

BOUNDING BY CANONICAL FUNCTIONS, WITH CH

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We show that the members of a certain class of semi-proper iterations do not add countable sets of ordinals. As a result, starting from suitable large cardinals one can obtain a model in which the Continuum Hypothesis holds and every function from ω_1 to ω_1 is bounded on a club by a canonical function for an ordinal less than ω_2 .

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1. Introduction

Given an ordinal γ , a function $f: \omega_1 \rightarrow \text{Ord}$ is a *canonical function* for γ if the empty condition (i.e. ω_1) in the forcing $\mathcal{P}(\omega_1)/NS_{\omega_1}$ forces that $j(f)(\omega_1^V) = \gamma$, where j is the elementary embedding induced by the generic. For each $\alpha < \omega_1$, the constant function with value α is the canonical function for α . For $\alpha \in [\omega_1, \omega_2)$, a canonical function f for α is obtained by taking a bijection $g: \omega_1 \rightarrow \alpha$ and letting $f(\beta)$ be the ordertype of $g[\beta]$. In this paper we let *Bounding* denote the statement that every function from ω_1 to ω_1 is bounded on a club subset of ω_1 by a canonical function for an ordinal less than ω_2 . It is fairly easy to see that if the nonstationary ideal on ω_1 (NS_{ω_1}) is saturated, then Bounding holds. The second author has shown [11] that given the existence of a Woodin cardinal there is a semi-proper forcing making NS_{ω_1} saturated, and it has been known for some time that there is a simpler forcing making Bounding hold from a weaker large cardinal hypothesis. The most quotable result in this paper is that this standard forcing to make every function from ω_1 to ω_1 bounded by a canonical function is (ω, ∞) -distributive (i.e. it does not add ω -sequences of ordinals), and so this statement is consistent with the

Continuum Hypothesis, even in the presence of large cardinals. This is in contrast with saturation, as Woodin [14] has shown that if NS_{ω_1} is saturated and sufficiently large cardinals exist, then there is a definable counterexample to CH. We give a more general theorem stating that the members of a certain class of semi-proper iterations are (ω, ∞) -distributive. This class includes the standard forcing to make Bounding hold, and is general enough to show that a generalization of Bounding for certain sets of reals is also consistent with CH, answering a question in [14].

The key construction used in the proof generalizes the notion of α -properness from [11, Chap. V] to semi-proper forcing. Briefly, a forcing is α -semi-proper if for any \in -chain of countable elementary submodels of length α , there is a condition which is simultaneously semi-generic for each model in the sequence. The problem in applying the method to show that a given improper iteration is (ω, ∞) -distributive is that for a given model N in the sequence, $N[G] \cap \kappa$ can be a proper superset of $N \cap \kappa$, where κ is the length of the iteration and G is generic for some initial segment, so that new steps appear. For the forcings in this paper, however, we have a good understanding of how to enlarge each such N , as well as how to produce the appropriate tower of models to overcome this.

This can be generalized further, getting the consistency of certain forcing axioms, using ideas from [11, Chaps. V and VIII] and [13]. The reader is referred to [10] for more on this topic and on RCS in particular.

Interest in this question derives also from the study of Woodin's \mathbb{P}_{\max} forcing [14], which produces a model in which CH fails and all forceable Π_2 sentences for $H(\omega_2)$ hold simultaneously. It is not known whether all such Π_2 sentences forceably consistent with CH can hold together with CH. The generalized form of bounding in this paper is a candidate for showing that this is impossible. Candidates for the other half of the incompatibility appear in [12, 15], and in [14, Section 10.6].

2. Skolem Hulls

Given a structure M with a predicate $<^*$ for a wellordering of the domain of M , and a subset X of the domain of M , we let $\text{Sk}_{(M, \in, <^*)}(X)$ denote the Skolem hull of X in $(M, \in, <^*)$. If X is a countable elementary substructure of some $H(\chi)$ and η is an ordinal in X , then we let D_η^X be the set of all $a \in [\eta]^{<\omega}$ such that $f(a) \in X$ for all $f: [\eta]^{|a|} \rightarrow \omega_1$ in X . The point is that if $<^*$ is a wellordering of $H(\chi)$, $X \prec (H(\chi), \in, <^*)$ is countable, and z is a subset of some ordinal $\eta \in X$, then

$$\text{Sk}_{(H(\chi), \in, <^*)}(X \cup z) = \{f(a) : a \in [z]^{<\omega} \wedge f \in H(\chi)^{([\eta]^{<\omega})} \cap X\}$$

and so if z is such that $[z]^{<\omega} \subset D_\eta^X$,

$$\text{Sk}_{(H(\chi), \in, <^*)}(X \cup z) \cap \omega_1 = X \cap \omega_1.$$

Note that if $\eta < \eta'$ are ordinals in an elementary submodel X , then

$$D_\eta^X = D_{\eta'}^X \cap [\eta]^{<\omega}.$$

In order to verify that our forcings satisfy the semi-properness condition we require, we need to repeatedly apply a certain simultaneous-extendibility property. The following lemma will be useful in this regard.

Lemma 2.1. *Say that M is a structure and $<^*$ is a wellordering of the domain of M . Let X be a countable elementary substructure of $(M, \in, <^*)$. Let η be an ordinal in X and let z_0, z_1 be countable subsets of η such that $[z_0]^{<\omega} \subset D_\eta^X$ and $[z_1]^{<\omega} \subset D_\eta^{X'}$, where $X' = \text{Sk}_{(M, \in, <^*)}(X \cup z_0)$. Then $[z_0 \cup z_1]^{<\omega} \subset D_\eta^X$.*

Proof. Let $a \in [z_0]^{<\omega}$ and $b \in [z_1]^{<\omega}$, and let $f: [\eta]^{|a|+|b|} \rightarrow \omega_1$. Then $f_a: [\eta]^{|b|} \rightarrow \omega_1$, defined by letting $f_a(y) = f(a \cup y)$, is a function in

$$X' = \text{Sk}_{(M, \in, <^*)}(X \cup z_0),$$

so $f(a \cup b) \in X' \cap \omega_1 = X \cap \omega_1$. □

Say that X, Y are countable elementary submodels of some $(H(\chi), \in, \leq_\chi)$ with $X \in Y$. Let η be an ordinal in X and let $\gamma \in Y \cap \eta$. Then even though $\text{Sk}_{(H(\chi), \in, \leq_\chi)}(X \cup \{\gamma\})$ is not directly definable in Y , the set itself is in Y , since it is equivalent to $\{f(\gamma) \mid f \in X \wedge f: \eta \rightarrow H(\chi)\}$.

Remark 2.2. Similarly, many of the arguments in this paper prove facts about sequences of elementary submodels by induction on the length of the sequence. Of course if X, Y are countable elementary submodels of $(H(\chi), \in, \leq_\chi)$ with $X \in Y$, Y does not see that X is an elementary submodel, and so the induction hypothesis cannot be applied directly in Y . However, the statement about whether an object exists with a certain relation to X , a semi-generic extending a certain condition, say, is formalizable in Y , and so if one exists in $H(\chi)$ then one exists in Y .

3. A Class of (ω, ∞) -Distributive Iterations

Each step of the iterations we are considering is a forcing which shoots a continuous increasing sequence of length ω_1 through a given stationary set of countable sets of ordinals. Under certain assumptions on the stationary set, such forcings are a typical example of improper forcings which preserve stationary subsets of ω_1 , and they have been well studied in recent years (see, for example [3]). For the iterations in this paper, we require that these stationary sets be definable from sets from the ground model with the help of functions from ω_1 to ω_1 added by initial segments of the iteration.

Definition 3.1. Let $\langle \lambda_\rho : \rho < \kappa \rangle$ be a continuous increasing sequence of ordinals, and let $\mathcal{A} = \langle A_\beta^\rho : \rho < \kappa, \beta < \omega_1 \rangle$ be such that each $A_\beta^\rho \subset [\lambda_{\rho+1}]^{<\omega_1}$. Given $\rho < \kappa$ and $f: \omega_1 \rightarrow \omega_1$, let $Q_{\rho, f}$ be the forcing whose conditions are countable, continuous, increasing sequences $\langle x_\beta : \beta \leq \gamma \rangle$ such that for each $\beta \leq \gamma$, $x_\beta \cap \omega_1 \in \omega_1$ and $x_\beta \in A_{f(x_\beta \cap \omega_1)}^\rho$, ordered by extension. Then an \mathcal{A} -iteration is a structure $\bar{Q} = \langle P_\rho, Q_\rho, f_\rho : \rho < \kappa \rangle$ such that

- (a) $\langle P_\rho, \tilde{Q}_\rho : \rho < \kappa \rangle$ is a revised countable support iteration,
- (b) each \tilde{f}_ρ is a P_ρ -name for a function from ω_1 to ω_1 ,
- (c) for all $\rho < \kappa$, $1_{P_\rho} \Vdash \tilde{Q}_\rho = \tilde{Q}_{\rho, \tilde{f}_\rho}$.

The stationary sets in our iterations must also satisfy certain extendibility conditions with respect to the countable elementary submodels of a sufficiently large initial segment of the universe.

Theorem 3.2. *Let $\langle \lambda_\rho : \rho < \kappa \rangle$ be a continuous increasing sequence of strong limit cardinals with supremum κ . Fix a regular cardinal $\chi > (2^\kappa)^+$, and let \leq_χ be a wellordering of $H(\chi)$. Let $\mathcal{A} = \langle A_\beta^\rho : \rho < \kappa, \beta < \omega_1 \rangle$ be such that each $A_\beta^\rho \subset [\lambda_{\rho+1}]^{<\omega_1}$, and let $\tilde{Q} = \langle P_\rho, \tilde{Q}_\rho, \tilde{f}_\rho : \rho < \kappa \rangle$ be an \mathcal{A} -iteration, such that the following hold.*

- (1) For all $\rho < \kappa$, $\beta < \omega_1$, if $E \in A_\beta^\rho$ and $E' \in [\lambda_{\rho+1}]^{<\omega_1}$ with $E \subset E'$ then $E' \in A_\beta^\rho$.
- (2) For all $\rho < \kappa$, 1_{P_ρ} forces that for all countable $X \prec (H(\chi)^{V^{P_\rho}}, \in, \leq_\chi)$ with $\lambda_{\rho+1}, \tilde{Q} \in X$, and for all $\beta < \omega_1$ there exists a countable $z \subset \lambda_{\rho+1}$ such that $[z]^{<\omega}$ is a subset of

$$\bigcap \{ D_{\lambda_{\rho+1}}^Z : Z \prec (H(\chi)^{V^{P_\rho}}, \in, \leq_\chi) \wedge \lambda_{\rho+1}, \tilde{Q} \in Z \in X \}$$

and, letting $Y = \text{Sk}_{(H(\chi)^{V^{P_\rho}}, \in, \leq_\chi)}(X \cup z)$,

- (i) $X \cap \omega_1 = Y \cap \omega_1$,
- (ii) $Y \cap \lambda_{\rho+1} \in A_\beta^\rho$.
- (3) For all countable $X \prec (H(\chi), \in, \leq_\chi)$ with $\tilde{Q} \in X$, and for all $\rho \in X \cap \kappa$, $\beta < \omega_1$, there exists a countable $Y \prec (H(\chi), \in, \leq_\chi)$ such that
 - (i) $X \subset Y$,
 - (ii) $X \cap V_{\lambda_{\rho+2}} = Y \cap V_{\lambda_{\rho+2}}$,
 - (iii) $Y \cap \lambda_{\rho+1} \in A_\beta^\rho$.

