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ON CLOSED P-SETS WITH ccc IN THE SPACE ω^*

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Abstract. It is proved that—consistently—there can be no ccc closed P-sets in the remainder space ω^*.

A subset $A \subseteq X$ of a topological space X is said to be a P-set if $A \subseteq \operatorname{int}(\bigcap R)$ holds for any countable family R of open neighborhoods of A. A point $x \in X$ is called a P-point if the one-element set $\{x\}$ is a P-set. In the case of the remainder space $\omega^* = \beta[\omega]\backslash\omega$ we may assume that R consists of open-closed neighborhoods. Rudin in [R] proved that if CH (the continuum hypothesis) is assumed, then P-points exist in the space ω^*. On the other hand, Shelah in [S] proved that—consistently—there can be no P-points in ω^*. In this paper we show how to construct a model of set theory in which there are no closed P-sets having ccc (every disjoint family of relatively open sets is countable) in the ultrafilter space ω^*. The problem of the existence of such sets (which are generalizations of P-points) has been known for some time and occurred explicity in [vM] (see also [JMPS] for more results on P-sets in this direction). In the proof we follow the method from [S] of the construction of a model with no P-points. We note here that the model from [S] does not work for our purpose. Actually, it is not difficult to see that each P-point from the ground model becomes a closed ccc P-set in the final model. A particular case of P-sets which are supports of measures on $P(\omega)$ with AP (the additive property) has been settled in [M], where the author shows that there can be no such measures on $P(\omega)$ (under CH, e.g., the Gleason space $\mathbb{G}(2^{\omega})$ of the Cantor space is a ccc P-set in ω^* which carries no measure with AP).

§1. It is well known that closed sets in ω^* have the form $\bigcap\{\bar{A}\backslash\omega : A \in F\}$, where F is a filter on the set ω ($\bar{A}$ is the closure of the set $A \subseteq \omega$ in $\beta[\omega]$). Filters F corresponding to closed P-sets are called P-filters and are characterized by the following condition:

$$\text{if } A_n \in F \text{ for } n < \omega, \text{ then there is an } A \in F \text{ such that } A \subseteq_* A_n \text{ for each } n < \omega$$

(here $A \subseteq_* B$ means $A \backslash B$ is finite).

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Thus, the dual ideal $I = \{\omega \setminus A: A \in F\}$ has the following property:

(1.1) if $A_n \in I$ for $n \in \omega$, then there is an $A \in I$ such that $A_n \subseteq_* A$ for each $n \in \omega$.

Further, the countable chain condition imposed upon F implies that I is fat in the following sense (see [F-Z]):

(1.2) if $A_n \in I$ for $n \in \omega$ and $\lim_n \min A_n = \infty$, then there is an infinite $Z \subseteq \omega$ such that $\bigcup_{n \in Z} A_n \in I$.

Indeed, let $e_n = A_n \setminus A$, where $A \in I$ is as in (1.1). Since $\min A_n$ are arbitrarily large, we can find an infinite set $Y \subseteq \omega$ such that the family $\{e_n: n \in Y\}$ is disjoint. If $\{Y_\alpha: \alpha < c\}$ is an almost disjoint family of subsets of Y, then the unions

$$S_\alpha = \bigcup \{e_n: n \in Y_\alpha\} \qquad \alpha < c,$$

are almost disjoint, and hence, the closures S_α^* in the space ω^* are disjoint. By ccc we have

$$S_\alpha^* \cap \bigcap \{B^*: B \in F\} = \varnothing$$

for some α and consequently $S_\alpha \in I$. It follows that the union

$$\bigcup_{n \in Y_\alpha} A_n = \bigcup_{n \in Y_\alpha} (A_n \cap A) \cup \bigcup_{n \in Y_\alpha} (A_n \setminus A)$$

is in I as a subset of $S_\alpha \cup A$.

