More on Stationary Coding

We here continue our investigations in [Sh1] on stationary coding sets (introduced and investigated by Zwicker [Z]) making some improvements and additions.

The various claims are not so connected. They include:

A If $\kappa = \kappa^{\aleph_0}$, $\lambda = \lambda^\kappa$ then there is a (κ^+,λ^+)-stationary coding (see 23)

B If $\lambda = \lambda^{\aleph_0}$ is regular, $S \subseteq \{\delta < \lambda^+ : cf\ \delta = \aleph_0\}$ is stationary but does not reflect *then* there is an $(\aleph_1,\lambda^+)$-stationary coding (see 24, 25)

C If $\lambda = \lambda^{\aleph_0}$ then $\diamondsuit\ (\mathcal{D}_{\aleph_1}(\lambda^+))$ (see 28); for more on diamonds see 13, 14, 15.

D We note that Martin Maximum implies that "there is no $(\aleph_1,\lambda)$-weak stationary coding for every λ" and we show that statement for $\lambda = \aleph_2$ when $2^{\aleph_0} \geq \aleph_3$, (see 3). We note also that for κ first inaccessible, strong stationary coding may not exist (see 4).

E We also give an elementary presentation of "a normal fine filter on λ (or $\mathcal{P}_{<\kappa}(\lambda)$) concentrating on the wrong cofinality is not λ^+-saturated" (see 6,7,8). $\diamondsuit\ (\mathcal{D})$ has an even stronger conclusion (see 17).

F On strong stationary coding see 18.

1. Notation:

1) If $<^* \restriction a$ well order the set a let $otp\ (a, <^*)$ be the order type. If a is a set of ordinals, $<^*$ the usual order then we write $otp\ (a)$.
Let *ord* be the class of ordinal.

2) $H_{<\kappa}(\alpha)$ is

$\{ a : |a| < \kappa$ and for every n and $x_1, \ldots, x_n$,

if $x_1 \in x_2 \in x_3 \cdots \in x_n \in a$,

then x_1 is an ordinal $< \alpha$ or

a set of power $< \kappa \}$

$H_{<\kappa}(0)$ is written $H(\kappa)$.

3) Observe that $|H_{<\kappa}(\alpha)| = |2+\alpha|^{<\kappa}$ when κ is regular.

4) For $\kappa \le \lambda$ let $\mathcal{B} = \mathcal{B}_{\kappa,\lambda}$ be a subset of $H_{<\kappa}(\lambda)$ of power λ, such that for some M^*, $M^* <_{es} (H((2^\lambda)^+), \in)$, $\kappa \in M^*$, $||M^*|| = \lambda$, $\lambda \in M^*$, $\lambda \subseteq M^*$ and $\mathcal{B} = M^* \cap H_{<\kappa}(\lambda)$, hence

(*i*) if $\lambda^{<\kappa} = \lambda$ then $\mathcal{B} = H_{<\kappa}(\lambda)$

(*ii*) if $\lambda^{<\kappa} > \lambda$ but there is $\mathcal{B} \subseteq H_{<\kappa}(\lambda)$,

$|\mathcal{B}| = \lambda$, $(\forall a \in \mathcal{P}_{<\kappa}(\lambda))$ $(\exists b \in \mathcal{B})$ $(a \subseteq b)$

then $\mathcal{B}$ satisfies this

5) Let $cd_{\kappa,\lambda}$ be a one-to-one function from $\mathcal{B}_{\kappa,\lambda}$ onto λ, and let $dcd_{\kappa,\lambda}$ be its inverse

Let $dcd``(a) = \{dcd(x) : x \in a\}$

6) Let $\mathcal{D}_\theta$ (θ an uncountable regular cardinal), be the filter generated by the closed unbounded subsets of θ, $\mathcal{D}^{cb}_\theta$ is the filter of co-bounded subsets of θ.

7) For $f, g : I \to ord$, $f <_D g$ means $\{t \in I : f(t) < g(t)\} \in D$, $f/\mathcal{D} < g/\mathcal{D}$ has the same meaning

8) If $\mathcal{D}$ is an $\aleph_1$-complete filter on a set I, $f: I \to ord$ then the $\mathcal{D}$-rank of f is denoted by $Rk(f, \mathcal{D})$, is an ordinal. We define it by defining by induction on α when $Rk(f, \mathcal{D}) = \alpha$:
$Rk(f, \mathcal{D}) = \alpha$ iff $\alpha = \cup\{\beta+1 : \beta < \alpha$, and for some $g/\mathcal{D} < f/\mathcal{D}$, $Rk(g, \mathcal{D}) = \alpha.\}$
9) If $Dom\ f = \theta$ a regular uncountable cardinal, let $Rk(f) = Rk(f, \mathcal{D}_\theta^{cb})$.
10) For $\mathcal{D}$ a fine normal filter on $\mathcal{P}_{<\kappa}(A)$, $B \subseteq A$ let

$$\mathcal{D} \restriction B = \left\{ \{a \cap B : a \in I\} : I \in \mathcal{D} \right\}$$

$\mathcal{D} \restriction B$ is a fine normal filter on $\mathcal{P}_{<\kappa}(B)$

2. Lemma:

1) The following are equivalent for a regular uncountable κ and stationary $T \subseteq \kappa$:
(i) there are function $g_\alpha (\alpha < \kappa^+), g$ from κ to κ, such that $(\forall i < \kappa)$ $g(i) < (\aleph_0 + |i|)^+$ and $g_\alpha/\mathcal{D}_\kappa < g_\beta/\mathcal{D}_\kappa$ for $\alpha < \beta < \kappa^+$ and $g/\mathcal{D}_\kappa \not\leq g_\alpha/\mathcal{D}_\kappa$
(v) *for any* cardinal μ such that $(\forall \delta \in T)\, [cf\, \delta > \mu \wedge |\delta|^\mu < \kappa]$, cardinal $\lambda > \kappa$ and subsets $P_i \subseteq \lambda\ (i < \mu)$ *there are* functions $g_i : \kappa \to \kappa\ (i < \mu)$ such that the following set is stationary (i.e., $\neq \phi \mod \mathcal{D}_{<\kappa}(\lambda)$)
$\{a \in \mathcal{P}_{<\kappa}(\lambda)$: $a \cap \kappa$ is an ordinal and for $i < \mu$ the order type of $a \cap P_i$ is $g_i(a \cap \kappa)$, and if δ is an accumulation point of a, $cf\, \delta \neq cf\,(a \cap \kappa)$ then $\delta \in a\}$

2) Assume (i) of 1) holds (for κ, T), $\lambda = \kappa^{+\alpha}$, $|\alpha|^+ < \kappa$ and $(\forall \gamma < \kappa)$ $[|\gamma|^{|\alpha|} < \kappa]$. If T is a set of inaccessibles (not necessarily strong limit) *then* there is a (κ,λ) $-$stationary coding.

2.A Remark:

Lemma 2 (1) says that in [Sh1] 12, 12A we can add condition (v) to the four equivalent conditions. Lemma 2 (2) says we can strengthen [Sh1] 13 (which uses the same assumption and deduce the existence of a (κ,λ)-weak stationary coding (with no additional condition on T)).

Proof:

1) We use [Sh1] 12A which has the same proof of [Sh1]12. Now (v) here implies (iv) there trivially. The proof there of (ii)$\Longrightarrow$ (iv) gives (ii)$\Longrightarrow$ (v).

2) Like the proof of [Sh1]13.