Let P be the Revised Countable Support limit of $\langle P_\rho, \tilde{Q}_\rho : \rho < \kappa \rangle$. Then P is (ω, ∞) -distributive.

Remark 3.3. By standard RCS arguments, if in the statement of Theorem 3.2, we assume in addition that κ is strongly inaccessible, then P is κ -c.c. (see Theorem 4.6).

Theorem 3.2(2) and (3) could easily be subsumed into one condition; indeed, in our applications we verify both conditions at the same time. The conditions correspond to separate parts of the proof, however. Condition (2) is needed to show that each successor step of the iteration is α -semi-proper for all countable α (Theorem 4.13), and could in fact be replaced by this requirement, though this would mean more work in applying the theorem. Condition (3) is used to construct the systems of models which we use to show that no countable sets of ordinals are added (Lemma 4.20).

Since in the end we show that P does not add ω -sequences, these are actually Countable Support iteration. Nonetheless, at this time we do not have a proof of (the corresponding version of) Theorem 3.2 which avoids RCS.

For the rest of this paper, sets denoted by κ , $\langle \lambda_\rho : \rho < \kappa \rangle$, χ , \leq_χ , $Q_{\rho,f}$, \bar{Q} , P , \mathcal{A} and $\langle A_\beta^\rho : \rho < \kappa, \beta < \omega_1 \rangle$ are supposed to have the properties given in the hypotheses of Theorem 3.2. This policy will be modified in two ways. First, in the application sections, we will add extra properties for these terms. On the other hand, in proving the main theorem we sometimes state our lemmas more generally in terms of certain properties of these objects, temporarily forgetting the others. We hope that it is clear when we are doing this, so that there will be no confusion.

4. Proof of Theorem 3.2

Proof of Theorem 3.2 breaks into two largely disjoint parts, which come together only in Lemma 4.21. In the first part, we define α -semi-proper forcing, where α is a countable ordinal, and show that for each countable ordinal α , α -semi-properness is preserved under RCS iterations. In Theorem 4.13, we use Theorem 3.2(2) to show that each successor step from one of these iterations is forced to be α -semi-proper for all countable α , which then carries over to the entire iteration. In the second part of the proof we use the end-extension property given in Theorem 3.2(3) to find a system of elementary submodels suitable for proving (ω, ∞) -distributivity, which is shown in Lemmas 4.17, 4.18, 4.20 and 4.21. By Theorem 4.7, the iteration is κ -c.c.

4.1. Revised countable support

The reader is referred to [10, 11] for the definition of Revised Countable Support and its basic analysis. Alternate presentations of RCS can be found in [2, 7]. We will show that the arguments here follow in all three versions. The only facts about RCS that we need outside of this section are Theorems 4.1, 4.7 and 4.11, and Property 4.3. These are (essentially see Remark 4.4) proved in [7, 8] for the version of RCS presented in [7]. We present proofs of these facts that work for the versions of RCS in [2, 11].

The following two theorems hold for all presentations of RCS. The first gives an important property of RCS which distinguishes it from Countable Support, that names for conditions are essentially conditions themselves.

Theorem 4.1. *Say that $\langle P_\alpha, Q_\alpha : \alpha < \kappa \rangle$ is an RCS iteration with RCS limit P . Fix $p \in P$ and $\gamma < \kappa$. Suppose that A is a maximal antichain in P_γ below $p \restriction \gamma$ and $f : A \rightarrow P$ is a function such that for each $a \in A$ $f(a) \leq p$ and $f(a) \restriction \gamma = a$. Then there is a condition $p' \leq p$ such that $p' \leq p$, $p \restriction \gamma = p' \restriction \gamma$ and each $a \in A$ forces that $p' \restriction [\gamma, \kappa) = f(a) \restriction [\gamma, \kappa)$.*

Theorem 4.2 follows from Theorem 4.1 and [11, Chap. X, Corollary 2.8] (and the corresponding theorem from [2]).

Theorem 4.2. *Suppose that $\langle P_\gamma, \tilde{Q}_\gamma : \gamma < \kappa \rangle$ is an RCS iteration with RCS limit P_κ such that each \tilde{Q}_γ is forced to be semi-proper and each \tilde{Q}_γ forces the corresponding P_γ to have cardinality \aleph_1 . Then the following hold.*

- (1) P_κ is semi-proper.
- (2) For all regular cardinals $\chi > 2^{|P_\kappa|}$, for all countable $X \prec H(\chi)$ with $P_\kappa \in X$, for all $\gamma < \delta$ in $X \cap (\kappa + 1)$, if
 - (a) $p \in P_\kappa$,
 - (b) q is an (X, P_γ) -semi-generic condition in P_γ below $p \upharpoonright \gamma$,
 - (c) q forces “there exists r in $P_\kappa \cap X$ such that $p \upharpoonright [\gamma, \kappa) = r \upharpoonright [\gamma, \kappa)$ ”, then there exists an (X, P_δ) -semi-generic condition $q' \in P_\delta$ such that $q' \upharpoonright \gamma = q$ and $q' \leq p \upharpoonright \delta$.

In [2], RCS is characterized by the following property.

Property 4.3. If $\langle P_\alpha : \alpha < \kappa \rangle$ is an RCS iteration with RCS limit P and $p \in P$, then for all $q \leq p$ in P there exist $\gamma < \kappa$, $r \in P_\gamma$ such that $r \leq q \upharpoonright \gamma$ and either $r \Vdash \text{cof}(\tilde{\kappa}) = \omega$ or $r \Vdash \forall \alpha \in (\tilde{\gamma}, \tilde{\kappa}) p(\alpha) = 1_{\tilde{Q}_\alpha}$.

If $\langle P_\alpha : \alpha < \kappa \rangle$ is an RCS iteration with RCS limit P then for each condition p in P there is an associated P -name $\text{supp}(p)$ for the support of p , the set of $\alpha < \kappa$ such that

$$p(\alpha) \neq 1_{\tilde{Q}_\alpha},$$

as decided by the P_γ -generic filter. One can easily prove by induction on κ that Property 4.3 implies that for each p in P $\text{supp}(p)$ is forced to be countable.

Remark 4.4. Property 4.3 is shown in [7] for the version of RCS in [7], for the special case where $\kappa = \omega_1$ (so the first possibility for r cannot hold), en route to (essentially) proving Theorem 4.6. Together these two facts give Property 4.3 for this version of RCS, since in the remaining case (where $|P_\alpha| < \text{cof}(\kappa)$ for all $\alpha < \kappa$) a maximal antichain deciding the supremum of $\text{supp}(p)$ for some condition p must have maximal restriction in some initial segment of the iteration.

In [11], we have the following property (see [11, Chap. X, Definition 1.1 and Claim 1.3(1)]).

Property 4.5. If $\langle P_\alpha : \alpha < \kappa \rangle$ is an RCS iteration of limit length with RCS limit P and $p \in P$, then there exist P -names τ_i ($i < \omega$) for ordinals less than κ such that

$$1_P \Vdash \text{supp}(\check{p}) = \{\tau_i : i < \omega\}$$

and such that for each $i < \omega$ and each $q \leq p$ in P there exist $\gamma < \kappa$, $r \in P_{\gamma+1}$ such that $r \leq q \upharpoonright (\gamma + 1)$ and $r \Vdash \tau_i = \check{\gamma}$.

If we assume that in addition each \tilde{Q}_α forces that the cardinality of the corresponding P_α is \aleph_1 , then Property 4.5 implies Property 4.3. We need only check the

limit case. Fix p and q , let $\{\tau_i : i < \omega\}$ be the sequence given by Property 4.5 with respect to p , and suppose that there is no pair γ, r such that

- $\gamma < \kappa$,
- $r \in P_\gamma$,
- $r \leq q \upharpoonright \gamma$,
- $r \Vdash \text{cof}(\check{\kappa}) = \omega$.

If some P_α has cardinality greater than the cofinality of κ , then (assuming that q is in the generic filter), κ will have cofinality ω_1 in the extension by $P_{\alpha+1}$. Then, working in the $P_{\alpha+1}$ -extension, let X be a countable elementary substructure of a large enough $H(\chi)$ with $P, p, q, \{\tau_i : i < \omega\} \in X$. Define $\zeta = \bigcup(X \cap \kappa)$ and let $q' \leq q \upharpoonright [\alpha + 1, \kappa)$ be $(X, P/P_{\alpha+1})$ -semi-generic. Then q' forces that $\text{supp}(p)$ will be contained in ζ . This means that for each $i < \omega$ every condition $q'' \leq q'$ in $P/P_{\alpha+1}$ will be compatible with an $r \in P_\zeta/P_{\alpha+1}$ forcing that $\tau_i < \zeta$. But then $q' \upharpoonright \zeta$ will also have this property, and thus $q' \upharpoonright \zeta$ also forces that $\text{supp}(p) \subset \zeta$. Therefore, there exists a condition $r \in P_\gamma$ for some $\gamma < \kappa$ such that $r \upharpoonright (\alpha + 1)$ forces that γ will have the properties of ζ as above, and that $r \upharpoonright [\alpha + 1, \gamma)$ will satisfy the properties of $q' \upharpoonright \zeta$. Such an r suffices.

On the other hand, if $\text{cof}(k) > |P_\alpha|$ for all α , then there is a sequence of pairs (γ_i, r_i) ($i < \omega$) such that

- the γ_i 's are increasing,
- each $r_i \in P_{\gamma_i}$ forces a bound below κ on τ_i ,
- for all $i < j$, $r_j \upharpoonright \gamma_i = r_i$,
- each $r_i \leq p \upharpoonright \gamma_i$.

