigcup_{i \in J} X_i$ is zero. We wish to apply Lemma 9 to construct a transversal of $(X_i)_{i \in I}$ which is unbounded in L_R . Clearly, it is harmless to assume that none of the sets X_i contains the zero ideal.

Since R has Krull dimension, L_R does not contain any subsets order-isomorphic to 2^ω , and consequently L_R satisfies the hypothesis of Lemma 9. We infer the existence of a finite subset $F \subseteq I$, and a family of ideals $(A_i)_{i \in F} \in \prod_{i \in F} X_i$ such that each ideal $A \in \bigcup_{i \in I \setminus F} X_i$ is tame relative to the intersection $B = \bigcap_{i \in I} A_i$, inside the dual ideal lattice of R .

Our initial assumption about the X_i forces the intersection of the ideals in $\bigcup_{i \in I \setminus F} X_i$ to be zero, and hence condition (5) provides us with a countable family $(A_n)_{n \in \omega}$ of ideals in $\bigcup_{i \in I \setminus F} X_i$ such that $\bigcap_{n \in \omega} A_n = 0$. Since each A_n is tame relative to B , we can, moreover, find infinite subsets $K_n \subseteq I$ and transversals $(A_{k,n})_{k \in K_n} \in \prod_{k \in K_n} X_k$ such that

$$B \cap \bigcap_{k \in S_n} A_{k,n} \subseteq A_n$$

for each infinite subset $S_n \subseteq K_n$. A standard diagonal technique then yields a family $(S_n)_{n \in \omega}$ of pairwise disjoint infinite sets $S_n \subseteq I$ such that S_n is contained in K_n for $n \in \omega$. Moreover, we define $S_{-1} = I \setminus \bigcup_{n \in \omega} S_n$ and, for each $i \in S_{-1}$, we pick an arbitrary ideal $A_{i,-1} \in X_i$. We shall check that the transversal $(A_{k,n})_{k \in S_n, n \geq -1}$ of $\prod_{i \in I} X_i$ is unbounded in L_R . Indeed, our construction entails that

$$B \cap \bigcap_{k \in S_n, n \geq -1} A_{k,n} \subseteq B \cap \bigcap_{k \in S_n, n \geq 0} A_{k,n} \subseteq \bigcap_{n \in \omega} A_n = 0.$$

But since B , being a finite intersection of non-zero ideals, is non-zero, this implies that $\bigcap_{k \in S_n, n \geq -1} A_{k,n} = 0$ as desired. Thus L_R is UTB, that is, condition (4) is satisfied.

4. Countable extensions of domains

Our aim is to prove the following, which generalizes (for the commutative case) Theorems 4.2 and 6.1 of [3].

THEOREM 10. *Suppose that $D \subseteq R$ is an extension of commutative domains such that R is countably generated over D . If D is productively bounded, then so is R .*

REMARK. The referee has pointed out that the implication of this theorem remains valid if D is a right Ore domain and the extension R is generated over D by countably many elements which are central in R .

Proof of Theorem 10. Let D be PB. By Theorem 4, it is enough to prove that R is wPB, that is, to show that if $(M_i)_{i \in I}$ is a family of R -modules with $\text{ann}_R(M_i) = 0$ for all $i \in I$, then $\prod_{i \in I} M_i$ is not torsion. Clearly, without loss of generality, we may assume that our family is countable, so I may be taken to be ω .

We first observe that we can reduce the situation to the case where R is finitely generated over D as a ring, as follows. If $R = D[x_n : n \in \omega]$, let $R_k = D[x_n : n < k]$. Moreover, we write $\omega = \coprod_{k \in \omega} N_k$, where the N_k are pairwise disjoint infinite sets. Assuming the result for the finitely generated case, we obtain, for each k , an element $(m_n)_{n \in N_k}$ in $\prod_{n \in N_k} M_n$ which is not torsion over R_k . Clearly, the transversal $(m_n)_{n \in \omega} \in \prod_{n \in \omega} M_n$ then fails to be a torsion element over R . (Note that the proof shows that the union of a countable chain of productively bounded domains is productively bounded.)