3. Fact:

1) If $2^{\aleph_0} > \aleph_2$ *then* there is a stationary $S \subseteq S_{\leq \aleph_0}(\aleph_2)$ which does not reflect, i.e., $S \neq \phi \bmod \mathcal{D}_{\leq \aleph_0}(\aleph_2)$ but for every $\alpha < \aleph_2$ (but $\geq \aleph_1$), $S \cap S_{\leq \aleph_0}(\alpha) = \phi \bmod \mathcal{D}_{\leq \aleph_0}(\alpha)$

2) If $S \subseteq \{a : a \subset \kappa^+, a \cap \kappa$ an ordinal, $|a| < \kappa\} \subseteq S_{<\kappa}(\kappa^+)$ is a stationary set which does not reflect κ regular uncountable, *then* for some $C \in \mathcal{D}_{\leq \kappa_0}(\kappa^+)$, $C \cap S$ is a weak (κ,κ^+)-stationary coding for (κ,κ^+)

Proof:

1) Let for an ordinal i, h_i be a one to one function from $|i|$ onto i. In $V' \stackrel{\text{def}}{=} L[\langle h_i : i < \omega_2 \rangle]$ there are at most $\aleph_2^V$ countable subsets of ω_2^V (and $\aleph_1^V = \aleph_1^{V'}$, $\aleph_2^V = \aleph_2^{V'}$ $[V' \models "a \in S_{\leq \aleph_0}(\aleph_2)" \Longrightarrow V \models "a \in S_{\leq \aleph_0}(\aleph_2)]$). But it is known that every $C \in \mathcal{D}_{\leq \aleph_0}(\aleph_2)$ has power $\leq 2^{\aleph_0} > \aleph_2$. So $S \stackrel{\text{def}}{=} \{a : a \in S_{\leq \aleph_1}(\aleph_2),\ a \notin V'\}$ is $\neq \phi \bmod \mathcal{D}_{\leq \aleph_0}(\aleph_0)$. But for every $\alpha < \aleph_2$ using h_α there is $C_\alpha \in \mathcal{D}_{\leq \aleph_0}(\alpha)$, $C_\alpha \subseteq V'$ (each member of C_α has the form $\{h_\alpha(i) : i < \delta\}$ for some $\delta < \omega_1$). So S does not reflect.

2) Let $S \subseteq S_{<\kappa}(\kappa^+)$ be $\neq \phi \bmod \mathcal{D}_{<\kappa}(\kappa^+)$, but $S \cap S_{<\kappa}(\alpha) = \phi \bmod \mathcal{D}_{<\kappa}(\alpha)$ when $\kappa \leq \alpha < \kappa^+$. Let h_β be a one to one function from $|\beta|$ onto β. When $\kappa \leq \alpha < \kappa^+$ let $\alpha = \bigcup_{i<\kappa} a_i^\alpha$, a_i^α increasing continuous in i, $a_i^\alpha \notin S$. (Possible by the choice of S). Let $C_\alpha = \{i < \kappa$: h_α maps i onto $a_i^\alpha\}$ so C_α is a club of κ. Let g_α: $\kappa \to \kappa$ be defined by $g_\alpha(i) = Min(C_\alpha - i)$.
Let $C^* = \{a \in S_{<\kappa}(\kappa^+)$: a is closed under h_α, h_α^{-1} and g_α and $a \cap \kappa$ is an ordinal$\}$

Obviously $C^* \in \mathcal{D}_{\leq \aleph_0}(\aleph_0)$. So $S \cap C^*$ is $\neq \phi \bmod \mathcal{D}_{\leq \aleph_0}(\kappa^+)$.

Suppose (*) $a, b \in S \cap C^*$, $a \subseteq b$, $a \cap w_1 = b \cap w_1$, $a \neq b$, and we shall get a contradiction.

Let $\delta = a \cap \kappa = b \cap \kappa$. If $\alpha \in a \cap b$, $\alpha \geq \kappa$, then $a \cap \alpha = \{h_\alpha(i)$: $i < \delta\} = b \cap \kappa$. We know $b - \neq \phi$, let $\beta = Min\ (b - a)$; by the previous sentence $a \subseteq \beta_1$ hence $a = b \cap \beta$. Now as b is closed by g_β, clearly $\delta \in C_\beta$, hence (using h_α and the definition of C_β) $a = a_\delta^\beta$, so $a \notin S$, contradiction.

So (*) is impossible hence $S \cap C^*$ is a weak (κ,κ^+)-stationary coding.

Remark: The proof is similar to some proofs in [FMS].

4. Fact:

It is consistent that e.g. the first inaccessible cardinal λ, is a strong limit and for no (regular uncountable) $\kappa < \lambda$, a strong (κ, λ)-stationary coding exists (assuming the consistency of suitable large cardinals)

Proof: Woodin constructs a model of set theory in which the first inaccessible λ is strong limit and $\diamondsuit_\lambda$ fail. By [Sh1] 7*A* for $\kappa < \lambda$, strong (κ,λ)-stationary coding does not exists.

Why 7*A* holds? By the known (folk?) proof that club implies diamond i.e.

4.A Fact: (= 7A of [Sh1])

If there is a strong (κ,λ)-stationary coding, $\kappa < \lambda$, $\lambda = \lambda^{<\lambda} > 2^{<\kappa}$ then $\diamondsuit_{\{\delta < \lambda : cf\, \delta < \kappa\}}$

Proof: As $\lambda = \lambda^{<\lambda}$ let $\{A_i : i < \lambda\}$ be a list of all bounded subset of κ. Let $\{a_\delta : \delta \in S\}$ be a strong (κ, λ)-stationary coding, for some stationary $S \subseteq \{\delta < \lambda : cf\, \delta < \kappa\} \subseteq \lambda$, $\delta = \sup a_\delta$ and $|a_\delta| < \kappa$. Let $\mathcal{P}_\delta = \{\bigcup_{i \in b} A_i : b \subseteq a\}$, so for $\delta \in S$, $\mathcal{P}_\delta$ is a family of $\leq 2^{<\kappa}$ subsets of δ. Now we shall prove that $\langle \mathcal{P}_\delta : \delta \in S \rangle$ satisfies

(*) for $X \subseteq A$, $\{\delta < \lambda : X \cap \delta \in \mathcal{P}_\delta\}$ is a stationary subset of λ.

For let $h : \lambda \to \lambda$ be defined by

$$h(i) = \text{Min}\,\{j : A_j \cap i = X \cap i\}$$

So for stationarily many δ's, a_δ is closed under h hence $X \cap \delta = \bigcup_{i \in a_\delta} (X \cap i) = \bigcup_{i \in a_\delta} A_{h(i)} = \bigcup \{A_\gamma : \gamma \in a,\ \gamma \in \text{Rang}\,(h \restriction a)\} \in \mathcal{P}_\delta$. As $2^{<\kappa} < \lambda$ we are finished by a Theorem of Kunen.

5. Lemma:

1) It is consistent (in fact follows from the axiom from Foreman Magidor and Shelah [FMS] Martin Maximum) *that:* for no $\lambda > \aleph_1$ is there an $(\aleph_1, \lambda)$-weak stationary coding

2) It suffice to assume that $\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturated, and for every stationary $S \subseteq \mathcal{P}_{<\aleph_1}(\lambda)$, $\{A \in \mathcal{P}_{<\aleph_2}(\lambda): \quad S \cap \mathcal{P}_{<\aleph_1}(A) \neq \phi \quad mod\ \mathcal{D}_{<\aleph_1}(A)\} \neq \phi$ $mod\ \mathcal{D}_{<\aleph_2}(\lambda)$.

Proof:

1) We prove 1) by 2), the assumptions of 2) holds by [FMS], and for 2) we may repeat [Sh1] 20.

Alternatively assume S is a weak $(\aleph_1,\lambda)$-stationary coding, let I_1 be the family of $T \subseteq \omega_1$ such that: there is an increasing continuous sequence $\langle a_i : i < \omega_1 \rangle$ of countable subsets of λ satisfying:
$\{i < \omega_1$: if $i \in T$ then $(\exists\, b \in S)\ [i = a_i \cap \omega_1 \subseteq b \subseteq a_i]\}$ contains a club C.
For $T \in I_1$ let $\langle a_i(T) : i < \omega_1 \rangle$, $C(T)$ be witnesses. Now I_1 is a normal ideal on ω_1, hence modulo the non-stationary ideal on ω_1 has a maximal member T^* (as $\mathcal{D}_{\omega_1}$ is $\aleph_2$-saturated).
If $T^* = \omega_1$ (or just contains a club), then

$$S^* = \{b \in \mathcal{P}_{<\aleph_1}(\lambda) : \ (\exists i)\ [b \cap \bigcup_{j<\omega_1} a_j(T^*) = a_i(T^*) \wedge i \in C(T^*)]$$

$$\text{and } b \not\subseteq \bigcup_{j<\omega_1} a_j(T^*)\ \}$$

is a club of $\mathcal{P}_{<\aleph_1}(\lambda)$, and any member of $S^* \cap S$ contradict the assumption "S is a weak $(\aleph_1,\lambda)$-stationary coding", but such an element exists.