Then the limit of the r_i 's is the desired condition. To find γ_{i+1}, r_{i+1} , find a maximal antichain A in P_{γ_i} below r_i such that for each $a \in A$, there is a $\gamma_a < \kappa$ and an $r_a \in P_{\gamma_a}$ such that

- $r_a \upharpoonright \gamma_i = a$,
- $r_a \leq p$,
- r_a decides τ_i .

Then apply Theorem 4.1 to A and the function $a \mapsto r_a$ to find r_{i+1} , and let $\gamma_{i+1} = \sup\{\gamma_a : a \in A\}$, which must be below κ since $\text{cof}(\kappa) > |P_{\gamma_i}|$.

Similar considerations give the following (see also [5, Lemma 36.5]). The point is that the RCS limit of an iteration of cofinality ω_1 is just the direct limit, and the ordinals of cofinality ω_1 are stationary below $\text{cof}(\kappa)$ as below. Then for any maximal antichain A in P as below there is some $\gamma < \kappa$ of cofinality ω_1 such that $A \cap P_\gamma$ is a maximal antichain in P_γ . But then $|A| \leq |P_\gamma|$.

Theorem 4.6. *Say that $\langle P_\gamma, \check{Q}_\gamma : \gamma < \kappa \rangle$ is an RCS iteration with RCS limit P such that $\text{cof}(\kappa) > |P_\gamma|$ for all $\gamma < \kappa$, and such that each \check{Q}_γ forces the corresponding P_γ to have cardinality \aleph_1 . Then P is $\text{cof}(\kappa)$ -c.c.*

Applying Theorems 4.2 and 4.6, we have reduced the proof of Theorem 3.2 to showing that each P_β ($\beta < \kappa$) is (ω, ∞) -distributive.

Theorem 4.7. *Let $\langle P_\beta, \tilde{Q}_\beta : \beta < \kappa \rangle$ be an RCS iteration of strongly inaccessible length with RCS limit P_κ such that*

- each 1_{P_β} forces the corresponding \tilde{Q}_β to be semi-proper,
- each \tilde{Q}_β makes the corresponding P_β have cardinality \aleph_1 ,
- each $|P_\beta| < \kappa$,
- each P_β is (ω, ∞) -distributive.

Then for all $\rho < \beta \leq \kappa$, P_β/P_ρ is semi-proper and κ -c.c. Therefore P_κ is (ω, ∞) -distributive.

4.2. Semi-generics for sequences

The following are generalizations of ideas from [11, Chaps. V, X and XII].

Definition 4.8. Let α be a countable ordinal.

- (1) The set $\text{SEQ}_\alpha(\chi)$ consists of all $\bar{N} = \langle N_\beta : \beta < \alpha \rangle$ such that
 - (a) each N_β is a countable elementary substructure of $(H(\chi), \in, \leq_\chi)$,
 - (b) for each $\gamma < \alpha$, $\bar{N} \upharpoonright \gamma = \langle N_\beta : \beta < \gamma \rangle \in N_\gamma$.
- (2) Let $\bar{N} \in \text{SEQ}_\alpha(\chi)$ and let P be a forcing construction in N_0 . A condition $p \in P$ is (\bar{N}, P) -semi-generic if

$$p \Vdash \check{N}_\delta[\check{G}_P] \cap \omega_1 = \check{N}_\delta \cap \omega_1$$

for all $\delta < \alpha$, where \check{G}_P is a P -name for the generic filter. The condition p is $(\bar{N} \upharpoonright [\gamma, \beta], P)$ -semi-generic if this holds for all $\delta \in [\gamma, \beta]$.

- (3) A forcing construction P is α -semi-proper if for every $\bar{N} \in \text{SEQ}_\alpha(\chi)$ with $P \in N_0$ and for all $p \in P \cap N_0$ there is an (\bar{N}, P) -semi generic $q \in P$ such that $p \geq q$.

Given $\bar{N} \in \text{SEQ}_\alpha(\chi)$ and $P \in N_0$, if $G \subset P$ is a generic filter then $\bar{N}[G]$ is the sequence $\langle N_\beta[G] : \beta < \alpha \rangle$. Both parts of the following lemma are immediate. Note that in the second part there is no need to find a $p \in \check{G}_P$ which is (\bar{N}, P) -semi-generic.

Lemma 4.9. *Let α be a countable ordinal and fix $\bar{N} \in \text{SEQ}_\alpha(\chi)$.*

- (1) *Let P be an α -semi-proper forcing in N_0 , and let \check{Q} be a P -name in N_0 for an α -semi-proper forcing. Let $q \in P$ be (\bar{N}, P) -semi-generic and let \check{p} be a P -name for a condition in $(\bigcup \bar{N}[\check{G}_P]) \cap \check{Q}$. Then there is a P -name \check{r} for a condition in \check{Q} such that $1_P \Vdash \check{r} \leq \check{p}$ and (q, \check{r}) is $(\bar{N}, P * \check{Q})$ -semi-generic.*
- (2) $1 \Vdash_P \check{N}[\check{G}_P] \in \text{SEQ}_\alpha(\check{\chi})^{V[\check{G}_P]}$.

Lemma 4.10. *Let α be a countable ordinal. Say that P_κ is the RCS limit of an RCS iteration $\langle P_\rho, \mathcal{Q}_\rho : \rho < \kappa \rangle$ in a set forcing extension V^* such that each 1_{P_ρ} forces the corresponding \mathcal{Q}_ρ to be semi-proper, and each \mathcal{Q}_ρ forces P_ρ to have cardinality \aleph_1 . Then the following hold in V^* .*

- (1) P_κ is α -semi-proper.
- (2) Fix $\beta \leq \gamma \leq \kappa$, and fix $\bar{N} \in \text{SEQ}_\alpha(\chi)$ with P_κ and $\beta, \gamma \in \bigcup \bar{N}$. Let $p \in P_\kappa$, $q \in P_\beta$ and $\delta < \alpha$ be such that

- $\beta \in N_\delta$,
- p is $(\bar{N} \upharpoonright \delta, P_\kappa)$ -semi-generic,
- q is $(\bar{N} \upharpoonright [\delta, \alpha], P_\beta)$ -semi-generic,
- $p \upharpoonright \beta \geq q$,
- q forces that for some $r \in P_\kappa \cap N_\delta$, $p \upharpoonright [\beta, \kappa) = r \upharpoonright [\beta, \kappa)$.

Let $\delta^* \in [\delta, \alpha)$ be such that $\gamma \in N_{\delta^*}$. Then there exist $q^* \in P_\gamma$ and $p^* \in P_\kappa$ such that

- p^* is $(\bar{N} \upharpoonright \delta^*, P_\kappa)$ -semi-generic,
- q^* is $(\bar{N} \upharpoonright [\delta^*, \alpha], P_\gamma)$ -semi-generic,
- $q^* \leq p^* \upharpoonright \gamma$,
- $p^* \leq p$,
- $q^* \upharpoonright \beta = q$,
- q^* forces that for some $r \in P_\kappa \cap N_{\delta^*}$, $p^* \upharpoonright [\gamma, \kappa) = r \upharpoonright [\gamma, \kappa)$.

Proof. We prove (2) by induction on α , working in V^* . The first part follows immediately. We fix the notation that G_ρ is the generic filter for P_ρ , for $\rho < \kappa$. The case $\alpha = 1$ is given by Theorem 4.2. For the case where $\alpha = \alpha' + 1$, there are two subcases. If $\delta^* < \alpha'$, then we may assume by the induction hypothesis that $\delta^* = \delta$ and that there is a $(\bar{N} \upharpoonright [\delta^*, \alpha'), P_\gamma)$ -semi-generic $q' \in P_\gamma$ such that $q' \upharpoonright \beta = q$ and $q' \leq p \upharpoonright \gamma$. Then since q forces that $N_{\alpha'}[G_\beta]$ will be elementary in $H(\chi)^{V^*[G_\beta]}$, where G_β is the generic filter for P_β , we can replace q' with a q'' with the additional property that q forces that $q'' \upharpoonright [\beta, \gamma)$ will be equal to $r \upharpoonright [\beta, \gamma)$ for some $r \in N_{\alpha'} \cap P_\kappa$. By Theorem 4.2(2), then, there is a $q^* \in P_\gamma$ such that $q^* \upharpoonright \beta = q$, $q^* \leq q''$ and q^* is $(N_{\alpha'}, P_\gamma)$ -semi-generic. Such a q^* suffices.

For the subcase $\delta^* = \alpha'$, by the induction hypothesis there is a $(\bar{N} \upharpoonright \alpha', P_\kappa)$ -semi-generic $q' \in P_\gamma$ such that $q' \upharpoonright \beta = q$ and $q' \leq p$. Then since q forces that $N_{\alpha'}[G_\beta]$ will be elementary in $H(\chi)^{V^*[G_\beta]}$, where G_β is the generic filter for P_β , we can replace q' with a q'' with the additional property that q forces that $q'' \upharpoonright [\beta, \kappa)$ will be equal to $r \upharpoonright [\beta, \kappa)$ for some $r \in N_{\alpha'} \cap P_\kappa$, and we can let $p^* = q''$. Then as in the previous paragraph, by Theorem 4.2(2), there is a q^* as desired.

The case where α is a limit and $\kappa = \kappa' + 1$ is similar, now fixing α and inducting on κ . By the induction hypothesis for α , we may assume that $\delta = \delta^*$ and by the induction hypothesis for κ we may assume that there is a $(\bar{N} \upharpoonright [\delta^*, \alpha], P_{\kappa'})$ -semi-generic $q' \in P_{\kappa'}$ such that $q' \upharpoonright \beta = q$ and $q' \leq p \upharpoonright \kappa'$. Since $\mathcal{Q}_{\kappa'}$ is forced to be

α -semi-proper, by Theorem 4.1 (or Lemma 4.9(1)), there is a (\bar{N}, P_κ) -semi-generic q^* below p such that $q^* \upharpoonright \kappa = q'$.

