On decomposable sentences for finite models

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A Definition : Suppose $\psi = \psi(\bar{P},\bar{Q})$ (i.e. ψ is first order depending on the predicates $\bar{P} = \langle P_\ell : \ell < n \rangle$, $\bar{Q} = \langle Q_\ell : \ell < n \rangle$, If the truth value of $(A,\bar{P},\bar{Q}) \models \psi(\bar{P},\bar{Q})$ depend on the isomorphism types of $(A,\bar{P})$ and $(A,\bar{Q})$ only, we call $\psi(\bar{P},\bar{Q})$ decomposable.

If this holds for all finite models we call $\psi(\bar{P},\bar{Q})$ finitely decomposable.

Let $K_\psi = \{(A,\bar{Q});\ \exists \bar{P}$ such that $(A,\bar{P},\bar{Q}) \models \psi\}$

B. Claim: If $\psi(\bar{P},\bar{Q})$ is decomposable then there are $\psi_\ell(\bar{P})$, $\psi^\ell(\bar{Q})$ such that we can compute the truth value of $(A,\bar{P},\bar{Q}) \models \psi$ from the truth values of $(A,\bar{P}) \models \psi_\ell(\bar{P})$ and $(A,\bar{Q}) \models \psi^\ell(\bar{Q})$.

Proof : Use saturated models.

C. Conclusion: If $\psi(\bar{P},\bar{Q})$ is decomposable then there are $\vartheta_m(\bar{Q})(m < m_0)$ such that each $K_\psi^\lambda = \{M \in K_\psi : ||M|| = \lambda\}$ is the class of models of $\vartheta_m(\bar{Q})$ where m depends on λ (and is the same for all infinite cardinals).

1. Example: We deal with models with universe $n = \{0,1,\dots,n-1\}$, ($n < \omega$ arbitrary).

We shall find sentences $\psi(\bar{P},\bar{Q}),\varphi(\bar{P})$ (not depending on n) such that

1) the truth value of $(n,\bar{P},\bar{Q}) \models \psi(\bar{P},\bar{Q})$ depend on the isomorphism type of $(n,\bar{P})$ and $(n,\bar{Q})$ only

2) $\psi(\bar{P},\bar{Q}) \rightarrow \varphi(\bar{P})$

3) in each finite power $\varphi(\bar{P})$ has a unique model.

4) For $n < \omega$ (quite large), the set $K_n = \{(n,\bar{Q}) : (\exists \bar{P})[(n,\bar{P},\bar{Q}) \models \psi(\bar{P},\bar{Q})]\}$

is not definable (among models of the right signature and power n) by any first order sentence of size $= {}^{200}\sqrt{n}$ (and even such quantifier depth.)

Remark: We do not try to improve the bounds appearing here, clearly $n^{(1/2+\varepsilon)}$ suffices (for any positive ε).

2. Construction: Let $\varphi(\bar{P})$ just say that (n,P) is a model $(n,+,\times,0,1,<)$ satisfying the reasonable rules of arithmetic (addition , product) (but not necessarily the standard ones). Let $\psi = \psi_0(\bar{Q})$ be such that

$(A,Q_0,Q_1,Q_2,Q_3,F_1,F_2,+',\times',0',1') \models \psi_0$ *iff* Q_0,Q_1,Q_2 are monadic relations which form a partition of A, Q_3 a monadic relation, $Q_3 \subseteq Q_1$, also $\varphi^{Q_2}(+',\times' \cdots)$ hold , F_1,F_2 are one place function from Q_1 onto Q_3. (so $F_\ell(x)$ is undefined for $x \notin Q_1$), and:

$$(\forall x \in Q_3)[x = F_1(x) = F_2(x)]$$
$$(\forall x,y \in Q_1)[x = y \equiv (F_2(x) = F_1(y) \wedge F_2(x) = F_2(y))]$$
$$(\forall x,y \in Q_3)\ (\exists z \in Q_1)[\ F_1(z) = x \wedge F_2(z) = y\,]$$

Let $K_n = \{M : \|M\| = n,\ M \models \psi_0,\ |Q_0^M|^{100} < |Q_1|$ and $|Q_0|$ is even$\}$ (we can replace "even" by anything reasonable.

Before we shall define a ψ, such that $K_n = K_\psi^n$ we have to deal with

3. Question: If $(n,\bar{P}) \models \varphi(\bar{P})$, $Q \subseteq M$, can we define (by a short formula) $|Q|$ in $(n,\bar{P},Q)$, i.e. we want as formula $\vartheta(x,\bar{P},Q)$ such that:

$$(n,\bar{P}) \models \varphi(\bar{P}),\ Q \subseteq n \Longrightarrow (n,\bar{P},,Q) \models (\forall x)\ [\,|\{y : y < x\}| = |Q| \equiv \vartheta(x,\bar{P},Q)]$$

The following approximation (and more) for this appeared in Deneberg Gurevich and Shelah [2], and is included for completeness.

4. Fact: There is a formula $\vartheta(x,\bar{P},Q)$ such that for every n and $\bar{P}$, if $(n,\bar{P}) \models \varphi(\bar{P})$ and $Q \subseteq n$ then $(n,\bar{P},Q) \models (\exists x)\vartheta(x,\bar{P},Q)$ and $\models \vartheta(x,\bar{P},\bar{Q})$ implies

$$|Q| \le |\{y : y < x\}| \le |Q|^2 |\ln n|^2 + 10$$

Proof : Let $\vartheta_0(x,\bar{P},Q)$ says that x is the first prime number such that for every $y \neq z \in Q$: $y \not\equiv z \ mod \ x$ (all arithmetic statements are interpreted by $\bar{P}$).