If $\omega_1 - T^*$ is stationary, $S_1 = \{b \in S : \ b \cap [\bigcup_{j<\omega_1} a_j(T^*)] = a_i(T^*)$ for some $i \notin T^*\}$ cannot be stationary otherwise by the second hypothesis of 5(2) we get contradiction to the maximalety of T^*. So for some $C_1 \in \mathcal{D}_{<\aleph_1}(\lambda)$, $C \cap S_1 = \phi$
Let $C_2 = \{b \in \mathcal{P}_{<\aleph_1}(\lambda)$: $b \not\subseteq \bigcup_{j<\omega_1} a_j(T^*)$, and

$b \cap \cap[\bigcup_{j<\omega_1} a_j(T^*]$ is $a_i(T^*)$ for some $i < \omega_1\}$.

Clearly $C_2 \in \mathcal{D}_{<\aleph_1}(\lambda)$. Hence $C_1 \cap C_2 \in \mathcal{D}_{<\aleph_1}(\lambda)$ hence there is $b \in C_1 \cap C_2 \cap S$. As $b \in C_1$ we know $b \notin S_1$, and as $b \in C_2$ for some $i < \omega_1$ $b \cap [\bigcup_{j<\omega_1} a_j(T^*)] = a_i(T^*)$, $b \neq a_i(T^*)$. This implies as $b \notin S_1$ by the definition of S_1 that $i \in T^*$, hence there is $a_i \in S_1$ $a_i(T^*) \cap \omega_1 \subseteq a \subseteq a_i(T^*)$. As by the choice of b and i $b \cap \omega_1 \subseteq a_i(T^*) \subseteq b$, $a_i(T^*) \neq b$ we get a,b contradicting "S is a weak $(\aleph_1,\lambda)$-stationary coding".

* * *

We give an elementary (i.e. with no forcing) presentation of the proof of [Sh1] 14.

6. Theorem:

If $\mathcal{D}$ is a fine normal filter on $I = \{a \subseteq \lambda : cf(\sup a) \neq cf\,|a|\}$, and λ is regular *then* there are functions f_i for $i < \lambda^+$ such that: Dom $f_i = I$, $f_i(a) \in a$ and for $i \neq j$, $\{a \in I : f_i(a) = f_j(a)\} = \phi \bmod \mathcal{D}$

Proof: We can find $A_i (i < \lambda^+)$ such that:

(*) A_i is a subset of λ, unbounded in λ, and for $j < i$, $A_i \cap A_j$ is bounded in λ

[just let $A_i(i<\lambda)$ be pairwise disjoint subsets of λ of power λ, and then define A_i $(\lambda \leq i < \lambda^+)$ by induction on i: for each i let $\{j : j < i\}$ be $\{j_\alpha : \alpha < \lambda\}$, and let $A_i = \{\gamma^i_\beta : \beta < \lambda\}$ where $\gamma^i_\beta =$ Min $(A_{j_\beta} - \bigcup_{\alpha<\beta} A_{j_\alpha})$,) it exists as $|A_{j_\beta} \cap A_{j_\alpha}| < \lambda$ for $\alpha < \beta$].

Let for $i < \lambda^+$, $g_i : i \to \lambda$ be such that $\{A_j - g_i(j) : j < i\}$ are pairwise disjoint. Let f_i be a strictly increasing function from λ onto A_i (for $i < \lambda^+$) hence $f_i(\alpha) \geq \alpha$. So $C_i = \{a : a$ is closed under $f_i\}$ belongs to $\mathcal{D}$. For each $a \in I$ let $a = \{x^a_\alpha : \alpha < |a|\}$.

Now for each $a \in C_i$, $a \cap A_i$ is unbounded is a, (by the definition of C_i) so for some $\alpha_i(a) < |a|$, $A_i \cap \{x^a_\alpha : \alpha < \alpha_i(a)\}$ is unbounded in a (as $cf(\sup a) \neq cf\,|a|$).

Next for $i < \lambda^+$ let h_i be a one-to-one function from λ onto $\lambda \cup \{j : j < i\}$ and define by induction on i:

$$
\begin{aligned}
C^1_i = \{a \subseteq i \cup \lambda :\ & a \text{ closed under } h_i\,,\ h_i^{-1}\,,\ a \cap \lambda \in I \\
& a \cap \lambda \text{ closed under } f_i\,,\ f_i^{-1}\,, \\
& a \text{ closed under } g_j\,,\ (j \in a \text{ or } j = i) \\
& \text{and for } j \in a\,,\ a \cap (j \cup \lambda) \in C^1_j\,\}
\end{aligned}
$$

Clearly $C^1_i \restriction \lambda = \{a \cap \lambda : a \in C^1_i\}$ is in $\mathcal{D}$, and for each $a \in I$ there is at most one $a' \in C^1_i$ satisfying $a' \cap \lambda = a$, namely $h_i``(a)$.

Now we define for $i < \lambda^+$ a function d_i with domain I.

$$d_i(a) = \begin{cases} \langle \alpha_i(a), otp(\{j \in h_i``(a) : \alpha_j(a) = \alpha_i(a)\} \rangle, & \text{if } h_i``(a) \cap \lambda = a \\ & \quad h_i``(a) \in C_i^1 \\ \text{Min } a & \text{otherwise} \end{cases}$$

Now we shall finish by showing:

A: for $i_1 \neq i_2$, $\{a \in I : d_{i_1}(a) = d_{i_2}(a)\} = \phi \bmod \mathcal{D}$

B: for $a \in I$, $\{d_i(a) : i < \lambda^+\}$ has cardinality $\leq a$

Why this suffice? As for each $a \in I$ we can find a one-to-one function e_a from $\{d_i(a) : i < \lambda^+\}$ into a and now use the λ^+ functions $\langle e_a(d_i(a)) : i < \lambda^+ \rangle$

Proof of A: *W.l.o.g.* $i_1 < i_2$ and $\lambda \leq i_1$ for notational simplicity. Clearly $R = \{a \in I : h_{i_2}``(a) \in C_{i_2}^1, \ i_1 \in h_{i_2}``(a)$ (hence $h_{i_1}``(a) = h_{i_2}``(a) \cap i_1 \in C_{i_1}^1$

belongs to $\mathcal{D}$. Let a be in it, and $d_{i_1}(a) = d_{i_2}(a)$. Clearly $d_{i_1}(a) \neq Min\ a$ hence by the first coordinale in $d_i(a)$, $\alpha_{i_1}(a) = \alpha_{i_2}(a)$. Now $\{\xi \in h_{i_1}``(a) : \alpha_\xi(a) = \alpha_{i_1}(a)\}$ is an initial segment of $\{\xi \in h_{i_2}``(a) : \alpha_\xi(a) = \alpha_{i_2}(a)\}$ (as $a \in R$) and a proper one (as i_1 belong to the latter but not the former). As the ordinals are well ordered, their order types are not equal. That means that the second coordinate in the $d_{i_1}(a)$, $d_{i_2}(a)$ are distinct. So $d_{i_1}(a) \neq d_i(a)$ is true for $i_1 \neq i_2$, $a \in R$, as required.

Proof of B: As the number of possible $\alpha_i(a)$ is $\leq |a|$, and the number of order types of well orderings of power $< |a|$ is $|a|$ it suffice to prove:

(*) for $i < \lambda^+$, $a \in C_i^1$, the set $u = \{j \in a : \alpha_j(a \cap \lambda) = \alpha_i(a \cap \lambda)\}$ has power $< |a|$

Why (*) holds? Because for $j \in u$ the set

$$A_j \cap \{x_\alpha^a : \alpha < \alpha_i(a \cap \lambda)\}$$

is unbound in $a \cap \lambda$

but $A_j \cap g_i(j)$ is bounded in $a \cap \lambda$ (as a is closed under g_i)

hence

$$r_j \overset{def}{=} (A_j \setminus g_i(j)) \cap \{x^a_\alpha : \alpha < \alpha_i(a \cap \lambda)\}$$

is an unbounded subset of $a \cap \lambda$, hence non empty.