For the case where α and κ are both limits there are two subcases (in each of which we may assume that $\delta = \delta^*$). If κ has cofinality ω or ω_1 , or if $|P_\rho| < \text{cof}(\kappa)$ for each $\rho < \kappa$, then we fix an increasing sequence $\langle \xi_i : i < \omega \rangle$ cofinal in α , with $\xi_0 = \delta$. If the cofinality of κ is countable, fix an increasing sequence of ordinals $\langle \beta_i : i < \omega \rangle \in N_0$ cofinal in κ , with $\beta_0 = \beta$. Otherwise, let β_i be any ordinal in N_{ξ_i} greater than $\sup(\bigcup \bar{N} \upharpoonright \xi_i \cap \kappa)$, again with $\beta_0 = \beta$. Let $p_0 = p$ and let $q_0 = q$. Alternately choose conditions p_{i+1}, q_{i+1} ($i < \omega$), such that

- (1) each p_{i+1} is a condition in P_κ ,
- (2) each q_{i+1} is a condition in $P_{\beta_{i+1}}$,
- (3) each $q_i \leq p_i \upharpoonright \beta_i$,
- (4) each $p_{i+1} \leq p_i$,
- (5) for all $i < j < \omega$, $q_j \upharpoonright \beta_i = q_i$,
- (6) each p_{i+1} is $(\bar{N} \upharpoonright \xi_{i+1}, P_\kappa)$ -semi-generic,
- (7) each q_{i+1} is $(\bar{N} \upharpoonright [\xi_{i+1}, \alpha], P_{\beta_{i+1}})$ -semi-generic,
- (8) each q_i forces that for some condition $r \in P_\kappa \cap N_{\xi_i}$, $r \upharpoonright [\beta_i, \kappa) = p_i \upharpoonright [\beta_i, \kappa)$.

For the case where no condition in any P_η makes $\text{cof}(\kappa) \leq \omega_1$, we modify condition (8) as follows:

- (8a) each q_i forces that for some condition $r \in P_\kappa \cap N_{\xi_i}$ such that for some $\gamma < \kappa$

$$1_{P_\kappa} \Vdash \text{supp}(\check{r}) \subset \check{\gamma}, r \upharpoonright [\beta_i, \kappa) = p_i \upharpoonright [\beta_i, \kappa).$$

That such conditions exist is immediate by the induction hypothesis ((8a) follows from Property 4.3). Then the limit of the q_i 's (call it q^*) will be the desired (\bar{N}, P_κ) -semi-generic, as long as it is below each p_i . This fact follows from the fact that q^* forces that $\{\beta_i : i < \omega\}$ will be cofinal in $\bigcup \{\text{supp}(p_i) : i < \omega\}$. This is clear if $\text{cof}(\kappa) = \omega$, and if $\text{cof}(\kappa) = \omega_1$, it follows from the fact that each $N_{\xi_i} \cap \kappa$ will be cofinal in $N_{\xi_i}[G_{\beta_i}] \cap \kappa$ since q_i is (N_{ξ_i}, P_{β_i}) -semi-generic. For the remaining case it follows from condition (8a).

If $|P_\rho| \geq \text{cof}(\kappa)$ for some $\rho < \kappa$, then we may assume that $\beta > \rho$ and so $\text{cof}(\kappa) \leq \omega_1$ in the P_β -extension. Then we may apply the previous argument in the P_β -extension, along with Theorem 4.1. □

Theorem 4.11. *Let α be a countable ordinal. Say that P_κ is the RCS limit of an RCS iteration $\langle P_\beta, Q_\beta : \beta < \kappa \rangle$ such that each $P_{\beta+1} = P_\beta * Q_\beta$ forces $|P_\beta| \leq \aleph_1$ and each 1_{P_β} forces that Q_β is α -semi-proper. Then for all $\beta < \gamma \leq \kappa$, P_γ/P_β is α -semi-proper for every $\alpha < \omega_1$.*

To show that each $Q_{\rho,f}$ is α -semi-proper for all $\alpha < \omega_1$, we show that we can extend sequences of models in a suitable way.

Lemma 4.12. *Assume that $\lambda < \chi$, \bar{Q} , and $\langle A_\beta : \beta < \omega_1 \rangle \subset [\lambda]^{<\omega_1}$ are such that*

- *for all $\beta < \omega_1$, $E \in A_\beta$, if $E \subset E'$ then $E' \in A_\beta$,*
- *for all countable $X \prec (H(\chi), \in, \leq_\chi)$ with $\lambda, \bar{Q} \in X$, and for all $\beta < \omega_1$ there exists a countable $z \subset \lambda$ such that $[z]^{<\omega}$ is a subset of*

$$\bigcap \{D_\lambda^Z : Z \prec (H(\chi), \in, \leq_\chi) \wedge \bar{Q}, \lambda \in Z \in X\}$$

and, letting $Y = \text{Sk}_{(H(\chi), \in, \leq_\chi)}(X \cup z)$,

- (i) $X \cap \omega_1 = Y \cap \omega_1$,
- (ii) $Y \cap \lambda \in A_\beta$.

Let $\bar{N} \in \text{SEQ}_\xi(\chi)$ (for some $\xi < \omega_1$) with \bar{Q} , $\lambda \in N_0$, and let $g: \omega_1 \rightarrow \omega_1$ be a function in N_0 . Then there exists $\bar{N}' \in \text{SEQ}_\xi(\chi)$ such that

- (a) *each $N_\eta \subset N'_\eta$,*
- (b) *each $N'_\eta \cap \omega_1 = N_\eta \cap \omega_1$,*
- (c) *each $N'_\eta \cap \lambda \in A_{g(N'_\eta \cap \omega_1)}$,*
- (d) *letting $N' = \bigcup \{N'_\eta : \eta \leq \xi\}$, $N' \cap \lambda \in A_{g(N' \cap \omega_1)}$.*

Proof. By induction on ξ . If \bar{N} has a last model, then we are done by the extension assumption on countable elementary submodels of $H(\chi)$. For the limit case, applying the induction hypothesis we may assume that every condition already holds, except Part (d). Let

$$N = \bigcup \{N_\eta : \eta \leq \xi\}.$$

Applying the end-extension assumption, let z be a countable subset of λ such that

$$[z]^{<\omega} \subset \bigcap \{D_\lambda^Z : Z \prec (H(\chi), \in, \leq_\chi) \wedge \lambda, \bar{Q} \in Z \in N\}$$

and such that, letting $N' = \text{Sk}_{(H(\chi), \in, \leq_\chi)}(N \cup z)$, we have

- (1) $N \subset N'$,
- (2) $N \cap \omega_1 = N' \cap \omega_1$,
- (3) $N' \cap \lambda \in A_{g(N' \cap \omega_1)}$.

Let $h: \omega \rightarrow z$ be a bijection, and fix an increasing sequence $\langle \zeta_i : i < \omega \rangle$ cofinal in ξ with $\zeta_0 = 0$. Now let each

$$N'_\eta = \text{Sk}_{(H(\chi), \in, \leq_\chi)}(N_\eta \cup h[i_\eta]),$$

where i_η is the largest i such that $\zeta_i \leq \eta$. Since these are all finite extensions, each initial sequence of the new sequence is an element of the later models. By the choice of z , each $N_\eta \cap \omega_1 = N'_\eta \cap \omega_1$, and so \bar{N}' is as desired. \square

Theorem 4.13. *If P_ρ is as in Theorem 3.2 and $f: \omega_1 \rightarrow \omega_1$ is a function added by P_ρ , then $Q_{\rho, f}$ is α -semi-proper in the extension by P_ρ for all $\alpha < \omega_1$.*

Proof. By induction on α , using the Lemma 4.12 and working in the extension by P_ρ . Fix $\bar{N} \in \text{SEQ}_\alpha(\chi)$ and $p \in Q_{\rho,f}$ with $p, \bar{Q}, \lambda \in N_0$, and let \bar{N}' be as in Lemma 4.12, with respect to $\langle A_\beta^\rho : \beta < \omega_1 \rangle$. If \bar{N}' has a final model (i.e. if α is a successor), then we can choose a $(\bar{N}' \upharpoonright (\alpha - 1), Q_{\rho,f})$ -semi-generic $p' \in N'_{\alpha-1}$ extending p by the induction hypothesis (see Remark 2.2). Then since

$$N'_{\alpha-1} \cap \lambda_{\rho+1} \in A_{f(N'_{\alpha-1} \cap \omega_1)}^\rho,$$

any $N'_{\alpha-1}$ -generic for $Q_{\rho,f}$ extending p' will suffice as the desired condition. For the case where α is a limit, we let \bar{N} and \bar{N}' be as in the proof of Lemma 4.12, and fix an increasing sequence ξ_i ($i < \omega$) of ordinals cofinal in α . Then letting p be p_0 , we can successively pick conditions p_i ($i < \omega$) such that each p_{i+1} is a $(\bar{N}' \upharpoonright \xi_{i+1}, Q_{\rho,f})$ -semi-generic in $N'_{\xi_{i+1}}$ extending p_i , and such that the last member of each p_{i+1} contains $N'_{\xi_i} \cap \lambda_{\rho+1}$. By Lemma 4.12(d), the limit of the p_i 's (adjoined by their union) will be a condition in $Q_{\rho,f}$, and it will be $(\bar{N}, Q_{\rho,f})$ -semi-generic since it extends an $(\bar{N} \upharpoonright \beta, Q_{\rho,f})$ -semi-generic for each $\beta < \alpha$. \square

4.3. Systems of models and conditions

To show that each initial segment P_β ($\beta < \kappa$) is (ω, ∞) -distributive, we choose a suitable pair (M, \bar{N}) where M is a countable elementary submodel of $H(\chi)$ and $\bar{N} \in \text{SEQ}_{o.t.(M \cap \beta)}(\chi)$, and find an (\bar{N}, P_β) -semi-generic condition in P_β which extends an M -generic filter. As defined below, a (\bar{Q}, ρ, β) -system is a partial construction of such an object (minus the M -genericity requirement), where ρ is the length of the initial segment for the iteration for which the desired condition has been constructed, and β is the target length.

Definition 4.14. Given ordinals $\rho \leq \beta \leq \kappa$, we say that (M, \bar{N}, p, q) is a (\bar{Q}, ρ, β) -system if (letting $\delta = o.t.(M \cap \rho)$ and $\gamma = o.t.(M \cap \beta)$):

- (1) $M \prec (H(\chi), \in, \leq_\chi)$,
- (2) \bar{Q}, ρ, β belong to M ,
- (3) $\bar{N} \in \text{SEQ}_\gamma(\chi)$ and $\bar{Q} \in N_0$,
- (4) $q \in P_\rho$ and q is $(\bar{N} \upharpoonright [\delta, \gamma], P_\rho)$ -semi-generic,
- (5) for all $\rho' \in \rho \cap M$, $q \upharpoonright \rho'$ is $(N_{o.t.(M \cap \rho')}, P_{\rho'})$ -semi-generic,
- (6) for all $\eta \in M \cap \beta$,

- (a) for all $\nu \in N_{o.t.(M \cap \eta)} \cap \omega_1$, $M \cap \lambda_{\eta+1} \in A_\nu^\eta$,
- (b) $M \cap V_{\lambda_{\eta+2}} \in N_{o.t.(M \cap \eta)}$,

- (7) $p \in P_\beta$, $p \upharpoonright \rho \geq q$ and for all $\eta \in [\rho, \beta) \cap M$, $p \upharpoonright \eta \in N_{o.t.(M \cap \eta)}$.