For the rest of the paper let us fix a given ccc P-filter F and its dual I. We shall define a forcing $\mathbb{P} = \mathbb{P}(F)$.

A partial ordering $(T, \leq_T)$, where $T \subseteq \omega$, will be called a tree if for each $i \in T$ the set of predecessors $\{j \in T: j \leq_T i\}$ is linearly ordered and

$$i \leq_T j \quad \text{implies} \quad i \leq j \text{ for all } i, j \in T.$$

We define a partial ordering for trees

$T \leq_t S$ iff $(S, \leq_S)$ is a subordering of $(T, \leq_T)$ and each branch of T contains cofinally a (unique) branch of S.

There is a tree T_0 such that $T_0 \in I$ and T_0 is order isomorphic to the full binary tree of height ω. Deleting the numbers $\leq n$ from T_0 we obtain a subtree denoted by $T_0^{(n)}$ (we have $T_0^{(n)} \leq_t T_0^{(m)}$ for $n \leq m$). Let $\mathscr{T}$ consist of all the trees $T \in I$ such that

$$T \leq_t T_0^{(n)} \quad \text{for some } n \in \omega.$$

Note that each tree $T \in \mathscr{T}$ has finitely many roots.

Definition. Elements of the forcing $\mathbb{P}$ are of the form $p = \langle T_p, f_p \rangle$, where $T_p \in \mathscr{T}$ and $f_p: T_p \to \{0, 1\}$. The ordering of $\mathbb{P}$ is defined thus

$$p \leq q \quad \text{iff} \quad T_p \leq_t T_q \text{ and } f_p \supseteq f_q.$$

Let $\{b_\alpha: \alpha < c\}$ be a fixed enumeration of all the branches of T_0 in V. For a generic $G \subseteq \mathbb{P}$ let $T_G = \bigcup_{p \in G} T_p$ and $f_G = \bigcup_{p \in G} f_p$. For each branch b_α there is a (unique) branch B_α of T_G containing b_α cofinally. We have $B_\alpha = \bigcup_{p \in G} b_\alpha^p$, where b_α^p denotes the branch of T_p extending b_α. Define

$$X_\alpha = \{i \in \omega: i \in B_\alpha \text{ and } f_G(i) = 1\}.$$

Since $T_p \in I$ for any $p \in \mathbb{P}$; hence, $\omega \backslash T_p \cap A$ is infinite for each $A \in F$. It follows that the sets

$$D_{n\varepsilon}^{A\alpha} = \{p \in \mathbb{P}: \exists i > n[i \in b_\alpha^p \text{ and } f_p(i) = \varepsilon]\}$$

are dense for each $A \in F$, $n \in \omega$, $\alpha < c$, and $\varepsilon = 0, 1$. Hence, X_α intersects infinitely each A in F, and $B_\alpha \backslash X_\alpha$ has the same property. Thus, $\mathbb{P}$ adds uncountably many almost disjoint Gregorieff-like sets.

§2. Let $\mathbb{Q} = \mathbb{Q}(F)$ be a countable product of $\mathbb{P} = \mathbb{P}(F)$. Thus, the elements $q \in \mathbb{Q}$ can be written in the form

$$q = \langle f_0^q, f_1^q, \ldots \rangle, \quad \text{where } \langle \mathrm{dm}(f_i^q), f_i^q \rangle \in \mathbb{P} \text{ for each } i < \omega.$$

By $q^{(n)}$ we denote the condition $\langle g_i : i < \omega \rangle$, where

$$g_i = \begin{cases} f_i^q \mid \mathrm{dm}(f_i^q)^{(n)} & \text{for } i < n, \\ f_i^q & \text{for } i \geq n. \end{cases}$$

Here $T^{(n)}$ is a tree obtained from T by deleting the numbers $\leq n$.

