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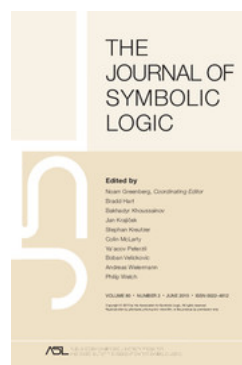
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Independence results

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INDEPENDENCE RESULTS

SAHARON SHELAH¹

Abstract. We prove independence results concerning the number of nonisomorphic models (using the S-chain condition and S-properness) and the consistency of "ZCF + $2^{\aleph_0} = \aleph_2$ + there is a universal linear order of power \aleph_1 ". Most of these results were announced in [Sh 4], [Sh 5].

In subsequent papers we shall prove an analog of MA for forcing which does not destroy stationary subsets of ω_1 , investigate \mathcal{D} -properness for various filters and prove the consistency with G.C.H. of an axiom implying SH (for \aleph_1), and connected results.

§1. Isomorphism of subsets of ω_2 . In this section, for sequences $\eta, \nu, \eta E_n^{\text{sq}} \nu$ iff $\eta \upharpoonright n = \nu \upharpoonright n$, so E_n^{sq} is an equivalence relation.

By [Sh 1, Theorem VIII, 1.8] if T is a complete \aleph_0 -unstable (first-order) theory, $T \subseteq T_1, |T_1| = \aleph_0$, then for $\aleph_0 < \lambda \leq 2^{\aleph_0}$, $I(\lambda, T_1, T) = 2^\lambda(I(\lambda, T_1, T))$, the number of nonisomorphic models of cardinality λ , which are in $\text{PC}(T_1, T)$ = the class of $L(T)$ -reducts of models of T_1). By [Sh 1] we can replace $|T_1| = \aleph_0$ by $|T_1| \leq \lambda$, $2^{\aleph_0} < 2^\lambda$ (even less). It was also proved in [Sh 1, Theorem VIII, 2.20] that for any complete \aleph_0 -unstable $T, \aleph_0 < \lambda \leq 2^{\aleph_0}, |T| \leq \lambda, I(\lambda, T) = 2^\lambda$. It was asked (see [Sh 1, VIII]) whether for any complete \aleph_0 -unstable $T, T \subseteq T_1, |T_1| \leq \lambda < 2^{\aleph_0}, I(\lambda, T_1, T) = 2^\lambda$, i.e. whether we can prove this in ZFC. We shall show that this is not the case.

THEOREM 1.1. (1) Let $T = \text{Th}(\langle \omega_2, E_0^{\text{sq}}, E_1^{\text{sq}}, \dots \rangle)$ (so T is countable complete, \aleph_0 -unstable but superstable). Then, if ZFC is consistent, it is consistent with ZFC, that for some $T_1, T \subseteq T_1, |T_1| = \aleph_1, I(\aleph_1, T_1, T) = 1$. Also T has a universal model in \aleph_1 .

Notation 1.2. Let T_1 be the following theory: it consists of (i) \bar{T} , (ii) _{n} a sentence saying that for any $x_0, y_0, n < \omega, z \mapsto F_n(x_0, y_0, z)$ is an automorphism of the $L(T)$ -reduction of the model taking x_0/E_n to y_0/E_n , and (iii) $z \mapsto F(x_0, z)$ is a one-to-one function from the model into $\{x: x E_n x_0 \text{ for all } n < \omega\}$, (iv) $\{E_n(c_\eta, c_\nu)^{\text{if } (\eta \upharpoonright n = \nu \upharpoonright n)}: \eta, \nu \in \omega_2, \eta, \nu \text{ are constructible (i.e. in } L)\}$ (c_η is an individual constant) where $\varphi^{\text{if (statement)}}$ is φ if the statement is true and $\neg\varphi$ otherwise. Let $\eta < \nu$ mean η is an initial segment of ν .

Claim 1.3. (1) T_1 is a theory (i.e. is consistent), $T \subseteq T_1, |T_1| = \aleph_1$ and every model in $\text{PC}(T_1, T)$ of cardinality \aleph_1 is isomorphic to a model of the form $(A \times \omega_1, E_0, E_1, \dots)$ where

- (i) $A \subseteq \omega_2, (\eta, \alpha) E_n(\nu, \beta)$ iff $\eta E_n^{\text{sq}} \nu$,

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(ii) for every $\eta, \nu \in {}^\omega 2$ there is an automorphism of $(A \cup {}^{>2}, <)$ taking η to ν ,

(iii) A includes all the constructible members of ${}^\omega 2$.

Claim 1.4. For proving 1.1, it suffices to prove the following. Suppose (for simplicity only), $V = L$, and find a forcing notion P , such that:

(1) P satisfies the countable chain condition, $|P| = \aleph_2$ hence in V^P , $2^{\aleph_0} = \aleph_2$;

(2) in V^P , R^L (the set of constructible reals) is of the second category;

(3) if $A, B \subseteq {}^\omega 2$ are everywhere of the second category, $|A| = |B| = \aleph_1$, then $(A \cup {}^{>2}, <) \cong (B \cup {}^{>2}, <)$.

Let us first concentrate on the induction step.

Claim 1.5. Suppose for every limit $\delta < \omega_1$, \mathcal{S}_δ is a countable family of subsets of δ .

Suppose W is a subset of ${}^{>\omega}$ closed under initial segments, and $W_\omega = \{\eta \in {}^\omega \omega : n < \omega \rightarrow \eta \upharpoonright n \in W\}$. So on W_ω a natural topology is defined ($\{\eta \in W_\omega : \nu < \eta\}$ for any $\nu \in W$ is a basic open set). Let P^0 be a countable set of partial automorphisms [i.e. functions f , with domain and range $\subseteq {}^\omega 2$, preserving $<$ and length] of $(W, <)$ such that

(a) for $f \in P^0$, $\text{Dom } f$ is closed under initial segments;

(b) if $f \in P_0, g$ a finite automorphism, $f \cup g$ a partial automorphism then $f \cup g \in P^0$.

Suppose A, B are subsets of W_ω everywhere of the second category, and $f \in P^0$, $\eta \in A, \nu \in B$ implies $\{\eta \upharpoonright n : n < \omega\} \cap \text{Dom } f, \{\nu \upharpoonright n : n < \omega\} \cap \text{Range } f$ are finite; then there is a forcing notion P , such that

(A) $|P| = \aleph_1$;

(B) $\emptyset \Vdash^P \text{“}(A \cup W, <), (B \cup W, <) \text{ are isomorphic”}$;

(C) there is a one-to-one function f from P onto ω_1 such that for any limit $\delta < \omega_1$, and $S \in \mathcal{S}_\delta$, $f^{-1}(S) = \{p \in P : f(p) \in S\}$ is predense in P , provided that it is predense in $f^{-1}(\delta)$;

(D) $P^0 \subseteq P$, moreover we can assume f extends a given one-to-one function f_0 from P_0 onto ω .

