Existence of Endo-Rigid Boolean Algebras

In [Sh 2] we answering a question of Monk have explicated the notion of "a Boolean algebra with no endomorphisms except the ones induced by ultrafilters on it" (see §2 here) and prove the existence of one with character density $\aleph_0$, assuming first $\diamondsuit_{\aleph_1}$ and then only CH. The idea was that if h is an endomorphism of B, not among the "trivial" ones, then there are pairwise disjoint $d_n \in B$ with $h(d_n) \not\subset d_n$. Then we can, for some $S \subset \omega$, add an element x such that $d_n \leq x$ for $n \in S$, $x \cap d_n = 0$ for $n \notin S$ while forbidding a solution for $\{y \cap h(d_n) = h(d_n) : n \in S\} \cup \{y \cap h(d_n) = 0 : n \notin S\}$. Further analysis showed that the point is that we are omitting positive quantifier free types. Continuing this Monk succeeds to prove in *ZFC* the existence of such Boolean algebras of cardinality $2^{\aleph_0}$ and density character $2^{\aleph_0}$. In his proof he

(a) replaces some uses of the countable density character by the $\aleph_1$-chain condition

(b) generally it is hard to omit $< 2^{\aleph_0}$ many types but because of the special character of the types and models involve, using $2^{\aleph_0}$ almost disjoint subsets of ω, he succeeds in doing this

(c) for another step in the proof (ensuring indecomposability - see Definition 2.1) he (and independently by Nyikos) find it is in fact easier to do this when for every countabe $I \subseteq B$ there is $x \in B$ free over it.

The question of the existence of such Boolean algebras in other cardinalities remains open (See [DMR] and a preliminary list of problems for the handbook of Boolean Algebras by Monk).

We shall prove (in ZFC) the existence of such B of density character λ and cardinality $\lambda^{\aleph_0}$ whenever $\lambda > \aleph_0$. We then conclude answers to some other questions from Monk's list, (combine 3.1 with 2.5). We use a combinatorial method from [Sh 3], [Sh 4] , it is represented in section 1.

In [Sh 1], [Sh 6] (and [Sh 7]) the author offers the opinion that the combinatorial proofs of [Sh 1], Ch. VIII (applied there for general first order theories) should be useful for proving the existence of many non-isomorphic, and/or pairwise non-embeddable structure which has few (or no) automorphism or endomorphism or direct decomposition etc. As an illumination in [Sh 6] a rigid Boolean algebra in every $\lambda > \aleph_0$ was constructed. The combinatorics we used here relay on [Sh 1], Ch. VIII 2.6 and it amounts to building a model of power $\lambda^{\aleph_0}$ omitting countable types along the way, the method is proved in *ZFC*, nevertheless it has features of the diamond. It has been used also in Gobel and Corner [CG] and Gobel and Shelah [GS1], [GS2]. See more on the method and on refinements of it in [Sh 4] and [Sh 3] and mainly [Sh 5].

§1 The combinatorial principle

Content: Let $\lambda > \kappa$ be fixed infinite cardinal.

We shall deal with the case $cf\ \lambda > \aleph_0$, $\lambda^{\aleph_0} = \lambda^{\kappa}$, and usually $\kappa = \aleph_0$. Let L be a set of function symbols, each with $\leq \kappa$ places, of power $\leq \lambda$. Let $\mathcal{M}$ be the L-algebra freely generated by $\mathbf{T} \overset{def}{=} {}^{\kappa>}\lambda(= \{\eta:\eta$ a sequence $< \omega$ of ordinals $< \lambda)$ (We could have as well considered $\mathbf{T}$ as a set of urelements, and let $\mathcal{M}$ be the family $H_{<\kappa^+}(\mathbf{T})$ of sets hereditarily of cardinality $\leq \kappa$ build from the urelements]. For $\eta \in \mathbf{T} \cup {}^{\kappa}\lambda$ let $orco(\eta) = \{\eta(i):i < \ell(\eta)\}$, for a sequence $\bar\eta = \langle \eta_i:i < \beta \rangle$ let $orco(\bar a) = \bigcup_{i<\beta} orco(\eta_i)$, for $a = \tau(\bar\eta) \in \mathcal{M}$ let $orco(\eta) = orco(\bar\eta)$ and $orco(\langle a_i:i < \beta \rangle) = \bigcup_{i<\beta} orco(a_i)$, and similarly for a set.

1.2 Explanation: We shall let B_0 be the Boolean Algebra freely generated by $\{\eta:\eta \in \mathbf{T}\}$, B_0^c its completion and we can interprete B_0^c as a subset of $\mathcal{M}$ (each $a \in B_0^c$ has the form $\bigcup_{n<\omega} \tau_n$ where τ_n is a Boolean combination of members of $\mathbf{T}$, so as we have in L $\aleph_0$-place function symbols there is no problem). As the $\eta \in \mathbf{T}$ may be over-used we replaced them for this purpose by x_η (e.g. let $F \in L$ be a monadic function symbol, $x_\eta = F(\eta)$).

Our desired Boolean Algebra B will be a subalgebra of B_0^c containing B_0.

1.3 Definition :

1) Let L_n be fixed vocabularies (= signatures), $|L_n| \leq \kappa$, $L_n \subseteq L_{n+1}$, (with each predicate function symbol finitary for simplicity, let $P_n \in L_{n+1} - L_n$ be monadic predicates.

2) Let $\mathcal{J}_n$ be the family of sets (or sequences) of the form $\{(f_\ell, N_\ell) : \ell \leq n\}$ satisfying

a) $f_\ell : {}^{\ell \geq} \kappa \to T$ is a *tree embedding* i.e.

(i) f_ℓ is length preserving i.e. η , $f_\ell(\eta)$ have the same length.

(ii) f_ℓ is order preserving i.e. for $\eta, \nu \in {}^{\ell \geq \kappa}$, $\eta \lessdot \nu$ iff $f_\ell(\eta) \lessdot f_\ell(\nu)$.

b) $f_{\ell+1}$ extend f_ℓ (when $\ell + 1 \leq n$)

c) N_ℓ is an L'_ℓ-model of power $\leq \kappa$, $|N_\ell| \subseteq |\mathcal{M}|$, where $L'_\ell \subseteq L_\ell$.

d) $L'_{\ell+1} \cap L_\ell = L'_\ell$ and $N_{\ell+1} \restriction L'_\ell$ extends N_ℓ

e) if $P_m \in L'_{m+1}$, then $P_m^{N_\ell} = |N_m|$ when $m < \ell \leq n$ and

f) Rang $(f_\ell) - \bigcup\limits_{m<\ell}$ Rang (f_m) is included in $|N_\ell| - \bigcup\limits_{m<\ell} |N_m|$.

3) Let $\mathcal{J}_\omega$ be the family of pairs (f, N) such that for some $(f_\ell, N_\ell)(\ell < \omega)$ the following holds:

(i) $\{(f_\ell, N_\ell) : \ell \leq n\}$ belongs to $\mathcal{J}_n$ for $n < \omega$.

(ii) $f = \bigcup\limits_{\ell<\omega} f_\ell$, $N = \bigcup\limits_{n<\omega} N_n$, (i. e. $|N| = \bigcup\limits_{n<\omega} |N_n|$, $L(N) = \bigcup\limits_{n} L(N_n)$, and $N \restriction L(N_n) = \bigcup\limits_{n<m<\omega} N_m \restriction L(N_n)$)

4) For any $(f, N) \in \mathcal{J}_\omega$ let (f_n, N_n) be as above (it is easy to show that (f_n, N_n) is uniquely determined - notice d),e) in (2),) so for (f^α, N^α) we get (f_n^α, N_n^α)

5) Let $\mathcal{J}_n = \{(f_n, N_n)$: for some $(f_\ell, N_\ell)(\ell < n), \{(f_\ell, N_\ell) : \ell \leq n\} \in \mathcal{J}_n$ and

we adopt conventions of 4).

6) A branch of Rang (f) or of f (for f as in (3)) is just $\eta \in {}^{\omega}\lambda$ such that for every $n < \omega$, $\eta \restriction n \in$ Rang (f).

