SOUSLIN PROPERTIES AND TREE TOPOLOGIES

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Abstract

Tree spaces provide a useful collection of 'standard examples' of topological spaces with specific properties: a fact which was already noticed by F. B. Jones in the 1930's. Certain properties of tree spaces depend upon the structural property of the tree concerned (as a partially ordered set). We obtain characterizations of those trees whose topologies are (a) normal, (b) collectionwise Hausdorff topologies. (Both results relate closely to the normal Moore space problem, and arose out of our research on that problem, the results of which appear elsewhere.) The structural properties of trees involved in these results are generalizations of the Souslin condition on trees.

1. Background

For topological background any standard text will suffice. For details on trees the reader should consult Jech's article [**3**]. We adopt the notations and conventions of contemporary set theory. In particular an ordinal is the set of its predecessors, lower case Greek letters are used to denote ordinals, cardinals are initial ordinals, ω is the first infinite ordinal, ω_α is the αth uncountable initial ordinal, and $\aleph_\alpha$ denotes ω_α regarded as a cardinal (with $\aleph_0$ denoting ω). The cardinality of a set X is denoted by $|X|$.

A *tree* is a poset $\boldsymbol{T} = (T, \leqslant_T)$ such that for every $x \in T$, the set $\hat{x} = \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$. The order-type of $\hat{x}$ under $<_T$ is the *height* of x in $\boldsymbol{T}$, $\mathrm{ht}(x)$. The αth *level* of $\boldsymbol{T}$ is the set

$$T_\alpha = \{x \in T \mid \mathrm{ht}(x) = \alpha\}.$$

We set $\boldsymbol{T} \restriction \alpha = (\bigcup_{\beta<\alpha} T_\beta, \leqslant_T \cap (T \restriction \alpha)^2)$. Define $T \restriction C = \bigcup_{\alpha \in C} T_\alpha$. The *height* of $\boldsymbol{T}$, $\mathrm{ht}(\boldsymbol{T})$, is the least α such that $T_\alpha = \emptyset$. A *branch* of $\boldsymbol{T}$ is a maximal chain of $\boldsymbol{T}$. If a branch has order-type α, it is called an α-*branch*. An *antichain* of $\boldsymbol{T}$ is a pairwise incomparable subset of $\boldsymbol{T}$.

An ω_1-*tree* is a tree $\boldsymbol{T}$ such that:

(i) $\mathrm{ht}(\boldsymbol{T}) = \omega_1$;

(ii) $(\forall \alpha < \omega_1)\,(|T_\alpha| \leqslant \aleph_0)$;

(iii) $(\forall \alpha < \beta < \omega_1)\,(\forall x \in T_\alpha)\,(\exists y_1, y_2 \in T_\beta)\,(y_1 \neq y_2 \;\&\; x <_T y_1 \;\&\; x <_T y_2)$;

(iv) $(\forall \alpha < \omega_1)\,(\forall x, y \in T_\alpha)\,(\lim(\alpha) \to (x = y \leftrightarrow \hat{x} = \hat{y}))$.

An *Aronszajn tree* is an ω_1-tree with no uncountable branch. A *Souslin tree* is an ω_1-tree with no uncountable antichain. It follows from condition (iii) above that every Souslin tree is an Aronszajn tree. The converse is false. An ω_1-tree is *special* if and only if it is a union of a countable family of antichains. Any special ω_1-tree is a non-Souslin, Aronszajn tree. (See Lemma 3.2 in this connection.) The following facts are standard.

Fact 1. There is a special Aronszajn tree. (Aronszajn.)

Fact 2. An ω_1-tree T is special if and only if there is an order-preserving map of T into $\mathbf{Q}$, the rationals.

Fact 3. It is consistent with the axioms of (Zermelo–Fraenkel) set theory plus the generalized continuum hypothesis (GCH) that every Aronszajn tree is special. (Jensen.)

Fact 4. It is consistent to assume there is no Souslin tree. (Solovay and Tennenbaum.)

Fact 5. If $V = L$ (the axiom of constructibility) be assumed, there is a Souslin tree. (Jensen.)

Proofs of all of these may be found in the book by Devlin and Johnsbråten [**1**]. (Facts 1, 4, and 5 are also proved in [**3**].)

Let T be an ω_1-tree. The *tree topology* on T is defined by taking as an open basis all intervals of the form

$$[0, t) = \{x \in T \mid x <_T t\}, \quad t \in T,$$

$$(s, t) = \{x \in T \mid s <_T x <_T t\}, \quad s, t \in T.$$

(This is just the tree analogue of the order topology for a totally ordered set with no greatest member.) A *tree topology* is any topological space of the above variety. If T is a special Aronszajn tree, its tree topology is often referred to as a *Jones space* (see [**4**]).

Proofs of the following facts may be found in or via our paper [**2**].

Fact 6. Any tree topology is 1st countable and T_3. (Trivial.)

Fact 7. An ω_1-tree T is a special Aronszajn tree if and only if its tree topology is a Moore space (that is, a T_3 space with a development). ($\rightarrow$ Jones; $\leftarrow$ Folklore.)

Fact 8. If T is a special Aronszajn tree, its tree topology fails to be a collectionwise Hausdorff topology, and is thus not metrizable. (A space is a collectionwise Hausdorff space if and only if every discrete collection of points can be simultaneously separated by disjoint open sets.) (Jones.)

Fact 9. Assume CH (or, more generally, $2^{\aleph_0} < 2^{\aleph_1}$). Then no special Aronszajn tree space is normal. (Devlin and Shelah.)

Fact 10. Assume Martin's axiom plus $2^{\aleph_0} > \aleph_1$. Then every Aronszajn tree is special and every Aronszajn tree space is normal. (Baumgarter and Kunen; Fleissner, respectively.)

2. Souslin trees

Recall that an ω_1-tree is a *Souslin* tree if and only if it has no uncountable antichains. We may reformulate this condition as follows. Let T be an ω_1-tree. A subset X of T will be called *unbounded* in T if and only if $\{\text{ht}(x) \mid x \in X\}$ is unbounded in ω_1. Since $T \restriction \alpha$ is countable for all $\alpha < \omega_1$, T will be a Souslin tree if and only if it has no unbounded antichain.

It has been known for many years that the topology determined by a Souslin tree is normal. The argument is essentially due to Rudin (see, for example, [7]). We understand that no proof of this fact itself has ever been published. We give a proof here, obtaining on the way a topological characterization of the Souslin condition on ω_1-trees.

2.1. THEOREM. *Let T be an ω_1-tree. T is a Souslin tree if and only if whenever A, B are disjoint closed subsets of the space T, $\hat{A} \cap \hat{B}$ is countable (where $\hat{A} = \bigcup_{x \in A} \hat{x}$, etc.).*

Proof. ($\rightarrow$) Let A, B be disjoint closed subsets of T. Suppose $C = \hat{A} \cap \hat{B}$ is uncountable. For $x \in T$, let $T^{(x)}$ denote the tree $\{y \in T \mid x \leqslant_T y\}$ (with the relativized ordering). We commence the proof with a claim.