PROOF OF 1.5. Let P_0 be the set of partial isomorphisms f from $(A \cup W, <)$ into $(B \cup W, <)$ such that for some $g \in P^0$, and $\eta_0, \dots, \eta_{k-1} \in A, g \subseteq f$ and

$$\text{Dom } f - \text{Dom } g \subseteq \{\eta_l \upharpoonright \alpha : l < k, \alpha \leq \omega\}.$$

So f determines $\eta_0, \dots, \eta_{k-1}$ uniquely. Let $m(f) = \min\{m < \omega : \text{for } l < k, \eta_l \upharpoonright m < \nu \in \text{Dom } f \text{ implies } \nu < \eta_l\}$ and let $f \upharpoonright m = f \upharpoonright \{\eta \in W : \text{for no } l, \eta < \eta_l, \text{ or } l(\eta) \leq m\}$ (so $f \upharpoonright m \in P^0$ when $m \geq m(f)$).

P_0 satisfies condition (B) but not necessarily (C) (notice (C) will assure us usually that P satisfies the countable chain condition, which does not hold here, as the number of possible images of any $\eta \in A$ is \aleph_1).

We shall define by induction on $i < \omega_1, f_i, A_i, B_i$ ($i < \omega_1$) such that:

(a) $A_i \subseteq A, B_i \subseteq B$ and both are countable;

(b) the A_i are pairwise disjoint, similarly the B_i ;

(c) A_i, B_i are dense (i.e., not disjoint to any basic subset of $A \cup W$);

(d) let $A = \{x_i^0 : i < \omega_1\}, B = \{x_i^1 : i < \omega_1\}$, then $x_\alpha^0 \in \bigcup_{i \leq 2\alpha} A_i, x_\alpha^1 \in \bigcup_{i \leq 2\alpha+1} B_i$;

(e) let $\bar{A}^\alpha = \langle A_i : i < \alpha \rangle, \bar{B}^\alpha = \langle B_i : i < \alpha \rangle$, and

$$P^\alpha = P(\bar{A}^\alpha, \bar{B}^\alpha) = \{g \in P_0 : \text{Dom } g \subseteq \bigcup_{i < \alpha} A_i \cup W, x \in A_j \Leftrightarrow g(x) \in B_j\},$$

Then f_α is a one-to-one function from P^α onto $\omega(1 + \alpha)$;

(f) f_i is increasing;

(g) if $\beta \leq \alpha$, $S \in \mathcal{S}_{\omega(1+\beta)}$, $f_\beta^{-1}(S)$ is predense in P^α , $f \in P^\alpha$, $g \in P^0$ then either for some $h \in f_\beta^{-1}(S)$, $f \cup g \cup h \in P^\alpha$ or for some $m < \omega$ for no $h \in f_\beta^{-1}(S)$, $(f \upharpoonright m) \cup g \cup h \in P^\alpha$.

Clearly it suffices to assume A_i, B_i ($i < \alpha$), f_α are defined, and are as required, and to define appropriately $A_\alpha, B_\alpha, f_{\alpha+1}$. For this let $W = \{\eta_l : l < \omega\}$ and we shall define by induction on $k < \omega$, sets A^k, B^k such that:

(i) $A^k \subseteq A - \bigcup_{i < \alpha} A_i$, $B^k \subseteq B - \bigcup_{i < \alpha} B_i$, A^k ($k < \omega$) is increasing and B^k ($k < \omega$) is increasing;

(ii) $|A^k| + |B^k| \leq k + 1$;

(iii) if α is even, let $j < \omega_1$ be minimal such that $x_j^0 \notin \bigcup_{i < \alpha} A_i$, and let $A^0 = \{x_j^0\}$, $B^0 = \emptyset$, and if α is odd let j be minimal such that $x_j^1 \notin \bigcup_{i < \alpha} A_i$ and let $A^0 = \emptyset$, $B^0 = \{x_j^1\}$;

(iv) if $k = 2l + 1$ for some $\eta \in A^k$, $\eta_l < \eta$, and if $k = 2l + 2$ for some $\eta \in B^k$, $\eta_l < \eta$;

(v) $P = P(\bar{A}^{\alpha^*} \langle A^k \rangle, \bar{B}^{\beta^*} \langle B^k \rangle)$ (defined as in (e)) satisfies (g).

If we succeed we shall let $A_\alpha = \bigcup_{k < \omega} A^k$, $B_\alpha = \bigcup_{k < \omega} B^k$. By (ii), A_α, B_α are countable; by (i) they are disjoint to $\bigcup_{i < \alpha} A_i, \bigcup_{i < \alpha} B_i$ resp.; by (iv) they are dense; by (iii) they satisfy (d). Last and most important, (g) is satisfied because

$$P^{\alpha+1} = \bigcup_{k < \omega} P(\bar{A}^{\alpha^*} \langle A^k \rangle, \bar{B}^{\beta^*} \langle B^k \rangle)$$

by the finite character of P_0 (and as being compatible in $P^{\alpha+1}$ is equivalent to being compatible in P_0).

Then we have to define $f_{\alpha+1}$ which presents no problems. So let us carry the induction on k .

Case 1. $k = 0$.

Define A^0, B^0 by (iii). Thus (i), (ii) hold trivially, (iv) is empty, and (v) immediate as by definition

$$P(\bar{A}^{\alpha^*} \langle A^0 \rangle, \bar{B}^{\alpha^*} \langle B^0 \rangle) = P(\bar{A}^\alpha, \bar{B}^\alpha) = P^\alpha$$

and P^α satisfies (g).

Case 2. $k = 2l + 1$ and we want to define for $k + 1$.

Same as Case 3.

Case 3. $k = 2l$, and we want to define for $k + 1$.

If we can find $\eta \in C_0 = \{\eta : \eta \in A - \bigcup_{i < \alpha} A_i \cup A^k, \eta_l < \eta\}$ such that $A^{k+1} = A^k \cup \{\eta\}$, $B^{k+1} = B^k$ satisfy (v) then (as (i)–(iv) hold trivially) we are finished. So for every $\eta \in C_0$ there are: $\beta = \beta_\eta$, $S_\eta \in \mathcal{S}_{\omega(1+\beta)}$ such that $f_\beta^{-1}(S_\eta)$ is predense in P^α (hence in $P(\bar{A}^{\alpha^*} \langle A^k \rangle, \bar{B}^{\alpha^*} \langle B^k \rangle)$), and $g_\eta \in P^0$, and $f_\eta \in P(\bar{A}^{\alpha^*} \langle A^k \rangle, \bar{B}^{\alpha^*} \langle B^k \rangle)$ and $\nu_\eta \in B^k$, such that: f'_η , the extension of f_η by $\eta \upharpoonright \alpha \mapsto \nu_\eta \upharpoonright \alpha$ ($\alpha \leq \omega$) belong to P_0 , is compatible with g_η , but not with $g_\eta \cup h$ for any $h \in f_\beta^{-1}(S_\eta)$, but for every $m < \omega$, for some $h \in f_\beta^{-1}(S_\eta)$, $f'_\eta \upharpoonright m \cup g_\eta \cup h \in P_0$.