1.4 Explanation of our Intended Plan (of Constructing e.g the Boolean Algebra)

We will be given $W = \{(f^{\alpha}, N^{\alpha}) : \alpha < \alpha(*)\}$, so that every branch η of f^{α} converge to some $\zeta(\alpha)$, $\zeta(\alpha)$ non-decreasing (in α). We have a free object generated by $\mathbf{T}$ (B_0 in our case) and by induction on α we define B_{α}, increasing continuous, such that $B_{\alpha+1}$ is an extension of B_{α}, $a_{\alpha} \in B_{\alpha+1} - B_{\alpha}$ (usually $B_{\alpha+1}$ is generated by B_{α} and a_{α}, and a_{α} is in the completion of B_0). Every element will depend on few ($\leq \kappa$) members of $\mathbf{T}$, and a_{α} "depends" in a peculiar way: the set $Y_{\alpha} \subseteq \mathbf{T}$ on which it "depends" is $Y^0_{\alpha} \cup Y^1_{\alpha}$ where Y^0_{α} is bounded below $\zeta(\alpha)$ (i.e. $Y^0_{\alpha} \subseteq {}^{\omega>}\zeta$ for some $\zeta < \zeta(\alpha)$) and Y^1_{α} is a branch of f^{α} or something similar. See more in 1.8.

1.5 Definition of the Game: We define for $W \subseteq \mathcal{J}_{\omega}$ a game $\mathbf{Gm}(W)$, which lasts ω-moves.

In the n-th move:

Player I: Choose f_n, a tree-embedding of ${}^{n\geq}\kappa$ into ${}^{n\geq}\lambda$, extending $\bigcup_{\ell<n} f_{\ell}$, such that Rang $(f_n) - \bigcup_{\ell<n}$ Rang (f_{ℓ}) is disjoint to $\bigcup_{\ell<n} |N_{\ell}|$; then

player II chooses N_n such that $\{(f_{\ell}, N_{\ell}) : \ell \leq n\} \in \mathcal{J}_n$.

In the end player I wins if $(\bigcup_{n<\omega} f_n, \bigcup_{n<\omega} N_n) \in W$.

1.6 Remark: We shall be interested in W such that player I wins (or at least does not lose) the game, but W is "thin". Sometimes we need a strengthening of the second player in two respects: he can force (in the n-th move) Rang (f_{n+1}) $-$ Rang (f_n) to be outside a "small" set, and in the zero move he can determine an arbitrary initial segment of the play.

1.7 Definition : We define, for $W \subseteq \mathcal{J}_{\omega}$, a game $\mathbf{Gm}'(W)$ which lasts ω-

moves.

In the zero move

player I choose f_0, a tree embedding of ${}^{0\geq}\kappa$ to ${}^{0\geq}\lambda$ (but there is only one choice).

player II chooses $k < \omega$ and $\{(f_\ell, N_\ell): \ell \leq k\} \in \mathfrak{I}_k$, and $X_0 \subset \mathbf{T}$, $|X_0| < \lambda$.

In the n-th move, $n > 0$:

player I chooses f_{k+n}, a tree embedding of ${}^{(k+n)\geq}\kappa$ into ${}^{(k+n)\geq}\lambda$, with Range $f_{k+n} - \bigcup_{\ell<k+n}$ Rang f_ℓ disjoint to $\bigcup_{\ell<k+n} N_\ell \cup \bigcup_{\ell<n} X_\ell$.

player II choose N_{k+n} such that $\{(f_\ell, N_\ell): \ell \leq k+n\} \in \mathfrak{I}_{k+n}$ and $X_n \subset \mathbf{T}$, $|X_n| < \lambda$.

1.8 Remark: What do we want from W?: First that by adding an element (to B_0) for each (f, N) we can "kill" every undesirable endomorphism, for this it has to encounter every possible endomorphism, and this will be served by "W a barrier". For this $W = \mathfrak{I}_\omega$ is O.K. but we also want W to be thin enough so that various demands will have small interaction, for this disjointness and more are demanded.

1.6 Definition : 1) We call $W \subseteq \mathfrak{I}_\omega$ a *strong barrier* if player I wins in $\mathbf{Gm}(W)$ and even $\mathbf{Gm}'(W)$ (which just means he has a winning strategy.)

2) We call W a *barrier* if player II does not win in $\mathbf{Gm}(W)$ and even does not win in $\mathbf{Gm}'(W)$.

3) We call W disjoint *if* for any distinct $(f^\ell, N^\ell) \in W$ $(\ell = 1,2)$, f^1 and f^2 has no common branch.

1.7 The Existence Theorem : 1) If $\lambda^{\aleph_0} = \lambda^\kappa$, $cf\ \lambda > \aleph_0$ then there is a strong disjoint barrier.

2) Suppose $\lambda^{\aleph_0} = \lambda^\kappa$, $cf\ \lambda > \aleph_0$. Then there is $W = \{(f^\alpha, N^\alpha): \alpha < \alpha^*\} \subseteq \mathfrak{I}_\omega$ and a function $\zeta : \alpha^* \to \lambda$ such that:

(a) W is a strong disjoint barrier, moreover for every stationary $S \subseteq \{\delta < \lambda : cf \ \delta = \aleph_0\}$ $\{(f^\alpha, N^\alpha) : \alpha < \alpha^*, \zeta(\alpha) \in S\}$ is a disjoint barrier.

(c) $cf \ (\zeta(\alpha)) = \aleph_0$ for $\alpha < \alpha^*$.

(d) Every branch of f^α is an increasing sequence converging to $\zeta(\alpha)$.

(e) If $\bar\eta$ is a sequence from $\mathbf{T}$ (of any length $\gamma < \kappa^+$), $\tau(\bar x)$ a term, $\ell(\bar x) = \gamma$ and $\tau(\bar\eta) \in N^\alpha$ then $\bar\eta \subseteq N^\alpha \cap \mathbf{T}$.

(f) If $\zeta(\alpha) = \zeta(\beta)$, $\alpha + \kappa^{\aleph_0} \leq \beta < \alpha^*$ and η is a branch of f^β then $\eta \restriction k \not\in N^\alpha$ for some $k < \omega$.

(g) If $\lambda = \lambda^\kappa$ we can demand: if η is a branch of f^α and $\eta \restriction k \in N^\beta$ for all $k < \omega$ (where $\alpha, \beta < \alpha^*$) then $N^\alpha \subseteq N^\beta$ (and even $N_n^\alpha \in N^\beta$ if $\mathfrak{M} = H_{<\kappa^+}(\mathbf{T})$.)

§2 Preliminaries on Boolean Algebras

We review here some easy material from [Sh 2].

2.1 Definition : 1) For any endomorphism h of a Boolean Algebra B, let $Ex\ Ker(h) = \{x_1 \cup x_2 : h(x_1) = 0$, and $h(y) = y$ for every $y \leq x_2\}$.

$Ex\ Ker^*(h) = \{x \in B :$ in $B / Ex\ Ker(h)$, below $x / Ex\ Ker(h)$, there are only finitely many elements$\}$.

2) A Boolean Algebra is endo-rigid *if* for every endomorphism h of B, $B / Ex\ Ker(h)$ is finite (equivalently: $1_B \in Ex\ Ker^*(h)$).

3) A Boolean algebra is indecomposable *if* there are no two disjoint ideal I_0, I_1 of B, each with no maximal member which generate a maximal ideal $(\{a_0 \cup a_1 : a_0 \in I_0, a_1 \in I_1\})$.

4) A Boolean algebra B is $\aleph_1$-compact if for pairwise disjoint $d_n \in B (n < \omega)$ for some $x \in B$, $x \cap d_{2n+1} = 0$, $x \cap d_{2n} = d_{2n}$.

2.2 Lemma : 1) A Boolean algebra B is endo-rigid *iff* every endomorphism of B is the endomorphism of some scheme (see Definition 2.3 below).

2) A Boolean algebra B is endo-rigid and indecomposable *iff* every endomorphism of B is the endomorphism of some simple scheme (see Def 2.3 below).

2.3 Definition : (1) A scheme of an endomorphism of B consists of a partition $a_0, a_1, b_0, \ldots, b_{n-1}, c_0, \ldots, c_{m-1}$ of 1, maximal nonprincipal ideal I_ℓ below b_ℓ for $\ell < n$, nonprincipal disjoint ideals I_ℓ^0, I_ℓ^0 below c_ℓ which generates a maximal ideal below c_ℓ for $\ell < m$, a number $k < n$, and a partition $b_0^*, \ldots, b_{n-1}^*, c_0^*, \ldots, c_{m-1}^*$ of $a_0 \cup b_0 \cup \cdots \cup b_{k-1}$. We assume also that $[k+m > 0 \Longrightarrow a_0 = 0]$, $[(n-k)+m > 0 \Longrightarrow a_1 = 0]$ and except in those cases there are no zero elements in the partition.

(2) the scheme is simple if $m = 0$.

(3) The endomorphism of the scheme is the unique endomorphism $T{:}B \to B$ such that:

(i) $Tx = 0$ when $x < a_0$ or $x \in I_\ell, \ell < k$, or $x \in I_\ell^0, \ell < m$.