By induction, the finitely generated case can obviously be reduced to the following: if D is PB and $R = D[x]$, then R is PB. So suppose that $R = D[x]$. We say that an element of R has degree $\leq k$ if and only if it can be written in the form $\sum_{i=0}^k d_i x^i$ for some $d_i \in D$. Once more, we decompose ω into infinitely many disjoint infinite subsets N_k . It clearly suffices to prove that for every $k \in \omega$, there is an element $(m_n)_{n \in N_k}$ in $\prod_{n \in N_k} M_n$ such that $\bigcap \{ \text{ann}_R(m_n) : n \in \bigcup_{\ell \leq k} N_\ell \}$ does

not contain any non-zero elements of degree $\leq k$. We construct such transversals $(m_n)_{n \in N_k}$ by induction on k . For $k = 0$, we obtain $(m_n)_{n \in N_0}$ as required from the fact that D is wPB; indeed, the latter implies that $\prod_{n \in N_0} M_n$ is not torsion as a D -module.

Suppose now that elements $(m_n)_{n \in N_\ell}$ with the desired properties have been defined for $\ell \leq k$. Set $A_k = \bigcap \{\text{ann}_R(m_n) : n \in \bigcup_{\ell \leq k} N_\ell\}$. In particular, A_k then does not contain any non-zero elements of degree $\leq k$. If A_k does not contain any elements of degree $\leq k + 1$, let $(m_n)_{n \in N_{k+1}}$ be the zero element. Otherwise, pick an element

$$r_{k+1} = d_{k+1}x^{k+1} + t_k$$

in A_k , where $d_{k+1} \in D \setminus \{0\}$ and $t_k \in R$ has degree $\leq k$. Since $\text{ann}_D(r_{k+1}M_n) = 0$ for all $n \in N_{k+1}$ and D is wPB, there is an element $(r_{k+1}m_n)_{n \in N_{k+1}} \in \prod_{n \in N_{k+1}} M_n$ which is not D -torsion. Hence

$$Dr_{k+1} \cap \bigcap \{\text{ann}_R(m_n) : n \in N_{k+1}\} = 0. \quad (4.1)$$

It follows that $\bigcap \{\text{ann}_R(m_n) : n \in \bigcup_{\ell \leq k+1} N_\ell\} = A_k \cap \bigcap \{\text{ann}_R(m_n) : n \in N_{k+1}\}$ does not contain any elements of degree $\leq k + 1$. Indeed, if $r = dx^{k+1} + t$ were such an element in the intersection (where $d \in D$ and t has degree $\leq k$), then $d_{k+1}r = dd_{k+1}x^{k+1} + dt_k$ because there is at most one element of the form $dd_{k+1}x^{k+1} + t'$ in A_k with t' of degree $\leq k$; keep in mind that A_k contains no non-zero elements of degree $\leq k$. Thus $d_{k+1}r \in Dr_{k+1}$, which contradicts (4.1).

This completes the inductive step, and hence the proof.

Combining Theorem 10 with a result of Bergman and Galvin, we obtain the following.

COROLLARY 11. *Every commutative domain which is countably generated over a noetherian subring is productively bounded.*

Proof. Use Theorem 10 above and Corollary 11 of [1].

In a similar manner one can prove the following.

THEOREM 12. *Suppose that $D \subseteq R$ is an extension of commutative domains such that R_D is countably generated. If the dual ideal lattice of D is uniformly transversally bounded, then the same is true for the dual ideal lattice of R .*

As a consequence of Theorems 8 and 12, we have the following.

COROLLARY 13. *Suppose that $D \subseteq R$ are commutative domains such that R is countably generated over D , and D has Krull dimension. Then weak productive boundedness of D implies uniform transversal boundedness of the lattice L_R .*

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- Note:* Prior to 1995, the second author published under the name 'Zimmermann-Huisgen'.

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