LEMMA 2.1. *For each decreasing sequence $p_0 \geq p_1 \geq \cdots$ there is a q and an infinite $Z \subseteq \omega$ such that*

$$q \leq p_n^{(n)} \quad \textit{for each } n \in Z.$$

PROOF. Let $T_{ni} = \mathrm{dm}(f_i^{p_n})$, where $p_n = \langle f_i^{p_n} : i < \omega \rangle$. Since $\min T_{ni}^{(n)} \geq n$, we may use (1.2) to define by induction a descending sequence $Z_0 \supseteq Z_1 \supseteq \cdots$ of infinite subsets of ω such that

$$\bigcup_{n \in Z_i} T_{ni}^{(n)} \quad \text{is in } I \text{ for each } i < \omega.$$

There is an infinite $Z \subseteq \omega$ such that $Z \subseteq_* Z_i$ for each $i < \omega$. Define

$$T_i = T_{ii} \cup \bigcup_{n \in Z} T_{ni}^{(n)}$$

and

$$f_i^q = f_i^{p_i} \cup \bigcup_{n \in Z} f_i^{p_n} \mid T_{ni}^{(n)}.$$

Then $\mathrm{dm}(f_i^q) = T_i$ and $q = \langle f_i^q : i < \omega \rangle$ is as required. Q.E.D.

For $q \in \mathbb{Q}$ and $n \in \omega$ let $S(q, n)$ be the set of all sequences $s = \langle s_0, \ldots, s_{n-1} \rangle$ satisfying the following properties:

(1) $s_0, \ldots, s_{n-1}$ are finite zero-one functions.

(2) The domains $t_0 = \mathrm{dm}(s_0), \ldots, t_{n-1} = \mathrm{dm}(s_{n-1})$ are finite trees such that

$$\max t_0 < \min T_0^{(n)}, \ldots, \max t_{n-1} < \min T_{n-1}^{(n)},$$

where $T_0 = \mathrm{dm}(f_0^q), \ldots, T_{n-1} = \mathrm{dm}(f_{n-1}^q)$.

(3) There are trees $Y_0, \ldots, Y_{n-1}$ in $\mathcal{T}$ such that

$$Y_0^{(n)} = T_0^{(n)}, \ldots, Y_{n-1}^{(n)} = T_{n-1}^{(n)}$$

and

$$Y_0^{(n)} \backslash T_0^{(n)} = t_0, \ldots, Y_{n-1}^{(n)} \backslash T_{n-1}^{(n)} = t_{n-1}.$$

Note that from the definition of $\mathscr{T}$ it follows that $S(q,n)$ is always finite. Let us denote

$$s * q^{(n)} = \langle s_0 \cup f_0^q \restriction [n,\infty), \ldots, s_{n-1} \cup f_{n-1}^q \restriction [n,\infty), f_n^q, \ldots \rangle$$

for q, n, s as above. Actually, $s * q^{(n)}$ depends also on the choice of the trees $Y_0, \ldots, Y_{n-1}$ as in (3) above. We omit this to simplify the notation. Obviously, we have

(2.2) the set $\{s * q^{(n)}: s \in S(q,n)\}$ is predense below $q^{(n)}$

(i.e., the Boolean sum $\sum_{s \in S(q,n)} s * q^{(n)} = q^{(n)}$).

Now, we easily obtain an analogue of [S, VI, 4.5].

(2.3) For arbitrary $p \in \mathbb{Q}$, $n < \omega$, and $\tau \in V^{(Q)}$ such that $\mathbb{Q} \Vdash$ "τ is an ordinal" there is a $q \leq p$ and ordinals $\{\alpha(s): s \in S(p,n)\}$ so that

$$q^{(n)} \Vdash \text{“}\bigvee_s \tau = \alpha(s)\text{”}.$$

Indeed, if $S(q,n) = \{s^0, \ldots, s^{m-1}\}$, then we define inductively conditions $p_0, \ldots, p_m$ so that $p_0 = p$ and $p_{k+1} \leq s^k * p_k^{(n)}$ is such that

$$p_{k+1} \Vdash \text{“}\tau = \alpha\text{”} \quad \text{for some ordinal } \alpha = \alpha(s^k).$$

Now $q = s * p_m^{(n)}$, where s is such that $p = s * p^{(k)}$ (we may assume $s \in S(p,n)$) satisfies (2.3).

