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Some notes concerning the homogeneity of Boolean algebras and Boolean spaces ☆

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Abstract

In this article we consider homogeneity properties of Boolean algebras that have nonprincipal ultrafilters which are countably generated.

It is shown that a Boolean algebra B is homogeneous if it is the union of countably generated nonprincipal ultrafilters and has a dense subset D such that for every $a \in D$ the relative algebra $B \restriction a := \{b \in B\colon b \leqslant a\}$ is isomorphic to B. In particular, the free product of countably many copies of an atomic Boolean algebra is homogeneous.

Moreover, a Boolean algebra B is homogeneous if it satisfies the following conditions:

(i) B has a countably generated ultrafilter,
(ii) B is not c.c.c., and
(iii) for every $a \in B \setminus \{0\}$ there are finitely many automorphisms $h_1, \dots, h_n$ of B such that $1 = h_1(a) \cup \cdots \cup h_n(a)$.

These results generalize theorems due to Motorov [Russian Math. Surveys 44 (16) (1989) 190–191] on the homogeneity of first countable Boolean spaces.

Finally, we provide three constructions of first countable homogeneous Boolean spaces that are linearly ordered. The first construction gives separable spaces of any prescribed weight in the interval $[\aleph_0, 2^{\aleph_0}]$. The second construction gives spaces of any prescribed weight in the interval $[\aleph_1, 2^{\aleph_0}]$

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that are not c.c.c. The third construction gives a space of weight $\aleph_1$ which is not c.c.c. and which is not a continuous image of any of the previously described examples.

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1. Introduction

A topological space X is *homogeneous* if for any two points $x, y \in X$ there is an autohomeomorphism of X mapping x to y. Among the most obvious examples of homogeneous spaces are topological groups. In the case of topological groups, translations can be used to show their homogeneity.

If we restrict our attention to zero-dimensional compact spaces, i.e., to Boolean spaces, topological groups are not interesting from the topological point of view since infinite compact zero-dimensional groups are all Cantor spaces, that is, they are homeomorphic to spaces of the form 2^κ where κ is a cardinal (see [4] or [5]).

Not too long ago, Dow and Pearl [2] proved a striking theorem concerning homogeneity, namely that for every first countable, zero-dimensional space X, X^ω is homogeneous. For Boolean spaces with a dense set of isolated points this was shown earlier by Motorov [12].

Apart from infinite powers, there is a surprising shortage of examples of homogeneous Boolean spaces. Interesting examples were provided by Maurice [11], who proved that for every indecomposable countable ordinal γ the lexicographically ordered space 2^γ is homogeneous. Here an ordinal γ is *indecomposable* if $\gamma = \alpha + \beta$ with $\beta > 0$ implies $\beta = \gamma$. If $\gamma > \omega$, then 2^γ ordered lexicographically does not satisfy the countable chain condition (c.c.c.) and therefore is not homeomorphic to a Cantor space.

Cantor spaces and the lexicographically ordered spaces 2^γ, γ countable and indecomposable, have the property that not only the spaces themselves, but also their dual Boolean algebras are homogeneous. A Boolean algebra B is *homogeneous* if for every $a \in B \setminus \{0\}$ the relative algebra $B \restriction a := \{b \in B\colon\ b \leqslant a\}$ is isomorphic to B. In general, there is no direct implication between the homogeneity of a Boolean algebra and the homogeneity of its Stone space. Van Douwen [13] constructed a first countable homogeneous Boolean space whose dual Boolean algebra is not homogeneous. And it is well known that the Boolean algebra $\mathcal{P}(\omega)/\mathit{fin}$ is homogeneous but its Stone space $\beta\omega \setminus \omega$ is not.

However, the homogeneity of first countable Boolean spaces follows from the homogeneity of their dual Boolean algebra. This was noticed independently by Motorov [12] and Koppelberg [7]. Motorov proved that the converse is also true in certain cases. He showed (in topological terms) that the homogeneity of a Boolean algebra follows from the homogeneity of its Stone space if the Boolean algebra is not c.c.c. and every ultrafilter is countably generated. Note that the last condition is equivalent to the first countability of the Stone space.

The main tool in Motorov's argument is

Theorem 1.1. *Let B be a Boolean algebra such that every ultrafilter of B is countably generated and B has a dense subset D such that for all $a \in D$, the algebra $B \restriction a$ is isomorphic to B. Then B is homogeneous.*

Unfortunately, published proofs of Motorov's results seem to be unavailable. We give the proofs of some generalizations of his theorems. The main observation is that in Theorem 1.1 the assumption "every ultrafilter of B is countably generated" can be weakened to "B is the union of countably generated ultrafilters" (which is equivalent to the Stone space of B having a dense set of points of countable character). This easily implies that the free product of infinitely many copies of an atomic Boolean algebra is homogeneous. Here a Boolean algebra B is *atomic* if the atoms are dense in B, i.e., if the Stone space of B has a dense set of isolated points.