As A is everywhere of the second category, and $\bigcup_{i < \alpha} A_i \cup A^k$ is countable clearly C_0 is of the second category. The number of possible S_η is countable (as $\bigcup_{\beta \leq \alpha} \mathcal{S}_{\omega(1+\beta)}$ is), and similarly f_η, ν_η, g_η . Hence for some $C \subseteq C_0$ of the second category, $S \in$

$S_{\omega(1+\beta)}$, $f_0 \in P(\bar{A}^\alpha \langle A^k \rangle, \bar{B}^\alpha \langle B^k \rangle)$, $g_0 \in P^0$, $\nu_0 \in B^k$ for every $\eta \in C$, $S_\eta = S$, $f_\eta = f_0$, $\nu_\eta = \nu_0$, and $g_\eta = g_0$.

We shall prove that C is nowhere dense, hence get a contradiction so finishing the induction on k and thus finishing the induction on α , and this will clearly finish the proof of 1.5; for (B) note the sets $D_{\eta,\nu} = \{f: \eta \in \text{Dom } f, \nu \in \text{Range } f\}$ are predense.

So suppose $\rho \in W$, and we shall find $\rho' \in W$, $\rho < \rho'$ such that $\{\eta \in W_\omega: \rho' < \eta\}$ is disjoint to C . If $\rho' = \rho$ is not as required, choose $\eta \in C$, $\rho < \eta$, let $f_1 = f_0 \cup \{\langle \eta \uparrow \gamma, \nu_0 \uparrow \gamma \rangle: \gamma \leq \omega\}$, so for every $m \geq m(f_1)$ for some $h_m \in f^{-1}_2(S)$ $((f_1 \uparrow m) \cup g_0 \cup h_m) \in P_0$. Let $l = l(\rho) + m(f_1) + 1$ and $m > l$ minimal such that $\nu_0 \uparrow m \notin \text{Range}((f_1 \uparrow l) \cup g_0 \cup h_l)$ and ρ^1 be that $((f_1 \uparrow l) \cup g_0 \cup h_l) (\rho^1 \uparrow (m-1)) = \nu_0 \uparrow (m-1)$, and $\rho^1 \notin \text{Dom}((f_1 \uparrow l) \cup g_0 \cup h_l)$. Clearly $\rho^1 < \eta_0 \in W_\omega \cap C$ implies

$$(f_0 \cup \{\langle \eta^0 \uparrow \gamma, \nu_0 \uparrow \gamma \rangle: \gamma \leq \omega\}) \cup g_0 \cup h_l \in P_0.$$

So $\rho' < \eta^1 \in W_\omega \rightarrow \eta^1 \notin C$.

Using the lemma we can choose \mathcal{S}_δ appropriately, assure P satisfy the countable chain condition and R^L remains of the second category (see below). However, if we repeat the iteration as in Solovay and Tenenbaum [ST], it is not clear why R^L remains of the second category. So we introduce:

DEFINITION 1.6. (1) A sequence $\mathbf{S} = \langle \mathcal{S}_\delta: \delta < \omega_1 \text{ limit} \rangle$ is an \aleph_1 -oracle if

(i) \mathcal{S}_δ is a transitive countable set, $\alpha + 1 \subseteq \mathcal{S}_\delta$, and \mathcal{S}_δ is quite closed, e.g. it is a model of ZFC except the power-set axiom;

(ii) for any $A \subseteq \omega_1$, $\{\delta < \omega_1: A \cap \delta \in \mathcal{S}_\delta\}$ is stationary.

(2) \mathbf{S}' is a proper extension of \mathbf{S} if: $\mathcal{S}'_\alpha \in \mathcal{S}'_\alpha$ and there is $f \in \mathcal{S}'_\alpha$ from ω onto \mathcal{S}'_α , for a closed unbounded set of α 's.

(3) We call \mathbf{S} a strong \aleph_1 -oracle if in (ii) the set contains a closed unbounded set.

Claim 1.7. (1) If \diamond_{\aleph_1} holds, then there is an \aleph_1 -oracle.

(2) Every \aleph_1 -oracle has a proper extension; in fact every \aleph_1 \aleph_1 -oracles has a common proper extension.

(3) If \mathbf{S} is an \aleph_1 -oracle, M_α a model with universe ω_1 and countable language then $\{\alpha: \langle M_i \uparrow \alpha: i < \alpha \rangle \in \mathcal{S}_\alpha\}$ is stationary.

DEFINITION 1.8. A forcing notion P satisfies the \mathbf{S} -chain condition (\mathbf{S} an \aleph_1 -oracle), if for every $P' \subseteq P$, $|P'| \leq \aleph_1$, there are P'' , $P' \subseteq P''$, $|P''| \leq \aleph_1$, and a one-to-one function f from P'' into ω_1 , and $A \subseteq \omega_1$, such that:

(*) if $A \cap \delta \in \mathcal{S}_\delta$ (δ limit $< \omega_1$), $S \in \mathcal{S}_\delta$, $S \subseteq \delta$, and $f^{-1}(S)$ is predense in $f^{-1}(\delta) = \{p \in P'': f(p) < \delta\}$ then $f^{-1}(S)$ is predense in P .

REMARK. Notice that for an \aleph_1 -oracle \mathbf{S} , $\mathcal{D}(\mathbf{S}) = \{\{\delta: A \cap \delta \in \mathcal{S}_\delta, \delta < \omega_1\}: A \subseteq \omega_1\}$ is a normal filter over ω_1 , so our demand is: "there is $B \in \mathcal{D}(\mathbf{S})$ such that for every $\delta \in B \dots$ ".

Claim 1.9. If P satisfies the \mathbf{S} -chain condition then

(1) for any one-to-one function f from P'' into ω_1 , for some $A \subseteq \omega_1$, (*) of Definition 1.7 holds;

(2) P satisfies the \aleph_1 -chain condition.

LEMMA 1.10. If \mathbf{S} is a proper extension of some \mathbf{S}' (both \aleph_1 -oracles), and P satisfy the \mathbf{S} -chain condition, then in V^P , R^V is still of the second category (similarly for any $(W_\omega)^V$ for W as in 1.5).