Remark 4.15. Note that if $\beta = \rho + 1$ and (M, \bar{N}, p, q) is a (\bar{Q}, ρ, β) -system, then (M, \bar{N}, p, q) is also a (\bar{Q}, β, β) -system, as (the second part of) (4) becomes vacuous in the second case, and (4) in the first case completes condition (5) in the second.

Definition 4.16. Let $M \prec (H(\chi), \in, \leq_\chi)$ and $\beta < \kappa$ with $P_\beta \in M$. A filter $g \subset P_\beta \cap M$ is M -generic if for all dense sets $D \subset P_\beta$ in M the intersection of g and

D is nonempty. A condition $p \in M$ is a *potential- M -generic* for P_β if there exists a $g \in P_\beta \cap M$ M -generic for P_β such that for all $\rho \in M \cap \beta$, 1_{P_ρ} forces that if

$$M \cap \lambda_{\rho+1} \in A_{\mathcal{F}_\rho(M \cap \omega_1)}^\rho$$

then $p(\rho) = g(\rho) \frown \langle \cup g(\rho) \rangle$, and if for all $\rho \notin M \cap \beta$, $p(\rho) = 1_{Q_\rho}$.

Note that if $g \in P_\beta$ is M -generic and p is in g then since $\text{supp}(p)$ is a P_β -name for a countable subset of β the support of p as determined by g is contained in $M \cap \beta$. We need to see that potential- M -generics exist in suitable generality.

Lemma 4.17. *For any $\beta \leq \kappa$, if $(M, \bar{N}, p, \emptyset)$ is a $(\bar{Q}, 0, \beta)$ -system with $p \in M$, then there is a potential- M -generic $\bar{p} \leq p$ such that $\bar{p} \upharpoonright \eta \in N_{o.t.(\eta \cap M)}$ for all $\eta \in M \cap \beta$.*

Proof. Given $g \in P \cap M$ M -generic for P_β , one can easily build a corresponding potential- M -generic p , letting $p(\rho)$ be the empty condition when

$$M \cap \lambda_{\rho+1} \notin A_{\mathcal{F}_\rho(M \cap \omega_1)}^\rho.$$

The point then is just to find an M -generic filter $g \subset M \cap P_\beta$ with $p \in g$, such that $g \cap P_\eta \in N_{o.t.(\eta \cap M)}$ for all $\eta \in M \cap \beta$.

Inducting primarily on $\beta \in M \cap \kappa$, and secondarily on $\gamma \in M \cap \beta$, we show that if $p^* \in P_\beta \cap M$ and $g^* \in N_{o.t.(M \cap \gamma)}$ is an M -generic filter for P_γ such that $p^* \upharpoonright \gamma \in g^*$ and $g^* \cap P_{\gamma'} \in N_{o.t.(M \cap \gamma')}$ for all $\gamma' \in M \cap \gamma$, then there exists an M -generic $g \subset P_\beta$ such that $g \cap P_\gamma = g^*$, $p^* \in g$ and $g \cap P_\eta \in N_{o.t.(M \cap \eta)}$ for all $\eta \in M \cap \beta$. Note that for each $\gamma \in M \cap \beta$, since

$$\mathcal{P}(\mathcal{P}(\lambda_\gamma)) \cap M \in N_{o.t.(M \cap \gamma)}$$

(and $|P_\gamma| \leq 2^{\lambda_\gamma}$), the M -genericity of g^* can be verified in $N_{o.t.(M \cap \gamma)}$ (M itself is not in any member of \bar{N} , so this is not automatic).

The base case $\beta = 0$ is trivial. For the successor step, we may assume that $\gamma + 1 = \beta$. Then since for the successor case there is no restriction on the last coordinate of the M -generic filter (other than M -genericity), g^* can be extended in any fashion to an M -generic for P_β .

For the limit, fix an increasing sequence η_i ($i < \omega$) cofinal in $\beta \cap M$, and let $p = p_0$. Now in ω many steps alternately pick

- M -generic $g_i \subset P_{\eta_i}$ in $N_{o.t.(M \cap \eta_i)}$ (applying the induction hypothesis plus the elementarity of $N_{o.t.(M \cap \eta_i)}$ plus the fact that $\bar{N} \upharpoonright o.t.(M \cap \eta_i) \in N_{o.t.(M \cap \eta_i)}$, as in Remark 2.2) with $p_i \upharpoonright \eta_i \in g_i$ such that
 - $g_i \cap P_\eta \in N_{o.t.(M \cap \eta)}$ for all $\eta \in M \cap \eta_i$,
 - $g_i \cap P_{\eta_j} = g_j$ for all $j < i$,
- $p_{i+1} \leq p_i$ in M meeting the i th dense set in M for P_β such that $p_{i+1} \upharpoonright \eta_i \in g_i$ (such a p_{i+1} exists because g_i is M -generic for P_{η_i}).

Since each $p_{i+1} \in M$, its initial segments are automatically in the corresponding N_ξ 's. Then $\{p_i : i < \omega\}$ generates an M -generic filter g for P_β , and for each η_i , $g \cap P_{\eta_i} = g_i$. Then by the induction hypothesis, $g \cap P_\eta \in N_{o.t.(M \cap \eta)}$ for all $\eta \in M \cap \beta$, since for all $i < \omega$ with $\eta_i \geq \eta$, $g \cap P_\eta = g_i \cap P_\eta$. \square

Lemma 4.18. *Fix $\beta \leq \kappa$. If for all $p_0 \in P_\beta$ there is a (\bar{Q}, β, β) -system (M, \bar{N}, p, q) with $p_0 \in M$ and $p \leq p_0$ a potential- M -generic, then the forcing P_β is (ω, ∞) -distributive.*

Proof. Towards a contradiction, fix the least β for which the lemma fails, and let p_0 be a condition in P_β forcing that P_β adds a new ω -sequence of ordinals. Let (M, \bar{N}, p, q) be a (\bar{Q}, β, β) -system with $p_0 \in M$ and $p \leq p_0$ a potential- M -generic, as given by the hypothesis of the lemma. Then there exists a P_β -name τ in M such that p_0 forces that τ will be a new ω -sequence of ordinals. Let $g \subset P_\beta \cap M$ be an M -generic filter witnessing that p is a potential- M -generic. We wish to see that q is below each member of g . Then we will be done, as for each integer i there is a member of g intersecting the antichain in P_β determining the i th member of τ . So we will show by induction on $\rho \in (\beta + 1) \cap M$ that $q \upharpoonright \rho \leq p' \upharpoonright \rho$ for all $p' \in g$ (simultaneously). The cases where $\rho = 0$ or ρ is a limit and $M \cap \rho$ is cofinal in ρ are clear.

For the successor step from ρ to $\rho + 1$, $\check{f}_\rho(M \cap \omega_1)$ is a P_ρ -name in $N_{o.t.(M \cap \rho)}$ for a countable ordinal, so by Definition 4.14(5) and (6), $q \upharpoonright \rho$ forces that

$$M \cap \lambda_{\rho+1} \in A_{\check{f}_\rho(M \cap \omega_1)}^\rho.$$

Then $q \upharpoonright \rho$ forces that $p(\rho) = g(\rho) \frown (\cup g(\rho))$. By Definition 4.14(7), $q \upharpoonright (\rho + 1) \leq p \upharpoonright (\rho + 1)$, so $q \upharpoonright \rho$ forces that $q(\rho)$ extends $p(\rho)$. Now fix $p' \in g$. Since $q \upharpoonright \rho$ forces that $p(\rho)$ extends $p'(\rho)$, we have that $q \upharpoonright (\rho + 1) \leq p' \upharpoonright (\rho + 1)$.

Lastly, for the case where ρ is a limit and $M \cap \rho$ is not cofinal in ρ , note that ρ has uncountable cofinality, and also that we have shown that each $P_{\rho'}$, $\rho' < \rho$ is (ω, ∞) -distributive. By Property 4.3 then, densely many conditions in P_ρ (and therefore $g \cap P_\rho$) are conditions in some $P_{\rho'}$, $\rho' < \rho$, and so densely many conditions in $g \cap P_\rho$ are in some $P_{\rho'}$ with $\rho' \in \rho \cap M$. \square

Given a countable $X \prec (H(\chi), \in, \leq_\chi)$ with $\lambda \in X$, and given $\eta < \lambda$, we say that $Y \prec (H(\chi), \in, \leq_\chi)$ is a *minimal (η, λ) -extension* of X if the following hold.

- (1) $X \cap \eta = Y \cap \eta$.
- (2) $Y = \text{Sk}_{(H(\chi), \in, \leq_\chi)}(X \cup A)$, for some $A \subset \lambda$.

The fact about these extensions that we will use is given in the following lemma.

Lemma 4.19. *Let $X \prec (H(\chi), \in, \leq_\chi)$, and let $\lambda < \gamma$ be ordinals in X with γ a regular cardinal. Let Y be a minimal (η, λ) -extension of X for some $\eta < \lambda$. Then $X \cap \gamma$ is cofinal in $Y \cap \gamma$.*

Proof. Since γ is regular, each $f: [\lambda]^{<\omega} \rightarrow \gamma$ has bounded range below γ . If f is in X then this bound exists in X . \square

Lemma 4.20. *For any set $x \in H(\chi)$ and any ordinal $\beta \leq \kappa$, there is a $(\bar{Q}, 0, \beta)$ -system $(M, \bar{N}, 1_{P_\beta}, \emptyset)$ with $x \in M$.*

Proof. Let $M_0 \prec (H(\chi), \in, \leq_\chi)$ be countable with $\{\bar{Q}, \beta, x\} \in M$. For some $\zeta \leq \omega_1$, we build $\langle M_\xi : \xi \leq \zeta \rangle$, $\langle \gamma_\xi : \xi \leq \zeta \rangle$ and $\langle N_\xi : \xi < \zeta \rangle$ satisfying the following conditions.