We recall now some notions from [S]. Let $H(\kappa)$ denote the family of all sets of hereditary power $< \kappa$. We say that a sequence $\langle N_\xi: \xi \leq \alpha \rangle$ is continuously increasing if $N_\xi \subseteq N_\eta$, for $\xi < \eta$ and $N_\beta = \bigcup_{\xi < \beta} N_\xi$ for any limit $\beta < \alpha$. Let $\alpha < \omega_1$. An arbitrary forcing $\mathbb{P}$ is called α-proper if for sufficiently large κ ($\kappa > \omega_1$ in our case) and any continuously increasing chain $\langle N_\xi: \xi \leq \alpha \rangle$ of countable elementary submodels of $H(\kappa)$ such that $\mathbb{P} \in N_0$ and $\xi \in N_\xi$, for every $\xi \leq \alpha$, $\langle N_\xi: \xi \leq \eta \rangle \in N_{\eta+1}$, all $\eta < \alpha$, for each $p \in \mathbb{P} \cap N_0$ there is a $q \leq p$ which is N_ξ-generic for all $\xi \leq \alpha$. Thus, 0-proper means proper.

We say that $\mathbb{P}$ has the strong PP-property if for any $p \in \mathbb{P}$ such that

$$p \Vdash \text{“}f: \omega \to \omega\text{”}$$

for some $f \in V^{(\mathbb{P})}$ and any $h: \omega \to \omega$ diverging to ∞ there is a $q \leq p$ and a perfect tree $T \subseteq \omega^{<\omega}$ such that $T \cap \omega^n$ is finite for all $n < \omega$ and $\operatorname{card}(T \cap \omega^n) \leq h(n)$ for infinitely many $n < \omega$ and $q \Vdash$ "$f \in \operatorname{Lim} T$". ($\operatorname{Lim} T$ denotes the set of all branches of T.)

From [S] we know that if $\mathbb{P}$ has strong PP-property, then it is ω^ω-bounding (i.e., the set of old functions: $\omega \to \omega$ dominates) and that an iteration of the length ω_2 with countable supports of ω-proper and ω^ω-bounding forcings is proper (and consequently does not collapse ω_1). Thus, what we need is the following

2.4. THEOREM. *$\mathbb{Q}$ is α-proper for every $\alpha < \omega_1$ and has the strong PP-property.*

PROOF. Let countable $N \prec H(\kappa)$ for sufficiently large κ be such that $\mathbb{Q} \in N$, and suppose that $p \in \mathbb{Q} \cap N$. To prove that $\mathbb{Q}$ is proper we have to find a $q \leq p$ which is N-generic. Let $\{\tau_n: n < \omega\}$ be an enumeration of all the $\mathbb{Q}$-names for ordinals such that $\tau_n \in N$ for $n < \omega$. Using (2.3), we define inductively a sequence $p_0 = p \geq p_1 \geq \cdots$ and ordinals $\alpha(n,s)$ so that

$$p_n^{(n)} \Vdash \text{“}\bigwedge_{i \le n} \bigvee_{s} \tau_i = \alpha(n,s)\text{”} \quad \text{for each } n < \omega$$

(i.e., in the nth step we apply (2.3) for all names $\tau_0, \ldots, \tau_n$). Note that the p's and α's can be found in N, since $N \prec H(\kappa)$. By Lemma 2.1 there is a q and an infinite $Z \subseteq \omega$ such that

$$q \le p_n^{(n)} \quad \text{for each } n \in Z.$$

Hence, $q^{(m)} \le p_n^{(n)}$ also holds for arbitrarily large n and all $m < \omega$, and thus

$$q^{(m)} \Vdash \text{“}\tau_n \in N\text{”}$$

for all $n, m < \omega$.