We also show that a Boolean algebra B which is not c.c.c. is homogeneous if it has at least one countably generated ultrafilter and the property that for all $a \in B \setminus \{0\}$ there are finitely many automorphisms $h_1, \ldots, h_n$ of B such that $1 = h_1(a) \cup \cdots \cup h_n(a)$. The latter property is equivalent to the property that every point of the Stone space X of B has a dense orbit with respect to the natural group action of the group $\mathrm{Aut}(X)$ of autohomeomorphisms of X.

Moreover, we provide three constructions leading to new examples of homogeneous Boolean spaces. In all cases we obtain first countable spaces which are linearly ordered. The first construction yields separable spaces of any prescribed weight in the interval $[\aleph_0, 2^{\aleph_0}]$. These spaces are constructed from nice suborders of $\mathbb{R}$. Note that the space of countable weight is homeomorphic to 2^ω since up to homeomorphism 2^ω is the only Boolean space of countable weight without isolated points.

The second construction uses an easy Löwenheim-Skolem argument and gives homogeneous continuous images of the lexicographically ordered spaces 2^γ, γ indecomposable with $\omega < \gamma < \omega_1$. The spaces obtained using this construction can have any prescribed weight in the interval $[\aleph_0, 2^{\aleph_0}]$, and their cellularity equals their weight. The third construction uses a linear order on an Aronszajn tree and yields a space of weight $\aleph_1$ which is not c.c.c. and not a continuous image of any of the lexicographically ordered spaces 2^γ, $\gamma < \omega_1$.

It should be pointed out that compact homogeneous spaces which are linearly ordered have to be first countable (see [1] or [10]).

2. Generalizing Motorov's results

As usual, the Stone space of a Boolean algebra B is denoted by $\mathrm{Ult}(B)$ and the Boolean algebra of clopen subsets of a Boolean space X is denoted by $\mathrm{Clop}(X)$. In the following, we will frequently switch between Boolean algebras and their Stone spaces, but our presentation will be mainly in topological terms.

Note that a Boolean algebra B is homogeneous if and only if every nonempty clopen subset of $\mathrm{Ult}(B)$ is homeomorphic to $\mathrm{Ult}(B)$.

Lemma 2.1. *Let X be a Boolean space such that* $\mathrm{Clop}(X)$ *is homogeneous. If $x, y \in X$ are points of countable character, then there is an autohomeomorphism of X mapping x to y. In particular, X is homogeneous if it is first countable.*

Proof. Assuming that X is infinite, it follows from the homogeneity of $\mathrm{Clop}(X)$ that X has no isolated points. Let $(A_n)_{n\in\omega}$ and $(B_n)_{n\in\omega}$ be clopen neighborhood bases of x and y, respectively. Since x and y are not isolated, we may assume that the sequences $(A_n)_{n\in\omega}$ and $(B_n)_{n\in\omega}$ are strictly decreasing. We may also assume $A_0 = B_0 = X$. For each $n \in \omega$ let $C_n := A_n \setminus A_{n+1}$ and $D_n := B_n \setminus B_{n+1}$ and fix an homeomorphism $h_n : C_n \to D_n$. It is easily checked that $h := \{(x, y)\} \cup \bigcup_{n\in\omega} h_n$ is an autohomeomorphism of X mapping x to y. □

In order to apply Lemma 2.1 we need a criterion for the homogeneity of Boolean algebras with first countable Stone spaces. A *π-base* of a topological space X is a family $\mathcal{F}$ of open subsets of X such that every nonempty open subset of X includes a member of $\mathcal{F}$. A family of clopen subsets of a Boolean space X is a π-base if and only if it is a dense subset of $\mathrm{Clop}(X)$.

Lemma 2.2. *Let X be a Boolean space with a dense set of points of countable character. Then* $\mathrm{Clop}(X)$ *is homogeneous if X has a π-base consisting of clopen sets which are homeomorphic to X.*

Proof. First note that if X has a π-base of clopen copies of X, then X is either a singleton or has no isolated points (tacitly assuming that X is nonempty). We may therefore assume that X has no isolated points.

We show that the nonempty clopen subsets of X are pairwise homeomorphic. Let A be a nonempty clopen subset of X. Let $x \in A$ be a point of countable character. As in the proof of Lemma 2.1, there is a disjoint family $(A_n)_{n\in\omega}$ of nonempty clopen subsets of A such that $A = \{x\} \cup \bigcup_{n\in\omega} A_n$.

Inductively we define sequences $(C_n)_{n\in\{-1\}\cup\omega}$ and $(B_n)_{n\in\omega}$ as follows: Let $C_{-1} := \emptyset$. Let $n \in \omega$ and suppose we have already defined C_{n-1}. Since the clopen subsets of X which are homeomorphic to X form a π-base of X, there is a clopen set $B_n \subseteq A_n$ such that B_n is homeomorphic to $X \setminus C_{n-1}$. With this choice, $C_{n-1} \cup B_n \cong X$. Let $C_n := A_n \setminus B_n$.