- (1) Each M_ξ is a countable elementary submodel of $(H(\chi), \in, \leq_\chi)$.
- (2) Each N_ξ is a countable elementary submodel of $(H(\chi), \in, \leq_\chi)$.
- (3) Each γ_ξ is the ξ th ordinal in M_ξ (0 being the 0th ordinal).
- (4) $\bar{Q} \in N_0$.
- (5) For all $\xi < \zeta$, $\langle N_\eta : \eta < \xi \rangle \in N_\xi$.
- (6) For all $\xi < \zeta$, $M \cap V_{\lambda_{\xi+2}} \in N_\xi$.
- (7) For all $\xi < \zeta$ and for all $\nu \in N_\xi \cap \omega_1$, $M_{\xi+1} \cap \lambda_{\gamma_\xi+1} \in A_\nu^{\gamma_\xi}$.
- (8) For each $\xi < \zeta$, $M_{\xi+1}$ is a minimal $(2^{2^{\lambda_{\gamma_\xi}}}, \lambda_{\gamma_\xi+1})$ -extension of M_ξ .
- (9) If ξ is a limit ordinal, then $M_\xi = \bigcup \{M_\eta : \eta < \xi\}$.
- (10) If there exists a $\xi < \omega_1$ such that $\gamma_\xi = \beta$, then ζ is the least such ξ ; otherwise $\zeta = \omega_1$.

Given M_ξ , by applying Theorem 3.2(3) repeatedly, once for each $A_\nu^{\gamma_\xi}$ with $\nu \in N_\xi \cap \omega_1$, we can choose each $M_{\xi+1}$ to meet (7) and (8). While the model Y resulting from this repeated application may not be a minimal $(2^{2^{\lambda_{\gamma_\xi}}}, \lambda_{\gamma_\xi+1})$ -extension of M_ξ ,

$$Y^* = \text{Sk}_{(H(\chi), \in, \leq_\chi)}(M_\xi \cup (Y \cap \lambda_{\gamma_\xi+1}))$$

will be, as $Y \cap V_{\lambda_{\gamma_\xi+1}} = Y^* \cap V_{\lambda_{\gamma_\xi+1}}$, so we can take this Y^* as our $M_{\xi+1}$. The rest of the construction is straightforward.

We claim that for each $\xi \leq \zeta$ the sequence $\langle \gamma_\eta : \eta \leq \xi \rangle$ lists the first $\xi + 1$ ordinals of M_ξ in increasing order. This follows by induction on ξ . It is clear when $\xi = 0$, and when $\xi = \xi' + 1$, it follows from (8) and the fact that

$$2^{2^{\lambda_{\gamma_{\xi'}}}} > \gamma_{\xi'}$$

(so in this case $\gamma_\xi = \gamma_{\xi'} + 1$). If ξ is a limit ordinal, we have from the fact that $M_\xi = \bigcup \{M_\eta : \eta < \xi\}$ and the induction hypothesis that $\langle \gamma_\eta : \eta < \xi \rangle$ lists the first ξ ordinals of M_ξ in increasing order. Then the definition of γ_ξ finishes the proof of the claim.

Next we claim that $\gamma_\xi = \beta$ for some $\xi < \omega_1$, and so $\zeta < \omega_1$. Given this we are done, letting $M = M_\zeta$ and $\bar{N} = \langle N_\xi : \xi < \zeta \rangle$. All the conditions of Definition 4.14 are satisfied trivially, aside from (6). Condition (6) is satisfied since for each $\eta \in M_\zeta \cap \beta$, there is some $\xi < \zeta$ such that $\eta = \gamma_\xi$ (and so $\xi = o.t.(M \cap \eta)$). Then N_ξ and M_ξ were chosen to satisfy Definition 4.14(6) and (7) of the construction, and this relationship was preserved for all later M_ξ by (8).

Assume to the contrary that $\zeta = \omega_1$. If $\gamma_{\omega_1} = \beta$, then since every ordinal in $M_{\omega_1} \cap \beta$ is equal to some γ_ξ , there is a limit ordinal $\xi < \omega_1$ such that

$$\{\gamma_\eta : \eta < \xi\} = M_\xi \cap \beta.$$

But then $\gamma_\xi = \beta$, contradicting $\zeta = \omega_1$. So we may assume that $\gamma_{\omega_1} < \beta$. We will show that the cofinality of $M_{\omega_1} \cap \gamma_{\omega_1}$ is countable, which is a contradiction since $\langle \gamma_\xi : \xi < \omega_1 \rangle$ is increasing and cofinal in it. If $\gamma_{\omega_1} \leq \lambda_{\gamma_\xi}$ for some γ_ξ with $\gamma_{\omega_1} \in M_\xi$, then $M \cap \gamma_{\omega_1} = M_\xi \cap \gamma_{\omega_1}$, which is countable. If not, $\gamma_{\omega_1} = \lambda_{\gamma_{\omega_1}}$. If γ_{ω_1} is singular, let ρ be the cofinality of γ_{ω_1} . Then $\rho < \lambda_{\gamma_\xi}$ for some γ_ξ with $\{\rho, \gamma_{\omega_1}\} \in M_\xi$, and there exists a cofinal map $f: \rho \rightarrow \gamma_{\omega_1}$ in M_ξ . Since $\rho \cap M_{\omega_1} = \rho \cap M_\xi$, $f[M_\xi \cap \rho]$ is a countable set cofinal in $M_{\omega_1} \cap \gamma_{\omega_1}$. The last remaining case is that γ_{ω_1} is a regular limit cardinal. Let ξ be least with $\gamma_{\omega_1} \in M_\xi$, and fix a cofinal sequence in $M_\xi \cap \gamma_{\omega_1}$. By Lemma 4.19 and (8) of the construction, this sequence is cofinal in $M_{\omega_1} \cap \gamma_{\omega_1}$. \square

We finish by applying the following lemma to the case $\rho = 0, \gamma = \beta$, from Lemma 4.20. By Lemma 4.18, then, we are done.

Lemma 4.21. *Fix $\beta \leq \kappa$. Let (M, \bar{N}, p, q) be a (\bar{Q}, ρ, β) -system with $\rho \leq \gamma \leq \beta$ in M , and assume that p is a potential- M -generic. Then there exists a condition $q' \in P_\gamma$ such that*

- (1) $q' \upharpoonright \rho = q$,
- (2) (M, \bar{N}, p, q') is a (\bar{Q}, γ, β) -system.

Proof. We first note that the lemma follows from the restricted version of the lemma where $\gamma = \beta$. To see this, fix $M, \bar{N}, p, q, \rho, \gamma, \beta$ as given by the hypothesis of the unrestricted version, and note that the restricted version then gives a $q^* \in P_\gamma$ such that $q^* \upharpoonright \rho = q$ and $(M, \bar{N} \upharpoonright o.t.(M \cap \gamma), p \upharpoonright \gamma, q^*)$ is a $(\bar{Q}, \gamma, \gamma)$ -system. Now, if $G_\rho \subset P_\rho$ is V -generic with $q \in G_\rho$, then $M[G_\rho] \cap \text{Ord} = M \cap \text{Ord}$ (since p is a potential- M -generic and $q \leq p$) and the following hold.

- $\forall \alpha < o.t.(\beta \cap M) \ N_\alpha[G_\rho] \cap \omega_1 = N_\alpha \cap \omega_1$,
- $\bar{N}[G_\rho] = \langle N_\alpha[G_\rho] : \alpha < o.t.(M \cap \beta) \rangle \in \text{SEQ}_{o.t.(M \cap \beta)}(\chi)^{V[G_\rho]}$.

Furthermore, in $V[G_\rho]$,

- $q^* \upharpoonright [\rho, \gamma] \in P_\gamma/P_\rho$,
- $\forall \gamma' \in (\gamma \setminus \rho) \cap M, q^* \upharpoonright [\rho, \gamma']$ is $(N_{o.t.(M \cap \gamma')}[G_\rho], P_{\gamma'}/P_\rho)$ -semi-generic,
- $p \upharpoonright [\rho, \gamma] \geq q^* \upharpoonright [\rho, \gamma]$.

Applying the elementarity of $N_{o.t.(M \cap \gamma)}[G_\rho]$ in $H(\chi)^{V[G_\rho]}$, we see that there exists a condition $\bar{q} \in N_{o.t.(M \cap \gamma)}[G_\rho] \cap P_\gamma/P_\rho$ satisfying these conditions. By Lemma 4.10(2), then, there is a condition $q' \in P_\gamma$ such that $q' \upharpoonright \rho = q$ and q forces that $q' \upharpoonright [\rho, \gamma]$ is a

$$(\bar{N}[G_\rho] \upharpoonright [o.t.(M \cap \gamma), o.t.(M \cap \beta)], P_\gamma/P_\rho)\text{-semi-generic}$$

condition extending such a \bar{q} . This q' suffices. We have Definition 4.14(5) and (7) by the properites listed above for $q^* \upharpoonright [\rho, \gamma)$, condition (4) by our extension, and the others by the assumptions of the lemma.

The restricted version of the lemma follows by induction on $o.t.((\beta \setminus \rho) \cap M)$. Note that our induction hypothesis entitles us (once we have fixed β and ρ) to assume that the unrestricted version holds whenever $\gamma < \beta$.

Now, when $\rho = \beta$ there is nothing to show. The argument for increasing $o.t.((\beta \setminus \rho) \cap M)$ by one follows from the case where $\beta = \rho + 1$, and this case is also trivial (see Remark 4.15).

The only remaining case then is when $\rho < \beta$ and $\beta = \gamma$ is a limit ordinal. Fix an increasing sequence η_i ($i < \omega$) cofinal in $M \cap \beta$ with $\eta_0 = \rho$, and successively apply the induction hypothesis (in V , as opposed to some submodel). That is, let $q_0 = q$, and given q_i , let $q_{i+1} \in P_{\eta_{i+1}}$ be such that $q_{i+1} \upharpoonright \eta_i = q_i$ and (M, \bar{N}, p, q_{i+1}) is a $(\bar{Q}, \eta_{i+1}, \beta)$ -system. Then each q_i is $(\bar{N} \upharpoonright [o.t.(M \cap \eta_i), o.t.(M \cap \beta)), P_{\eta_i})$ -semi-generic, and the limit of the q_i 's is the desired q' . Definition 4.14(4) is then easily satisfied (the second half being vacuous), and Definition 4.14(5) and (7), being local properties, are satisfied by the induction hypothesis. \square

5. Applications

5.1. Bounding

The following is a a minor modification of a standard fact.