Claim. For some $c \in C$, $T^{(c)} \subseteq C$.

Proof of claim. Suppose, on the contrary, that for all $c \in C$, $T^{(c)} \nsubseteq C$. Thus, for each $c \in C$ we can pick an $x \in T$ such that $c <_T x$ and $x \notin C$. By induction now, pick $c_\nu \in C$, $x_\nu \in T$ ($\nu < \omega_1$) so that

$$c_\nu <_T x_\nu, \quad \text{ht}(c_\nu) > \sup_{\tau < \nu} \text{ht}(x_\tau), \quad x_\nu \notin C.$$

Since C is an initial segment of T, $\{x_\nu \mid \nu < \omega_1\}$ is clearly an uncountable antichain of T, which is absurd. The claim is proved.

By the claim, pick $c \in C$ with $T^{(c)} \subseteq C$. Set $\tilde{T} = T^{(c)}$. $\tilde{T}$ is clearly a Souslin tree. Define functions σ_1, σ_2 from ω_1 into ω_1 so that:

(a) for each $x \in \tilde{T} \restriction \alpha$ there is a maximal pairwise incomparable subset A' of $A \cap \tilde{T}^{(x)}$ such that $A' \subseteq \tilde{T} \restriction \sigma_1(\alpha)$;

(b) for each $y \in \tilde{T} \restriction \alpha$ there is a maximal pairwise incomparable subset B' of $B \cap \tilde{T}^{(y)}$ such that $B' \subseteq \tilde{T} \restriction \sigma_2(\alpha)$.

Since $\tilde{T} \restriction \alpha$ is countable and $\tilde{T}$ is a Souslin tree, there is no problem in defining such σ_1, σ_2.

Set $E = \{\alpha \in \omega_1 \mid \lim(\alpha) \;\&\; (\forall \nu < \alpha)\, (\sigma_1(\nu) < \alpha \;\&\; \sigma_2(\nu) < \alpha)\}$. It is not hard to show that E is closed and unbounded in ω_1. Let α be the ωth member of E. Let $x \in \tilde{T}_\alpha$, and pick $a \in A$, $b \in B$ with $x <_T a$ and $x <_T b$. Now, c is the unique member of $\tilde{T}_0$, so $c <_T a$. So, as $a \in A \cap \tilde{T}^{(c)}$ and $\alpha \in E$, there must be an $a_1 \in A \cap \tilde{T} \restriction \alpha$ with $c <_T a_1 <_T a$. Notice that (by choice) $a_1 <_T x$. Again, $a_1 <_T b$ and $b \in B \cap \tilde{T}^{(a_1)}$, so as $a \in E$ there will be a $b_1 \in B \cap \tilde{T} \restriction \alpha$ with $a_1 <_T b_1 <_T b$. Notice that $b_1 <_T x$. Continuing in this manner we obtain a sequence $a_1 <_T b_1 <_T a_2 <_T b_2 <_T \ldots <_T x$ with $a_i \in A$, $b_i \in B$, for each i. Let $\gamma = \sup_{i<\omega} \mathrm{ht}(a_i) = \sup_{i<\omega} \mathrm{ht}(b_i)$. Let z be the unique predecessor of x on T_γ. (Possibly $z = x$.) Since $\{a_i \mid i < \omega\}$ is cofinal in z (in T), and since A is closed, $z \in A$. Similarly, since $\{b_i \mid i < \omega\}$ is cofinal in z and B is closed, $z \in B$. Thus $A \cap B \neq \emptyset$, a contradiction.

($\leftarrow$) Suppose T is not a Souslin tree. Let C be a maximal, uncountable antichain of T. Clearly, C is a discrete subset of the space (i.e. every point of the space has a neighbourhood which contains at most one point of C), so any subset of C is closed. We define disjoint subsets, A, B of C by induction. Suppose $A \cap T \restriction \alpha$, $B \cap T \restriction \alpha$ are defined. If every point of T_α were to extend a member of $C \cap T \restriction \alpha$, then C would be countable (because no element of T above level α could be incomparable with all members of $C \cap T \restriction \alpha$). Pick $x \in T_\alpha$ so that x does not extend any member of $C \cap T \restriction \alpha$, and such that x has at least two extensions in C. (If it were not possible to satisfy this last requirement, then since T_α is countable, C would have to be countable.) Let a, b be extensions of x in C. Pick β with $a, b \in T \restriction \beta$. Then put a into A and b into B to define $A \cap T \restriction \beta$, $B \cap T \restriction \beta$. Clearly, this procedure produces disjoint closed sets $A, B \subseteq C$, both uncountable, such that $\hat{A} \cap \hat{B}$ is uncountable.

Using Theorem 2.1, an essentially routine argument (see, for instance, [**5**, p. 113]) gives the normality of Souslin tree topologies:

2.2. Corollary. *If T is a Souslin tree, its topology is normal.*

Proof. Let A, B be disjoint, closed subsets of T. Pick $\alpha < \omega_1$ so that $\hat{A} \cap \hat{B} \subseteq T \restriction \alpha$. Set $A_0 = A \cap T \restriction \alpha+1$, $B_0 = B \cap T \restriction \alpha+1$. A_0, B_0 are closed subsets of T.

Let $(x_n)_{n<\omega}$ enumerate $A_0 \cup B_0$. Define a sequence $(X_n)_{n<\omega}$ of subsets of $T \restriction \alpha+1$ of the form $X_n = (y_n, x_n]$ as follows, by induction. Suppose $X_0, \ldots, X_{n-1}$ have been defined. If $\mathrm{ht}(x_n)$ is a successor ordinal, let $X_n = \{x_n\}$. If $x_n \in X_0 \cup \ldots \cup X_{n-1}$, take $X_n = \emptyset$. Finally, if neither of these is the case, we may pick a point $y_n <_T x_n$ such that

$$[y_n, x_n] \cap (X_0 \cup \ldots \cup X_{n-1}) = \emptyset$$

and such that $x_n \in A_0 \to [y_n, x_n] \cap B_0 = \emptyset$ and $x_n \in B_0 \to [y_n, x_n] \cap A_0 = \emptyset$. (Since, in this case, $\mathrm{ht}(x_n)$ is a limit ordinal, and since A_0, B_0 are closed and disjoint, this is clearly possible.) That completes the definition.

Set $U_0 = \bigcup\{X_n \mid x_n \in A_0\}$, $V_0 = \bigcup\{X_n \mid x_n \in B_0\}$. Notice that U_0, V_0 are disjoint, open sets, with $A_0 \subseteq U_0 \subseteq T \restriction \alpha+1$, $B_0 \subseteq V_0 \subseteq T \restriction \alpha+1$.