Now

$$A \setminus \{x\} = \bigcup_{n\in\omega} (B_n \cup C_n) = B_0 \cup \bigcup_{n\in\omega} (C_n \cup B_{n+1}).$$

By the choice of the B_n, $n \in \omega$, $\bigcup_{n\in\omega}(C_{n-1} \cup B_n)$ is homeomorphic to the disjoint union of $\aleph_0$ copies of X. It follows that A is the one-point compactification of the disjoint union of $\aleph_0$ copies of X. Since A was arbitrary, it follows that the nonempty clopen subsets of X are pairwise homeomorphic. □

Using Lemmas 2.2 and 2.1, we can give an easy proof of the homogeneity of the lexicographically ordered spaces 2^γ, γ countable and indecomposable. For every $\alpha < \gamma$

and every $x \in 2^{\alpha+1}$ the set $I_x := \{y \in 2^\gamma\colon\ x \subseteq y\}$ is a clopen interval in 2^γ. By the indecomposability of γ, each I_x is homeomorphic to 2^γ. Clearly,

$$\left\{I_x\colon\ \alpha < \gamma \wedge x \in 2^{\alpha+1}\right\}$$

is a π-base of 2^γ. Thus, $\mathrm{Clop}(2^\gamma)$ is homogeneous by Lemma 2.2. Now the homogeneity of 2^γ follows from Lemma 2.1.

Another corollary of Lemma 2.2 gives information about free products of atomic Boolean algebras.

Corollary 2.3. *Let X be a Boolean space with a dense set of isolated points. Then* $\mathrm{Clop}(X^\omega)$ *is homogeneous.*

Proof. Let D be the set of subsets of X^ω of the form $\{(x_0,\dots,x_{n-1})\} \times X^{\omega\setminus n}$ where each $x_i \in X$ is isolated. Clearly, D consists of clopen sets that are homeomorphic to X^ω. Since the isolated points are dense in X, D is a π-base of X^ω. Those sequences $(x_i)_{i\in\omega} \in X^\omega$ for which each x_i is isolated in X form a dense subset of X^ω, and each of these sequences is of countable character in X^ω. Now it follows from Lemma 2.2 that $\mathrm{Clop}(X^\omega)$ is homogeneous. □

Note that for every cardinal κ, $\mathrm{Clop}(X^\kappa)$ is isomorphic to the free product of κ copies of $\mathrm{Clop}(X)$. It is easily checked that $\mathrm{Clop}(X^\kappa)$ is homogeneous if there is a cardinal $\lambda \leqslant \kappa$ such that $\mathrm{Clop}(X^\lambda)$ is homogeneous. Therefore Corollary 2.3 implies that for a Boolean space X with a dense set of isolated points, for every infinite cardinal κ the Boolean algebra $\mathrm{Clop}(X^\kappa)$ is homogeneous. In other words, if B is an atomic Boolean algebra, then every free product of infinitely many copies of B is homogeneous.

To proceed we need a technical lemma relating the cellularity of a compact space with many autohomeomorphisms to the cellularities of its nonempty open subsets. For a topological space X let $c(X)$ denote the cellularity of X. Recall that $\mathrm{Aut}(X)$ is the group of autohomeomorphisms of X. For $x \in X$ the $\mathrm{Aut}(X)$-*orbit* of x is the set $\{h(x)\colon\ h \in \mathrm{Aut}(X)\}$.

Lemma 2.4. *Let X be compact and infinite. If every $x \in X$ has a dense* $\mathrm{Aut}(X)$*-orbit, then for every nonempty open subset O of X we have $c(O) = c(X)$.*

Proof. It is easily checked that all $\mathrm{Aut}(X)$-orbits are dense in X if and only if for every nonempty open set $O \subseteq X$, $\{h[O]\colon\ h \in \mathrm{Aut}(X)\}$ covers X. Let $O \subseteq X$ be open and nonempty. By the compactness of X, there are $n \in \omega$ and $h_1,\dots,h_n \in \mathrm{Aut}(X)$ such that $X = h_1[O] \cup \cdots \cup h_n[O]$.

Let $\mathcal{A}$ be an infinite family of pairwise disjoint subsets of X. For some $i \in \{1,\dots,n\}$, the set $\{A \in \mathcal{A} : A \cap h_i[O] \neq \emptyset\}$ is of size $|\mathcal{A}|$. It follows that $c(O) \geqslant |\mathcal{A}|$. This implies $c(O) = c(X)$. □

Now we have collected the necessary tools to show

Theorem 2.5. *Let X be a Boolean space which is not c.c.c. and has a point of countable character. Suppose every $x \in X$ has a dense* $\mathrm{Aut}(X)$*-orbit. Then* $\mathrm{Clop}(X)$ *is homogeneous.*

Proof. Since X is not c.c.c., X is infinite. Since every $\mathrm{Aut}(X)$-orbit is dense in X, X has no isolated points. Let $x_0 \in X$ be a point of countable character. Since the $\mathrm{Aut}(X)$-orbit of x_0 is dense in X, X has a dense set of points of countable character. By Lemma 2.2, it remains to show that X has a π-base consisting of clopen sets which are homeomorphic to X.

Let $(U_n)_{n\in\omega}$ be a neighborhood base of x_0 consisting of clopen sets. For every $n \in \omega$ there are $m \in \omega$ and $h_1, \ldots, h_m \in \mathrm{Aut}(X)$ such that $X = h_1[U_n] \cup \cdots \cup h_m[U_n]$. It follows that for each $n \in \omega$, X is homeomorphic to a disjoint union of finitely many copies of clopen subsets of U_n.