Theorem 5.1. *Let λ be a measurable cardinal, let θ be a regular cardinal such that $V_{\lambda+2} \subset H(\theta)$, and let \leq^* be a wellordering of $H(\theta)$. Then for every countable ordinal β and every $X \prec (H(\theta), \in, \leq^*)$, there is a $z \in [\lambda]^\beta$ such that*

$$[z]^{<\omega} \subset \bigcap \{D_\lambda^Z : Z \prec (H(\theta), \in, \leq^*) \wedge Z \in X\}, \quad (5.1)$$

and such that, letting

$$Y = \text{Sk}_{(H(\theta), \in, \leq^*)}(X \cup z),$$

$X \cap \lambda$ is a proper initial segment of $Y \cap \lambda$.

Proof. It suffices to prove the theorem for the case $\beta = 1$, since then we may repeat the construction any countable ordinal number of times. To see that (5.1) is preserved, note that if after repeating the construction some countable number of times we have the λ -end-extension X' of X and we have added $\gamma_0, \dots, \gamma_n$ (possibly among other ordinals) to z and $Z \prec (H(\theta), \in, \leq^*)$ is in X , then $Z' = \text{Sk}_{(H(\theta), \in, \leq^*)}(Z \cup \{\gamma_0, \dots, \gamma_n\}) \in X'$. Further, if $\gamma_{n+1} \in D_\lambda^{Z'}$, then by Lemma 2.1, $\{\gamma_0, \dots, \gamma_{n+1}\} \in D_\lambda^Z$.

Now, to prove the theorem for the case $\beta = 1$, let μ be the \leq^* -least normal measure on λ , and fix X as in the statement of the theorem. Let γ be any member of $\bigcap (X \cap \mu)$, and let

$$Y = \text{Sk}_{(H(\theta), \in, \leq^*)}(X \cup \{\gamma\}).$$

The key point is that if $f: \lambda \rightarrow \lambda$ is a regressive function in any elementary submodel of $(H(\theta), \in, \leq^*)$, then f is constant on a set $A \in \mu$, and since μ is \leq^* -least, μ and therefore A and the constant value are also in the model. Along with the fact that $\bigcap(X \cap \mu) \subset \bigcap(Z \cap \mu)$ for any $Z \prec (H(\theta), \in, \leq^*)$ with $Z \in X$, this shows that γ satisfies (5.1). To see that $Y \cap \lambda$ is a proper end-extension of $X \cap \lambda$, note that $\gamma \notin X$, and since any ordinal $\eta \in Y \cap \lambda$ is of the form $f(\gamma)$ for some $f: \lambda \rightarrow \lambda$ in X , if $\eta \in Y \cap \gamma$ then any corresponding f is regressive on a set in μ , and so $\eta \in X$, by the key point. \square

Putting together Theorems 3.2 and 5.1, we have the following.

Corollary 5.2. *Say that there exists a strongly inaccessible limit of measurable cardinals. Then there is a semi-proper forcing extension in which Bounding holds, along with the Continuum Hypothesis.*

Proof. Let $\langle \lambda_\rho : \rho < \kappa \rangle$ be a continuous increasing sequence of cardinals with supremum κ strongly inaccessible such that each $\lambda_{\rho+1}$ is a measurable cardinal. Fix a regular cardinal $\chi > (2^\kappa)^+$, and let \leq_χ be a wellordering of $H(\chi)$. Let

$$\mathcal{A} = \langle A_\beta^\rho \subset [\lambda_{\rho+1}]^{<\omega_1} : \rho < \kappa, \beta < \omega_1 \rangle$$

be such that each A_β^ρ is the set of countable subsets of $\lambda_{\rho+1}$ of ordertype greater than β . Then by Theorem 5.1, Theorem 3.2(1)–(3) are satisfied (condition (2) by the fact that small forcing preserves measurable cardinals), so any forcing P as in the statement of Theorem 3.2 is semi-proper and (ω, ∞) -distributive. We may further require that for every function f added by some initial stage of the iteration there is a $\rho < \kappa$ such that $f = \dot{f}_\rho$. Each $Q_{\rho,f}$ forces that $|\lambda_{\rho+1}| = \aleph_1$ and that any canonical function for $\lambda_{\rho+1}$ dominates f on a club, so P forces Bounding. Further, since $2^\omega < \kappa$ in V and P forces $\kappa = \omega_2$, P forces CH. \square

Deiser and Donder have recently shown [1] that Bounding is equiconsistent with a strongly inaccessible limit of measurable cardinals.

5.2. Suslin bounding

The previous application can be generalized to show that the following statement can be forced without adding reals, answering a question in [14].

Definition 5.3 ([4]). Suppose that $A \subset \mathbb{R}$. Then A is *universally Baire* if for any compact Hausdorff space X and any continuous function $\pi: X \rightarrow \mathbb{R}$, the set $\{a \in X \mid \pi(a) \in A\}$ has the property of Baire in X .

Definition 5.4. *Suslin bounding* is the following statement. Suppose that $A \subset \mathbb{R}$ is universally Baire, and fix a function $f: \omega_1 \rightarrow A$. Then there is a tree T on $\omega \times \omega_1$ such that $A = p[T]$ and such that $\{\alpha < \omega_1 \mid f(\alpha) \in p[T \upharpoonright (\omega \times \alpha)]\}$ contains a club.

Theorem 5.5. *Bounding is equivalent to Suslin bounding for Π_1^1 sets.*

Proof. For the forward direction, we adapt the proof that Π_1^1 sets are projections of trees on $\omega \times \omega_1$ (see [9]). Let S be a tree on $\omega \times \omega$, let $B = p[T]$ and let $A = \omega^\omega \setminus B$. Recall that for all $x \in A$,

$$S_x = \{\sigma \in \omega^{<\omega} \mid (x \upharpoonright |\sigma|, \sigma) \in S\}$$

is wellfounded. For each $x \in A$, let β_x be the least ordinal (necessarily countable) such that there exists a function $r_x: \omega^{<\omega} \rightarrow \beta_x$ with the property that if $\sigma, \sigma' \in S_x$ are such that σ is a proper initial segment of σ' , then $r_x(\sigma) > r_x(\sigma')$. Fix $f: \omega_1 \rightarrow A$, and for each $\alpha < \omega_1$ let $h(\alpha) = \beta_{f(\alpha)}$. Now fix $\gamma < \omega_2$ and a bijection $g: \omega_1 \rightarrow \beta$ such that for every α in a fixed club $C \subset \omega_1$, *o.t.*($g[\alpha]$) $> h(\alpha)$. Let $\sigma: \omega \rightarrow \omega^{<\omega}$ be a bijection such that $|\sigma(i)| \leq i + 1$ for all $i < \omega$. Let T be the tree on $\omega \times \omega_1$ consisting of all pairs (ρ, τ) such that if $i, j < |\rho| (= |\tau|)$ and $\sigma(i)$ is a proper initial segment of $\sigma(j)$ with

$$(\rho \upharpoonright |\sigma(i)|, \sigma(i)), (\rho \upharpoonright |\sigma(j)|, \sigma(j))$$

both in S , then $g(\tau(i)) > g(\tau(j))$. Then $p[T] = A$, and, for each $x \in A$, if δ_x is the least (again, necessarily countable) ordinal δ such that *o.t.*($g[\delta]$) $\geq \beta_x$ and i_x is an order preserving embedding of β_x into $g[\delta_x]$, then $(x, g^{-1} \circ i_x \circ r_x \circ \sigma)$ is a path through $T \upharpoonright (\omega \times \delta_x)$. Since $\delta_{f(\alpha)} \leq \alpha$ for all $\alpha \in C$, this shows that T satisfies the definition of Suslin bounding for f .

For the other direction, let $W \subset \omega^\omega$ be the set of functions coding wellorderings of ω under some fixed coding with the property that for each $g \in \omega^\omega$ and $n \in \omega$, $g \upharpoonright (n+1)$ codes how $n+1$ compares with each $m < n+1$. Each function from ω_1 to ω_1 induces a corresponding function from ω_1 into W . Let $f: \omega_1 \rightarrow W$ be such a function, and let T be the tree given by Suslin bounding, with C the witnessing club set. We have a partial, not necessarily transitive, order \leq_o on the sequences in T , where $\sigma_0 \leq_o \sigma_1$ if one extends the other, and, σ_i being the longer one, the first coordinate of the last element of σ_i codes that $|\sigma_0| < |\sigma_1|$ in the corresponding ordering. Let \leq_t be the least transitive ordering containing \leq_o . Then \leq_t is a wellfounded partial order. Seeing this requires checking that the resulting order is antireflexive and wellfounded. To see antireflexivity, assume that

$$\tau \leq_o \sigma_0 \leq_o \dots \leq_o \sigma_n \leq_o \tau$$

is the shortest possible counterexample. First note that σ_0 and σ_n cannot be comparable in \leq_o (or identical) since then there would be a shorter counterexample, removing the τ 's and placing either $\sigma(n)$ and the beginning of the sequence or σ_0 at the end if they are unequal. Therefore σ_0 and σ_n must be incompatible extensions of τ , and σ_1 must be comparable with τ and distinct from σ_n . But then depending on whether $\sigma_1 \geq_o \tau$ or $\tau \geq_o \sigma_1$, there is a shorter counterexample, removing either σ_0 or $\sigma_2 \dots \sigma_n$ from the original sequence. To see wellfoundedness, let σ_i ($i < \omega$) be a descending sequence in \leq_o . First note that if some $\tau \in T$ has infinitely many extensions in the sequence, then all but finitely many of them must be extensions of a fixed immediate successor of τ , since otherwise initial segments of τ are visited infinitely often by the sequence, which is impossible (using

antireflexivity of \leq_t), there being only finitely many of them. But the empty sequence has infinitely many extension in the sequence, which means that we can build an infinite chain though T all of whose members have this property. We claim that infinitely many members of this chain must be in the sequence, which gives a contradiction since the chain codes a wellordering. To see the claim, fix a member τ_0 of the chain, and an arbitrary integer n . Then there is some σ_i , $i > n$, extending τ_0 , and a member τ_1 of the chain with length greater than $|\sigma_i|$. Let σ_j be an extension of τ_1 , for some $j > i$. Then if σ_i is not on the chain, there must be some k in the interval (i, j) such that σ_k is an initial segment of τ_1 . Since n was arbitrary, the claim follows.