For any $x \in T$ now with $\mathrm{ht}(x) > \alpha$, let $p(x)$ be the unique $p \in T_\alpha$ with $p <_T x$. Let

$$U = U_0 \cup \bigcup\{(p(a), a] \mid a \in A \ \&\ \mathrm{ht}(a) > \alpha\},$$
$$V = V_0 \cup \bigcup\{(p(b), b] \mid b \in B \ \&\ \mathrm{ht}(b) > \alpha\}.$$

Since $\hat{A} \cap \hat{B} \subseteq T \restriction \alpha$, $U \cap V = \emptyset$. Moreover, U, V are open sets, and $A \subseteq U$, $B \subseteq V$. Thus T is normal.

In § 4 we consider a strengthening of Corollary 2.2.

3. Almost Souslin trees

Let T be an ω_1-tree. A subset X of T will be called *stationary* if and only if $\{\mathrm{ht}(x) \mid x \in X\}$ is stationary in ω_1. An ω_1-tree is an *almost Souslin* tree if and only if it has no stationary antichain.

Clearly, every Souslin tree is an almost Souslin tree. But as our first result below shows, the two concepts are not identical. We need a classic result on stationary sets.

FODOR'S THEOREM. *Let $E \subseteq \omega_1$ be stationary, and let $f: E \to \omega_1$ be regressive (that is, $f(\alpha) < \alpha$ for all $\alpha \in E$). Then for some stationary set $E' \subseteq E$,* f is constant on E'.

3.1. LEMMA. *If there exists an almost Souslin tree, then there exists an almost Souslin tree which is not a Souslin tree.*

Proof. Let T be an almost Souslin tree. If T is not a Souslin tree, there is nothing more to be done, so let us assume T is a Souslin tree. By discarding T_0 and renumbering the levels, if necessary, we may assume that T_0 contains at least two elements.

For each non-zero $\alpha < \omega_1$, pick $x_\alpha \in T_\alpha$. Let

$$T^* = T \cup \{(t, \alpha) \mid t \in T \ \&\ 0 < \alpha < \omega_1\},$$

and define a partial ordering on T^* by

$$s, t \in T \to [s <^* t \leftrightarrow s < t];$$
$$x \leqslant x_\alpha \ \&\ t \in T \to x <^* (t, \alpha),$$
$$s, t \in T \to [(s, \alpha) <^* (t, \alpha) \leftrightarrow s < t].$$

In all other cases there is no ordering between elements. It is easily seen that $T^* = (T^*, <^*)$ is an Aronszajn tree: indeed, T^* just consists of T together with a copy of T 'grafted' on above x_α for each $\alpha < \omega_1$. If

$t_0 \in T_0$, the collection $A = \{(t_0, \alpha) \mid \alpha < \omega_1\}$ is clearly an antichain of T^*. Hence T^* is not a Souslin tree.

We show that T^* is an almost Souslin tree. Suppose otherwise, and let $\{z_\alpha \mid \alpha \in E\}$ be an antichain of T^* such that $z_\alpha \in T_\alpha$, where E is stationary. Assume, without loss of generality, that $0 \notin E$. Define a function $f: E \to \omega_1$ as follows:

$$f(\alpha) = 0, \quad \text{if } z_\alpha \in T,$$

$$f(\alpha) = \gamma, \quad \text{if } z_\alpha = (t, \gamma) \text{ for some } t \in T.$$

Since $\mathrm{ht}(x_\alpha) = \alpha$ for all α, f is clearly regressive. Hence by Fodor's Theorem there is a stationary set $E' \subseteq E$ such that f is constant on E', say with value γ. If $\gamma = 0$, then $\{z_\alpha \mid \alpha \in E'\}$ is an uncountable antichain of T, which is absurd. But if $\gamma > 0$, then $z_\alpha = (t_\alpha, \gamma)$ for all $\alpha \in E'$, and $\{t_\alpha \mid \alpha \in E'\}$ is an uncountable antichain of T, again a contradiction.

The reader may have noticed that we prefaced the statement of Lemma 3.1 by an existence assumption. This is necessary because the existence of an almost Souslin tree cannot be proved in set theory, even if we assume GCH. This follows from Fact 3 (see § 1) and the following easy result:

3.2. Lemma. *If T is a special Aronszajn tree, then T is not an almost Souslin tree.*

Proof. Let A_n, with $n < \omega$, be antichains of T with $T = \bigcup_{n<\omega} A_n$. We may assume that the sets A_n are disjoint. Pick $x_\alpha \in T_\alpha$ for each $\alpha < \omega_1$. Define $f: \omega_1 \to \omega$ by setting $f(\alpha) = n$ if and only if $x_\alpha \in A_n$. Then f is regressive on $\omega_1 \setminus \omega$, so for some stationary set $E \subseteq \omega_1 \setminus \omega$, f is constant on E, say with value n_0. Since $\{x_\alpha \mid \alpha \in E\} \subseteq A_{n_0}$, this set is our stationary antichain, proving that T is not an almost Souslin tree.

Our next result provides us with an exact characterization of almost Souslin trees in topological terms. Recall that a topological space is a *collectionwise Hausdorff* space if and only if whenever X is a discrete subset of the space, there is a disjoint collection $\{U_x \mid x \in X\}$ of open sets U_x such that $x \in U_x$ (such a collection being called a *separation* of X).

3.3. Theorem. *Let T be an ω_1-tree. T is an almost Souslin tree if and only if its tree topology is a collectionwise Hausdorff topology.*

Proof. ($\to$) Let X be a discrete subset of T. Let $H = \{\mathrm{ht}(x) \mid x \in X\}$.
Claim. H is non-stationary in ω_1.

Proof of claim. Suppose the claim is not true. Let

$$X' = \{x \in X \mid \text{ht}(x) \text{ is a limit ordinal}\},$$

and set

$$H' = \{\text{ht}(x) \mid x \in X'\} = \{\alpha \in H \mid \alpha \text{ is a limit ordinal}\}.$$

Thus H' is stationary in ω_1. For each $x \in X'$ there is a neighbourhood of x disjoint from $X' \setminus \{x\}$, and hence we can pick $f(x) <_T x$ such that $(f(x), x] \cap X' = \{x\}$. For each $\alpha \in H'$ now, pick $x_\alpha \in X' \cap T_\alpha$ and set $h(\alpha) = \text{ht}_T(f(x_\alpha))$. Then h is a regressive function on H' so for some stationary set $H'' \subseteq H'$, h is constant on H'', say with value γ. Thus for all $\alpha \in H''$, $f(x_\alpha) \in T_\gamma$. Since T_γ is countable we can find a stationary set $H''' \subseteq H''$ such that for some fixed $z \in T_\gamma$, $f(x_\alpha) = z$ for all $\alpha \in H'''$. Let $X'' = \{x_\alpha \mid \alpha \in H'''\}$. For any $x \in X''$, we have $z <_T x$ and $(z, x) \cap X'' = \emptyset$. Hence X'' is an antichain of T. But $H''' = \{\text{ht}(x) \mid x \in X''\}$ is stationary, so we have a contradiction. This proves the claim.