By [S, III, 2.6] each $q^{(m)}$ is N-generic. To see that $\mathbb{Q}$ is α-proper let $\langle N_\xi : \xi \le \alpha\rangle$ be a continuous sequence of elementary countable submodels of $H(\kappa)$ such that $\mathbb{Q} \in N_0$ and

$$\langle N_\xi : \xi \le \eta\rangle \in N_{\eta+1} \quad \text{for each } \eta < \alpha.$$

Assume that $\mathbb{Q}$ is β-proper for each $\beta < \alpha$, and let $q_0 \in \mathbb{Q} \cap N_0$. If $\alpha = \beta + 1$, we have a $q \le q_0$ which is N_ξ-generic for each $\xi \le \beta$, and we may assume that the $q^{(n)}$ have the same property for all $n < \omega$. Since $N_\alpha \prec H(\kappa)$ and all the parameters are in N_α, such a q can be found in N_α and as above we construct a $q_\alpha \le q$ which is N_α-generic and so are the $q_\alpha^{(n)}$ for $n < \omega$. Thus, q_α and all the $q_\alpha^{(n)}$ are N_ξ-generic for all $\xi \le \alpha$. If α is a limit ordinal, we fix an increasing sequence $\langle \xi_n : n < \omega\rangle$ such that $\alpha = \sup_{n<\omega} \xi_n$ and by the inductive hypothesis there is a sequence $q_0 \ge q_{\xi_0} \ge q_{\xi_1} \ge \cdots$ such that for each $n < \omega$ q_{ξ_n} is N_ξ-generic for each $\xi \le \xi_n$ and $q_{\xi_n} \in N_{\xi_n+1}$ and that $q_{\xi_n}^{(m)}$ have the same property for each $m < \omega$. By Lemma 2.1 there is a $q \in \mathbb{Q}$ such that $q \le q_{\xi_n}^{(n)}$ for infinitely many $n < \omega$. Thus, $q \le q_0$ and q is N_ξ-generic for each $\xi < \alpha$ and, hence, also for each $\xi \le \alpha$.

Finally, to prove the PP-property let $h: \omega \to \omega$ diverge to infinity and suppose that $p \Vdash$ “$f: \omega \to \omega$”. Define

$$k_n = \min\{i : h(i) > 2^n \cdot \operatorname{card} S(p,n)\} \quad \text{for } n < \omega$$

and using (2.3) define inductively the sequence $p = p_0 \ge p_1 \ge \cdots$ such that

$$p_n^{(n)} \Vdash \text{“}\bigwedge_{i<k_n} \bigvee_{s \in S(p_i,i)} f(i) = \alpha(s,i)\text{”}$$

for each $n < \omega$ and some integers $\alpha(s,i) < \omega$. Let T be the tree built up of sequences of integers

$$\{\alpha(s,i) : i < \omega \text{ and } s \in S(p_i,i)\}.$$

If $q \le p_n^{(n)}$ for infinitely many n, then we have $q \Vdash$ “$f \in \operatorname{Lim} T$” and $T \cap \omega^{k_n}$ has less elements than $h(k_n)$ for all $n < \omega$, which finishes the proof. Q.E.D.

The last point to be discussed is how $\mathbb{Q} = \mathbb{Q}(F)$ acts in the course of iteration.