But $\langle r_j : j \in a, \alpha_j(a \cap \lambda) = \alpha_i(a \cap \lambda) \rangle$ is a sequence of pairwise disjoint subsets of $\{x^a_\alpha : \alpha < \alpha_i(a \cap \lambda)\}$ (by the choice of g_i). As they are non empty their number is $\leq |\{x^a_\alpha : \alpha < \alpha_i(a \cap \lambda)\}| < |a|$.

7. Claim:

Let $\mathcal{D}$ be a fine normal filter on $I \subseteq \mathcal{P}_{<\kappa}(\lambda)$, λ singular and $(\forall a \in I)$ $(|a| \geq cf\ \lambda \wedge \quad cf\ |a| \neq cf\ \lambda \quad \wedge cf\ \lambda = \sup(cf\ \lambda \cap a))$ and $Rk(|a|, \mathcal{D}^{cb}_{cf\lambda}) \leq |a|^+$

Then there are functions f_i for $i < \lambda^+$, Dom $f_i = I$, $(\forall a \in I)[f_i(a) \in a]$ and for $i \neq j$ $\{a \in I : f_i(a) = f_j(a)\} = \phi \bmod \mathcal{D}$

Proof: Let $\sigma = cf\ \lambda$, $\lambda = \sum_{\zeta < \sigma} \lambda_\zeta$, each λ_ζ regular, $\sum_{\xi < \zeta} \lambda_\xi < \lambda_\zeta < \lambda$ for $\zeta < \sigma$. We can find for $i < \lambda^+$ functions A_i from σ to λ, $\sum_{\xi < \zeta} \lambda_\xi < A_i(\zeta) < \lambda_\zeta$ such that for $i < j < \lambda^+$ there is $\xi < \sigma$ such that

$$\xi \leq \zeta < \sigma \Longrightarrow A_i(\zeta) < A_j(\zeta)$$

Let again $a = \{x^a_\alpha : \alpha < |a|\}$, so for each $i < \lambda^+$, $a \in I$ if Range A_i is unbounded in a then for some $\alpha_i(a) < a$, (Range A_i) $\cap \{x^a_\alpha : \alpha < \alpha_i(a)\}$ is unbounded in a (and $\alpha_i(a) =$ Min a otherwise).

Now for $i < \lambda^+$ we define a function d_i with domain I (h_i- a one-to-one function from λ onto $i \cup \lambda$):

$$d_i(a) = \begin{cases} \langle \alpha_i(a),\ otp\{j \in h_i``(a) : \alpha_j(a) = \alpha_i(a)\} \rangle & \text{if } a = h_i``(a) \cap \lambda, \\ & (\forall \zeta \in (a \cap cf\ \lambda)) A(\zeta) \in a \\ & \text{and } (\forall j \in a) \\ & a = h_j``(a) \cap \lambda \\ \text{Min } a & \text{otherwise} \end{cases}$$

We finish as in 6.

7A Remark:

1) Really we use $Rk(|a|, \mathcal{D}^{cb}_\sigma) \leq |a|^+$ (where $\sigma = cf\ \lambda$) just to get, that for every $\zeta < |a|$ for some $\xi_\zeta < |a|^+$

(*) there are no $f_i : \sigma \to \zeta$ for $i < \xi_\zeta$, $[i<j \Longrightarrow f_i <_{D^{cb}_\sigma} f_j]$

We should observe that for $a \in I$, $a \cap \sigma$ has order type σ.

Note that if for each $\zeta < |a|$ there is such ξ_ζ then $\xi(*) = \bigcup_{\zeta < |a|} \xi_\zeta$ is $< |a|^+$ and work for all ζs.

Similar remark apply to 8.

8. Claim:

Suppose $\kappa \leq \sigma = cf\ \lambda < \lambda$,

$I \subseteq \{a \in \mathcal{P}_{<\kappa}(\lambda)$: $cf\,|a| \neq cf\,(\sup\,(a \cap \sigma))$, and $Rk(|a|, \mathcal{D}^{cb}_{cf(\sup(a \cap \sigma))}) \leq |a|^+$ when $cf\,(\sup a) > \aleph_0$ and $|a|^{\aleph_0} = |a|$ when $cf\,(\sup a) = \aleph_0\}$,

and $\mathcal{D}$ a normal fine filter on I.

Then there are for $i < \lambda^+$ functions $f_i : I \to \lambda$, $f_i(a) \in a$ and for $i \neq j$ $\{a \in I : f_i(a) = f_j(a)\} = \phi \bmod \mathcal{D}$.

Proof: Let A_i, λ_ζ be as in the proof of 7, $a = \{x^a_\alpha : \alpha < |a|\}$. Let h_i be a one-to-one function from λ onto $\lambda \cup \{j : j < i\}$. For each i the set $C^1_i \overset{def}{=} \{a \in I$: a is closed under A_i, and (Range A_i) $\cap\, a$ is unbounded in a, $h_i``(a) \cap \lambda = a$ and $a \in C^1_j$ for $j \in h_i``(a)$ and $cf\,(\sup a) = cf\,(\sup\,(a \cap \sigma))\}$ belongs to $\mathcal{D}$, and for $a \in C^1_i$ let $\alpha_i(a) < |a|$ be minimal such that (Range A_i) $\cap \{x^a_\alpha : \alpha < \alpha_i(a)\}$ is unbounded in a. We then let

$$d_i(a) = \begin{cases} \langle \alpha_i(a),\ otp\{j : j \in h_i''(a),\ \alpha_j(a) = \alpha_i(a)\} \rangle & \text{if } a \in C^1_i, \\ \text{Min } a & \text{otherwise} \end{cases}$$

and we proceed as in the proof of 6, 7 (and see 7A).

9. Definition:

1) For $\kappa < \lambda$, κ regular, and a model N with universe $|N|$ which is an ordinal $< \kappa$, two place relation R^N_1, R^N_2, a three place relation R^N_3 and a partial one place function F^N (if one of them is not memtioned this means it is empty), let (see notation 1(5)):

$T_{\kappa,\lambda}(N) = \{a \in \mathcal{P}_{<\kappa}(\lambda)$: $dcd``(a) \cap \lambda = a$, and there are b_s (for $s \in |N|$) such that:

(i) $b_s \subseteq a$, $a = \bigcup_{s \in N} b_s$, $b_s \in dcd``(a)$ (equivalently $cd_{\kappa,\lambda}(b_s) \in a$)

(ii) $sR_1^N t$ implies $b_s \subseteq b_t$

(iii) $sR_2 t$ implies $cd_{\kappa,\lambda}(b_s) \in b_t$

(iv) for each t, $cd\{<\alpha, cd_{\kappa,\lambda}(b_s)> : \alpha \in N, R_3^N(\alpha, s, t)\} \in b_t$

(v) for $t \in \mathrm{Dom} F^N$, $|b_t| \leq F(t)$

2) For K a family of models N, $T_{\kappa,\lambda}(K) = \bigcup_{N \in K} T_{\kappa,\lambda}(N)$

3) $N_{\boldsymbol{\theta}}^0 = (\boldsymbol{\theta})$ (so R_1, R_2, R_3, F are empty)

$N_{\boldsymbol{\theta}}^1 = (\boldsymbol{\theta}, <)$ (so R_2, R_3, F are empty)

$N_{\boldsymbol{\theta}}^2 = (N, <, <)$ (so R_3, F are empty)

$N_{\boldsymbol{\theta}}^3 = (\boldsymbol{\theta}, <, <, R_3)$ where $R_3 = \{<\alpha, \alpha, \gamma> : \alpha < \gamma < \boldsymbol{\theta}\}$ (so F is empty)

* * *

We now show that [Sh1] 13 (and 12) is applicable sometimes. (see 2, 2A above for what they say). This is when $\kappa = \lambda$ in 10.

10. Claim:

Suppose $\kappa = \mu^+ \leq \lambda$, $\boldsymbol{\theta}$ regular, $\aleph_0 < \boldsymbol{\theta} < \mu$, and $Rk(\mu^+, \mathcal{D}_{\boldsymbol{\theta}}^{cb}) = \mu^+$. Then there is a function g from $T = T_{\kappa,\lambda}(N_{\boldsymbol{\theta}}^1)$ to κ such that for every well ordering $<^*$ of λ

$$\{a \in \mathcal{P}_{<\kappa}(\lambda) : otp(a, <^*) < g(a)\} \supseteq T \bmod \mathcal{D}_\kappa(\lambda)$$

11. Remark:

1) We can use other N's, but then have to change accordingly the filter by which we define the rank.