By the claim we can find a closed unbounded set $C \subseteq \omega_1$ such that $X \cap T \restriction C = \emptyset$. We may assume that $0 \in C$ and that

$$\alpha \in C \;\&\; \alpha > 0 \rightarrow \lim(\alpha).$$

Let $\langle \alpha_\nu \mid \nu < \omega_1 \rangle$ be the monotone enumeration of C. For each $\nu < \omega_1$, let $\langle x_j^\nu \mid j < \omega \rangle$ be a one-to-one enumeration of $X \cap [T \restriction \alpha_{\nu+1} - T \restriction \alpha_\nu]$. We define a sequence $\langle U_j^\nu \mid \nu < \omega_1 \;\&\; j < \omega \rangle$ of disjoint open sets as follows. If we have defined U_j^τ for all $\tau < \nu$, all $j < \omega$, we pick points $y_j^\nu <_T x_j^\nu$ inductively on j so that $\text{ht}(y_j^\nu) \geqslant \alpha_\nu$ and

$$(y_j^\nu, x_j^\nu] \cap [\{x_k^\nu \mid k > j\} \cup (y_0^\nu, x_0^\nu] \cup \ldots \cup (y_{j-1}^\nu, x_{j-1}^\nu]] = \emptyset,$$

and set $U_j^\nu = (y_j^\nu, x_j^\nu]$ for each j. Clearly, (U_j^ν) is a separation of X.

($\leftarrow$) Suppose T were not an almost Souslin tree, and let $\{x_\alpha \mid \alpha \in E\}$ be an antichain of T such that $x_\alpha \in T_\alpha$ and E is stationary in ω_1. We may assume that $\alpha \in E \rightarrow \lim(\alpha)$, of course. Being an antichain of T, $\{x_\alpha \mid \alpha \in E\}$ is a discrete subset of the tree topology on T. Hence we can separate the x_α's by a family of disjoint open sets. It follows that we can separate the x_α's by disjoint basic open sets of the form $(f(x_\alpha), x_\alpha]$, where $f(x_\alpha) <_T x_\alpha$ for all $\alpha \in E$. Let $h(\alpha) = \text{ht}(f(x_\alpha))$, for each α. Then h is regressive on E, so as E is stationary there is a stationary set $E' \subseteq E$ such that h is constant on E', say with value δ. Thus for all $\alpha \in E'$, $(f(x_\alpha), x_\alpha] \cap T_{\delta+1} \neq \emptyset$. Let z_α be the unique predecessor of x_α on $T_{\delta+1}$, for each $\alpha \in E'$. Since $T_{\delta+1}$ is countable we can find $\alpha, \beta \in E'$, with $\alpha \neq \beta$, such that $z_\alpha = z_\beta$. Then $(f(x_\alpha), x_\alpha] \cap (f(x_\beta), x_\beta] \neq \emptyset$, a contradiction. The proof is complete.

4. Normality of tree spaces

It should be clear from the results stated in § 1 that any characterization of normality of tree spaces in terms of structural tree properties will require additional axioms of set theory for its proof. (Indeed, if Martin's axiom plus $2^{\aleph_0} > \aleph_1$ be assumed, *all* Aronszajn tree spaces are normal.) We present here a characterization assuming $V = L$, the axiom of constructibility. The tree property involved is a slight strengthening of the almost Souslin concept.

Let T be an ω_1-tree. We shall say T has *property* γ if and only if, whenever A is an antichain of T, there is a closed unbounded set $C \subseteq \omega_1$ such that $T \setminus T \upharpoonright C$ contains a closed neighbourhood of A.

Clearly, any Souslin tree will have property γ. (If T is a Souslin tree and A is an antichain of T, then $A \subseteq T \upharpoonright \alpha$ for some α, so $T \setminus T \upharpoonright C$ is a closed neighbourhood of A where $C = \omega_1 - (\alpha + 1)$.) Also, any tree with property γ will be an almost Souslin tree. (Indeed, a tree T is an almost Souslin tree if and only if whenever A is an antichain of T there is a closed unbounded set $C \subseteq \omega_1$ such that $T \setminus T \upharpoonright C$ contains *some* neighbourhood of A, so property γ is just a 'slight' (?) strengthening of the almost Souslin property.) We prove later that property γ lies strictly between the Souslin property and the almost Souslin property. First we show how property γ relates to normality of the tree space.

4.1. THEOREM. *Let T be an ω_1-tree. If T has property γ, then T is normal.*

Proof. Let A, B be disjoint, closed subsets of T. We define, by induction, subsets C_n, D_n of T as follows.

Let

$$D_0 = T, \quad C_0 = \{a \in A \mid a \text{ is minimal in } A\},$$

and, in general,

$$D_{m+1} = \{d \in T \mid (\exists c \in C_m)\,(c <_T d)\},$$

$$C_{2n} = \{a \in A \mid a \text{ is minimal in } A \cap D_{2n}\},$$

$$C_{2n+1} = \{b \in B \mid b \text{ is minimal in } B \cap D_{2n+1}\}.$$

The following facts are immediate:

(i) each C_m is an antichain of T;

(ii) $D_0 \supset D_1 \supset D_2 \supset \dots$;

(iii) if $d \in D_m$, there are unique $c_0^d <_T \dots <_T c_{m-1}^d <_T d$ with $c_i^d \in C_i$.

Claim. $\bigcap_{m<\omega} D_m = \emptyset$.

Proof of claim. Suppose $d \in \bigcap_{m<\omega} D_m$. Then there are unique $c_0^d <_T c_1^d <_T c_2^d <_T \dots <_T d$ with $c_i^d \in C_i$. Since $C_{2n} \subseteq A$ for all n and A is closed, the unique predecessor $\dot{d}$ of d on level $\sup_{m<\omega} \mathrm{ht}(c_m^d)$ of T lies in A. Since $C_{2n+1} \subseteq B$ for all n, we likewise have $\dot{d} \in B$. This contradicts $A \cap B = \emptyset$. The claim is proved.