Let $(n,\bar{P})$ be for notational simplicity the usual arithmetic. So clearly there is at most one such x and $|Q| \leq x$. Suppose that $T < n$ and for every prime $|Q| \leq p$ $|Q| \leq p \leq T$, there is a pair $y \neq z \in Q$ so that p divides $y - z$.

Then $A = \prod_{\substack{y,z \in Q \\ z > y}} (z - y)$ is divisible by $B = \Pi\{p : p \text{ prime}, |Q| \leq p < T\}$. Hence $B \leq A$; but $A \leq n^{|Q|^2}$, whereas $B \geq |Q|^{\pi}$, where π is the number of primes in ($|Q|,T)$, So $e^{|Q|^2 \ln n} = n^{|Q|^2} \geq |Q|^{T/\ln T - |Q|/\ln(Q)} = (e^{T \ln|Q|/\ln T})e^{-|Q|}$, hence

$$|Q|^2 \ln n + |Q| \geq T \ln|Q|/\ln T$$

Hence if e.g. $T = |Q|^2(\ln n)^2$,$n \geq 10$ we get contradiction.

5. Fact: In 4) we can also define a one to one function from Q into $\{y : y < x\}$, and then we can do the same analysis on the image, replacing n by $\{y : y < x\}$) (or even if you want, $T = |Q|^2(\ln n)^2$); so we get a new bound

$$|Q| \leq |\{y : y < x'\}| \leq |Q|^2 (\ln T)^2$$

So if e.g. $|Q| \leq {}^3\sqrt{\ln n}$, we can find a one to one map from Q onto an initial segment: as by the previous analysis w.l.o.g. $Q \subset {}^2\sqrt{\ln n}$, the funcion $q: Q \to n$, $q(x) = |\{y \in Q: y < x\}|$ is represnted in $(n,\bar{P})$.

6. Fact: There is a formula $\vartheta(x,y,\bar{P},\bar{Q})$ such that if $(n,\bar{P},\bar{Q}) \models \varphi(\bar{P},\bar{Q})$, $\varphi(\bar{P}) \wedge \psi_0(\bar{P},\bar{Q})$, then $\vartheta(x,y,\bar{P},\bar{Q})$ defines an isomorphism from $(Q_2,+',\times',\cdots)$ onto an initial segment of $(n,\bar{P})$.

Proof : By (5) we can do this for large enough initial segment, of power $k = \sqrt{\ln n}$; then we know that in a model of finite arithmetic, 2^k is definable as well as the representation of every $\ell \leq 2^k$ by a subset of k (using binary representation). Doing it twice we finish.

7. The sentence ψ: So we have to describe the sentence ψ such that $K_n = K_\psi^n$ for every finite n. It will be the conjunction of $\varphi(\bar{P})$, $\psi_0(\bar{Q})$ and another sentence which we describe what it says, rather than write it down.

So let $M = (A, \bar{P}, \bar{Q}) \models \psi_0(\bar{Q}) \wedge \varphi(\bar{P})$, $|A| = n$. For simplicity we ignore the case some Q_ℓ is empty. W.l.o.g. $(A, \bar{P})$ is the standard model. All considerations are uniform in the sense they do not depend on n.

By (6) we can define the number $|Q_2|$ hence the numbers $|Q_0| + |Q_1| = n - |Q_2|$. By (4) we can define an x such that:

$$|Q_0| \leq x \leq |Q_0|^2 (\ln n)^2$$

We can also define the number $\ln n$. We recall that $|Q_1|$ is a perfect square (by the functions F_0, F_1). So there is a number $y < n$, $y^2 = |Q_1|$. Can we define y in M?

It satisfies:

(*) $n - |Q_2| - y^2 \leq x \leq (n - |Q_2| - y^2)^2 (\ln n)^2$

We have already defined all numbers appearing here (by suitable formulas) except y. So it sufices to show that (*) has a unique solution when $M \in K_n$ (as then we can define it and write our demand on $|Q_0|$ which is $n - |Q_2| - y^2$); if however there are two solutions, then $M \notin K_n$).

Now if $M \in K_n$, $|Q_0|^{100} < |Q_1|$ and $y_1 \neq y_2$ are solutions, we get a contradiction or $y \leq (\ln n)^{10}$, but then we can define $|Q_0|$ directly.

8. Non definability of K_n:

It is well known that two models of the theory of equality of power $> n$ satisfies the same first order sentence of quantifiers depth n. So by the Feferman Vaught theorem (see [CK]), if $M \upharpoonright Q_2 = N \upharpoonright Q_2$, $M \upharpoonright Q_1 = N \upharpoonright Q_1$ and $|Q_0^N| = |Q_0^M| + 1$, (and M,N are finite) then M,N, satisfy the same first order sentences of quantifier depth $< |Q_0^N|$, but $M \in \bigcup_{n<\omega} K_n \Longleftrightarrow N \notin \bigcup_{n,\omega} K_n$.

So we finish.

References

[1] C. C. Cang and H. J. Keisler, Model Theory, North Holland Publ. Co.

[2] L. Deneberg, Y. Gurevich and S. Shelah, Cardinalities definable by constant depth polynomial size circuits. Information and Control.