PROOF. Suppose not. Thus in V^P there is $A = \bigcup A_n \supseteq R^V$, A_n closed nowhere dense sets of reals. So $A = \langle A_n : n < \omega \rangle$ has a name $\mathcal{A} = \langle \mathcal{A}_n : n < \omega \rangle$ and there are $p_{\nu,n}^k, \rho_{\nu,n}^k$ ($k < \omega, \nu \in {}^{\omega}2, n < \omega$) such that:

- (a) $\{p_{\nu,n}^k : k < \omega\}$ is a maximal antichain in P ;
- (b) $p_{\nu,n}^k \Vdash^P \text{“}\mathcal{A}_n \text{ is disjoint to } \{\eta \in {}^{\omega}2 : \rho_{\nu,n}^k < \eta\}\text{”}$;
- (c) $\nu < \rho_{\nu,n}^k \in {}^{\omega}2$.

Apply Definition 1.7 for $P' = \{p_{\nu,n}^k : k, n < \omega, \nu \in {}^{\omega}2\}$ so we have appropriate $P'', P' \subseteq P'' \subseteq P, |P''| \leq \aleph_1, f: P'' \rightarrow \omega_1$, and $A \subseteq \omega_1$. As S' is an \aleph_1 -oracle and, S a proper extension, for some limit $\delta < \omega_1$, $A \cap \delta \in \mathcal{S}'_\delta$, and $\{\langle k, n, \nu, \rho_{\nu,n}^k \rangle : k, n < \omega, \nu \in {}^{\omega}2\} \in \mathcal{S}'_\delta$ and $\{(f(p), f(q)) : p, q \in P'', p \leq q\} \in \mathcal{S}'_\delta$ and $\mathcal{S}'_\delta \in \mathcal{S}_\delta$.

Every closed nowhere dense subset of ${}^{\omega}2$ is representable by a real, so there are only countable many such sets representable in \mathcal{S}'_δ , so as \mathcal{S}_δ “consider” \mathcal{S}'_δ as a countable set there is $\eta^* \in \mathcal{S}_\delta^*$ which is in no nowhere dense subsets of ${}^{\omega}2$ representable in \mathcal{S}'_δ . Let

$$Q_n = \{p_{\nu,n}^k : k < \omega, \rho_{\nu,n}^k < \eta^*, \nu \in W\}.$$

So clearly $f(Q_n) \in \mathcal{S}_\delta$: If it is predense in P , then $\phi \Vdash^P \text{“}\eta^* \notin \mathcal{A}_n\text{”}$ for every n and it is predense in $f^{-1}(\delta)$ by the choice of η^* . So we finish by the S -chain condition.

LEMMA 1.11. *If P_α ($\alpha < \delta$) is an increasing continuous sequence of forcing notions satisfying the S -chain condition, then their direct limit satisfies it too.*

LEMMA 1.12. *Suppose P satisfies the S -chain condition, $|P| \leq \aleph_1$. Then in V^P there is an \aleph_1 -oracle S' (defined uniformly) such that if in V^P Q satisfies the S' -chain condition, then $P * Q$ satisfies the S -chain condition in V .*

PROOF. W.l.o.g. the set of elements of P is ω_1 , such that for limit δ , if $p, q < \delta$ have a common upper bound, then they have a common upper bound $< \delta$. Let $U = \{\delta : \delta < \omega_1, \text{ limit, and every } S \in \mathcal{S}_\delta \text{ which is predense in } P \upharpoonright \delta \text{ is predense in } P\}$.

We define \mathcal{S}'_δ as $\mathcal{S}_\delta \cup \{a \in V^P : a, \text{ and every member of its transitive closure, have a name (in } V) \text{ which is in } \mathcal{S}_\delta\}$ if $\delta \in U$, and as \mathcal{S}_δ otherwise. We leave the details to the reader.

PROOF OF THEOREM 1.1, CLAIM 1.4. Just combine all claims and lemmas. Note that by Claim 1.5 for a given \aleph_1 -oracle S we can construct P as there with the S -chain condition.

REMARK. Noting 1.7(2), clearly instead of working with an S , iterating \aleph_2 times, we can at each stage add one more S . Thus we can make 1.4 true not only for ${}^{\omega}2$, but for any W as in 1.5.

REMARK. It may be instructive here to remember Baumgartner [B]. He proved the consistency of $ZFC + 2^{\aleph_0} = \aleph_2$ with “every two subsets of R (reals, in the usual sense) of cardinality \aleph_1 which are \aleph_1 -dense (i.e. any interval has cardinality \aleph_1) are isomorphic”. He makes iterated forcing satisfying the countable chain condition. Given two such sets of reals, A, B the obvious forcing (finite isomorphism), fail to satisfy the countable chain condition so he decomposes each to \aleph_1 pairwise disjoint countable dense sets A_α ($\alpha < \omega_1$), B_α ($\alpha < \omega_1$) and looks only at finite isomorphisms $f, x \in A_\alpha = f(x) \in B_\alpha$. Now the obvious reason for the failure of the countable chain condition disappears, and using CH he proves we can find a decomposition in

which it holds. But as he iterates ω_2 times, CH is still true in any intermediate step, so he can finish.

We use similar decompositions, but in their construction we use the fact that A, B are of the second category and we have to preserve the fact that $(\omega_2)^L$ remains of the second category. In particular, it is not obvious why this will be preserved at limit stages. So we use the method of [Sh 2], represented differently, i.e. the S-chain condition. Note that we could have alternatively used forcing as in the proof of the consistency of the nonexistence of P -points (see Wimmers [W]).

However, Baumgartner's forcing P cannot satisfy an S-chain condition. This will be discussed somewhere else. For this, and a more detailed exposition of forcing with the oracle chain condition, see [Sh 3].

§2. Nonsuperstable theories. By [Sh 1, VII, §2], if T is not superstable, $T \subseteq T_1$, $\lambda \geq |T_1| + \aleph_1$ then $I(\lambda, T_1, T) = 2^\lambda$, except maybe when all the following conditions hold: $\lambda = |T_1|$, $T_1 \neq T$, $\lambda^{\aleph_0} > \lambda$, and even $\lambda < \sum_{\mu < \lambda} \mu^{\aleph_0}$. It was asked whether we can omit those restrictions, and we shall show that we cannot.

THEOREM 2.1. *Let $T = \text{TH}(\omega_1)E_{\delta^{\aleph_0}}, E_{\aleph_1^{\aleph_0}}, \dots$ (so T is a countable, complete, stable but not superstable theory). Then (if ZFC is consistent) it is consistent with ZFC that for some T_1 , $T \subseteq T_1$, $|T_1| = \aleph_1$ and $I(\aleph_1, T_1, T) = 1$. Also T has a universal model of power \aleph_1 .*

Notation 2.2. Let T_1 be the following theory: it consists of T , for $n < \omega$ a sentence saying that for any $x_0, y_0, z \mapsto F_n(x_0, y_0, z)$ is a mapping from x_0/E_n onto y_0/E_n , preserving all the E_m 's and $z \mapsto F(x_0, z)$ is a one-to-one function from the model into $\{x: xE_n x_0 \text{ for all } n < \omega\}$, and $\{E_n(c_\eta, c_\nu)^{\text{if } (\eta \upharpoonright n = \nu \upharpoonright n)}: \eta, \nu \in \omega_2 \text{ are constructible}\}$ (c_η an individual constant).