By the claim, $T = \bigcup_{n<\omega}(D_n \setminus D_{n+1})$. It suffices to define, for each n, a function f_n: $(A \cup B) \cap (D_n \setminus D_{n+1}) \to T$ such that $f_n(x) <_T x$ and for $a \in A$, $b \in B$, $a, b \in D_n \setminus D_{n+1}$, $(f_n(a), a] \cap (f_n(b), b] = \emptyset$. For suppose such f_n are defined. Define f: $(A \cup B) \to T$ by letting

$$f(d) = \begin{cases} f_0(d), & \text{if } d \in D_0 \setminus D_1, \\ \text{the } T\text{-maximum of } c_n^d, f_n(d), & \text{if } d \in D_n \setminus D_{n+1}. \end{cases}$$

Clearly, $f(x) <_T x$ for all $x \in A \cup B$, and if $a \in A$, $b \in B$, then

$$(f(a), a] \cap (f(b), b] = \emptyset.$$

Let $U = \bigcup_{a \in A}(f(a), a]$, $V = \bigcup_{b \in B}(f(b), b]$. Then U, V are disjoint open sets and $A \subseteq U$ and $B \subseteq V$, so we have finished.

We consider first the case where $n = 0$. Now,

$$A \cap (D_0 \setminus D_1) = A \setminus D_1 = C_0,$$

an antichain of T. By the γ property, let $E \subseteq \omega_1$ be closed and unbounded so that $T \setminus T \restriction E$ contains a closed neighbourhood of C_0. Let $E = \{e_\nu \mid \nu < \omega_1\}$. Let U be an open set containing C_0 such that $U^- \subseteq T \setminus T \restriction E$. Fix now $\nu < \omega_1$. Let $(x_n^\nu)_{n<\omega}$ enumerate $(A \cup B) \cap (D_0 \setminus D_1) \cap (T \restriction e_{\nu+1} \setminus T \restriction e_\nu)$. By induction on n we define $y_n^\nu <_T x_n^\nu$, as follows. Suppose $x_n^\nu \in A$. Thus $\mathrm{ht}(x_n^\nu) > e_\nu$. Now x_n^ν cannot be a limit point of B, so we can find a $y_n^\nu <_T x_n^\nu$, with $\mathrm{ht}(y_n^\nu) \geqslant e_\nu$, such that $(y_n^\nu, x_n^\nu] \subseteq U$, $(y_n^\nu, x_n^\nu] \cap B = \emptyset$, and for all $m < n$, if $x_m^\nu \in B$, then $(y_m^\nu, x_m^\nu] \cap (y_n^\nu, x_n^\nu] = \emptyset$. (If x_n^ν lies on a successor level of T, then we can take for y_n^ν the immediate predecessor of x_n^ν; otherwise, there is enough room below x_n^ν to pick y_n^ν as required. Either way, y_n^ν may be found.) Now suppose $x_n^\nu \in B \cap T_{e_\nu}$. Since $U^- \cap T_{e_\nu} = \emptyset$, we can find $y_n^\nu <_T x_n^\nu$ so that $(y_n^\nu, x_n^\nu] \cap U = \emptyset$. Suppose finally that $x_n^\nu \in B$, $\mathrm{ht}(x_n^\nu) > e_\nu$. Then we may pick y_n^ν just as for the case $x_n^\nu \in A$ above (only with the roles of A and B interchanged now, of course, and without the requirement that $(y_n^\nu, x_n^\nu] \subseteq U$). Now suppose $x \in (A \cup B) \cap (D_0 \setminus D_1)$. For some unique $\nu, n, x = x_n^\nu$. Set $f_0(x) = y_n^\nu$. Clearly, this defines f_0 as required.

Now consider the general case, $n > 0$. Let $c \in C_{n-1}$. By the same procedure as above we can define f_n^c: $T^{(c)} \cap (A \cup B) \cap (D_n \setminus D_{n+1}) \to T^{(c)}$ of the appropriate kind. (Above c, the situation is the same as for the case where $n = 0$.) Set $f_n = \bigcup_{c \in C_{n-1}} f_n^c$. Clearly, f_n is as required. The proof is complete.

In order to obtain the converse to Theorem 4.1, we need to assume $V = L$. Our argument is an improvement of an argument due to Fleissner (see [**6**]).

4.2. THEOREM. *Assume $V = L$. Let T be an ω_1-tree. Then T is normal if and only if it satisfies property γ.*

Proof. By Theorem 4.1, if T satisfies property γ, it is normal. Suppose now that T fails to satisfy property γ. Then there is an antichain A of T such that for no closed unbounded set $C \subseteq \omega_1$ does $T \setminus T \restriction C$ contain a closed neighbourhood of A. Clearly, A must be uncountable. There are two cases to consider.

Case 1. A is non-stationary (that is, $\{\mathrm{ht}(a) \mid a \in A\}$ is not stationary in ω_1). Let C be a closed unbounded set such that $a \in A \rightarrow \mathrm{ht}(a) \notin C$. Let $B = T \restriction C$. Clearly, B is a closed set. And of course, being an antichain, A is closed. Suppose there were open sets U, V with $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$. Then $T \setminus V$ is a closed neighbourhood of A contained in $T \setminus T \restriction C$, which is impossible. Hence as A, B are disjoint closed sets, T cannot be normal.

Case 2. A is stationary. Let E be a stationary set of limit ordinals such that $E \subseteq \{\mathrm{ht}(a) \mid a \in A\}$. For each $\alpha \in E$, pick $a_\alpha \in A \cap T_\alpha$. We may identify T with ω_1 in such a way that $\alpha <_T \beta \rightarrow \alpha < \beta$ and for $\alpha \in E$, a_α is identified with α.

Since $V = L$ holds, there is a sequence $\langle f_\alpha \mid \alpha \in E \rangle$ such that $f_\alpha : \alpha \rightarrow 2$ and for any $f : \omega_1 \rightarrow 2$ there is an $\alpha \in E$ such that $f \restriction \alpha = f_\alpha$.

Let H be the set of all $\alpha \in E$ such that for some $\beta <_T \alpha$, $f_\alpha(\xi) = 0$ for all ξ such that $\beta <_T \xi <_T \alpha$, and set $K = E \setminus H$.

Since E is an antichain, H, K are disjoint closed subsets of T. Suppose there were disjoint open sets U, V with $H \subseteq U$, $K \subseteq V$. Define $f : \omega_1 \rightarrow 2$ by $f(\xi) = 0 \leftrightarrow \xi \in V$. Pick $\alpha \in E$ so that $f \restriction \alpha = f_\alpha$. Now if $\alpha \in H$, then since $H \subseteq U$ and U is open, there is $\beta <_T \alpha$ such that $(\beta, \alpha] \subseteq U$. By definition of f, $\beta <_T \xi <_T \alpha \rightarrow f(\xi) = 1$. So as $f \restriction \alpha = f_\alpha$, we have $\alpha \in K$, a contradiction. Similarly the assumption $\alpha \in K$ is contradictory. This completes the proof.

We finish by proving that the γ property is weaker than the Souslin condition but stronger than the almost Souslin condition.