(ii) $Tx = x$ when $x \leq a_1$ or $x \in I_\ell, k \leq \ell < n$ or $x \in I_\ell^1, \ell < m$.

(iii) $T(b_\ell) = b_\ell^*$ when $\ell < k$.

(iv) $T(b_\ell) = b_\ell \cup b_\ell^*$ when $k \leq \ell < n$.

(v) $T(c_\ell) = c_\ell \cup c_\ell^*$ when $\ell < m$.

2.4 Claim: If h is an endomorphism of a Boolean Algebra B,and $B/\, Ex\ Ker(h)$ is infinite *then* there are pairwise disjoint $d_n \in B(n < \omega)$ such that $h(d_n) \not\subseteq d_n$. By easy manipulation we can assume that $h(d_n) \cap d_{n+1} \neq 0$, and if B satisfies the c.c.c. then $\{d_n : n < \omega\}$ is a maximal antichain.

2.5 Lemma : 1) Every endo-rigid Boolean Algebra B is Hopfian and and dual Hopfian. Even $B + B$ is Hopfian (and dual Hopfian) but not rigid.

Proof : Easy to check using 2.2, 2.3.

§3 The Construction.

3.1 Main Theorem : Suppose $\lambda > \aleph_0$. Then there is a B.A. (Boolean Algebra) B such that:

1) B satisfies the c.c.c.

2) B has power $\lambda^{\aleph_0}$, and density character λ.

3) B is endo-rigid and indecomposable.

Proof: We concentrate on the case $cf\ (\lambda) \geq \aleph_1$ (on the case $cf\ \lambda = \aleph_0$ see [Sh 5, §2, §3]) we shall use Theorem 1.3, and let $W = \{(f^\alpha, N^\alpha) : \alpha < \alpha^*\}$, the function ζ, $\mathfrak{M}$ and $\mathrm{T} = {}^{\omega>}\lambda$ be as there.

Stage A: Let B_0 be the B.A. freely generated by $\{x_\eta : \eta \in \mathrm{T}\}$, let $x_\eta = a_\eta$ and B_0^c be its completion. For $A \subseteq B_0^c$ let $\langle A \rangle_{B_0^c}$ be the Boolean subalgebra A generates. As B_0 satisfies the c.c.c every element of B_0^c can be represented as a countable union of members of B_0, so w.l.o.g. $B_0^c \subseteq \mathfrak{M}$. We say $x \in B_0^c$ is based on $J \subseteq {}^{\omega>}\lambda$ if it is based on $\{x_\nu : \nu \in J\}$ [i.e. $X = \bigcup_n y_n$, each y_n is in the subalgebra generated by $\{x_\nu : \nu \in J\}$] and let $\underline{d}(x)$ be the minimal such I. We shall now define by induction on $\alpha < \alpha^*$, the truth value of "$\alpha \in J$ ", η_α, and members a_α, b_n^α, $c_m^\alpha, d_m^\alpha, \tau_m^\alpha$ of B_0^c such that , letting $B_\alpha = \langle B_0\ ,\ a_i : i < \alpha, i \in J \rangle_{B_0^c}$:

1) η_α is a branch of Rang (f^α), $\eta_\alpha \neq \eta_\beta$ for $\beta < \alpha$.

2) if $\alpha \in J$, then for some $\xi < \zeta(\alpha)$:

$a_\alpha = \bigcup_m (\tau_m^\alpha \cap d_m^\alpha)$ where $\langle d_m^\alpha : m < \omega \rangle$ is a maximal antichain of non zero elements (of B_0^c) $\bigcup_m \underline{d}(d_m^\alpha) \subseteq {}^{\omega>}\xi$, $\tau_m^\alpha \in \langle x_\rho : \eta_\alpha \restriction m \leq \rho, \rho \in \mathrm{T} \rangle_{B_0^c}$, and $\tau_m^\alpha \cap d_m^\alpha > 0$.

3) if $\alpha \in J$, $b_n^\alpha, d_n^\alpha \in N_0^\alpha$, $c_n^\alpha, \tau_m^\alpha \in N^\alpha$ (hence each is based on $\{x_\nu : \nu \in {}^{\omega>}\lambda, \nu \in N^\alpha\}$), and $b_n^\alpha \cap b_m^\alpha = 0$ for $n \neq m$.

4) for $\beta < \alpha$, $\beta \in J$, B_α omit $p_\beta = \{x \cap b_n^\alpha = c_n^\alpha : n < \omega\}$.

Remark: Many times we shall write $\beta < \alpha < \alpha^*$ or $w \subseteq \alpha < \alpha^*$ intead $\beta \in \alpha \cap J$, $w \subseteq \alpha \cap J$.

Before we carry the construction note:

3.2 Crucial Fact: For any $x \in B_{\alpha}$ there are $k, \xi < \zeta$, and $\alpha_0 < \cdots < \alpha_k$ such that $\zeta(\alpha_0) = \zeta(\alpha_1) = \zeta(\alpha_2) = \cdots = \zeta(\alpha_k) = \zeta$, x is based on $\{x_\nu : \nu \in {}^{\omega>}\xi \text{ or } \nu \in \underline{d}(\tau_m^{\alpha_\ell})$, for some $\ell \le k$, $m < \omega\}$, but for some $\xi_1 < \zeta$, and for no $m < \omega$ and $\ell \in \{0, \ldots, k\}$ is x based on $\{x_\nu : \eta_{\alpha_\ell} \restriction m \not\trianglelefteq \nu \in {}^{\omega>}\lambda\}$.

Stage B. Let us carry the construction. For $\xi < \lambda, w \subseteq \alpha^*$ let

$$I_{\xi,w} = \{\nu : \nu \in {}^{w>}\xi \quad or \quad \nu \in \bigcup_{\substack{m<\omega \\ \gamma\in w}} \underline{d}(\tau_m^{\gamma})\}$$

We let $\alpha \in J$ *iff* $|N^{\alpha}| \subseteq B_{\alpha}, N^{\alpha} = (B_0^c \restriction |N^{\alpha}|, h_{\alpha})$ where h_{α} is an endomorphism of $B_0^c \restriction |N^{\alpha}|$ (hence maps N_n^{α} into N_n^{α} for $n < \omega$) and there are $d_m^{\alpha} \in N_0^{\alpha}$ for $m < \omega$, $d_m^{\alpha} \ne 0$, $d_m^{\alpha} \cap d_\ell^{\beta} = 0$ for $m \ne \ell$, such that for some $\xi < \zeta(\alpha)$ each d_m^{α} is based on ${}^{\omega>}\xi$, and there are a branch η_{α} of Rang (f^{α}) and $\tau_m^{\alpha} \in N^{\alpha}(m < \omega)$ as in 1), 2) above, such that if we add $\bigcup_{n<\omega} (\tau_n^{\alpha} \cap d_\ell^{\beta})$ to B_{α}, each $p_{\beta}(\beta < \alpha)$ is still omitted as well as $\{x \cap h_{\alpha}(d_m^{\alpha}) = h_{\alpha}(d_m^{\alpha} \cap \tau_m^{\alpha}) : m < \omega\}$ and $\langle d_m^{\alpha} : m < \omega \rangle$ is a maximal antichain.

If $\alpha \in J$ we choose $\eta^{\alpha}, d_n^{\alpha}, \tau_m^{\alpha}$, satisfying the above and let $b_m^{\alpha} = h_{\alpha}(d_m^{\alpha})$, $c_m^{\alpha} = h_{\alpha}(d_m^{\alpha} \cap \tau_m^{\alpha})$.

The Boolean algebra B is B_{α^*}. We shall investigate it and eventually prove it is endo-rigid (in 3.11) and indecomposable (in 3.12) (3.1(1), 3.1(2) are trivial).

Note also

3.3 Fact: 1) For $\nu \in {}^{\omega>}\lambda$, x_ν is free over $\{x_\eta : \eta \in {}^{\omega>}\lambda, \eta \ne \nu\}$ hence also over the subalgebra of B_0^c of those elements based on $\{x_\eta : \eta \in {}^{\omega>}\lambda, \eta \ne \nu\}$.

2) For every branch η of f^{α} such that $\eta \ne \eta_\beta$ for $\beta < \alpha, \xi < \zeta(\alpha)$; and finite $w \subseteq \alpha$ *there is* k such that $\{\rho : \eta \restriction k \trianglelefteq \rho \in \mathbf{T}\}$ is disjoint to ${}^{\omega>}\xi \cup \bigcup\{N^{\beta} \cap \mathbf{T} : \beta \in w, \beta + 2^{\aleph_0} \le \alpha\} \cup \bigcup\{\underline{d}(\tau_n^{\beta}) : n < \omega, \beta \in w\}$.