Claim 2.3. (1) T_1 is a theory (i.e. consistent) and every model in $\text{PC}(T_1, T)$ of cardinality \aleph_1 is isomorphic to a model of the form $(A \times \omega_1, E_0, E_1, \dots)$ where

- (i) $A \subseteq {}^\omega \omega_1$;
- (ii) $(\eta, i)E_n(\nu, j)$ iff $\eta \upharpoonright n = \nu \upharpoonright n$.

Claim 2.4. For proving 2.1, it suffices to prove the following.

Suppose (for simplicity only) that $V = L$, and find a forcing notion P such that

- (1) $|P| = \aleph_2$, P does not collapse \aleph_1, \aleph_2 and in V^P , $2^{\aleph_0} = \aleph_2$;
- (2) in V^P , $({}^\omega \omega_1) \cap L$ is a subset of ${}^\omega \omega_1$ of the second category;
- (3) if ${}^\omega \omega_1 \cap L \subseteq A, B \subseteq {}^\omega \omega_1$, and for each limit $\delta < \omega_1$, $\eta^* \in {}^{\omega > \delta} \delta$, $A \cap \{\eta \in {}^{\omega \delta}: \eta \notin \bigcup_{\alpha < \delta} {}^\omega \alpha, \eta^* < \eta\}$, $B \cap \{\eta \in {}^{\omega \delta}: \eta \notin \bigcup_{\alpha < \delta} {}^\omega \alpha, \eta^* < \eta\}$ are of the second category, then $(A \cup {}^{\omega > \omega_1} \omega_1, <) \cong (B \cup {}^{\omega > \omega_1} \omega_1, <)$.

§3. Proper forcing.

DEFINITION 3.1. (1) A forcing notion P is λ -proper, where λ is a regular cardinal (if $\lambda = \aleph_1$ we omit it) provided that the following holds.

If $I_\alpha = \{p_\alpha^i: i < i_\alpha\}$ is predense in P for $\alpha < \alpha_0$, $i^* = \bigcup_{\alpha < \alpha_0} i_\alpha \cup \alpha_0$, $p \in P$ then $U = \{U: U \subseteq i^*, |U| < \lambda, U \cap \lambda \text{ is an initial segment of } \lambda \text{ and there is a } q_U \in P, q_U \geq p, \text{ above which } I_\beta^U = \{p_\beta^i: i \in U\} \text{ is predense, for each } \beta \in U\}$ contains a closed unbounded subset of $\mathbf{P}_{< \lambda}(i^*) = \{U: U \subseteq i^*, |U| < \lambda\}$.

(2) U is a closed unbounded subset of $\mathbf{P}_{< \lambda}(i^*)$ if for some model M with countable language with universe i^* , $U = \{N: N < M, \|N\| < \lambda, |N| \cap \lambda \text{ an initial segment of } \lambda\}$. The last phrase is not needed for $\lambda = \aleph_1$. An alternative definition is: U is

unbounded $[\forall A \in \mathbf{P}_{<\lambda}(i^*)][\exists B \in \mathbf{P}_{<\lambda}(i^*)(A \subseteq B)]$ and closed under union of increasing sequences of length λ .

(3) In (2) we shall say M exemplifies \mathbf{U} is closed unbounded (or for (1) contains a closed unbounded set).

Claim 3.2. (1) In Definition 3.1 it suffices to assume it holds for I_α maximal antichains.

(2) If Definition 3.1 holds, then for any ordinal $\alpha_0, p \in P$ and predense subsets $I_t = \{p'_t: i \in A_t \subseteq \alpha_0\}$ of P , ($t \in A \subseteq {}^{\omega}>\alpha_0$) for a closed unbounded set of $U \in \mathbf{P}_{<\lambda}(\alpha_0)$, for some $p_U \in P$, $I'_t = \{p'_t: i \in A_t \cap U\}$ is predense above p_U for every $t \in ({}^{\omega}>U) \cap A$, and $p_U \geq p$.

PROOF. (1) We assume the statement in Definition 3.1 holds for antichains, suppose $I_\alpha = \{p'_\alpha: i < i_\alpha\}$ ($\alpha < \alpha_0$) are predense in P , and show \mathbf{U} contains a closed unbounded subset of $\mathbf{P}_{<\lambda}(i^*)$. (The other direction is trivial.)

For each α , among the antichains $\{q_j: j < j'\} \subseteq P$ satisfying: for each $j < j'$ for some $i < i_\alpha, p'_i \leq q_j$, choose a maximal one $J_\alpha = \{q'_j: j < j_\alpha\}$ and let $f_\alpha: j_\alpha \rightarrow i_\alpha$ be such that $p'_{f_\alpha(j)} \leq q'_j$. Clearly J_α exist by Zorn's Lemma. It is maximal (among antichains in general) as I_α is predense.

By the hypothesis there is a model M_0 with countable language, such that for any $N < M_0, \|N\| < \lambda, |N| \cap \lambda$ an ordinal there is $r_N \geq p, r_N \in P$, above which $J'_\alpha = \{q'_j: j < j_\alpha, j \in N\}$ is predense for each $\alpha \in N$. Let f be a two-place function on $\alpha_0, f(\alpha, j) = f_\alpha(j)$ when defined and zero otherwise. $M = (M_0, f)$ is a model exemplifying the properness of P for the I_α .

(2) Easy.

LEMMA 3.3. *Suppose λ is regular, P is λ -proper (in V), and Q is λ -proper in V^P . Then $P * Q$ is λ -proper in V .*

PROOF. Remember that elements of $P * Q$, are pairs $(p, q), \phi \Vdash^P \text{“}q \in Q\text{”}$, and $(p_1, q_1) \leq (p_2, q_2)$ iff $P \models p_1 \leq p_2$ and $p_2 \Vdash^P \text{“}q_1 \leq q_2\text{”}$. So let $I_\alpha = \{(p'_i, q'_i): i < i_\alpha\}$ be maximal antichains in $P * Q$ for $\alpha < \alpha_0$. Let G be a generic set for P , and $A_\alpha = \{i < i_\alpha: p'_i \in G\}$, $J_\alpha^0 = \{q'_i: i \in A_\alpha\}$. As Q is λ -proper, there is a model M_1 as in Definition 3.1, and w.l.o.g. it has Skolem functions $F_n(\bar{x}_n)$. Let $\mathbf{M}_1, F_n(\bar{x}_n)$ be their names in V and for notational simplicity let $i^* \geq |P|$. So for every $\bar{a} \in |\mathbf{M}_1| = i^*$ there is a maximal antichain $J_\alpha^{\bar{a}} = \{p'_{\bar{a},n}: i < i_{\bar{a},n} \leq i^*\} \subseteq P$, and a function f such that $p'_{\bar{a},n} \Vdash^P \text{“}F_n(\bar{a}) = f(\bar{a}, n, i)\text{”}$ (and $f(\bar{a}, n, i)$ is an ordinal $< i^*$).