Now let O be a nonempty open subset of X. By Lemma 2.4, there is an uncountable family $\mathcal{A}$ of pairwise disjoint nonempty open subsets of O. For every $A \in \mathcal{A}$ let $h_A \in \mathrm{Aut}(X)$ be such that $h_A(x_0) \in A$. h_A exists since the $\mathrm{Aut}(X)$-orbit of x_0 is dense. For every $A \in \mathcal{A}$ there is $n(A) \in \omega$ such that $h[U_{n(A)}] \subseteq A$. Since $\mathcal{A}$ is uncountable, there is $n_0 \in \omega$ such that for uncountably many $A \in \mathcal{A}$, $n(A) = n_0$. It follows that O includes uncountably many pairwise disjoint open copies of U_{n_0}. But since X is homeomorphic to a disjoint union of finitely many copies of clopen subsets of U_{n_0}, O includes a clopen copy of X. □

Corollary 2.6. *Let X be a first countable Boolean space of uncountable cellularity. If every point in X has a dense* $\mathrm{Aut}(X)$*-orbit, then* $\mathrm{Clop}(X)$ *and X are both homogeneous. In particular, X is homogeneous if and only if* $\mathrm{Clop}(X)$ *is.*

Proof. The homogeneity of $\mathrm{Clop}(X)$ follows immediately from Theorem 2.5. The homogeneity of X now follows from Lemma 2.1. □

3. Examples of homogeneous Boolean spaces

The homogeneous Boolean spaces we are going to construct will be Stone spaces of interval algebras of certain linear orders. As usual, if $(L, \leqslant)$ is a linear order, we use $<$ to denote $\leqslant \setminus =$. Similarly, if $<$ is transitive and irreflexive, we use $\leqslant$ to denote $< \cup =$.

Definition 3.1. Let $(L, \leqslant)$ be a linear order. The *interval algebra* $B(L)$ of L is the subalgebra of $\mathcal{P}(L)$ generated by the intervals $[x, y)$, $x, y \in L \cup \{\infty\}$, $x < y$.

Every element of $B(L)$ is a finite union of intervals of the form $[x, y)$, $x, y \in L \cup \{\infty\}$, $x < y$, and of the form $(-\infty, x)$, $x \in L \cup \{\infty\}$ (see [6]).

The Stone space of an interval algebra $B(L)$ is homeomorphic to the linear order of initial segments of L (see [6]). Using this fact, we can characterize those linear orders whose interval algebras have a first countable Stone space.

Call a subset S of a linear order L *coinitial* if for all $a \in L$ there is $b \in S$ such that $b \leqslant a$. The *coinitiality* of L is the least size of a coinitial subset of L, which is the same as the cofinality of the reversed order.

Lemma 3.2. *The Stone space of an interval algebra $B(L)$ is first countable if and only if every initial segment of L has a countable cofinality and every final segment has a countable coinitiality.*

Proof. Let X be the set of initial segments of the linear order L. X itself is linearly ordered by $\subseteq$. Suppose X is first countable. We show that every initial segment of L is of countable cofinality. The proof that every final segment of L is of countable coinitiality is symmetric.

By the first countability of X, for every $x \in X$, the set $\{y \in X\colon\ y \subsetneq x\}$ is of countable cofinality. Let $x \in X$ be nonempty and assume that x has no last element. Then the set $\{(-\infty, a]\colon a \in x\}$ is cofinal in $\{y \in X\colon\ y \subsetneq x\}$. Therefore, $\{(-\infty, a]\colon a \in x\}$ is of countable cofinality. This implies that x is of countable cofinality.

Now suppose that every initial segment of L is of countable cofinality and that every final segment of L is of countable coinitiality. To show the first countability of X, it suffices to prove that for all $x \in X$ the following two conditions hold:

(1) If in X there is no largest element below x, then x is the first element of X or there is a countable sequence in X converging to x from the left.
(2) If in X there is no smallest element above x, then x is the last element of X or there is a countable sequence in X converging to x from the right.

We show only the first condition since the proof of the second condition is symmetric. Suppose that there is no largest element in X which is below $x \in X$. Assume further that x is not the first element of X. Then x as a subset of L is nonempty and does not have a last element. By our assumption on L, there is a sequence $(a_n)_{n\in\omega}$ which is cofinal in x. We may assume that $(a_n)_{n\in\omega}$ is strictly increasing. The sequence $((-\infty, a_n])_{n\in\omega}$ of initial segments of L converges to x from the left. □

Lemma 3.2 easily implies:

Corollary 3.3. *If the linear order L has no uncountable sequences* (*indexed by ordinals*) *which are strictly increasing or strictly decreasing, then the Stone space of $B(L)$ is first countable. In particular, the Stone space of $B(L)$ is first countable if L has a countable dense subset.*

Proof. If L has an initial segment of uncountable cofinality, then it has a strictly increasing sequence of length ω_1. Similarly, if L has a final segment of uncountable coinitiality, then it has a strictly decreasing sequence of length ω_1.

If L has a strictly increasing or strictly decreasing sequence of length ω_1, then it is not c.c.c. and therefore cannot have a countable dense subset. □

The following lemma provides an easy criterion for the homogeneity of an interval algebra.