From 3.2 we can conclude:

3.4 Fact: If $\xi < \zeta(\beta), \beta < \alpha$, $I \subseteq \mathbf{T}$ finite then every element of B_{α}, based on $I \cup {}^{\omega>}\xi$ is in B_β.

3.5 Notation: 1) Let B^ξ be the set of $a \in B_0^c$ supported by ${}^{\omega>}\xi$

2) For $a \in B_0^c$, $\xi < \lambda$ let $pr_\xi(x) = \bigcap \{a \in B^\xi : x \le a\}$.

3) For $\xi < \lambda$ let $\varepsilon(\xi) = Min\{\gamma{:}\zeta(\gamma) > \xi\}$.

4) For $\gamma < \alpha^*$ let $B_{<\gamma>} = \left\langle \{x_\eta{:}\eta \in {}^{\omega>}\zeta(\gamma)\} \cup \{x_\beta{:}\beta < \gamma\}\right\rangle_{B_0^c}$.

5) For $I \subseteq {}^{\omega>}\lambda$, $w \subseteq \alpha^*$, let $B(I,w) = \left\langle \{x_\eta : \eta \in I\} \cup \{x_\beta{:}\beta \in w \cap J\}\right\rangle$.

6) For $\xi < \lambda$ let $B_{[\xi]} = \left\langle \{x_\eta : \eta \in {}^{\omega>}\xi\} \cup \{x_\beta : \zeta(\beta) \le \xi\}\right\rangle_{B_0^c}$.

3.6 Fact: 1) B^ξ is a complete Boolean subalgebra of B_0^c.

2) $pr_\xi(x)$ is well defined for $x \in B_0^c$.

3) if $\xi_0 < \xi_1 < \lambda$, $x \in B_0^c$ then $pr_{\xi_0}(pr_{\xi_1}(x)) = pr_{\xi_0}(x)$.

4) If $\xi < \lambda$, $w \subseteq \mathrm{T}$ is finite then for the function $pr_{\xi,w}(x) = \bigcap \{y \in \left\langle B^\xi \cup \{x_\nu : \nu \in w\}\right\rangle : x \le y\}$ is well defined.

3.7 Fact: 1) For $x \in B_{\alpha^*}$, $\xi < \lambda$, the element $pr_\xi(x)$ belong to $\left\langle B^\xi \cup \{x_\nu : \nu \in w\}\right\rangle$.

2) For $x \in B_{\alpha^*}$, $\xi < \lambda$, $w \subseteq {}^{\omega>}(\xi+1)$, the element $pr_{\xi,w}(x)$ belongs to $B({}^{\omega>}\xi, w)$.

Proof: 1) We prove this for $x \in B_\alpha$, by induction on α (for all ξ).

Note that $pr_\xi(\bigcup\limits_{\ell<n} x_\ell) = \bigcup\limits_{\ell<n} pr_\xi(x_\ell)$.

Case i: $\alpha = 0$, or even $(\forall \beta < \alpha)\ [\zeta(\beta) \le \xi]$.

Easy; if $x = \tau(a_0, \ldots, a_{n-1}, x_{\nu_0}, \ldots, x_{\nu_{m-1}})$ where τ is a Boolean term, $a_\ell \in B_{[\xi]}$, $\nu_\ell \in {}^{\omega>}\lambda - {}^{\omega>}\xi$; by the remarks above w.l.o.g. $x = \bigcap\limits_{\ell<n+m} \tau_\ell$, $\tau_\ell \in \{a_\ell, 1-a_\ell\}$ when $\ell < n$, $\tau_\ell \in \{x_{\nu_{\ell-n}}, 1-x_{\nu_{\ell-n}}\}$ when $n \le \ell < n+m$, and the sequence $\left\langle x_{\nu_0}, \ldots, x_{\nu_{n-1}}\right\rangle$ is with no repetition, then clearly $pr_\xi(x) = \bigcap\limits_{\ell<n} \tau_\ell \in B_{[\xi]}$;

Case ii: α limit.

Trivial as $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$.

Case iii: $\alpha = \beta + 1$.

By the induction hypothesis w.l.o.g. $x \notin B_\beta$. As $x \in B_\alpha$ there are disjoint $e_0, e_1, e_2 \in B_\beta$ such that $x = e_0 \cup (e_1 \cap a_\beta) \cup (e_2 - a_\beta)$. It suffices to prove that $pr_\xi(e_0)$, $pr_\xi(e_1 \cap a_\beta)$, $pr_\xi(e_2 - a_\beta) \in B_{[\xi]}$, the first is trivial and w.l.o.g. we concentrate on the second. There are $\xi_0 < \zeta(\beta)$ and $k < \omega$ such that e_1 is based on $J \stackrel{def}{=} {}^{\omega>}\lambda - \{\rho : \eta_\beta \restriction k \trianglelefteq \rho \in {}^{\omega>}\lambda\}$and each $d_n^\beta (n < \omega)$ is based on ${}^{\omega>}\xi_0$. By Case i, we can assume $\xi < \zeta(\beta)$ hence w.l.o.g. $\xi < \xi_0$, and by the induction hypothesis and 3.6(3) it suffices to prove $pr_{\xi_0}(e_1 \cap a_\beta) \in B_{[\xi]}$. W.l.o.g. $e_1 \cap d_m^\alpha = 0$ for $m < k$ and now clearly $pr_{\xi_0}(e_1 \cap a_\beta) = e_1$ as $pr_{\xi_0}(e_1 \cap d_m^\alpha \cap \tau_m^\alpha) = e_1 \cap d_m^\alpha$ for $m \geq k$, (because d_m^α, e_1 are based on J, ${}^{\omega>}\xi_0 \subseteq J$ and τ_m^α is based on ${}^{\omega>}\lambda - J$ and is > 0).

2) Same proof.

3.8 Lemma : Suppose I,w satisfies:

$(*)_{I,w}$ $I \in {}^{\omega>}\lambda$, $w \subseteq \alpha^*$, I is closed under initial segments, and for every $\alpha < \alpha^*$ *if* $\bigwedge_{m<\omega} (\eta_\alpha \restriction m \in I)$ then $\tau_m^\alpha, d_m^\alpha$ are based on I and belong to $B(I,w)$.

Then for any countable $C \subseteq B_{\alpha^*}$ there is a projection from $\langle B(I,w), C \rangle_{B_0^c}$ onto $B(I,w)$.

Proof : We can easily find $I(*), w(*)$ such that $w \subseteq w(*) \subseteq \alpha^*$, $|w(*) - w| \leq \aleph_0$, $I \subseteq I(*) \subseteq {}^{\omega>}\lambda$, $|I(*) - I| \leq \aleph_0$ and if $\alpha \in w(*) - w$, then $\tau_m^\alpha, d_m^\alpha \in B(I(*), w(*))$. Let $w(*) - w = \{\alpha_\ell : \ell < \omega\}$, and we define by induction on ℓ a natural number $k_\ell < \omega$, such that the sets $\{\nu \in {}^{\omega>}\lambda : \nu$ appears is $\tau_m^{\alpha_\ell}$ for some $m > k_\ell\}$ are pairwise disjoint and disjoint to I. Now we can extend the identity on $B(I,w)$ to a projection h_0 from $B(I(*),w)$ onto $B(I,w)$ such that if $\ell < \omega, m > k_\ell$, then $h_0(\tau_m^{\alpha_\ell} \cap d_m^{\alpha_\ell}) = 0$. Now we can define by induction on $\alpha \in (w(*) - w) \cup \{0, \lambda\}$ a projection h_α from $B(I(*), w \cup (w(*) \cap \alpha))$ onto

$B(I,w)$ extending h_β for $\beta < \alpha$ (and $\beta \in (w(*)-w) \cup \{0\}$). For $\alpha = 0$ we have defined, for $\alpha = \lambda$ we get the conclusion, and in limit stages takes the union. In successive stages there is no problem by the choice of h_0, and the k_ℓ's.)

3.9 Claim: If B' is an uncountable subalgebra of B_{α^*} *then* there is an antichain $\{d_n : n < \omega\} \subseteq B'$ and for no $x \in B$, $x \cap d_{2n} = 0$, $x \cap d_{2n+1} = d_n$ for every n provided that

(*) no one countable $I \subseteq {}^{\omega>}\lambda$ is a support for every $a \in B'$.

Proof : We now define by induction on $\alpha < \omega_1, d_\alpha, I_\alpha$, such that:

(i) $I_\alpha \subseteq {}^{\omega>}\lambda$ is countable.