Now we apply Definition 3.1 (or more exactly Claim 3.2(2)) to the family:

$$I'_\alpha = \{p'_i: i < i_\alpha\}, \quad J_{\bar{a}}^n(\alpha < \alpha_0, \bar{a} \in {}^{\omega}>(i^*), n < \omega)$$

and get a model $M_0, |M_0| = i^*$. Let $M = (M_0, f)$ and we shall prove it exemplifies the demand from Definition 3.1 for the I_α .

So let $N < (M, f), \|N\| < \lambda, |N| \cap \lambda$ be an ordinal. By its definition, there is $p_N \in P, p_N \geq p$, such that above it

$$\{p'_i: i < i_\alpha, i \in N\}, \quad \{p'_{\bar{a},n}: i < i_{\bar{a},n}, i \in N\}$$

are predense, for $\alpha \in N, \bar{a} \in N$. By the definition of $J_{\bar{a}}^n$, and as $|N|$ is closed under f, p_N force $F_n(\bar{a})$ to be in $|N|$. Let $G \subseteq P$ be generic, $p_N \in G$. As M_1 (in V^P) has Skolem functions, it forces $|N|$ to be the universe of an elementary submodel of M_1 . So in $V^P = V[G]$ there is $q_N \in Q$ such that above it $J^0 = \{q'_i: p'_i \in G, i \in |N|\}$

is predense for $\alpha \in N$ and $q_N \geq q$. It is now easy to finish.

LEMMA 3.4. *Suppose $P_{n+1} = P_n * Q_n$, P_0 trivial, and Q_{n+1} is proper in V^{P_n} . Then the inverse limit P of the P_n is proper.*

PROOF. Clearly an element of P has the form $\bar{q} = \langle q_n : n < \omega \rangle$, $q \upharpoonright k \Vdash^{P_{k+1}}$ “ $q_k \in Q_k$ ”. So suppose $I_\alpha = \{\bar{q}_i^\alpha : i < i_\alpha\} \subseteq P$ ($\alpha < \alpha_0$) are predense in P . For regular $\bar{\kappa}$ big enough let $M_0 = (H(\bar{\kappa}), \in, P, \Vdash)$ (“big enough” means $P \in H(\bar{\kappa})$, and names of elements which will be in $H(\bar{\kappa})$ are in $H(\bar{\kappa})$).

Let $N < M_0$, $\|N\| < \lambda$, $|N| \cap \lambda$ be an ordinal, $\bar{p} \in P \cap N$ and we shall prove that there is $\bar{q} \in P$ above which $I_\alpha \cap N$ ($\alpha \in N$) are predense. This is sufficient, for first expand M_0 to M_1 by adding Skolem functions, then expand M_1 to M_2 by adding for each formula a relation equivalent to it, reduce M_2 to M_3 by throwing away the functions, and M will be the submodel of M_3 with universe $i^* = \bigcup_{\alpha < \alpha_0} i_\alpha \cup \alpha_0$.

Let the ordinals in N be $\{\gamma(l) : l < \omega\}$.

We now define by induction on $n < \omega$, $r_n \in Q_n$ (i.e., $\phi \Vdash^{P_n}$ “ $r_n \in Q_n$ ”) $r_n^* \in P \cap N$ and P_n -names ϱ_n such that

- (a) above $\langle r_0, \dots, r_{n-1} \rangle \in P_n$, $\{\bar{q}_i^\alpha \upharpoonright n : i < i_\alpha, i \in N\}$ is predense, for each $\alpha \in N$;
- (b) $r_n^* \leq r_{n+1}^*$, $r_n^* \upharpoonright n \leq \langle r_0, \dots, r_{n-1} \rangle$;
- (c) ϱ_n is the following P_n -name: the first $i \in N$ such that $\bar{q}_i^{\gamma(n)} \upharpoonright n$ is in the generic set of P_n if there is such an i , and i^* otherwise. So by (a), $\langle r_0, \dots, r_{n-1} \rangle \Vdash^{P_n}$ “ $\varrho_n < i^*$ ”;

- (d) $\langle r_0, \dots, r_{n-1} \rangle \Vdash^{P_n}$ “ $\bar{q}_{\sigma_n}^{\gamma(n)} \leq r_n^*$ ”.

It is easy to see that if we succeed, $\bar{r} = \langle r_0, r_1, \dots \rangle \in P$, and for each $\alpha \in N$, for some $l < \omega$, $\alpha = \gamma(l)$, and so $\bar{r} \Vdash^P$ “ $\sigma_l \in N$ and $\bar{q}_{\sigma_l}^{\gamma(l)}$ will be in the generic set”.

So we have to do the induction step. For $n = 0$ there is no problem. So suppose r_n^* , r_0, \dots, r_{n-1} , $\varrho_0, \dots, \varrho_{n-1}$ are defined, as we shall define r_n , ϱ_n .

We define ϱ_n , r_n^* by (b)-(d). Now let G_n be a generic set for P_n to which $\langle r_0, \dots, r_{n-1} \rangle$ belongs, and we work for a while in $V[G_n]$. So now $\sigma_0, \dots, \sigma_n$ are ordinals $< i^*$, in fact $\in N$. So $Q_n \in N[G]$ and $I_\alpha^n = \{\bar{q}_i^\alpha(n) : \bar{q}_i^\alpha \upharpoonright n \in G_n\} \in N[G]$ as well as $\langle I_\alpha^n : \alpha < \alpha_0 \rangle$, and I_α^n is predense in Q_n .

Now in $M_0[G_n]$ there is a model M^1 with universe i^* (and countable language) with Skolem functions w.l.o.g. such that if $N^1 < M^1$, $\|N^1\| = \aleph_0$, $|N^1| \cap \omega_1$ is an ordinal, then for some $r \in Q_n$, $\{\bar{q}_i^\alpha(n) \in I_\alpha^n : i \in N^1\}$ is predense over r for every $\alpha \in N^1$.