4.3. LEMMA. *If there is a tree with property γ, then there is a tree with property γ which is not a Souslin tree.*

Proof. Let T^0 be a tree with property γ. If T^0 is not a Souslin tree, we have finished, so suppose otherwise. Let x_α be any element of T^0_α for each non-zero $\alpha < \omega_1$, and (assuming, as we may, that $|T^0_0| \geqslant 2$) obtain $(T^0)^*$ from T^0 as in Lemma 3.1. From now on, let T denote $(T^0)^*$ (so T^0 is a subtree of T) and let T^α denote the part of T above x_α. We show that T has property γ. (As in Lemma 3.1, T is not a Souslin tree.)

Let A be an antichain of T. Let

$$C = \{\alpha \in \omega_1 \mid \lim(\alpha) \ \& \ (\forall \beta < \alpha)\ (A \cap T^\beta \subseteq T \restriction \alpha)\}.$$

Since each T^β is a Souslin tree, it is easily seen that C is closed and unbounded in ω_1. Let C' be the set of all limit points of C. We show that $T \setminus T \restriction C'$ contains a closed neighbourhood of A. Let $a \in A$. Clearly, $a \notin T \restriction C$, so $a \notin T \restriction C'$. Let $\alpha(a) = \sup(C \cap \mathrm{ht}(a))$, and let $p(a)$ be the unique predecessor of a on $T_{\alpha(a)}$. Let $U = \bigcup_{a \in A}(p(a), a]$, a neighbourhood of A. We show that $U^- \cap T \restriction C' = \emptyset$. Suppose not, and let $x \in U^- \cap T_\delta$, where $\delta \in C'$. For some α, $x \in T^\alpha$, let β be the least member of C above α. Thus $\beta < \delta$ and $A \cap T^\alpha \subseteq T \restriction \beta$. Let $y <_T x$, $\mathrm{ht}(y) > \beta$. Since $x \in U^-$, $(y, x] \cap U \neq \emptyset$, so for some $a \in A$, $(p(a), a] \cap (y, x] \neq \emptyset$. But for this to be the case, we must have $a \in T^\alpha$, and then $(p(a), a] \subseteq T \restriction \beta$, so we have a contradiction.

Our next result shows that it is not the case that every almost Souslin tree has the γ property.

4.4. THEOREM. *Assume $V = L$. Then there is an almost Souslin, Aronszajn tree T which is not normal.*

Proof. We define a tree T by induction on the levels. The elements of T_α will be strictly increasing, continuous $(\alpha+1)$-sequences of non-negative real numbers, and the ordering will be inclusion. We shall set

$$B = \bigcup_{\alpha < \omega_1} T_{\omega^2 \cdot \alpha},$$

and, as we define $T_{\omega\alpha+\omega}$ we shall also define a set

$$A_{\omega\alpha+\omega} \subseteq T_{\omega\alpha+\omega}.$$

The idea is that, with $A = \bigcup_{\alpha < \omega_1} A_{\omega\alpha+\omega}$, A and B will be a pair of disjoint, closed sets which cannot be separated. The construction is carried out to ensure that T satisfies the following conditions:

(i) if $\alpha < \beta < \omega_1$ and $x \in T_\alpha$ and $\varepsilon > 0$, there is a $y \in T_\beta$ such that $x <_T y$, $\max(y) < \max(x) + \varepsilon$, and $(x, y] \cap A = \emptyset$;

(ii) if $x \in T_\alpha$ and $\varepsilon > 0$, there is $y \in T_{\alpha+\omega} \cap A$ such that $x <_T y$ and $\max(y) < \max(x) + \varepsilon$;

(iii) if $x \in T_{\omega^2 \cdot \alpha}$, then there is $y <_T x$ such that $(y, x] \cap A = \emptyset$.

To aid the construction, we make use of the function $f: \omega_1 \to \omega_1$ defined by $f(\alpha)$ = the least γ such that

$$L_\gamma \models \text{`}\alpha \text{ is countable'} \quad \text{and} \quad T \upharpoonright \alpha, \bigcup_{\omega\beta+\omega<\alpha} A_{\omega\beta+\omega} \in L_\gamma.$$

To commence the construction we set $T_0 = \{\langle 0, 0\rangle\}$. Now suppose T_α is defined. We set

$$T_{\alpha+1} = \{s \cup \{\langle q, \alpha+1\rangle\} \mid s \in T_\alpha \;\&\; q \in \mathbf{Q} \;\&\; q > \max(s)\},$$

where $\mathbf{Q}$ is the set of all rationals. This definition clearly preserves conditions (i)–(iii) above. Finally, we must show how to define T_α when $\lim(\alpha)$ and $T \upharpoonright \alpha$ has been defined.

Case 1. $\alpha = \omega\beta + \omega$. Let $((x_m, q_m))_{m<\omega}$ enumerate all pairs (x, q) such that $x \in T \upharpoonright \alpha$ and $q \in \mathbf{Q}$, with $q > 0$. By induction on m we define elements y_m^0, y_m^1 of T_α so that $x_m <_T y_m^i$ and $\max(y_m^i) < \max(x_m) + q_m$. Suppose we have defined $y_0^i, \ldots, y_{m-1}^i$. (This will include the case where $m = 0$, at the start of the construction, as a special case.) Let $(\alpha_n)_{n<\omega}$ be strictly increasing and cofinal in α with $x_m \in T \upharpoonright \alpha_0$. Let $(\varepsilon_n)_{n<\omega}$ be a strictly increasing sequence of positive rationals with limit $\frac{1}{2}q_m$. By the definition of the successor levels of T, together with property (i) (which $T \upharpoonright \alpha$ is assumed to satisfy), we can find $x_0^0, x_0^1 \in T_{\alpha_0}$, $x_0^i >_T x$, with

$$\max(x_0^i) < \max(x_m) + \varepsilon_0, \quad (x, x_0^i] \cap A = \emptyset,$$

such that for all $k = 0, \ldots, m-1$, $x_0^i \not\leq y_k^0, y_k^1$. By induction now, using property (i), pick $x_{n+1}^i \in T_{\alpha_{n+1}}$, $x_{n+1}^i >_T x_n^i$, with

$$\max(x_{n+1}^i) < \max(x_n^i) + (\varepsilon_{n+1} - \varepsilon_n) \quad \text{and} \quad (x_n^i, x_{n+1}^i] \cap A = \emptyset.$$

Let $y_m^i = (\bigcup_{n<\omega} x_n^i) \cup \{\langle \sup_{n<\omega} x_n^i, \alpha+1\rangle\}$. That completes the definition. Notice that for all m, $(x_m, y_m^i) \cap A = \emptyset$, and for $m \neq n$, $y_m^i \neq y_n^j$. Let T_α consist of all these y_m^i. Set $A_\alpha = \{y_m^1 \mid m < \omega\}$. Clearly, $T \upharpoonright \alpha + 1$ satisfies (i)–(iii).