Lemma 3.4. *Let L be a dense linear order with the property that every two nonempty open intervals of L are isomorphic. Then $B(L)$ is homogeneous.*

Proof. By the density of L, for all $x, y \in L$ with $x < y$ the interval $[x, y)$ is infinite. Since every two nonempty open intervals of L are isomorphic, L has no end points. Moreover, the intervals of the form $[x, y)$ with $x \in L$, $y \in L \cup \{\infty\}$, $x < y$, are pairwise isomorphic. It follows that for all $n \in \omega$ and all $x_0, \ldots, x_{2n+1} \in L \cup \{-\infty, \infty\}$ with $x_0 < \cdots < x_{2n+1}$, $(x_0, x_1) \cup \bigcup_{i=1}^{n} [x_{2i}, x_{2i+1})$ is isomorphic to L.

Let $a \in B(L) \setminus \{\emptyset\}$. Then for some $n \in \omega$ there are $x_0, \ldots, x_{2n+1} \in L \cup \{-\infty, \infty\}$ with $x_0 < \cdots < x_{2n+1}$ such that either $x_0 = -\infty$ and $a = (x_0, x_1) \cup \bigcup_{i=1}^{n} [x_{2i}, x_{2i+1})$ or $x_0 \in L$ and $a = [x_0, x_1) \cup \cdots \cup [x_{2n}, x_{2n+1})$. In either case, it is easily checked that $B(L) \restriction x$ is isomorphic to the interval algebra of $(x_0, x_1) \cup \bigcup_{i=1}^{n} [x_{2i}, x_{2i+1})$. Since $(x_0, x_1) \cup \bigcup_{i=1}^{n} [x_{2i}, x_{2i+1})$ is isomorphic to L, $B(L) \restriction a$ is isomorphic to $B(L)$. □

Combining the information we have gathered so far we obtain

Lemma 3.5. *Let L be a dense linear order such that every nonempty open interval of L isomorphic to L itself. Then* $\mathrm{Ult}(B(L))$ *is homogeneous if and only if L has no strictly increasing or strictly decreasing sequences of length ω_1. In particular,* $\mathrm{Ult}(B(L))$ *is homogeneous if L is separable.*

Proof. By Lemma 3.4, $B(L)$ is homogeneous. If L has no strictly increasing or strictly decreasing sequences of uncountable length, then $\mathrm{Ult}(B(L))$ is first countable by Corollary 3.3 and homogeneous by Lemma 2.1. Now suppose that L has a strictly increasing or strictly decreasing sequence of uncountable length. Then the Stone space of $B(L)$, being homeomorphic to the linear order of initial segments of L, is not first countable and therefore cannot be homogeneous, as mentioned in the introduction. □

It remains to construct linear orders with the properties required in Lemma 3.5. We first construct some separable linear orders.

Lemma 3.6. *For every cardinal $\kappa \in [\aleph_0, 2^{\aleph_0}]$ there is a separable dense linear order L of size κ which is isomorphic to every one of its nonempty open intervals.*

Proof. Let L be a subfield of $\mathbb{R}$ of size κ. Then L is separable since it contains the rationals. L is dense since for all $x, y \in L$ we have $\frac{1}{2}(x + y) \in L$. Since L is a subfield of $\mathbb{R}$, for all $x \in L$ it is closed under the order isomorphisms

$$f_x : (x, \infty) \to \mathbb{R}; \qquad y \mapsto (x - y)^{-1} + y - x$$

and

$$g_x : (-\infty, x) \to \mathbb{R}; \qquad y \mapsto (x - y)^{-1} + y - x$$

and their inverses. This implies that for all $x \in L$, L is order-isomorphic to $(-\infty, x) \cap L$ and to $(x, \infty) \cap L$. It follows that L is order-isomorphic to every one of its nonempty open intervals. □

Corollary 3.7. *Let $\kappa \in [\aleph_0, 2^{\aleph_0}]$ be a cardinal. Then there is a homogeneous Boolean space of weight κ which is separable and first countable.*

Proof. Let L be a linear order of size κ as in Lemma 3.6. By Corollary 3.3, the Stone space of $B(L)$ is first countable. Let $D \subseteq L$ be countable and dense. Then for all $a \in D$, the set of those $x \in B(L)$ which contain a is an ultrafilter F_a. Since D is dense in L and $B(L)$ does not contain any singletons by the density of L, $B(L)$ is the union of the F_a, $a \in D$, i.e., $B(L)$ is σ-centered. This is equivalent to the separability of $\mathrm{Ult}(B(L))$. Finally, $\mathrm{Ult}(B(L))$ is homogeneous by Lemma 3.5. □

As the referee pointed out, the spaces in Corollary 3.7 can be obtained as variants of the double arrow space and have already been mentioned in Section 2.5 of [3]. The description in [3] is as follows: In order to obtain a space of weight $2^{\aleph_0}$ consider the double arrow space, i.e., $[0,1] \times 2$ ordered lexicographically. After removing the endpoints, we are left with a homogeneous Boolean space X of weight $2^{\aleph_0}$ which is separable and first countable. X is homeomorphic to the Stone space of $B(\mathbb{R})$. Spaces of smaller weight are obtained by replacing $[0,1]$ in the construction above by $[0,1] \cap K$ where K is a subfield of $\mathbb{R}$ of suitable size (namely the desired weight of the space).