2) In [Sh2] various sufficient conditions for $R\kappa(\mu^+,\mathcal{D}^{cb}_{\theta}) = \mu^+$ are given:

(When $cf\,\mu \neq \theta$):

$$(\forall \sigma < \mu)\ [\sigma^{\theta} \leq \mu]$$

and

$$"\mu > 2^{\theta} \text{ and } \mu \leq (\sup\{\sigma : \sigma^{\theta} \leq \mu\})"$$

4) As for $a \in T$ $\{\beta : \beta < g(a)\}$ has power μ, and $C' = \{a \in \mathcal{P}_{<\kappa}(\lambda): |a| = \mu\} \in \mathcal{D}_{<\kappa}(\lambda)$, we can deduce that:

If the conclusion of 10 holds for T then there are functions $g_i : T \to \lambda$ (for $i < \lambda^+$) $g(a) \in a$ such that for $i \neq j$ $\{a \in T: g_i(a) = g_j(a)\} = \phi \bmod \mathcal{D}_{<\kappa}(\lambda)$

Proof of 10: For each well ordering $<^*$ of λ let

$C[<^*] = \{a \in \mathcal{P}_{<\kappa}(\lambda)$: for each $i \in a$, $cd_{\kappa,\lambda}(i) \cap \lambda \subseteq a$ and

$$otp(cd_{\kappa,\lambda}(i) \cap \lambda, <^*) < otp(a)\}$$

It is clearly closed unbounded, i.e., belongs to $\mathcal{D}_{<\kappa}(\lambda)$. Now if $a \in T(N^1_{\theta}) \cap C[<^*]$, let $\langle b_\alpha : \alpha < \theta\rangle$ witness "$a \in T$" (i.e., $i_\alpha \in a$, $i_\alpha = cd_{\kappa,\lambda}(b_\alpha) \in \mathcal{B}_{\kappa,\lambda}$, $a = \bigcup_{\alpha<\theta} b_\alpha$, b_α is increasing in α), so $otp(b_{i_\alpha}, <^*) < otp(a)$ for each α. So clearly it suffices to prove:

12. Fact:

If θ is regular cardinal, $\aleph_0 < \theta < \mu$, $\theta \neq cf\,\mu$ and $Rk(\mu^+,\mathcal{D}^{cb}_{\theta}) = \mu^+$ *then* for every $\xi < \mu^+$ there is $\zeta_\xi < \mu^+$ such that: if $\zeta = \bigcup_{i<\theta} A_i$, A_i increasing, then for some $i < \theta$ $otp(A_i) \geq \xi$

Proof:

Suppose $otp(A_i) < \xi$ for $i < \theta$, A_i increasing, and $\zeta = \bigcup_{i<\theta} A_i$. Define for $\gamma < \zeta$ a function $h_\gamma : \theta \to \xi$ by: $h_\gamma(i) = otp(A_i \cap \gamma)$. So each h_γ is a function from θ to ordinals, and for $\beta < \gamma$ $(\forall i < \theta)$ $[h_\beta(i) \leq h_\gamma(i)]$, moreover for some $j < \theta$ $\beta \in A_j$ hence $(\forall i)$ $[j < i < \theta \to h_\beta(i) < h_\gamma(i)]$. This clearly implies $Rk(\xi, \mathcal{D}^{cb}_{\theta}) \geq \zeta$ but $Rk(\xi, \mathcal{D}^{cb}_{\theta}) < \mu^+$.

13. Definition

For $\kappa \le \lambda$, κ regular, $\mathcal{D}$ a normal fine filter on $I \subseteq \mathcal{P}_{<\kappa}(\lambda)$,

1) $\diamondsuit(\mathcal{D})$ means that there are $\langle A_a : a \in I \rangle$, $A_a \subseteq a$, such that for every $A \subseteq \lambda$, $\{a \in I : A \cap a = A_a\} \neq \phi \bmod \mathcal{D}$

2) $\diamondsuit^*(\mathcal{D})$ means that there are $\langle \mathcal{P}_a : a \in I \rangle$, $\mathcal{P}_a$ a family of $\le |a|$ subsets of a, such that for every $A \subseteq \lambda$ $\{a \in I : A \cap a \in \mathcal{P}_a\} \in \mathcal{D}$

3) We replace $\mathcal{D}$ by I when $\mathcal{D}$ is the filter generated by the family of closed unbounded subsets of I. We write $I, \mathcal{D}$ instead of $\mathcal{D} + I$,

14. Remark:

We implicitly assume $I \neq \phi \bmod \mathcal{D}_{<\kappa}(\lambda)$,

15. Fact:

1) For $I \subseteq J \subseteq \mathcal{P}_{<\kappa}(\lambda)$, $\mathcal{D}_1 \subseteq \mathcal{D}_2$ normal fine filter on $\mathcal{P}_{<\kappa}(\lambda)$,

i) $\diamondsuit^*(\mathcal{D}_1+J) \Longrightarrow \diamondsuit^*(\mathcal{D}_2+I)$

ii) $\diamondsuit^*(\mathcal{D}_1+J) \Longrightarrow \diamondsuit(\mathcal{D}_1)$

iii) $\diamondsuit(\mathcal{D}_2+I) \Longrightarrow \diamondsuit(\mathcal{D}_1+J)$

iv) $\diamondsuit^*(\mathcal{D}_1+J) \Longrightarrow \diamondsuit(\mathcal{D}_2+I)$

(remember $\mathcal{D}_{<\kappa}(\lambda)+I \subseteq \mathcal{D}$ for any fine normal filter on I)

2) Suppose $\kappa < \lambda = \lambda^{<\kappa}$,

$T = \{a$: for some θ, $a \in T_{\kappa,\lambda}(N^0_\theta)$, $|a|^\theta = |a|$

or $a \in T_{\kappa,\lambda}(N^1_\theta)$, and $cf\,|a| \neq \theta$ $(\forall \sigma < |a|)\sigma^\theta \le |a|$

or $(\exists \chi,\sigma,\alpha)$ $(2^\chi \le \lambda \wedge \lambda = \chi^{+\alpha}$ $\bigwedge |a|^{<\sigma} = |a| \wedge$ $(\forall \gamma < \alpha)$ $[cf\,(a \cap \chi^{+(\gamma+1)} < \sigma] \wedge \alpha < \sigma)\}$

Suppose further $T \neq \phi \bmod \mathcal{D}_{<\kappa}(\lambda)$. Then $\diamondsuit^*(T, \mathcal{D}_\kappa(\lambda))$

Proof: By straightforward generalization of the proof for the case $\lambda = \kappa$, due to Kunen for 1, (i.e., 1(ii), the rest being trivial) Gregory and Shelah for 2) (see e.g. [Sh3]). I.e., for 1)(ii), suppose $\langle \mathcal{P}_a : a \in \mathcal{P}_{<\kappa}(\lambda) \rangle$ exemplifies $\diamondsuit^*(\mathcal{D}_1 + J)$. Let $\mathcal{P}_a = \{A^a_i : i \in a\}$. Let $<\ ,\ >$ be a pairing function on λ, and for each $i < \lambda$, $a \in \mathcal{P}_{<\kappa}(\lambda)$ let

$$B^i_a = \{\alpha : \alpha \in a \ , \ <\alpha,i> \ \in A^a_i\}$$

So $B^i_a \subseteq a_i$; is $\langle B^i_a : a \in \mathcal{P}_{<\kappa}(\lambda) \rangle$ a $\diamondsuit(\mathcal{D}_1)$-sequence for some i? If yes we finish, if not let $B^i \subseteq \lambda$ exemplify this i.e.,

$$C^i = \{a \in \mathcal{P}_{<\kappa}(\lambda) : \ B^i \cap a \neq B^i_a\} \ \in \mathcal{D}_1$$

Hence

$$C = \{a \in \mathcal{P}_{<\kappa}(\lambda) : \ (\forall i \in a) \ a \in C^i \ , \ \text{and } a \text{ is closed under } < \ , >\} \ \in \mathcal{D}$$

and let

$$A = \{<\alpha,i> \ : \alpha \in B^i\} \ .$$

So for some $a \in C$, $A \cap a \in \mathcal{P}_a$ hence for some $i \in A$, $A \cap a = A^a_i$ hence $B^i \cap a = B^i_a$ contradiction.