(ii) $\bigcup_{\beta<\alpha} I_\beta \subseteq I_\alpha$ and for α limit, equality holds.

(iii) $d_\alpha \in B'$ is supported by $I_{\alpha+1}$ but not by I_α.

There is no problem in this.

By (iii) for each α there are $\tau_\alpha^0 \in \langle a_\eta : \eta \in I_\alpha \rangle_{B_0^c}$, $\tau_\alpha^1, \tau_\alpha^2 \in \langle a_\eta : \eta \in I_{\alpha+1} - I_\alpha \rangle_{B_0^c}$ such that $\tau_\alpha^1 \cap \tau_\alpha^2 = 0$, $\tau_\alpha^0 \cap \tau_\alpha^1 \leq d_\alpha$, $\tau_\alpha^0 \cap \tau_\alpha^2 \leq 1 - d_\alpha$.

By Fodour's lemma w.l.o.g. $\tau_\alpha^0 = \tau^0$ (i.e. does not depend on α). For each α there is $n(\alpha) < \omega$ such that

$$\tau_\alpha^0 \in \langle a_\eta : \eta \in I_\alpha \cap {}^{n(\alpha)\geq}\lambda \rangle_{B_0^c}, \tau_\alpha^1, \tau_\alpha^2 \in \langle a_\eta : \eta \in (I_{\alpha+1} - I_\alpha) \cap {}^{n(\alpha)\geq}\lambda \rangle_{B_0^c}$$

Again by renaming w.l.o.g. $n(\alpha) = n(*)$ for every α. Let for $n < \omega$, $d^n = d_n - \bigcup_{\ell<n} d_\ell$, $\tau^n = \tau^0 \cap \bigcap_{\ell<n} \tau_\ell^2 \cap \tau_n^1$, so easily $d^n \in B'$, $\langle d^n : n < \omega \rangle$ is an antichain, $\tau^n \leq d^n$ and $\tau^n \in \langle a_\eta : \eta \in {}^{n(*)\geq}\lambda \rangle_{B_0^c}$. Suppose $x \in B$ $x \in B, x \cap d^{2n} = 0$, $x \cap d^{2n+1} = d^{2n+1}$. Hence for $n < \omega$, $x \cap \tau^{2n} = 0$, $x \cap \tau^{2n+1} = \tau^{2n+1}$. But by 3.8 (for $I = {}^{n(*)\geq}\lambda$), there is such x in $\langle a_\eta : \eta \in {}^{n(*)\geq}\lambda \rangle_{B_0^c}$, an easy contradiction.

So we have proven that for every $\aleph_1$-compact $B' \subseteq B_{\alpha^*}$, some countable

$I \subseteq {}^{\omega>}\lambda$ support every $x \in B'$.

3.10 Claim: No infinite subalgebra B' of B_{α^*} is $\aleph_1$-compact.

Proof : Suppose there is such B' , and let ξ be minimal such that there is such $B' \subseteq B_{[\xi]}$.

Part I: if (*) a) $B' \subseteq B_{\alpha^*}$ is $\aleph_1$-compact and infinite and

b) $B' \subseteq B_{[\xi]}$,

then

c) for every $\zeta < \xi$ and $x \in B' - \{y : \{z \in B' : z \le y\}$ is finite$\}$, there is $x_1 \in B', x_1 \le x$ such that for no $y \in B_{[\zeta]}$, $y \cap x = x_1$.

So assume B', satisfies a) and b) but they fail c) for $\zeta < \xi$ and $x \in B'$, where $\{y : y \le x, y \in B'\}$ is infinite. So for every $z \in B'$, there is $g(z) \in B_{[\zeta]}$ such that $g(z) \cap x = z \cap x$ (use $x_1 = z \cap x$). Let B^a be the subalgebra of $B_{[\zeta]}$ generated by $\{g(z) : z \in B'\}$. Clearly $\{y \in B' : y \le x\} = \{t \cap x : t \in B^a\}$. Let $x^* = pr_\zeta(x)$, (it is in $B_{[\zeta]}$ by 3.7(1)) and let $B^b = \{t \cap x^* : t \in B^a\} \cup \{t \cup (1-x^*) : t \in B^a\}$. Clearly B^b is a subalgebra of $B_{[\zeta]}$, and $1-x^*$ is an atom of B^b;B^b is infinite as there are in B' distinct $x_n \le x$, so $g(x_n) \in B^a$ hence $g(x_n) \cap x^* \in B^b$, as $x \le x^*$ and $[n \ne m \Longrightarrow g(x_n) \cap x \ne g(x_m) \cap x]$ clearly $[n \ne m \Longrightarrow g(x_n) \cap x^* \ne g(x_n) \cap x^*]$. We shall prove that B^b is $\aleph_1$-compact, thus contradicting the choice of ξ. Let $d_n \in B^b$ be pairwise disjoint, and we want to find $t \in B^b$, $t \cap d_{2n} = 0$, $t \cap d_{2n+1} = d_{2n+1}$ (for $n < \omega$). Clearly w.l.o.g. $d_n \le x^*$ (as $1-x^*$ is an atom of B^b). So $d_n = t_n \cap x^*$ for some $t_n \in B^a$, hence easily $t_n \cap x \in B'$ so for some $x_n \in B'$, $x_n \le x$ and $t_n \cap x = x_n \cap x = x_n$. So $x_n = g(x_n) \cap x$. For $n \ne m$,

$$x_n \cap x_m = (t_n \cap x) \cap (t_m \cap x) \le (t_n \cap x^*) \cap (t_m \cap x^*) = d_n \cap d_m = 0$$

As B' is $\aleph_1$-compact there is $y \in B'$, $y \cap x_{2n} = 0$, $y \cap x_{2n+1} = x_{2n+1}$. Now $g(y), d_n, t_n$ belongs to $B_{[\zeta]}$ and (as $x_n \le x \le x^*$):

(i) $g(y) \cap d_{2n} \cap x = g(y) \cap t_{2n} \cap x =$
$g(y) \cap x_{2n} \cap x = y \cap x_{2n} \cap x = 0.$

(ii) $g(y) \cap d_{2n+1} \cap x = g(y) \cap t_{2n+1} \cap x = g(y) \cap x_{2n+1} \cap x = y \cap x_{2n+1} \cap x = x_{2n+1} \cap x = t_{2n+1} \cap x = d_{2n+1} \cap x$.

Now by the definition of $x^* = pr_\xi(x)$, $[\tau \in B_{[\xi]} \wedge \tau \cap x = 0 \Longrightarrow \tau \cap x^* = 0]$ (as $1-\tau \in B_{[\xi]}$, $x \leq 1-\tau$) hence by (i) (for $\tau = g(y) \cap d_{2n}$):

(iii) $g(y) \cap d_{2n} \cap x^* = 0$.

Also by the definition of $x^* = pr_\xi(x)$:

$$\tau_1, \tau_2 \in B_{[\xi]} \wedge \tau_1 \cap x = \tau_2 \cap x \Longrightarrow \tau_1 \cap x^* = \tau_2 \cap x^*$$

(as $\tau_1 - \tau_2 \in B_{[\xi]}, x \leq 1-(\tau_1 - \tau_2)$) hence by (ii)

(iv) $g(y) \cap d_{2n+1} \cap x^* = d_{2n+1} \cap x^*$.

But $d_n \leq x^*$, so from (iii) and (iv) $(g(y) \cap x^*) \cap d_{2n} = 0$, $(g(y) \cap x^*) \cap d_{2n+1} = d_{2n+1}$, and $g(y) \in B^a$ hence $g(y) \cap x^* \in B^b$. So B^b is $\aleph_1$-compact this contradicts the minimality of ξ, so we finish Part I.

Part II: if B^1 is $\aleph_1$-compact $B^1 \subseteq B^2$, $B^2 = \langle B^1 \cup \{z\} \rangle$ then B^2 is $\aleph_1$-compact.

The proof is straightforward. [If $d_n \in B^2$ are pairwise disjoint, let $d_n = (d_n^1 \cap z) \cup (d_n^2 - z)$ for some $d_n^1, d_n^2 \in B^1$. Now w.l.o.g $d_n^1 \cap d_m^1 = 0$ for $n \neq m$ - otherwise replace then by $d_n^1 - \bigcup_{\ell < n} d_\ell^1$; Similarly $d_n^2 \cap d_m^2 = 0$, for $n \neq m$ So there are $y^\ell \in B^1$, $y^\ell \cap d_{2n}^\ell = 0$, $y^\ell \cap d_{2n+1}^\ell = d_{2n+1}^\ell = d_{2n+1}^\ell$, and $(y^1 \cap z) \cup (y^2 - z)$ is the solution.]