Note that all these spaces are continuous images of X. So, in the end, a Löwenheim-Skolem argument is used (when choosing subfields of $\mathbb{R}$) to obtain homogeneous zero-dimensional quotients of X of a prescribed weight.

We carry out this type of argument once more in a more explicit way in order to construct first countable homogeneous Boolean spaces of a given weight in the interval $[\aleph_1, 2^{\aleph_0}]$ that are not c.c.c. Starting with an indecomposable countable ordinal $\gamma > \omega$ we apply the downward Löwenheim-Skolem theorem to get a continuous image of the lexicographically ordered space 2^γ with the right properties.

Theorem 3.8. *For every cardinal $\kappa \in [\aleph_0, 2^{\aleph_0}]$ there is a first countable homogeneous Boolean space X of weight and cellularity κ.*

Proof. Let $\gamma > \omega$ be a countable indecomposable ordinal. Then the lexicographically ordered space 2^γ is a first countable homogeneous Boolean space. Since 2^γ is linearly ordered, it is the Stone space of an interval algebra (see [6]). Let L be a linear order with $2^\gamma \cong \mathrm{Ult}(B(L))$.

Since $\mathrm{Ult}(B(L))$ is first countable, in L there is no strictly increasing or strictly decreasing sequence of length ω_1. The cellularity of 2^γ is $2^{\aleph_0}$, as is the weight. By Corollary 2.6, $B(L)$ is a homogeneous Boolean algebra.

Let λ be a sufficiently large cardinal and consider the structure $(H_\lambda, \in)$ where H_λ is the family of sets whose transitive closure is of size $< \lambda$. Fix an antichain $\mathcal{A} \subseteq B(L)$ of size $2^{\aleph_0}$ and let M be an elementary submodel of $(H_\lambda, \in)$ of size κ such that $L, \kappa, \mathcal{A} \in M$ and $\kappa \subseteq M$.

Let $B := B(L) \cap M$. By elementarity, $B = B(L \cap M)$. $L \cap M$ is a linear order without strictly increasing or strictly decreasing sequences of length ω_1. Thus, $X := \mathrm{Ult}(B)$ is first countable. Again by elementarity, B is homogeneous. By Lemma 2.1, X is homogeneous, too.

Since $\kappa \subseteq M$, B is of size κ, and so is $\mathcal{A} \cap M$. It follows that the cellularity of X is κ. □

There is another interesting example of a first countable homogeneous Boolean space. This one is constructed from a linear order on an Aronszajn tree and is not the continuous image of any of the first countable homogeneous Boolean linear orders mentioned so far.

Recall that a tree is *Aronszajn* if it is of height ω_1, has only countable levels, and does not include an uncountable chain. If T is a tree ordered by $\supseteq$, we will always assume that incomparable elements of T are disjoint.

Lemma 3.9. *There is a dense linear order L of size $\aleph_1$ without endpoints and with the following properties*:

(i) *L has no strictly increasing or strictly decreasing sequences of length ω_1.*
(ii) *L is isomorphic to every one of its nonempty open intervals.*
(iii) *L is not c.c.c.*
(iv) *$B(L)$ has a subset which is an Aronszajn tree* (*ordered by $\supseteq$*).

Proof. For two functions f and g with the same domain we write $f =^* g$ if f and g agree on all but finitely many points of their common domain. Using the construction of an Aronszajn tree given in [9], we obtain a sequence $(f_\alpha)_{\alpha<\omega_1}$ such that each f_α is a 1–1 function from α into $\mathbb{Q}\cap(0,1)$ and for all $\alpha,\beta<\omega_1$ with $\alpha<\beta$, $f_\alpha =^* f_\beta\restriction\alpha$.

Now for each $\alpha<\omega_1$ let

$$S_\alpha := \{f\in{}^\alpha\mathbb{Q}\colon\ f \text{ is } 1{-}1 \text{ and } f =^* f_\alpha\}$$

and $T_\alpha := \bigcup_{\beta<\alpha} S_\beta$. $T := \bigcup_{\alpha<\omega_1} T_\alpha$ ordered by inclusion is an Aronszajn tree.

We define a linear order on T. Let $x,y\in T$ be such that $x\neq y$. If x and y are incomparable with respect to $\subseteq$, put $\Delta(x,y) := \min\{\nu\in\operatorname{dom}(x)\colon\ x(\nu)\neq y(\nu)\}$ and let $x<y$ if $x(\Delta(x,y))<y(\Delta(x,y))$. If $x\subseteq y$ and $\operatorname{dom}(x)=\alpha$, let $x<y$ if $y(\alpha)>\pi$ and $y<x$ if $y(\alpha)<\pi$.

In other words, T is ordered lexicographically after identifying each $x\in T$ with the function $x^\frown\pi$ where $^\frown$ denotes the concatenation of sequences and π is identified with the sequence of length one with value $\pi\in\mathbb{R}$.