16. Remark:

We can enlarge T in 15(2) to:
the set of $a \in \mathcal{P}_{<\kappa}(\lambda)$ satisfying:
(*) there is a family H of $\leq |a|$ functions from a to a such that: *for any* $h : a \to a$, *for some* $b \subseteq a$, $h \restriction b \in H$ and $a \subseteq \bigcup_{i \in b} dcd_{\kappa,\lambda}(i)$

* * *

Now 15(2) can be combined with (15(ii) and):

17. Observation:

If $\mathcal{D}$ is a fine normal filter on $\mathcal{P}_{<\kappa}(\lambda)$, and $\diamondsuit(\mathcal{D})$ holds, *then:* there are $J_\alpha \subseteq \mathcal{P}_{<\kappa}(\lambda)$ for $\alpha < 2^\lambda$ such that:

$$J_\alpha \neq \phi \ mod \ \mathcal{D} \ , \quad J_\alpha \cap J_\beta = \phi \ mod \ \mathcal{D} \ \text{ for } \alpha \neq \beta$$

18. Conclusion:

Suppose $\lambda = \lambda^{<\kappa}$, $\theta < \kappa$, κ regular, λ regular, and there is a strong (κ,λ)-stationary coding set S^* such that $(\forall a \in S^*)$ $[cf(\sup a) = \theta]$ and $\diamondsuit(\mathcal{D}_\kappa(\lambda)+S^*)$. Then there are $S_\alpha \subseteq \{\delta < \lambda: cf\ \delta = \theta\}$ for $\alpha < 2^\lambda$, each stationary, the intersection of any two non-stationary (any normal filter $\mathcal{D}$ on λ will satisfy this if $\{\sup a : a \in S^*\} \neq \phi\ mod\ \mathcal{D}$ and $\diamondsuit(\mathcal{D}'+S^*)$ where $\mathcal{D}' = \mathcal{D}+\{\{a : \sup a \in A\} : A \in \mathcal{D}\}$).

19. Conclusion:

If $\theta < \kappa \leq \lambda$, $\kappa = \mu^+$, $\mu^\theta = \mu$, *then* for $T = T_{\kappa,\lambda}(N^0_\theta)$, $T \neq \phi\ mod\ \mathcal{D}_\kappa(\lambda)$ and $\diamondsuit\ (T, \mathcal{D}_\kappa(\lambda))$.

Remark: This is closely related to [Sh6], [Sh7], (see particularly last section of [Sh7]) which continues [Sh4] VIII 2.6.

Proof: By 15(2).

20. Lemma:

1) Suppose $\theta < \kappa \leq \chi \leq \lambda$, $T \subseteq P_{<\chi^+}(\lambda)$, $T \neq \phi\ mod\ \mathcal{D}_{\chi^+}(\lambda)$, $\diamondsuit\ (T, \mathcal{D}_{\chi^+}(\lambda))$ and for each $a \in T$, $\chi \subsetneq a$ and:

$$(i)\quad (\exists b \subseteq a)[\,|b| < \kappa \wedge a = \bigcup_{\alpha \in b} dcd_{\chi^+,\lambda}(\alpha)]$$

Then we can find $T_1 \subseteq P_{<\kappa}(\lambda)$, $T_1 \neq \phi\ mod\ \mathcal{D}_\kappa(\lambda)$ such that $\diamondsuit\ (T_1, \mathcal{D}_\kappa(\lambda))$ holds.

2) Suppose in addition that for $a \in T$:

$$(ii)\quad (\forall c \subseteq a)[\,|c| < \kappa \rightarrow cd_{\chi^+,\lambda}(c) \in a]$$

Then we can demand $T_1 \subseteq T_{\kappa,\chi}(N^3_\theta)$

Proof: 1) As in the proof of claim 7 in [Sh1].
As $\diamondsuit\ (T_1, \mathcal{D}_{\chi^+}(\lambda))$, we can find $\langle M_a : a \in T \rangle$ such that M_a is a model with universe a and countably many (finitary) functions, and for every model M with universe λ and countably many functions $\{a : M_a = M \restriction a\}$ $\neq \phi\ mod\ \mathcal{D}_{\chi^+}(\lambda)$

For $a \in T$ we can find $b_a \subseteq a$, $|b_a| < \kappa$ such that b_a is closed under the functions of M_a and $a \subseteq \bigcup_{\alpha \in b_a} dcd_{\chi^+,\lambda}(\alpha)$. By the last condition, and as $[\alpha \in a \Longrightarrow dcd_{\chi^+,\lambda}(\alpha) \subseteq a]$ clearly $[a_1 \neq a_2 \Longrightarrow b_{a_1} \neq b_{a_2}]$. We define $N_{b_a} = M_a \restriction b_a$, and let $T_1 = \{b_a : a \in T\}$. So $N_b\,(b \in T_1)$ is well defined. Now

(i) $T_1 \subseteq \mathcal{P}_{<\kappa}(\chi)$,

(ii) $T_1 \neq \phi \bmod \mathcal{D}_\kappa(\chi)$ [if M is a model with universe λ and countably many functions, for some $a \in T$ $M_a = M \restriction a$, so b_a is closed under the functions of M and $b_a \in T_1$

(iii) For every model M with universe λ and countably many functions, for some $b \in T_1$, $N_b = M \restriction b$. [same proof as in (ii)]. Hence $\diamondsuit\ (T_1, \mathcal{D}_\kappa(\lambda))$ holds.

2) Easy from the proof of 1), choosing b_a in $T_{\kappa,\lambda}(N^3_\theta)$

21. Lemma:

Suppose $\mathcal{D}_1$ is a fine normal filter on $\mathcal{P}_{<\kappa}(\lambda_1)$, $\kappa \leq \lambda_1 < \lambda$. Let $\mathcal{D}$ be the normal fine filter on $\mathcal{P}_{<\kappa}(\lambda)$ generated by $\left\{\{a \in \mathcal{P}_{<\kappa}(\lambda) : a \cap \lambda_1 \in S\} : S \in \mathcal{D}_1\right\}$. Suppose further that $T_1 \subseteq \mathcal{P}_{<\kappa}(\lambda_1)$, $T_1 \neq \phi \bmod \mathcal{D}_1$, $\diamondsuit\ (T_1, \mathcal{D}_1)$ and T_1 is a (κ,λ_1)-weak stationary coding.

Lastly suppose $NSi(\kappa,\lambda)$ holds (see [Sh1] Def.8) or at least: for some algebra M will universe λ and countably many functions, M has no isomorphic but distinct subalgebras $M_1 \subseteq M_2$, $M_1 \cap \lambda_1 = M_2 \cap \lambda_1 \in T$

Then there is a (κ,λ)-weak stationary coding set T, for which $\diamondsuit\ (T, \mathcal{D})$ holds.

Proof: Just like 10 of [Sh1].

Remark: We can combine 21 or 22 with 23 or 24, getting existence for many cardinals.

22. Lemma:

Suppose in the previous lemma, κ is a strongly Mahlo cardinal, T is a (κ,λ_1)-stationary coding. Suppose further that if $b \subset a$ are in T then for every subset c of a of power $\leq |b|$, $cd_{\kappa,\lambda}(c) \in a$. Then $\mathcal{P}_{<\kappa}(\lambda)$ has a (κ,λ)-

stationary coding.

23. Lemma:

1) Suppose $\aleph_0 < \kappa < \lambda$, κ is regular, $\lambda^{<\kappa} = \lambda$ and $(\forall \sigma < \kappa)\, \sigma^{\aleph_0} < \kappa$, (hence $2^{\aleph_0} < \kappa$).

Then there is a (κ,λ^+)-stationary coding set T.