Part III. ξ cannot be a successor ordinal.

Proof: Let B' satisfy (*).

Suppose $\xi = \zeta + 1$, and by 3.9 there is a countable $I \subseteq {}^{\omega>}\xi$ which support every $a \in B'$. w.l.o.g. I is closed under initial segments and $k = |I - {}^{\omega>}\zeta|$ is minimal. Now Part I can be applied with $\langle B_{[\zeta]}, \{a_\eta : \eta \in w\} \rangle_{B_0^\xi}$, for any finite $w \subseteq I$ of power $< k$ instead $B_{[\xi]}$ (using 3.7(2) instead 3.7(1)). So by applying Part I (to $\langle B_{[\zeta]}, \{a_\eta : \eta \in w\} \rangle_{B_0^\xi}$) we can add to its conclusion:

d) *for every* finite $w \subseteq I$, $|w| < |I - {}^{\omega>}\zeta|$ and $x \in B'$ for which $\{y \in B' : y \leq x\}$ is infinite, *there is* $x_1 \in B', x_1 \leq x$ *such that for* no $y \in \langle B_{[\zeta]} \cup \{a_\eta : \eta \in w\}\rangle_{B_0^c}$, $y \cap x = x_1$.

Now $I - {}^{\omega>}\zeta$ is infinite [otherwise let $B'' = \langle B' \cup \{a_\eta : \eta \in I - {}^{\omega>}\zeta\}\rangle_{B_0^c}$, easily it is infinite and $\aleph_1$-compact by Part II and then we apply Part I : for $I - {}^{\omega>}\zeta = \{\eta_0, \ldots, \eta_{k-1}\}$ and for $u \subseteq \{0, \ldots, k-1\}$, let $x_u \stackrel{def}{=} \bigcap \{x_{\eta_\ell} : \ell \in u\} \cap \{1 - x_{\eta_\ell} : \ell < k, \ell \notin u\}$ so $x_u \in B''$, $1 = \bigcup\{x_u : u \subseteq \{0, \ldots, k-1\}\}$, hence for some u, $\{y \in B'' : y \leq x_u\}$ is infinite; ζ, x_u contradict the conclusion of Part I.

As B' is $\aleph_1$-compact, for any $x \in B'$ such that $\{y \in B' : y \leq x\}$ is infinite, x can be splitted in B' to two elements satisfying the same i.e. $x = x^1 \cup x^2$; $x^1 \cap x^2 = 0$, $\{y \in B' : y \leq x^\ell\}$ is infinite for the $\ell = 1,2$. Let $I - {}^{\omega>}\zeta = \{\eta_\ell : \ell < \omega\}$, so we can find pairwise disjoint $e_n \in B'$, $\{y \in B' : y \leq e_n\}$ is infinite; now by d) above for each n we can find d_{2n}, d_{2n+1}, such that $e_n = d_{2n} \cup d_{2n+1}$, $d_{2n} \cap d_{2n+1} = 0$ and that for no $y \in \langle B_{[\zeta]} \cup \{a_{\eta_\ell} : \ell < n\}\rangle$, $y \cap (d_{2n} \cup d_{2n+1}) = d_{2n+1}$. As B' is $\aleph_1$-compact there is $y \in B'$ such that $y \cap (d_{2n} \cup d_{2n+1}) = d_{2n+1}$ for every n. So for no n $y \in \langle B_{[\zeta]} \cup \{a_{\eta_\ell} : \ell < n\}\rangle_{B_0^c}$.

As $y \in B'$ clearly $y \in B_{[\zeta+1]}$, but y is based on ${}^{\omega>}\zeta \cup \{a_{\eta_\ell} : \ell < \omega\}$ so by 3.7(2) $y \in \langle B_{[\zeta]} \cup \{a_{\eta_\ell} : \ell < \omega\}\rangle_{B_0^c}$, hence by Stage B for some n, $y \in \langle B_{[\zeta]} \cup \{a_{\eta_\ell} : \ell < n\}\rangle_{B_0^c}$, contradiction to $y \cap (d_{2n} \cup d_{2n+1}) = d_{2n+1}$.

Part IV: Let B', satisfy (*) of Part I. By 3.9 for some countable $I \subseteq {}^{\omega>}\xi$, every $b \in B'$ is based on I. By Part III ξ is not a successor ordinal, so necessarily $cf(\xi) = \aleph_0$, let $Fi(B') = \{x \in B' : \{y \in B' : y \leq x\}$ is finite$\}$. Next we shall show:

(**) for some finite $w \subseteq \{\gamma : \zeta(\gamma) = \xi\}$ and $x^* \in B' - Fi(B')$ for every $y < x^*$ from B', for some $z \in \langle \bigcup_{\zeta<\xi} B_{[\zeta]} \cup \{a_\alpha : \alpha \in w\}\rangle_{B_0^c}$, $z \cap x^* = y$.

Suppose (**) fail, and we define by induction $n < \omega$, x_n, y_n, w_n such that :

(i) $x_n \in B'$,

(ii) $1-\bigcup_{i<n} x_i \notin Fi(B')$

(iii) $w_n \subseteq \{\gamma : \zeta(\gamma) = \xi\}$ is finite.

(iv) $w_n \subseteq w_{n+1}$

(v) $y_n \leq x_n, y_n \in B'$.

(vi) for no $z \in \left\langle \bigcup_{\zeta<\xi} B_{[\zeta]} \cup \{a_\alpha : \alpha \in w_n\}\right\rangle_{B_0^c}$ is $z \cap x_n = y_n$.

For $n = 0$ $1 \notin Fi(B')$.

For every n let w_n be a finite subset of $\{\gamma : \zeta(\gamma) = \xi\}$ extending $\bigcup_{\ell<n} w_\ell$, such that for every $\ell < n$, $x_\ell, y_\ell \in \left\langle \bigcup_{\zeta<\xi} B_{[\zeta]} \cup \{a_\alpha : \alpha \in w_n\}\right\rangle_{B_0^c}$. Then as $1-\bigcup_{\ell<n} x_i \notin Fi(B')$, and as B' is $\aleph_1$-compact, there is $x_n \leq 1-\bigcup_{i<n} x_i$, $x_n \in B'$, $1-\bigcup_{\ell\leq n} x_i \notin Fi(B')$ and $x_n \notin Fi(B')$. Now as (**) fails, w_n, x_n does not satisfy the requirements on w, x^* in (**), so there is $y_n \in B'$, $y_n \leq x_n$ such that for no $z \in \left\langle \bigcup_{\zeta<\xi} B_{[\zeta]} \cup \{a_\alpha : \alpha \in w_n\}\right\rangle_{B_0^c}$ is $z \cap x_n = y_n$.

As B' is $\aleph_1$-compact, for some $z^* \in B'$, $z^* \cap x_n = y_n$ for every n. As $z^* \in B'$ for some finite $w^* \subseteq \varepsilon(\xi)$, $z^* \in \left\langle \bigcup_{\zeta<\xi} B_{[\zeta]} \cup \{a_\alpha : \alpha \in w^*\}\right\rangle_{B_0^c}$. As w^* is finite, for some $n(*) < \omega$, $w^* \cap (\bigcup_{n<\omega} w_n) \subseteq w_{n(*)}$. Let $\zeta < \xi$ be such that: $\underline{d}(d_n^\alpha) \subseteq {}^{\omega>}\zeta$ for $\alpha \in w_{n(*)+1} \cup w^*$, $n < \omega$ and $x_n, y_n \in \left\langle B_{[\zeta]} \cup \{a_\alpha : \alpha \in w_{n(*)+1}\}\right\rangle_{B_0^c}$, for $n \leq n(*) + 1$ and $z^* \in \left\langle B_{[\zeta]} \cup \{a_\alpha : \alpha \in w^*\}\right\rangle_{B_0^c}$. By 3.8 we can easily get a contradiction to (vi). So (**) holds.

Let $t_0, \dots, t_m \in B_{[\xi]}$ be such that $\bigcup_{\ell=1}^{m} t_\ell = 1$ and $(\forall \ell \leq m)(\forall \alpha \in w)$ $[t_\ell \leq a_\alpha \vee t_\ell \cap a_\alpha = 0]$. There is an $\ell \leq m$ such that $\{y \cap t_\ell : y \leq x^*$ and $y \in B'\}$ is infinite. It is clear (by Part II) that $B'' = \langle B', t_\ell \rangle_{B_0^c}$ is $\aleph_1$-compact: also $x^* \cap t_\ell \in B'' - Fi(B'')$. Now if $y \in B'', y \leq x^* \cap t_\ell$ then for some $y' \in B', y = y' \cap t_\ell$ and w.l.o.g. $y' \leq x^*$, so for some $z \in \left\langle \bigcup_{\zeta<\xi} B_{[\zeta]} \cup \{a_\alpha : \alpha \in w\}\right\rangle_{B_0^c}$ $z \cap x^* = y'$ hence $z \cap (x^* \cap t_\ell) = y$, and by the choice of t_ℓ, for some $z' \in \bigcup_{\zeta<\xi} B_{[\zeta]}$, the equation $z' \cap (x^* \cap t_\ell) = z \cap (x^* \cap t_\ell) = y$ holds.