Let $F = \langle F_m : m < \omega \rangle$ be the functions of N^1 , so $M^1, F \in N$ (more exactly such names are in N), so for every $\bar{a} \in i^* \cap N$, $n < \omega$, $\{p \in P_n : p \Vdash F_n(\bar{a}) = \beta$ for some $\beta\}$ is predense in P_n , belongs to N . Hence its intersection with N is predense above $\langle r_0, \dots, r_{n-1} \rangle$; hence $\langle r_0, \dots, r_{n-1} \rangle \Vdash$ “ $F_n(\bar{a}) \in N$ ”. So in $V[G_n]$, $|N| \cap i^*$ is the universe of an elementary submodel of M^1 . As $\langle r_0, \dots, r_{n-1} \rangle \in G_n$, also $r_n^*(n) \in Q_n$, and we can relativize the above discussion to $\{r \in Q_n : r \geq r_n^*(n)\}$.

So there is $r_n \in Q_n$, $r_n^*(n) \leq r_n$, above which $\{\bar{q}_i^\alpha(n) \in I_\alpha^n : i \in N\}$ is predense for $\alpha \in N$. Now r_n (its name) is as required.

LEMMA 3.5. *Suppose P_0 is trivial, $P_{i+1} = P_i * Q_i^*$ ($i < \alpha$) for limit $\delta \leq \alpha$, P_δ is the inverse limit if cf $\delta = \aleph_0$, and the direct limit otherwise.*

If $\phi \Vdash^{P_i}$ “ Q_i is proper” then P_α is proper.

PROOF. Like 3.4 (if $N < M_0$ in $N \cap \alpha$, there is a last element, trivial; otherwise

there are $\gamma_n < \gamma_{n+1} \in N \cap \alpha$, unbounded in $N \cap \alpha$, and repeat the previous argument (proving simultaneously, by induction on α , that if p_{n+1} is a P_{γ_n} -name of a member of $P_{\gamma_{n+1}}$, $p_{n+1} \upharpoonright \gamma_n = \dot{\phi}$ then there is q in P_δ above all p_n , i.e. $q \upharpoonright \gamma_n$ force $p_{n+1} \leq q$).

Conclusion 3.6. Assume $2^{\aleph_1} = \aleph_2$, $2^{\aleph_0} = \aleph_1$, then there is a forcing notion P , $|P| = \aleph_2$, satisfying the \aleph_2 -chain condition, such that in V^P , for any proper Q , $|Q| = \aleph_1$, and dense sets $\mathcal{D}_i \subseteq Q$ ($i < \omega_1$) there is $G \subseteq Q$ generic for the \mathcal{D}_i 's.

PROOF. We iterate with countable support ω_2 times forcings with universe ω_1 . Then by 3.5, P is proper, so \aleph_1 is not collapsed. As to the \aleph_2 -chain condition note that even if $|P| = \aleph_1$, $V^P \models |Q| = \aleph_1$, $P * Q$ is not of power \aleph_1 (as we have many names) however by the properness, above any condition there is a condition with countable history, and then we can apply the Δ -system argument. The rest is trivial.

REMARK. (1) For a full development of proper forcing see the forthcoming survey of E. Wimmers [W].

(2) If one is left unconvinced by the proof of the \aleph_2 -chain condition, iterate κ times with countable support, κ the first strongly inaccessible.

§4. Now we combine the methods of §1 (S-chain condition) and §3 (properness) to prove Theorem 2.1.

DEFINITION 4.1. A forcing notion P is S-proper, for an \aleph_1 -oracle S (see Definition 1.6) if for every $P' \subseteq P$, $|P'| \leq \aleph_1$, $p^* \in P$, there are P'' , $P' \subseteq P''$, $|P''| = \aleph_1$, and a one-to-one function from P'' into ω_1 , and $A \subseteq \omega_1$ such that

(*) if $A \cap \delta \in \mathcal{S}_\delta$ then there is $p \in P$, $p \geq p^*$, such that: if $S \in \mathcal{S}_\delta$, $S \subseteq \delta$, $f^{-1}(S)$ is predense in $f^{-1}(\delta)$, then $f^{-1}(S)$ is predense above p .

Claim 4.2. If P is S-proper then

(1) in Definition 4.1, we can replace f by any other one-to-one function f' from P'' into ω_1 , and then we need to change A only;

(2) if S is a strong \aleph_1 -oracle, then P is proper, provided that $|P| = \aleph_1$.

LEMMA 4.3. Suppose S is a proper extension of S' , both \aleph_1 -oracles and P are S-proper. Then in V^P , $(\omega > 2) \cap V$ is still of the second category (and similarly for any W as in 1.5).

PROOF. Like 1.10.

LEMMA 4.4. Suppose P is S-proper, $|P| \leq \aleph_1$. Then in V^P there is an \aleph_1 -oracle S' (defined uniformly so we denote it by S^P or $S[G]$ if $G \subseteq P$ is generic) such that for any S'-proper $Q \in V^P$, $P * Q$ is S-proper. It is done in such a way that $S^{P*Q} = (S^P)^Q$.

PROOF. Like 1.12. We assume the set of elements of P is ω_1 . By the S-properness of P , and few manipulations for some $A \subseteq \omega_1$, $A \cap \delta \in \mathcal{S}_\delta$ implies that for every $p < \delta$, there is $p' > p$, such that: $S \in \mathcal{S}_\delta$, $S \subseteq \delta$, S predense in $P \cap \delta$ above p implies S is predense in P above p' . Let $G \subseteq P$ be generic, then clearly

$$U = \{\delta < \omega_1: \delta \text{ limit and for some } p_\delta \in G, \text{ if } S \in \mathcal{S}_\delta, S \subseteq \delta \text{ is predense in } P \cap \delta \text{ then it is predense in } P \text{ above } p_\delta \text{ (hence } S \cap G \neq \emptyset)\}$$

is nonempty. Moreover if $B \subseteq \omega_1$, $B \in V$, $B \neq \emptyset \pmod{\mathcal{D}_S}$, then $B \cap U \neq \emptyset$. We define $\mathcal{S}'_\delta = \mathcal{S}_\delta[G \cap \delta]$ for $\delta \in U$, and arbitrarily otherwise. Now $S' = \langle \mathcal{S}'_\delta: \delta < \omega_1 \rangle$ shows our conclusion. Note that $G \cap \delta$ is generic for $P \upharpoonright \delta$ in \mathcal{S}_δ , for every $\delta \in U$, so \mathcal{S}'_δ is of the right form.

Also \mathbb{S} has the “diamond property”. For if $B \subseteq \omega_1$ in $V[G]$, $\underline{B} \in V$ is a name. Let $I(\underline{B}, \alpha)$ be $\{p: p \Vdash^P \alpha \in \underline{B} \text{ or } p \Vdash^P \alpha \notin \underline{B}\}$. Clearly $I(\underline{B}, \alpha)$ is predense, and (in V)

$$C = \{\delta: \text{for every } \alpha < \delta, I(\underline{B}, \alpha) \cap \delta \text{ is predense in } P \upharpoonright \delta\}$$

is closed unbounded. So $\delta \in C \cap U$ implies $A \cap \delta \in \mathcal{S}_\delta[G \cap \delta] = \mathcal{S}'_\delta$.