2.5. LEMMA. *If $\mathbb{R}$ is ω^ω-bounding and $\mathbb{Q}(F)$ is a complete subforcing of $\mathbb{R}$, then in $V^{(\mathbb{R})}$ the filter F cannot be extended to a ccc P-filter.*

PROOF. Let X_α^n be the α-th set added by nth factor of the product $\mathbb{Q} = \mathbb{P}^\omega$. Suppose that for some $r \in \mathbb{R}$ and a ccc P-filter $E \in V^{(\mathbb{R})}$ we have

$$r \Vdash \text{“}F \subseteq E\text{”}.$$

Note that for each $n < \omega$ the relation $X^n_\alpha \in E$ holds for at most countably many α's, since E is ccc. Hence, there is an α such that for all $n < \omega$ we have $\omega\backslash X^n_\alpha \in E$, and since E is a P-filter, there is an $A \in E$ and a function g so that $A \subseteq \bigcap_{n<\omega}(\omega\backslash X^n_\alpha) \cup [0, g(n))$, i.e., for some $r_1 \leq r$ we have

$$r_1 \Vdash \text{“} \bigcap_{n<\omega} (\omega\backslash X^n_\alpha) \cup [0, g(n)) \in E\text{”}. \tag{2.6}$$

Since $\mathbb{R}$ is ω^ω-bounding, we may assume that $g \in V$. By the assumption $\mathbb{Q}$ is a complete subforcing of $\mathbb{R}$, and hence, there is a $q \in \mathbb{Q}$ such that r is compatible with each $q' \leq q$. On the other hand, since $T_n = \mathrm{dm}(f^q_n) \in \mathscr{T}$, there is a set $B \in I$ and an increasing sequence $a_0 < a_1 < \cdots$ such that $T_n\backslash[0, a_n) \subseteq B, g(n) < a_n$, and $[a_n, a_{n+1})\backslash B \neq \varnothing$ for each $n < \omega$. Define $q' \leq q$ as follows. For a given n extend T_n by adding elements of $[a_n, a_{n+1})\backslash B$ on the αth branch b^q_α and put $f^{q'}_n(i) = 1$ for each $i \in [a_n, a_{n+1})\backslash B$. Obviously, we have

$$q' \Vdash \text{“}(\omega\backslash X^n_\alpha) \cup [0, g(n)) \cap [a_n, a_{n+1})\backslash B = \varnothing\text{”} \quad \text{for each } n,$$

and hence,

$$q' \Vdash \text{“} \bigcap_{n<\omega} (\omega\backslash X^n_\alpha) \cup [0, g(n))\backslash B \cap \bigcup_{n<\omega} [a_n, a_{n+1}) = \varnothing\text{”}.$$

Consequently, $q' \Vdash$ “$\bigcap_{n<\omega}(\omega\backslash X^n_\alpha) \cup [0, g(n)) \subseteq_* B$” which contradicts (2.6).
Q.E.D.

The rest of the proof is routine. Beginning with a model V of $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$, we iterate with countable supports the forcings $\mathbb{Q}(F)$ for all ccc P-filters F booked at each stage $\alpha < \omega_2$ of the iteration. From [S, V.4] we know that the resulting forcing $\mathbb{R}$ (obtained after ω_2 stages) is proper and ω^ω-bounding. Hence, in $V[G]$ there are no ccc P-sets.

REFERENCES

[F-Z] R. Frankiewicz and P. Zbierski, *Strongly discrete subsets in* ω^*, ***Fundamenta Mathematicae***, vol. 129 (1988), pp. 173–180.

[JMPS] W. Just, A. R. D. Mathias, K. Kirby, and P. Simon, *On the existence of large P-ideals*, this Journal, vol. 55 (1990), pp. 457–465.

[M] A. H. Mekler, *Finitely additive measures on N and the additive property*, ***Proceedings of the American Mathematical Society***, vol. 92 (1984), pp. 439–444.

[vM] J. van Mill, *Introductions to* $\beta\omega$, ***Handbook of set-theoretic topology*** (Kunen and Vaughan, editors), North-Holland, Amsterdam, 1984.

[R] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, ***Duke Mathematical Journal***, vol. 23 (1956), pp. 409–419.

[S] S. Shelah, *Proper forcing*, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin and New York, 1982.

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