So B'', $x^{**} \overset{\mathrm{def}}{=} x^* \cap t_\ell$ satisfy the requirements in (*). Now we use (c) of Part I. As $cf(\xi) = \aleph_0$, let $\xi = \bigcup_{n<\omega} \zeta_n$, and we define by induction on $n < \omega, x_n, y_n$ such that :

(i) $x_n \in B''$, $x_n \leq x^{**}$

(ii) $x^{**} - \bigcup_{\ell<n} x_i \notin Fi(B'')$

(iii) $y_n \in B'$, $y_n \leq x_n$

(iv) for no $z \in B_{[\zeta_n]}$, $z \cap x_n = y_n$.

As B'' is $\aleph_1$-compact, for some $z^* \in B''$, $z^* \cap x_n = y_n$ for each n.

Now as B'', x^{**} satisfy (**), for some $z^{**} \in \bigcup_{\zeta<\xi} B_{[\zeta]}$ $z^* \cap x^{**} = z^{**} \cap x^{**}$. So for some n $z^{**} \in B_{[\zeta_n]}$, contradicting (iv) above. Thus we have finished the proof of 3.9.

3.11 Claim: B_{α^*} is endo-rigid.

Proof: Suppose h is as counterexample, i.e. h is an endomorphism of B_{α^*} but $B_{\alpha^*}/\ Ex\ Ker(h)$ is infinite, and we shall get a contradiction.

Clearly if for some α, $N^\alpha = (|N^*|, h \restriction N^\alpha)$, h maps $N^* \cap B_{\alpha^*}$ into itself and $\alpha \in J$ (see Stage B) then $h(a_\alpha)$ realizes the type p_α, contradiction (by stage A, B_{α^*} omits p_α.) So we shall try to find such α which satisfy the requirements in Stage B for belonging to J. We assume $N^\alpha = (|N^\alpha|, h_\alpha)$, $|N^\alpha| \subseteq B_\alpha$, $h_\alpha = h \restriction N^\alpha$, h_α maps $N^\alpha \cap B_\alpha$ onto itself, and N_0^α contains some elements we need and somewhat more (see latter). As W is a barrier this is possible. We then will choose η_α, an ω-branch of f^α, distinct from η_β for $\beta < \alpha$ [if $\beta + 2^{\aleph_0} \leq \alpha$ this follows, the rest exclude $< 2^{\aleph_0}$ branches of f^α but there are $2^{\aleph_0}$ such branches], a maximal antichain $\langle d_n : n < \omega \rangle$ of B_α, $d_n \in N_0^\alpha$, and $\tau_n \in N^\alpha$ in $\langle x_\nu : \eta_\alpha \restriction n \underline{\prec} \nu \in \mathrm{T} \rangle_{B_0^c}$, and let $b_n = h(d_n)$, $c_n = h(d_n \cap \tau_n)$,

$p_\alpha = \{x \cap b_n = c_n : n < \omega\}$, and $a_\alpha = \bigcup_{n<\omega} (d_n \cap \tau_n) \in B_0^c$. All should have superscript $\bar{d},\bar{\tau}$ (where $\bar{d} = \langle d_n : n < \omega \rangle$, $\bar{\tau} = \langle \tau_n : n < \omega \rangle$) but we usually omit them or write $a_\alpha[\bar{\tau},\bar{d}]$, $p_\alpha[\bar{\tau},\bar{d}]$ etc.

The choice of $\bar{d},\bar{\tau}$ (and η_α which is determined by $\bar{\tau}$) is done by listing the demands on them (see Stage B) and showing a solution exists. The only problematic one is (a) (omitting p_β for $\beta \le \alpha$) and we partition it to three cases :

(I) $\zeta(\beta) < \zeta(\alpha)$ or $\zeta(\beta) = \zeta(\alpha)$, $\beta + 2^{\aleph_0} \le \alpha$,

(II) $\zeta(\beta) = \zeta(\alpha)$, $\beta < \alpha < \beta + 2^{\aleph_0}$.

(III) $\beta = \alpha$.

We shall prove that every $\bar{\tau},\bar{d}$ are O.K. for (I), that for any family $\{(\bar{d}^i,\eta^i,\bar{\tau}^i) : i < 2^{\aleph_0}\}$ (η a branch of f^α, etc.) with pairwise distinct η^i's, all except $< 2^{\aleph_0}$ many are O.K. for instance of (II), and that there is a family of $2^{\aleph_0}$ triples $(\bar{d},\eta,\bar{\tau})$ satisfying (III) with pairwise distinct η^i's. This clearly suffices.

Case I: $\zeta(\beta) < \zeta(\alpha)$ or $\zeta(\beta) = \zeta(\alpha), \beta + 2^{\aleph_0} \le \alpha$.

Suppose some $x \in \langle B_\alpha, a_\alpha[\bar{\tau},\bar{d}] \rangle_{B_0^c}$ realizes p_β. Clearly there is a partition $\langle e_\ell : \ell < 4 \rangle$ of 1 (in B_α) such that $x = e_0 \cup (e_1 \cap a_\alpha[\bar{\tau},\bar{d}]) \cup (e_2 - a_\alpha[\bar{\tau},\bar{d}])$. Choose $\xi < \zeta(\alpha)$ large enough and finite $w \subseteq \alpha$ so that $[\zeta(\beta) < \zeta(\alpha) \Longrightarrow \zeta(\beta) < \xi]$, $d_n, h_\alpha(d_n)$ b_n^β, are based on $\{x_\nu : \nu \in {}^{\omega >}\xi\}$ (for $n < \omega$) and c_ℓ^β $(\ell < \omega)$, e_0, e_1, e_2, e_3 are based on $J = \{\nu \in \mathbf{T} : \eta_\alpha \restriction k \not\trianglelefteq \nu\}$, where $k < \omega$ also satisfies such that $\eta_\alpha(k) > \xi$, $\eta_\alpha \restriction k \not\in N_\beta$.

We claim:

(*) there is $m < \omega$ such that $b_m^\beta \cap (e_1 \cup e_2) - \bigcup_{n \le k} d_n \ne 0$.

For suppose (*) fail, then as $a_\alpha[\bar{\tau},\bar{d}] \cap (\bigcup_{n \le k} d_n) \in B_\alpha$, w.l.o.g. $(e_1 \cup e_2) \cap \bigcup_{n \le k} d_n = 0$ (otherwise let

$$e_0' = e_0 \cup (e_1 \cap a_\alpha[\bar{\tau},\bar{d}] \cap \bigcup_{n\le k} d_n) \cup (e_2 \cap \bigcup_{n\le k} d_n - a_\alpha[\bar{\tau},\bar{d}])$$

$$e_1' = e_1 - \bigcup_{n\le k} d_n,$$

$$e_1' = e_2 - \bigcup_{n\le k} d_n).$$

so if x realizes p_β then so does e_0, but $e_0 \in B_\alpha$ contradicting an induction hypothesis . So (*) holds.

Now as $\langle d_n : n < \omega \rangle$ is a maximal antichain in B_α, for some $\ell < \omega$, $d_\ell \cap (b_m^\beta \cap (e_1 \cup e_2 - \bigcup_{n\le k} d_n)) \ne 0$. Necessarily $\ell > k$. So for some $\varepsilon \in \{1,2\}$, $d_\ell \cap b_m^\beta \cap e_\varepsilon \ne 0$. As x realizes p_β, $x \cap (d_\ell \cap b_m^\beta \cap e_\varepsilon) = d_\ell \cap c_n^\beta \cap e_\varepsilon$ which is based on J. But we know that $x \cap (d_\ell \cap b_\varepsilon^\beta \cap e_\varepsilon)$ is $d_\ell \cap b_m^\beta \cap e_1 \cap a_\alpha[\bar{\tau},\bar{d}] = d_\ell \cap b_m^\beta \cap e_1 \cap \tau_\ell$ (if $\varepsilon=1$) or $d_\ell \cap b_m^\beta \cap e_2 \cap (1-a_\alpha[\bar{\tau},\bar{d}]) = d_\ell \cap b_m^\beta \cap e_2 \cap 1-\tau_\ell$ (if $\varepsilon=2$). As $d_\ell \cap b_m^\beta \cap e_\varepsilon \ne 0$ is based on J, $\ell > k, \eta_\alpha(k) > \xi$, τ_ℓ is free over J, (see Fact 3.3(2)) necessarily $x \cap (d_\ell \cap b_m^\beta \cap e_\varepsilon)$ is not based on J, contradiction.