We leave the rest to the reader.

LEMMA 4.5. *Suppose $P_{n+1} = P_n * Q_n$, P_0 trivial, Q_n is \mathbb{S}^{P_n} -proper for $n < \omega$. Then the inverse limit P of the P_n is \mathbb{S} -proper.*

PROOF. Like 3, 4, with a little more checking.

LEMMA 4.6. *Suppose P_0 is trivial, $P_{i+1} = P_i * Q_i$ ($i < \delta$) for limit $\delta \leq \alpha$, cf $\delta = \omega$, P_δ is the inverse limit of P_i ($i < \delta$), and for limit $\delta \leq \alpha$, cf $\delta > \omega$, P_δ is the direct limit of P_i .*

If Q_i is \mathbb{S}^{P_i} -proper $|Q_i| = \aleph_1$, then P_α is \mathbb{S} -proper.

PROOF. Like 3.5.

Now we return to Theorem 2.1.

PROOF OF THEOREM 2.1. Clearly it suffices to prove 2.4 and assume $V = L$. It suffices to prove:

(*) For an oracle \mathbb{S} , and $A, B \subseteq {}^\omega\omega_1$ as in 2.4(3) there is an \mathbb{S} -proper forcing Q , $|Q| = \aleph_1$,

$$\Vdash^Q (A \cup {}^{>\omega_1}, <) \cong (B \cup {}^{>\omega_1}, <).$$

For then we iterate the forcing Q_i, P_i ($i < \omega_2$), as in 4.6. $|P_i| = \aleph_1$ each time dealing with one isomorphism, and so $\bigcup_{i < \omega_2} P_i$ (as in the proof of 3.6 it satisfies the \aleph_2 -chain condition) is as required in 2.4. Hence prove 2.1 (by 2.4).

PROOF OF (*). We let Q^* be the set of partial isomorphisms from $(A \cup {}^{>\omega_1}, < \upharpoonright)$ onto $(B \cup {}^{>\omega_1}, < \upharpoonright)$, $\text{Dom } f = X \cap (A \cup {}^{>\omega_1})$, such that for some limit or zero $\alpha < \omega_1$, and finite $A \subseteq {}^{>\omega_1}$, $f \upharpoonright (\text{Dom } f - A)$ is an isomorphism from ${}^{>\alpha}A \cup ({}^\omega\alpha \cap A)$ onto ${}^{>\alpha}B \cup ({}^\omega\alpha \cap B)$. The α is unique and we denote it by $\alpha(f)$.

Trivially for $\eta \in A \cup {}^{>\omega_1}$, $\{f \in Q^*: \eta \in \text{Dom } f\}$ is dense in Q^* , and for $\eta \in B \cup {}^{>\omega_1}$, $\{f \in Q^*: \eta \in \text{Range } f\}$ is dense in Q^* . So we have to prove only that Q^* is \mathbb{S} -proper. For this we do a preliminary \mathbb{S} -proper forcing (iterating ω_1 times) ensuring that

(*) Let $\delta < \omega_1$ be limit, $P^0 \subseteq Q^*$ be countable, and suppose:

(a) for every $f \in P^0$ for some $\alpha < \delta$, $(\text{Dom } f) \cup (\text{Range } f) \subseteq {}^{>\alpha}$;

(b) every finite $f \in Q^*$, $(\text{Dom } f) \cup (\text{Range } f) \subseteq {}^{>\delta}$ is in P^0 ;

(c) if $f, g \in P^0$, $f \cup g \in Q^*$ then $f \cup g \in P^0$;

(d) if $f \in P^0$, $\alpha < \delta$, then for some $g, f \subseteq g \in P^0$, $\alpha(g) \geq \alpha$.

Then there is $f \in Q^*$, $\alpha(f) = \delta$, such that for every $\alpha < \delta$, $f \upharpoonright ({}^{>\alpha}) \in P^0$.

To ensure one instance of (*) we use 1.5 (we have to replace ${}^\omega\omega$ by ${}^\omega\delta$, but this is trivial). So after ω_1 times “we catch our tail”, i.e. consider all possible P^0 's (possible by the \mathbb{S} -properness, and we can iterate by the previous lemmas, or even §1) as the \mathbb{S} -chain condition implies \mathbb{S} -properness).

Now (*) trivially ensures that Q^* is \mathbb{S} -proper for every oracle \mathbb{S} , so we are finished.

THEOREM 4.7. *It is consistent with $ZFC + 2^{\aleph_0} = \aleph_2$ (if ZFC is consistent) that there is a universal (linear) order of power \aleph_1 .*

REMARKS. The situation for the existence of saturated model of a first-order theory T of power λ , is known, see [Sh 1, VIII 4.7]. As for universal models, it was known that $\lambda = \lambda^{>2} > |T|$ is sufficient, and it was natural to assume that for unstable theories this condition is also necessary. Note that if we add \aleph_2 Cohen reals there is no universal order of power \aleph_1 .

Problem. (1) For which T can we prove the analog of 4.7?

(2) Is there a condition ensuring T has a universal model of power λ iff $\lambda = \lambda^{>2}$?

(3) Generalize 4.7 to larger cardinals, even $> 2^{\aleph_0}$.

DEFINITION 4.8. Let K be the class of orders M of power \aleph_1 satisfying:

(1) M has cofinality \aleph_1 , moreover for every $a \in M$, $\{b : b < a\}$ has cofinality \aleph_1 .

(2) $M = \bigcup_{i < \omega_1} M_i$, M_i increasing, continuous, each M_i has the order type of the rationals, and for every $a \in M - M_i$, $\{b \in M : (\forall x \in M_i) (a < x \equiv b < x)\}$ has cofinality \aleph_1 and the set of Dedekind cuts of M_i realized by members of $M - M_i$, is of the second category.

(1)*, (2)* The same as (1), and (2) for the inverse order.

PROOF OF 4.7. Clearly every order of power \aleph_1 , can be embedded into a member of K , provided that there is a set of reals of the second category of power \aleph_1 . So we imitate the proof of 2.4 (in §4), making any two members of K isomorphic. Given $M, N \in K$, let $M = \bigcup_{i < \omega_1} M_i$, $N = \bigcup_{i < \omega_1} N_i$ so that $\langle M_i : i < \omega_1 \rangle$, $\langle N_i : i < \omega_1 \rangle$ are as required in (2), and (2)* simultaneously. We want to use $Q^* = \{f : f \text{ as an isomorphism from } M_i \text{ onto } N_i, \text{ for some } i < \omega_1, \text{ such that: for every } x \in M - M_i \text{ there is } y \in N - N_i \text{ and for every } y \in N - N_i \text{ there is } x \in M - M_i \text{ such that } (\forall a \in M_i)(a < x \equiv f(a) < y)\}$.

Again we have to do preliminary forcing to make Q^* proper.

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