Claim 3.10. *$(T,\leqslant)$ is not c.c.c.*

It is well known that T ordered by reverse inclusion is not c.c.c. For example, for every $\alpha<\omega_1$ let $x\in S_{\alpha+1}$ be such that $x(\alpha)=0$. Since the x_α, $\alpha<\omega_1$, are 1–1, $(x_\alpha)_{\alpha<\omega_1}$ is an antichain in $(T,\supseteq)$. Note that for all $x\in T$, $\mathbb{Q}\setminus\operatorname{ran}(x)$ is infinite since otherwise the construction of the f_α, $\alpha<\omega_1$, would break down at some point. Thus, for every $\alpha<\omega_1$ there are $p_\alpha,q_\alpha\in\mathbb{Q}\setminus\operatorname{ran}(x_\alpha)$ with $p_\alpha<q_\alpha$. Now for each $\alpha<\omega_1$, $x_\alpha{}^\frown q_\alpha$ and $x_\alpha{}^\frown p_\alpha$ are elements of T. Clearly, $\{(x_\alpha{}^\frown p_\alpha, x_\alpha{}^\frown q_\alpha)\colon\ \alpha<\omega_1\}$ is an uncountable family of pairwise disjoint nonempty open intervals of $(T,\leqslant)$.

Claim 3.11. *In $(T,\leqslant)$ there is no strictly increasing or strictly decreasing sequence of length ω_1.*

This claim seems to be folklore. A more general statement about so-called *squashings* of Aronszajn trees is proved in [8, Theorem 4.10]. $(T,\leqslant)$ is a squashing of $(T,\subseteq)$.

In order to prove that $(T, \leqslant)$ is isomorphic to each of its nonempty open intervals it suffices to show

Claim 3.12. *For every $x \in T$, $(T, \leqslant)$ is isomorphic to (x, ∞) and to $(-\infty, x)$.*

We only show $T \cong (x, \infty)$ since the proof of $T \cong (-\infty, x)$ is symmetric. Let $\delta := \mathrm{dom}(x) + 1$.

For each $y \in S_\delta$ let $\mathrm{ext}(y) := \{z \in T\colon\ \delta \subseteq \mathrm{dom}(z) \wedge z \restriction \delta = y\}$. As before, $\mathrm{ext}(y)$ is a convex subset of T. For $y, z \in S_\delta$, $(\mathrm{ext}(y), \leqslant\restriction \mathrm{ext}(y))$ and $(\mathrm{ext}(z), \leqslant\restriction \mathrm{ext}(z))$ are isomorphic by the isomorphism mapping every $y' \in \mathrm{ext}(y)$ to $z \cup y' \restriction (\mathrm{dom}(y') \setminus \delta)$.

As suborders of $(T, \leqslant)$, $T_{\delta+1}$ and $T_{\delta+1} \cap (x, \infty)$ both are countable dense linear orders without endpoints. S_δ is a dense and co-dense subset of $T_{\delta+1}$ and $S_\delta \cap (x, \infty)$ is a dense and co-dense subset of $T_{\delta+1} \cap (x, \infty)$. By the usual back-and-forth argument, there is an isomorphism φ between $T_{\delta+1}$ and $T_{\delta+1} \cap (x, \infty)$ mapping S_δ onto $S_\delta \cap (x, \infty)$.

For every $y \in S_\delta$ let φ_y be an isomorphism between $\mathrm{ext}(y)$ and $\mathrm{ext}(\varphi(y))$. Now we can construct an isomorphism ψ between T and (x, ∞) by letting $\psi(y) := \varphi(y)$ for every $y \in T_\delta$ and $\psi(y) := \varphi_{y\restriction\delta}(y)$ for every $y \in T \setminus T_\delta$.

Finally, we have to find an Aronszajn tree inside $B(T, \leqslant)$. Let T' be the subtree of T consisting of those elements of T whose ranges are subsets of $(0, 1)$. As $(T, \subseteq)$, $(T', \subseteq)$ is an Aronszajn tree. This is the place where we take advantage of the fact that the ranges of the f_α, $\alpha < \omega_1$, are subsets of $(0, 1)$. Mapping every $x \in T'$ to the interval $[x^\frown 0, x^\frown 1)$ of $(T, \leqslant)$, we obtain an embedding of $(T', \subseteq)$ into $(B(T, \leqslant), \supseteq)$. Note that this embedding maps incomparable elements of T' to disjoint members of $B(T, \leqslant)$. □

Theorem 3.13. *There is a homogeneous Boolean space of weight $\aleph_1$ which is first countable, not c.c.c., and not a continuous image of any of the lexicographically ordered spaces 2^γ, $\gamma < \omega_1$.*

Proof. Let L be a linear order as in Lemma 3.9. Then $\mathrm{Ult}(B(L))$ is first countable by Corollary 3.3. $\mathrm{Ult}(B(L))$ is homogeneous by Lemma 3.5 and $B(L)$ is of size $\aleph_1$ since L is. $B(L)$ is not c.c.c. since L is not c.c.c.