Case 2. Otherwise (thus $\alpha = \omega^2.\beta$ for some β). Let $x \in T \upharpoonright \alpha$, $\varepsilon \in \mathbf{Q}$, $\varepsilon > 0$. Let $(\alpha_n)_{n<\omega}$ be a strictly increasing sequence of limit ordinals, cofinal in α, with $x \in T \upharpoonright \alpha_0$. Let $(S_n)_{n<\omega}$ enumerate the (countable) set of all antichains of $T \upharpoonright \alpha$ lying in $L_{f(\alpha)}$. Let (U_α, V_α) be the $<_L$-least pair of disjoint open subsets of $T \upharpoonright \alpha$ such that $A \subseteq U_\alpha$ and $B \subseteq V_\alpha$. (Since $T \upharpoonright \alpha$ is countable, and since A and B are disjoint closed subsets of $T \upharpoonright \alpha$, such a pair (U_α, V_α) must exist: this requires just the argument used in Corollary 2.2.) Pick $y_0 \in T_{\alpha_0}$, $x <_T y_0$, with

$$\max(y_0) < \max(x) + \tfrac{1}{2}\varepsilon \quad \text{and} \quad (x, y_0] \cap A = \emptyset.$$

This is possible by (i). Now pick $z_0 \in T_{\alpha_0+\omega} \cap A$, $y_0 <_T z_0$, with

$$\max(z_0) < \max(y_0) + \tfrac{1}{2}\varepsilon < \max(x) + \varepsilon.$$

This is possible by (ii). Since $A \subseteq U_\alpha$ and U_α is open we can pick $x_0 \in U_\alpha$ such that $y_0 <_T x_0 <_T z_0$. Notice that because $\alpha_0 < \mathrm{ht}(x_0) < \alpha_0 + \omega$, $(x, x_0] \cap A = \emptyset$. Let $\varepsilon_1 = \varepsilon - [\max(x_0) - \max(x)]$. Since $\max(x_0) < \max(z_0)$, $\max(x_0) - \max(x) < \max(z_0) - \max(x) < \varepsilon$, so $\varepsilon_1 > 0$. If there is an $x_1 \in T \restriction \alpha$ such that $(x_0, x_1] \cap A = \emptyset$ and $\max(x_1) < \max(x_0) + \varepsilon_1$ and $(\exists w \in S_0)\ (w <_T x_1)$, pick such an x_1. Otherwise, set $x_1 = x_0$. Let $\varepsilon_2 = \varepsilon_1 - [\max(x_1) - \max(x_0)]$. Let $\beta_1 = \max(\alpha_1, \mathrm{ht}(x_1)) + \omega$. In general, suppose $x_{2n-1}, \varepsilon_{2n}$ have been defined. Let $\beta_n = \max(\alpha_n, \mathrm{ht}(x_{2n-1})) + \omega$. Pick $y_{2n} \in T_{\beta_n}$, $x <_T y_{2n}$ with

$$\max(y_{2n}) < \max(x_{2n-1}) + \tfrac{1}{2}\varepsilon_{2n} \quad \text{and} \quad (x_{2n-1}, y_{2n}] \cap A = \emptyset.$$

(Possible by (i).) Now pick $z_{2n} \in T_{\beta_n + \omega} \cap A$, $y_{2n} <_T z_{2n}$, with

$$\max(z_{2n}) < \max(y_{2n}) + \tfrac{1}{2}\varepsilon_{2n} < \max(x_{2n-1}) + \varepsilon_{2n}.$$

(Possible by (ii).) Now pick $x_{2n} \in U_\alpha$ with $y_{2n} <_T x_{2n} <_T z_{2n}$. Set $\varepsilon_{2n+1} = \varepsilon_{2n} - [\max(x_{2n}) - \max(x_{2n-1})]$. Then $\varepsilon_{2n+1} > 0$. If there is an $x_{2n+1} \in T \restriction \alpha$ such that

$$(x_{2n}, x_{2n+1}] \cap A = \emptyset \quad \text{and} \quad \max(x_{2n+1}) < \max(x_{2n}) + \varepsilon_{2n+1}$$

and

$$(\exists w \in S_n)\ (w <_T x_{2n+1}),$$

pick such an x_{2n+1}. Otherwise, set $x_{2n+1} = x_{2n}$. Set

$$\varepsilon_{2n+2} = \varepsilon_{2n+1} - [\max(x_{2n+1}) - \max(x_{2n})].$$

The construction completed, set

$$u(x, \varepsilon) = \Big(\bigcup_{n<\omega} x_n\Big) \cup \{\langle \sup_{n<\omega} x_n, \alpha + 1\rangle\}.$$

Clearly, $x \subset u(x, \varepsilon)$ and $(x, u(x, \varepsilon)) \cap A = \emptyset$ (in $T \restriction \alpha$), and

$$\max(u(x, \varepsilon)) \leqslant \max(x) + \varepsilon.$$

We set $T_\alpha = \{u(x, \varepsilon) \mid x \in T \restriction \alpha \ \&\ \varepsilon \in \mathbf{Q},\ \varepsilon > 0\}$, with each $u(x, \varepsilon)$ defined as above. $T \restriction \alpha + 1$ clearly satisfies (i)–(iii). That defines T. We must show that T is as required. Clearly, T must be an Aronszajn tree: otherwise we would have a strictly increasing ω_1-sequence of real numbers, which is impossible!

We show that T is not normal. The set B is clearly a closed subset of T. And by condition (iii), A is a closed subset of T, disjoint from B. (A limit point of A would have to have height of the form $\omega^2.\alpha$, and (iii) prohibits this. Hence, not only is A closed, it is, indeed, discrete.) Suppose T were normal. Then we can find a pair (U, V) of disjoint open sets with $A \subseteq U$, $B \subseteq V$. Let (U, V) be, in fact, the $<_L$-least such pair.

Let $M \prec L_{\omega_2}$ be countable with $T, A, B, U, V \in M$. Let $\alpha = M \cap \omega_1$. Let $\pi: M \simeq L_\beta$. Then

$$\pi \restriction \alpha = \mathrm{id} \restriction \alpha, \quad \pi(\omega_1) = \alpha, \quad \pi(T) = T \restriction \alpha, \quad \pi(A) = A \cap T \restriction \alpha,$$
$$\pi(B) = B \cap T \restriction \alpha, \quad \pi(U) = U \cap T \restriction \alpha, \quad \pi(V) = V \cap T \restriction \alpha.$$

Since

$L_{\omega_2} \models$ '(U, V) is the $<_L$-least pair of disjoint open sets with

$A \subseteq U$ and $B \subseteq V$',

we have

$L_\beta \models$ '$(U \cap T \restriction \alpha,\ V \cap T \restriction \alpha)$ is the $<_L$-least pair of disjoint open

sets with $A \cap T \restriction \alpha \subseteq U \cap T \restriction \alpha$ and $B \cap T \restriction \alpha \subseteq V \cap T \restriction \alpha$'.