2) Also we can have $\diamondsuit\ (T, \mathcal{D}_\kappa(\lambda))$

3) Suppose that $\lambda^{<\kappa} \le \lambda^+$, $\mathcal{D}_1$ is a normal fine filter on $\mathcal{P}_{<\kappa}(\lambda)$, $T^* \in \mathcal{D}_1$, T^* has cardinality λ, and

$(a)\quad (\forall a \in T^*)\,(\forall b \subseteq a)\,[\,|b| \le \aleph_0 \rightarrow cd_{\kappa,\lambda}(b) \in a\,]$

Let $\mathcal{D}$ be the minimal normal fine filter on $\mathcal{D}_{<\kappa}(\lambda^+)$ such that $\mathcal{D} \restriction \lambda = \mathcal{D}_1$. *Then* for some $\mathcal{D}$-stationary T, $(\mathcal{D} + T) \restriction \lambda = \mathcal{D}_1$, and T is a stationary coding set.

4) For 3) if $\lambda = \lambda^{\aleph_1}$, $\lambda^{<\kappa} \le \lambda^+$ and for some $T_0 \subseteq \mathcal{P}_{<\kappa}(\lambda)$

$$|T_0| = \lambda \wedge (\forall a \in \mathcal{P}_{<\kappa}(\lambda))(\exists b \in T_0)\,[a \subseteq b]$$

then $\mathcal{D}_{<\kappa}(\lambda) + T$ is as required where
$T = \{a \in \mathcal{P}_{<\kappa}(\lambda)$: there are $b_i \in T_0$ $(i < \omega_1)$ increasing $a = \bigcup_{i<\omega_1} b_i$,

$$a = \lambda \cap dcd_{\kappa,\lambda}``(a),\ cd_{\kappa,\lambda}(b_i) \in a\}$$

Proof:

1) Let $\mathcal{P}_{<\kappa}(\lambda) = \{b_i:\ i < i(*)\}$, $i(*) \le \lambda^+$, and let for $i < i(*)$ $S_i \subseteq S^* = \{\delta < \lambda^+: cf\ \delta = \aleph_0\}$ be pairwise disjoint stationary subsets of λ^+, $S^* = \bigcup_i S_i$. For $\delta \in \bigcup_{i<i(*)} S_i$ let $i(\delta)$ be the unique i such that $\delta \in S_i$.

Let f, g be such that: f, g two place functions from λ^+ to λ^+, for $i < \lambda^+$, $i = \{j : j < i\} = \{f(i,j): j < |i|\}$ and for $j < |i| < \lambda^+$ $g(i, f(i,j)) = j$.

23.A. Observation:

If $a \in \mathcal{P}_{<\kappa}(\lambda^+)$ is closed under f and g, $w \subseteq a$ is unbounded in a and $a \cap \lambda = b_i$ *then* a is totally determined by w and i, and we write $a = a_i[w]$.

Let for $\delta \in S_i$

$$T^i_\delta = \{a \in \mathcal{P}_{<\kappa}(\lambda^+) : \sup a = \delta,\ a \cap \lambda = b_i,\ \text{a closed under f and g,}$$
$$\text{and for any bound countable } w \subseteq a,\ \text{with } \sup w \in S^*,$$
$$cd_{\kappa,\lambda^+}(a_{i(\sup w)}[w]) \in a\}$$

$$T^i = \bigcup_{\delta \in S_i} T^i_\delta$$

$$T = \bigcup_{i < i(*)} T^i$$

23.B. Observation:

If $c \subseteq d, d \neq c$ and $c, d \in T$ then $cd_{\kappa,\lambda^+}(c) \in d$

Proof: Let $d \cap \lambda = b_i$, $c \cap \lambda = b_j$, $w \subseteq c$ a countable subset of c with $\sup w = \sup c$ (w exists as for each $a \in T$, $cf(\sup a) = \aleph_0$).) As $c \in T$, $c \cap \lambda = bj$ necessarily $\sup w \in S_j$. If $i = j$ then $d \cap \lambda = c \cap \lambda$ and w is an unbounded subset of both so $d = c = a_i[w]$ contradiction. So assume $i \neq j$, so necessarily $\sup w \neq \sup a$ hence $\sup w < \sup a$ hence $a_{i(sup\ w)}[w] = c$ but as $d \in T$ by the definition of the T^i_δ's we know that $a_{i(sup\ w)}(w)) \in d$. So $cd_{\kappa,\lambda^+}(c) \in d$.

23.C. Observation: $T \neq \phi \mod \mathcal{D}_\kappa(\lambda)$

Proof: By Rubin and Shelah [RS]. (see proof of 24 after 24A)

Continuation of the proof of 23.

The observations above finishes the proof of 23(1).

2) We let $\{(b_i, M_i): i < i(*)\}$ list all pairs (b, M), where $b \in \mathcal{P}_{<\kappa}(\lambda)$, $M = (\alpha^M, A^M)$, $\alpha^M < \kappa$, $A^M \subseteq \alpha$. We use $\{b_i : i < i(*)\}$ as above and for $a \in T$, $\sup a \in T_i$, let $A_a = \{\xi \in A: otp(a \cap \xi) \in A^{M_i}\}$. Now $\langle A_a : a \in T \rangle$ is a wittness for $\diamondsuit(T, \mathcal{D}_\kappa(\lambda))$.

3) Same proof.

4) Left to the reader.

24. Lemma:

Suppose $\aleph_0 < \kappa \leq \lambda$, κ regular, $S^* \subseteq \{\delta < \lambda^+ : cf\ \delta = \aleph_0\}$, and $\mathcal{D}$ is a normal fine filter on $\mathcal{P}_{<\kappa}(\lambda)$ such that:

(i) $\lambda^+ = (\lambda^+)^{<\kappa}$

(ii) there is $Y^* \in \mathcal{D}$ of power λ

(iii) if $\lambda < \alpha < \lambda^+$, and $\mathcal{D}_\alpha$ is the unique normal fine filter on α such that $\mathcal{D}_\alpha \restriction \lambda = \mathcal{D}$ then:

$$\{a \in \mathcal{P}_{<\kappa}(\alpha) : \text{there is } \delta \in S^* \cap \alpha - a \text{ such that } \delta = \sup(\delta \cap a)\} = \phi \bmod D_\alpha$$

(iv) $2^{<\kappa} \leq \lambda$

Let $\mathcal{D}_1$ be the minimal normal fine filter on

$$\mathcal{P}_{<\kappa}(\lambda^+) \text{ such that } \mathcal{D}_1 \restriction \lambda = \mathcal{D}$$

Then there is $T \subseteq \mathcal{P}_{<\kappa}(\lambda^+)$, such that T is a (κ,λ^+)-stationary coding, $(\mathcal{D}_1 + T) \restriction \lambda = \mathcal{D}$ and $\diamondsuit\ (T, \mathcal{D}_1)$

Proof: Let $\{(b_i, M_i) : i < i(*)\}$ (where $i(*) \in \{\lambda, \lambda^+\}$) list the pairs (b, M), $b \in Y^*$, $M = (\alpha^M, A^M)$, $\alpha^M < \kappa$, $A^M \subseteq \alpha^M$ (by (i) this is possible). Let $S_i \subseteq S^*$ (for $i < i(*)$) be pairwise disjoint stationary subsets of λ^+, $S^* = \bigcup_{i < i(*)} S_i$. For $\delta \in S^*$ let $i(\delta)$ be the unique $i < i(*)$ such that $\delta \in S_i$. Let f, g be two-place functions on λ^+ such that for $i < \lambda^+$ $i = \{f(i,j) : j < |i|\}$ and for $j < |i|$ $g(i, f(i,j)) = j$. Let $C_0 = \{a \in \mathcal{P}_{<\kappa}(\lambda^+) : a \text{ closed under } f \text{ and } g \text{ and } x+1\}$

For $w \subseteq \lambda^+$ countable with $\sup w \in S^*$ let set $[w]$ be the closure of $w \cup b_{i(\sup w)}$ under f and g. For $i < i(*)$, $\delta \in S_i$ let

$$T^i_\delta \stackrel{def}{=} \{a \in \mathcal{P}_{<\kappa}(\lambda^+) : \sup a = \delta,\ a \cap \lambda = b_{i(\delta)}$$

a is a closed under f and g ,

and for any bounded countable

$w \subseteq a$: if $\sup w \in S^*$ (and

set $[w] \cap \lambda = b_{i(\sup w)}$) *then* $cd_{\kappa,\lambda}$ (set $[w]) \in a$ }

$$T^i \stackrel{def}{=} \bigcup_{\delta \in S_i} T^i_\delta$$

For $a \in T^i$ let h_a be the unique order preserving function from a onto the ordinal $otp(a)$ (= the order type of a). Let $A_a = \{j \in a : h_a(j) \in A^{M_i}\}$, so A_a is a subset of a.

$$T \stackrel{def}{=} \bigcup_{i < i(*)} T^i$$

24A. Observation: If $c \subseteq d$, $c \neq d$ both are in T then $cd_{\kappa,\lambda^+}(c) \in d$
As in the previous proof (i.e., see 23A).