Now suppose that $\mathrm{Ult}(B(L))$ is a continuous image of the lexicographically ordered space 2^γ for some $\gamma < \omega_1$. Then $B(L)$ embeds into $\mathrm{Clop}(2^\gamma)$. By condition (iv) of Lemma 3.9, this implies that $\mathrm{Clop}(2^\gamma)$ has a subset T such that $(T, \supseteq)$ is an Aronszajn tree. This contradicts

Claim 3.14. *Let $T \subseteq \mathrm{Clop}(2^\gamma)$ be such that $(T, \supseteq)$ is a tree whose levels are all countable. Then T is countable.*

Every element a of $\mathrm{Clop}(2^\gamma)$ can be uniquely written as a finite union of clopen intervals that are maximal convex subsets of a. We may assume that γ is infinite. If $a \in \mathrm{Clop}(2^\gamma)$ is nonempty, let $\mathrm{depth}(a)$ be the least ordinal $\alpha < \gamma$ such that there are $x, y \in a$ with $x < y$ and $\Delta(x, y) = \alpha$ such that the closed interval $[x, y]$ is a maximal convex subset of a.

For every $a \in T$ let $\mathrm{height}(a)$ be the ordertype of $(\{b \in T\colon\ b \supsetneqq a\}, \supseteq)$. We show that for every $a \in T$ and every $\alpha < \omega_1$ the following statement holds:

$(*)_{a,\alpha}$ The set $\{b \in T\colon\ b \subseteq a \wedge \operatorname{depth}(b) \leqslant \operatorname{depth}(a) + \alpha\}$ is countable.

The claim follows from this since there is no $b \in T$ with $\operatorname{depth}(b) > \gamma$.

We show $(*)_{a,\alpha}$ by induction on $\alpha < \omega_1$ simultaneously for all $a \in T$. Let $a \in T$. We start with proving $(*)_{a,1}$ since $(*)_{a,0}$ is trivial, i.e., there is no $b \in T$ with $b \subseteq a$ and $\operatorname{depth}(b) < \operatorname{depth}(a)$.

If $b \in T$ is such that $b \subseteq a$ and $\operatorname{depth}(b) = \operatorname{depth}(a)$, then there are $x, y \in b$ such that $x < y$, $[x, y]$ is a maximal convex subset of b, and $\Delta(x, y) = \operatorname{depth}(a)$. It is easily checked that there is $z \in a$ such that either

(a) $z < x$, $[z, y]$ is a maximal convex subset of a, and $\operatorname{depth}(a) = \Delta(z, y)$, or
(b) $y < z$, $[x, z]$ is a maximal convex subset of a, and $\operatorname{depth}(a) = \Delta(x, z)$.

Suppose that $\{b \in T\colon\ b \subseteq a \wedge \operatorname{depth}(b) = \operatorname{depth}(a)\}$ is uncountable. Then there is $p \in a$ such that for uncountably many $b \in T$ with $b \subseteq a$, p occurs as y in a) or as x in b). This implies that $\{b \in T\colon\ b \subseteq a \wedge p \in b\}$ is uncountable. By our assumption on trees ordered by $\supseteq$, any two elements of T are either disjoint or comparable, and thus $\{b \in T\colon\ b \subseteq a \wedge p \in b\}$ is a chain in T. But this contradicts the fact that $\operatorname{Clop}(2^\gamma)$ does not include any uncountable well-ordered chain. This finishes the proof of $(*)_{a,1}$.

Now let $\alpha = \beta + 1$ for some $\beta < \omega_1$ and suppose we have $(*)_{b,\beta}$ for all $b \in T$. Let $a \in T$. By $(*)_{a,\beta}$, there are only countably many $b \in T$ with $a \supseteq b$ and $\operatorname{depth}(b) < \operatorname{depth}(a) + \beta$. Let $\delta < \omega_1$ be a bound for the heights of such b. If $b \in T$ is minimal (with respect to the order $\supseteq$ on T) with $a \supseteq b$ and $\operatorname{depth}(b) = \operatorname{depth}(a) + \beta$, then $\operatorname{height}(b) \leqslant \delta + 1$. It follows that there are only countably many $b \in T$ that are minimal with $a \supseteq b$ and $\operatorname{depth}(b) = \operatorname{depth}(a) + \beta$. Since $(T, \supseteq)$ is a tree, every $b \in T$ with $a \supseteq b$ and $\operatorname{depth}(b) = \operatorname{depth}(a) + \beta$ is above a minimal element of T with these properties. Applying $(*)_{b,1}$ for every minimal $b \in T$ with $a \supseteq b$ and $\operatorname{depth}(b) = \operatorname{depth}(a) + \beta$, we obtain $(*)_{a,\beta+1}$.

Finally suppose that $\alpha < \omega_1$ is a limit ordinal and for all $a \in T$ and all $\beta < \alpha$ we have $(*)_{a,\beta}$. Then for all $a \in T$ the set

$$\begin{aligned}&\{b \in T\colon\ b \subseteq a \wedge \operatorname{depth}(b) < \operatorname{depth}(a) + \alpha\}\\ &\quad = \bigcup_{\beta<\alpha} \{b \in T\colon\ b \subseteq \alpha \wedge \operatorname{depth}(b) < \operatorname{depth}(a) + \beta\}\end{aligned}$$

is countable, which shows $(*)_{a,\alpha}$. □

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