Now extend \leq_t to a wellordering \leq_T of T , and let γ be the length of \leq_T . Let $h: \omega_1 \rightarrow T$ be a bijection, and define $g: \omega_1 \rightarrow \omega_1$ by letting $g(\alpha)$ be the ordertype of \leq_T restricted to $h[\alpha]$. Then g is a canonical function for γ . Furthermore, for a club $C' \subset C$ of $\alpha < \omega_1$, $T \upharpoonright (\omega \times \alpha) = h[\alpha]$. For these α , $g(\alpha)$ is greater than the ordertype of every wellordering in the projection of $T \upharpoonright (\omega \times \alpha)$, and thus greater than $f(\alpha)$. □

Remark 5.6. Note that Suslin bounding for Π_1^1 sets and Suslin bounding for \mathbb{P}_1^1 sets are identical.

Instead of working directly with the definition of universal Baireness, we will consider the equivalent (in the presence of large cardinals) form given by Theorem 5.8. Given a set X , we let $m(X)$ denote the set of countably complete ultrafilters on X . Recall that a sequence of measures $\langle \mu_i : i < \omega \rangle$ such that each μ_i concentrates on κ^i is a *countably complete tower* if for any sequence $\langle A_i : i < \omega \rangle$ such that each $A_i \in \mu_i$ there is a sequence $z \in \kappa^\omega$ such that each $z \upharpoonright i \in A_i$. See [6, 14] for more detail.

Definition 5.7. Suppose that κ is a nonzero ordinal and that T is a tree on $\omega \times \kappa$. Then T is δ -homogeneous if there is a partial function $\pi: \omega^{<\omega} \rightarrow m(\kappa^{<\omega})$ such that

- (1) if $s \in \text{dom}(\pi)$ then $\pi(s)$ is a δ -complete measure on $\kappa^{|s|}$ and $\pi(s)(T_s) = 1$, where $T_s = \{t \in \kappa^{|s|} : (s, t) \in T\}$,
- (2) for all $x \in \omega^\omega$, $x \in p[T]$ if and only if
 - (a) $\{x \upharpoonright k : k \in \omega\} \subset \text{dom}(\pi)$,
 - (b) $\langle \pi(x \upharpoonright k) : k \in \omega \rangle$ is a countably complete tower.

A set $A \subset \mathbb{R}$ is δ -homogeneously Suslin if $A = p[T]$ for some δ -homogeneous tree T . A is $^\infty$ -homogeneously Suslin if it is δ -homogeneously Suslin for arbitrarily large δ .

Theorem 5.8 ([4]). Suppose that there is a proper class of Woodin cardinals and that $A \subset \mathbb{R}$. Then the following are equivalent.

- (1) A is universally Baire.
- (2) A is $^\infty$ -homogeneously Suslin.

We use the following fact to ensure that our forcing iteration considers all universally Baire sets.

Lemma 5.9. *Let P be an (ω, ∞) -distributive partial order, and let $G \subset P$ be V -generic. Then in $V[G]$, for all $A \subset \mathbb{R}$ and all $\gamma > (2^{|P|})^+$, A is γ -homogeneously Suslin if and only if $A \in V$ and A is γ -homogeneously Suslin in V .*

Proof. This follows from the following standard facts about measures, where $V[G]$ is an extension by a forcing P such that $(2^{|P|})^+ < \gamma \leq \kappa$.

- (1) For every γ -complete measure U on $\kappa^{<\omega}$ in $V[G]$, $U \cap V \in V$. (Otherwise, densely often in P there is a set in V whose membership in the measure is undecided; by genericity, then there will be a subset of the measure of size $\leq |P|$ with empty intersection.)
- (2) For every γ -complete measure on $\kappa^{<\omega}$ in $V[G]$, every positive set contains a positive set in V . (For each P -name for a positive set and each condition in P , consider the set of sequences p forces into the positive set.)
- (3) Every γ -complete measure on $\kappa^{<\omega}$ in V extends to one in $V[G]$. (All sets containing positive sets from the ground model.)

For the forward direction, let T be a γ -homogeneous tree on $\omega \times \kappa$ in $V[G]$ such that $p[T] = A$, for some $\gamma > (2^{|P|})^+$. Let $\pi: \omega^{<\omega} \rightarrow m(\kappa^{<\omega})$ witness that T is γ -homogeneous. Each $\pi(\sigma)$ extends a measure in V on $\kappa^{<\omega}$, and since P is (ω, ∞) -distributive, the corresponding function π' taking each σ to the restriction of $\pi(\sigma)$ to V exists in V . For each $x \in \mathbb{R} \setminus p[T]$, let $\langle A_k^x : k < \omega \rangle$ be a witness to the fact that $\langle \pi(x \upharpoonright k) : k < \omega \rangle$ is not countably complete. For each $s \in \omega^{<\omega}$, let

$$B_s = \bigcap \{A_{|s|}^x \mid x \in \mathbb{R} \setminus p[T] \wedge s \subset x\}.$$

Since every positive set for each $\pi(x \upharpoonright k)$ contains one from V , and since P is (ω, ∞) -distributive, we can assume by shrinking if necessary that $\langle B_s \mid s \in \omega^{<\omega} \rangle$ is in V . Now let $T' \in V$ be the set of pairs s, t such that $t \in B_s$. Since the measures are all γ -complete, each B_s is positive for $\pi(s)$. Then T' is γ -homogeneous (with π' as a witness) with the same projection as T .

For the other direction, assume that T (on $\omega \times \kappa$) and π in V witness that A is γ -homogeneously Suslin. Extend the $\pi(\sigma)$'s to $V[G]$ -measures, inducing a function $\pi': \omega^{<\omega} \rightarrow m(\kappa^{<\omega})$. Since each positive set in $V[G]$ contains one in V , and since no ω -sequences of ordinals have been added by P , for each $x \in \omega^\omega$ the countable completeness of the corresponding tower is not changed by P . Since no countable sets of ordinals have been added the projection of T is the same, so T (along with π') witnesses in $V[G]$ that A is γ -homogeneously Suslin. \square

Let T be a tree on $\omega \times \kappa$ and let $f: \omega_1 \rightarrow p[T]$. Our one-step forcing $R_{f,T}$ is the set of continuous increasing sequences $\langle x_\alpha \in [\kappa]^{<\omega_1} : \alpha \leq \beta \rangle$ of countable length such that for all $\alpha \leq \beta$, $x_\alpha \cap \omega_1 \in \omega_1$ and $f(x_\alpha \cap \omega_1) \in p[T \upharpoonright (\omega \times x_\alpha)]$, ordered by extension.

Given a δ^+ -homogeneous set of reals, we use the measures witnessing homogeneity to suitably expand countable elementary substructures of $H(\chi)$.

Lemma 5.10. *Fix $\delta \geq \omega_1$, and let T be a δ^+ -homogeneous tree on $\omega \times \kappa$, witnessed by $\pi: \omega^{<\omega} \rightarrow m(\kappa^{<\omega})$. Let $\chi > 2^\kappa$ be a regular cardinal with \leq_χ a wellordering of $H(\chi)$, and let $X \prec (H(\chi), \in, \leq_\chi)$ with $T, \pi \in X$. Then for any countable $a \subset p[T]$ there exists a countable $z \subset \kappa$ such that letting $Y = \text{Sk}_{(H(\chi), \in, \leq_\chi)}(X \cup z)$, we have that*

- (1) $X \cap \delta = Y \cap \delta$,
- (2) $a \subset p[T \upharpoonright (\omega \times (Y \cap \kappa))]$,
- (3) $[z]^{<\omega} \subset \bigcap \{D_\kappa^Z : Z \prec (H(\chi), \in, \leq_\chi) \wedge \{T, \pi\} \in Z \in X\}$.

Proof. Fix T, π, X and $a = \{a_i : i < \omega\}$. By Lemma 2.1, it suffices to show that we can deal with a_0 , as we can just repeat the process ω times. For each $k < \omega$, let $A_k = \bigcap (X \cap \pi(a_0 \upharpoonright k))$. Then by the definition of δ^+ -homogeneity (i.e. countable completeness) there exists $z \in (\kappa \setminus \delta)^\omega$ such that for all $k < \omega$, $(a_0 \upharpoonright k, z \upharpoonright k) \in T$ and $z \upharpoonright k \in A_k$. Now z is as desired, since by the δ^+ -completeness of each $\pi(a_0 \upharpoonright k)$, if $k \in \omega$ and $h: \kappa^k \rightarrow \delta$ then h is constant on a set $C_h \in \pi(a_0 \upharpoonright k)$, and so this constant value must be an element of any elementary submodel of $(H(\chi), \in, \leq_\chi)$ with h and π as members. Further, if h is in X , then $z \upharpoonright k \in C_h$, so $h(z \upharpoonright k)$ is the corresponding constant value. □

Given Y as in Lemma 5.10 where $f(Y \cap \omega_1) \in a$, the union of any Y -generic for $R_{f,T}$, adjoined by its union, is a condition.

To get the consistency of Suslin bounding from Theorem 3.2, we start from a proper class of Woodin cardinals, and let κ be a strongly inaccessible cardinal such that every $^\infty$ -homogeneously Suslin set of reals is $^\infty$ -homogeneously Suslin in V_κ . The assumption of the Woodin cardinals is just to make $^\infty$ -homogeneous Suslinity equal to universal Baireness. Let $F: \omega_1 \rightarrow \mathbb{R}$ be a wellordering of the reals. Let our bookkeeping for P be such that each pair (A, f) , where A is a universally Baire set (from the ground model) and $f: \omega_1 \rightarrow A$ is added by some initial segment of the iteration, is associated to some ρ greater than the stage at which f was added, such that A is the projection of a $|\mathcal{P}(\mathcal{P}(\lambda_\rho))|^{+}$ -homogeneously Suslin tree $T \in V_\kappa$ on $\omega \times \kappa$ for some $\kappa \leq \lambda_{\rho+1}$. For this ρ , we let each A_β^ρ be the set of countable $x \subset \lambda_{\rho+1}$ such that $F(\beta) \in p[T \upharpoonright (\omega \times (x \cap \kappa))]$, so that $Q_{\rho, (F^{-1} \circ f)} = R_{f,T}$. By Lemma 5.10, these sets also satisfy Theorem 3.2(1)–(3). This scheme then gives the following corollary.

Corollary 5.11. *Suppose that there is a proper class of Woodin cardinals and let κ strongly inaccessible be such that every $^\infty$ -homogeneously Suslin set of reals is $^\infty$ -homogeneously Suslin in V_κ . Then there is a semi-proper (ω, ∞) -distributive forcing of size κ in whose extension the Continuum Hypothesis and Suslin bounding hold.*

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