Now let M be an algebra with universe λ^+ and countably many functions including f, g and $A \subseteq \lambda^+$, and let $Y \subseteq \mathcal{P}_{<\kappa}(\lambda)$, $Y \neq \phi \mod \mathcal{D}$. We shall find $a \in T$, $a \cap \lambda \in Y$ and a is a subalgebra of M such that $A \cap a = A_a$. This will prove $T \neq \phi \mod \mathcal{D}_1$, $(\mathcal{D}_1 + T) \restriction \lambda = \mathcal{D}$ and $\Diamond\ (T, \mathcal{D}_\kappa(\lambda))$.

We imitate Rubin and Shelah [RSh]: We define a game $\mathcal{G}$ which lasts ω moves. In the n^{th} move player I chooses $a_n \in \mathcal{P}_{<\kappa}(\lambda)$ and then player II chooses an ordinal α_n, which satisfies:

(I) (i) a_n is a subalgebra of M

(ii) $a_n \cap \lambda \in Y$

(iii) $a_n \cap \alpha_{n-1} = a_{n-1}$ when $n > 0$

(iv) there is no $\delta \in (\sup a_n) \cap S^* - a_n$, $\delta = \sup (a_n \cap \delta)$

(II) (i) $\alpha_n > \sup a_n$, $\alpha_n > \lambda$ and when $n > 0$, $\alpha_n > \alpha_{n-1}$

The game is determined being closed. If player I has a winning strategy, a_0 his first move, let $b_0 = a_0$ and simulate a play $\langle a_n, \alpha_n : n < \omega \rangle$ in which player I uses his winning strategy and $\cup \alpha_n \in S_i$. Now $a \stackrel{def}{=} \bigcup_{n<\omega} a_n$ is in T

and is a subalgebra of M. What about $A_a = A \cap a$? For each $\alpha < \kappa$, $B \subseteq \alpha$ we define a game $\mathcal{G}(\alpha,B)$, similar to $\mathcal{G}$, but player I also choose in his n^{th} move an order preserving $h_n : a_n \to \alpha$, $\bigcup_{m<n} h_m \subseteq h_n$ and $(\forall \alpha \in a_n)$ $(\alpha \in A \equiv h_n(\alpha) \in B)$. If for some α, B player I has a winning strategy, we have no problem. If not then (as the games are closed hence determined) player II has a winning strategy $F_{\alpha,B}$ for $\mathcal{G}(\alpha,B)$ for each $\alpha < \kappa$, $B \subseteq \alpha$. Now we define a strategy for player II in G:

$F(a_0, \ldots, a_n) = \bigcup \{F_{\alpha,B}(a_0, h_0, a_n, h_1, \ldots, a_n, h_m)+1$: for $l \leq n$, h_l a function from a_c into α, $\alpha < \kappa$, $B \subseteq \alpha\}$

Clearly this gives a legal move for player II, and in the end we can define $\alpha = otp(\bigcup_{m<\omega} a_n)$, $B = \{otp(\xi \cap \bigcup_m a_n)$: $\xi \in A \cap \bigcup_{n<\omega} a_n\}$, and define $h_m : a_m \to \alpha$ by $h_m(\gamma) = otp(\gamma \cap C_m)$ and get contradiction.

So it is enough to prove that player I wins $\mathcal{G}$, or equivalently that player II has no winning strategy. So suppose F is a winning strategy. Now by assumption (ii) of 24 w.l.o.g. $|Y| = \lambda$ and (by 24 (iv)) $\{a \cap \kappa : a \in Y\}| \leq \lambda$. Now let for $\zeta < \kappa\omega$ M_ζ be an elementary submodel of $H((2^{\lambda^+})^+, \in)$ to which S^*, $\mathcal{D}$, M, F, Y belongs, $\{i{:}i<\lambda\} \subseteq M_\zeta$, $\langle M_\xi{:}\xi \leq \zeta\rangle \in M_{\zeta+1}$, $||M_\zeta|| = \lambda$. Let $\beta_\zeta = \sup(M_\zeta \cap \lambda^+) = \mathrm{Min}(\lambda^+ - |M_\zeta|)$, and let $\beta = \bigcup_\zeta \beta_\zeta$. So M_ζ is increasing. Choose $a \subseteq (\bigcup_{\zeta<\kappa\omega} M_\zeta) \cap \lambda^+$, $a \cap \lambda \in Y$ and $a \cap \{\beta_{\kappa m+\xi}{:}\ \xi < \kappa\} = \{\beta_{\kappa m+\xi}{:}\ \xi \in a \cap \kappa\}$, a is closed under f, g, and there is no $\delta \in S^* \cap \beta - a$, $\delta = \sup(a \cap \delta)$. (This demand "$a \cap \lambda \in Y$" restrict ourselves to a positive set $mod\ \mathcal{D}_\beta$, the rest to a member of $\mathcal{D}_\beta$ (the last demand by (iii) of 24) so there is such a.)

As $a \cap \lambda \in Y$, clearly for each ζ $a \cap \lambda \in M_\zeta$, and as $a \cap \{\beta_{\kappa m+\xi}{:}\ \xi < \kappa\} = \{\beta_{\kappa m+\xi}{:}\ \xi < \kappa, \xi \in a\}$, and $a \cap \kappa \in M_\zeta$, (by the restriction on Y) and $f, g \in M$, and $\langle M_\zeta{:}\ \zeta < \kappa m + (\sup \kappa \cap a)\rangle \in M_{\kappa m+1}$ (as for $\sup(\kappa \cap a) < a$) clearly we get $a \cap M_{\kappa(m+1)} \in M_{\kappa(m+1)}$. Now we can simulate a play of the game in which player II uses his winning strategy F, whereas player I choose $a_n = a \cap M_{\kappa(n+1)}$. By what we say above $F(a_0, \ldots, a_n) \in M_{\kappa(n+1)}$ hence $F(a_0, \ldots, a_n) < \beta_{\kappa(n+1)}$, so actually player I wins the play, contradiction.

25. Conclusion:

Suppose κ is regular $> \aleph_0$, $\lambda = \lambda^{<\kappa}$, and $S^* \subseteq \{\delta < \lambda^+ : cf\ \delta = \aleph_0\}$ is stationary, but for no $\delta < \lambda^+$ of cofinality κ is $S^* \cap \delta$ stationary in δ. *Then*, there is a (κ, λ^+)-stationary coding $T \subseteq T_{\kappa, \lambda^+}(N^2_\omega)$ and even $\diamondsuit\ (T, \mathcal{D}_\kappa(\lambda))$ holds.

26. Remark:

1) When does such a S^* exist? It follows from the existence of square on $\{\delta < \lambda^+ : cf\ \delta < \kappa\}$, which $\neg 0^\#$ implies holds when $\kappa < \lambda$ (and even for many $\kappa = \lambda$'s (see Magidor's work).

2) We can weaken the non-reflection as in 7 of [Sh1].

27. Claim:

In 24 if we do not require $\diamondsuit\ (T, \mathcal{D}_1)$ then we can omit (i) and (iv).
We can deduce from the proof of 24 also:

28. Lemma:

1) $\diamondsuit\ (\mathcal{D}_{<\aleph_1}(\lambda^+))$ when $\lambda = \lambda^{\aleph_0}$

2) If $\mathcal{D}$ is normal fine filter on $\mathcal{P}_{<\aleph_1}(\lambda^+)$, $\mathcal{D}_1$ is the minimal normal fine filter on $\mathcal{P}_{<\aleph_1}(\lambda^+)$ such that $\mathcal{D}_1 \restriction \lambda = \mathcal{D}$ and $\lambda = \lambda^{\aleph_0}$ *then* $\diamondsuit\ (\mathcal{D}_1)$

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