By absoluteness, the above sentence is really valid, so we conclude that (since $\pi(\omega_1) = \alpha$ implies $\omega^2.\alpha = \alpha$) $U \cap T \restriction \alpha = U_\alpha$ and $V \cap T \restriction \alpha = V_\alpha$.

Now if we pick any x_0 in T_α, our construction guarantees that x_0 extends an α-branch of $T \restriction \alpha$ which meets U_α cofinally often (in defining $u(x, \varepsilon)$ from x, ε, we constructed a sequence $(x_n)_{n<\omega}$ which had $u(x, \varepsilon)$ as its limit, such that $x_{2n} \in U_\alpha$ for all n). But $x_0 \in B \subseteq V$, so as V is open, there must be a $y <_T x_0$ with $(y, x_0] \subseteq V$. Hence $U_\alpha \cap V \neq \varnothing$, giving $U \cap V \neq \varnothing$, a contradiction.

Finally we show that T is an almost Souslin tree. Suppose it is not. Let S be an antichain of T such that $E = \{\mathrm{ht}(s) \mid s \in S\}$ is stationary in ω_1.

Let $M_0 \prec L_{\omega_2}$ be countable with $T, A, S \in M_0$, and set $\alpha_0 = \omega_1 \cap M_0$. By induction, let $M_{\nu+1} \prec L_{\omega_2}$ be countable with $\alpha_\nu \in M_{\nu+1}$ and set $\alpha_{\nu+1} = \omega_1 \cap M_{\nu+1}$, and for $\lim(\lambda)$, let $M_\lambda = \bigcup_{\nu<\lambda} M_\nu$, $\alpha_\lambda = \sup_{\nu<\lambda} \alpha_\nu = \omega_1 \cap M_\lambda$. Thus $M_0 \prec M_1 \prec \ldots \prec M_\nu \prec \ldots \prec L_{\omega_2}$.

Now, $\{\alpha_\nu \mid \nu < \omega_1\}$ is closed and unbounded in ω_1, so we can find an $\alpha \in E$ such that $\alpha_\alpha = \alpha$. Let $\pi: M_\alpha \cong L_\beta$. Then

$$\pi \restriction \alpha = \mathrm{id} \restriction \alpha, \quad \pi(\omega_1) = \alpha, \quad \pi(T) = T \restriction \alpha, \quad \pi(S) = S \cap T \restriction \alpha.$$

Now, $L_\beta \models$ 'α is uncountable' (since $\alpha = \pi(\omega_1)$) and $L_{f(\alpha)} \models$ 'α is countable', so $\beta < f(\alpha)$. Hence $\mathrm{ran}(\pi) = L_\beta \subseteq L_{f(\alpha)}$. In particular, $S \cap T \restriction \alpha \in L_{f(\alpha)}$. So, as $S \cap T \restriction \alpha$ is an antichain of $T \restriction \alpha$, $S \cap T \restriction \alpha = S_n$ for some n. (We adopt the notation of the construction in Case 2 now; since $\alpha = \pi(\omega_1)$, $\omega^2\alpha = \alpha$, of course.)

Since $\alpha \in E$ there is an $s \in T_\alpha \cap S$. For some $x \in T \restriction \alpha$, $\varepsilon \in \mathbf{Q}$, $\varepsilon > 0$, $s = u(x, \varepsilon)$. Now, since $(x, s] \cap A = \varnothing$, we certainly have $(x_{2n}, s] \cap A = \varnothing$. Moreover, since $\max(s) \leqslant \max(x) + \varepsilon$, we have $\max(s) \leqslant \max(x_{2n}) + \varepsilon_{2n+1}$.

Thus,

$$L_{\omega_2} \models \text{'there is a } z \in S \text{ such that } x_{2n} <_T z \text{ and } (x_{2n}, z] \cap A = \emptyset$$
$$\text{and } \max(z) \leqslant \max(x_{2n}) + \varepsilon_{2n+1}\text{'}.$$

But $x_{2n}, \varepsilon_{2n+1}, A, S \in M_\alpha$. (This is true for x_{2n}, because $x_{2n} \in T \restriction \alpha \subseteq M_\alpha$. For ε_{2n+1}, notice that as $\varepsilon, x, x_0, \ldots, x_{2n+1}, x_{2n} \in M_\alpha$, an easy induction gives $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n+1} \in M_\alpha$. And, of course, $A, S \in M_0$.) Thus,

$$M_\alpha \models \text{'there is a } z \in S \text{ such that } x_{2n} <_T z \text{ and } (x_{2n}, z] \cap A = \emptyset$$
$$\text{and } \max(z) \leqslant \max(x_{2n}) + \varepsilon_{2n+1}\text{'}.$$

Applying $\pi: M_\alpha \cong L_\beta$, and observing that the above sentence will be absolute, we see that there will be a $z \in S_n = S \cap T \restriction \alpha$ such that $x_{2n} <_T z$ and $(x_{2n}, z] \cap A = \emptyset$ and $\max(z) \leqslant \max(x_{2n}) + \varepsilon_{2n+1}$. (We clearly have $\pi(x_{2n}) = x_{2n}$, $\pi(\varepsilon_{2n+1}) = \varepsilon_{2n+1}$.) Thus, our construction will have ensured that x_{2n+1} extends some member of S_n. Thus $s = u(x, \varepsilon) >_T x_{2n+1}$ extends some member of S_n. But $S_n = S \cap T \restriction \alpha$, so this is in contradiction with the fact that S is an antichain of T. The proof is complete.

Let us remark that the full power of $V = L$ is not really required in the above: the combinatorial principle $\diamondsuit^*$ (see [**1**]) suffices.

5. Further remarks

Since we have been concerned with stationary antichains, it is natural to ask about trees with club antichains. Here the situation is fairly clear. Assuming $\diamondsuit$, it is easy to construct a special Aronszajn tree with no club antichain. (The argument resembles Theorem 4.4 somewhat, but is much, much easier.) On the other hand, if Martin's axiom plus $2^{\aleph_0} > \aleph_1$ be assumed, then every Aronszajn tree is the union of a countable family of club antichains (indeed, each of these antichains can be taken to intersect each infinite level of the tree). Perhaps the easiest way to see this is to consider the c.c.c. poset of all finite order-preserving maps of the tree concerned into $\mathbf{Q}$. For each infinite level α and each $q \in \mathbf{Q}$, the set of all those conditions which take the value q somewhere on T_α is dense. Hence the generic map of the tree into $\mathbf{Q}$ will attain the value q on every infinite level of T. Also, by the argument of [**1**], it is consistent with ZFC + GCH that every Aronszajn tree is the union of a countable family of club antichains. This is because the method used in [**1**] to destroy Souslin trees is to generically embed them in $\mathbf{Q}$, and the genericity ensures that each rational is attained on each infinite level of the tree concerned. We leave it to the reader to check the details for himself.

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