Case II: $\beta < \alpha < \beta + 2^{\aleph_0}$.

We shall prove that if $\eta^\ell, \bar{\tau}^\ell$ are appropriate (for $\ell = 1, 2$) and $\eta^1 \ne \eta^2$ then p_β cannot be realized in both $\langle B_\alpha, a[\bar{\tau}_\ell,\bar{d}] \rangle_{B_0^c}$.

As there is a perfect set of appropriate η's it will suffice to prove that for each ω-branch η of Rang (f^α) for some appropriate $\bar{\tau}$ $\langle B_\alpha, a^{\bar{\tau}} \rangle_{B_0^c}$ omit $p_\alpha = p_\alpha[\bar{\tau},\bar{d}]$ which will be in done in Case III.

Note that $I_\beta^\alpha = \{e \in B_\alpha :$ for some $x \le e$ for every n $x \cap b_\beta^n \cap e = c_\beta^n \cap e\}$ is an ideal.

The details are easy.

Case III: $\beta = \alpha$.

This case is splitted into several subcases. Let η_α be any ω-branch of f^α, $\eta_\alpha \ne \eta_\beta$ whenever $\beta < \alpha < \beta + 2^{\aleph_0}$. Let $I^* = \bigcup\{\underline{d}(h(x)) : x \in B_\alpha\}$. We shall assume $|I^*| \leq \aleph_0 \Longrightarrow I^* \subseteq N_0^\alpha$, so in this case p_α is omitted by $B_{\alpha+1}$ or B_{α^*} iff it is omitted by B_α (by 3.7(1)). As accomplishing our aim is easier we shall ignore this case (work as in III 4 and use quite arbitrary p_β.

Subcase III 1.: For some $\rho^* \in T$, and $a^* \in B_\alpha - Ex\ Ker^*(h)$ for every $\rho, \rho^* \leq \rho \in T$ for some $\tau \in \langle x_\eta : \rho \leq \eta \in T \rangle_{B_0^\alpha}$, $\tau \cap a^* \neq 0 = h(\tau \cap a^*)$.

As we are interested not in (f^α, N^α) itself, but in h, by using $Gm'(W)$, w.l.o.g. $\rho^* \in$ Range (f^α). By 3.9 (for Rang (h), which by assumption, is infinite) and easy manipulations (see 2.4 and [Sh 2]) there is a maximal antichain $\langle d_n : n < \omega \rangle$ of B_{α^*} such that for no $x \in B_\alpha$, $x \cap h(d_{2n}) = h(d_{2n})$ and $x \cap h(d_{2n+1}) = 0$. W.l.o.g. $\{d_n : n < \omega\} \subseteq N_0^\alpha$.

It suffices to prove the conclusion for any ω-branch η_α of Range (f^α), $\rho^* < \eta_\alpha \not\in \{\eta_\beta : \beta < \alpha\}$. We define by induction on n, $\tau_n \in N_n^\alpha$, $\tau_n \in \langle x_\eta : \eta_\alpha \restriction n \leq \eta \rangle_{B_0^\alpha}$, $\tau_n \neq 0,1$ and $h(\tau_{2n}) = 1, h(\tau_{2n+1}) = 0$. (possible by the assumption of subcase III 1), so we finish this subcase.

Subcase III 2. For some $a^* \in B_\alpha$, $\{h(x) - a^* : x \in B_*, x \leq a^*\}$ is infinite.

Clearly $B^a = \{h(x) - a^* : x \in B_{\alpha^*}, x \leq a^*\} \cup \{1 - (h(x) - a^*) : x \in B_{\alpha^*}, x \leq a_\alpha\}$ is a subalgebra of B_{α^*} (with a^* an atom). By assumption (of this subcase) B^a is infinite. So by 3.9 there are $e_n \in B^\alpha$, pairwise disjoint, and $\neg(\exists x \in B_a) \bigwedge_n (x \geq e_{2n} \wedge x \cap e_{2n+1} = 0)$. As a^* is an atom of B^α w.l.o.g. $e_n \leq 1 - a^*$, hence there is $d_n \leq a^*$ (in B_{α^*}), such that $h(d_n) = e_n$. Clearly $h(d_n - \bigcup_{\ell<n} d_n) = e_n - \bigcup_{\ell<n} e_\ell = e_n$, so w.l.o.g. the d_n are pairwise disjoint. So by easy manipulation for some $\langle d_n : n < \omega \rangle$ the following holds:

(i) $d_0 = 1 - a^*$

(ii) $\langle d_n : n < \omega \rangle$ is a maximal antichain of B_{α^*}.

(iii) for no $x \leq 1 - a^*$, $x \cap h(d_{2n+2}) - a^* = h(d_{2n+2}) - a^*$,

$$x \cap h(d_{2n+1}) - a^* = 0$$

We can assume that d_n, $h(d_n) \in N_0^\alpha$.

Let $\bar{\tau}^0 = \langle \tau_n^0 : n < \omega \rangle$ be a suitable sequence, (for our η_α) then so are $\bar{\tau}^\ell = \langle \bar{\tau}_n^\ell : n < \omega \rangle$, for $\ell < 4$ where

$$\tau_{2n}^1 = 1 - \tau_{2n}^0,\ \tau_{2n+1}^1 = \tau_{2n+1}^0;$$

$$\tau_{2n}^2 = \tau_{2n}^0, \qquad \tau_{2n+1}^2 = 1 - \tau_{2n+1}^0;$$

$$\tau_{2n}^3 = 1 - \tau_{2n}^0, \quad \tau_{2n+1}^3 = 1 - \tau_{2n+1}^0.$$

Suppose for each $\ell < 4$, in $\langle B_\alpha, a_\alpha[\bar{\tau}^\ell, \bar{d}] \rangle_{B_0^c}$ there is an element y^ℓ which satisfies $y^\ell \cap h(d^\ell) - a^* = h(\tau_n^\ell \cap d_n) - a^*$ for $1 \le n < \omega$. W.l.o.g. $y^\ell \le 1 - a^* = d_0$ hence $y^\ell \in B_\alpha$. Now $(y^0 \cup y^1) \cap (y^2 \cup y^3) \in B_\alpha$ contradict (iii) above.

Subcase III 3. For some $a^* \in B_{\alpha^*} - Ex\ Ker^*(h)$, and $\rho^* \in \mathrm{T}$, for every ρ, $\rho^* \lessdot \rho \in \mathrm{T}$ there is $\tau \in \langle x_\nu : \rho \lessdot \nu \in \mathrm{T} \rangle_{B_0^c}$ such that $h(\tau \cap a^*) \cap a^* = \tau \cap a^*$.

Clearly the function $h' : B_{\alpha^*} \restriction a^* \to B_{\alpha^*} \restriction a^*$ defined by $h'(x) = h(x) \cap a^*$ is an endomorphism; W.l.o.g. the assumption of subcase III 2 fail hence $\{h(x) - a^* : x \le a^*\}$ is finite, hence the range of h' is infinite (as $a^* \notin Ex\ Ker^*(h)$, so by 2.4 there is $x \le a^*$ such that $h(x) \cap a^* - x \ne 0$; we know that $\underline{d}(x)$ is countable, hence for some $\rho^{**}, \rho^* \lessdot \rho^{**} \in T$ and $\{\nu : \rho^{**} \lessdot \nu \in \mathrm{T}\}$ is disjoint to $\underline{d}(a^*) \cup \underline{d}(x) \cap \underline{d}(h(x))$. Now by the hypothesis of subcase III 3 we can easily find $\tau_n \in \langle x_\nu : \rho^{**} \lessdot \nu \in \mathrm{T} \rangle_{B_0^c}$, with pairwise disjoint $\underline{d}(\tau_n)$ and $h(\tau_n \cap a^*) \cap a^* = \tau_n \cap a^*$. So

$$h(\tau_n \cap x) \cap (a^* - x) =$$
$$h((\tau_n \cap a^*) \cap x) \cap (a^* - x) = h(\tau_n \cap a^*) \cap h(x) \cap (a^* - x) =$$
$$(h(\tau_n \cap a^*) \cap a^*) \cap h(x) \cap (a^* - x) = (\tau_n \cap a^*) \cap h(x) \cap (a^* - x) =$$
$$= \tau_n \cap h(x) \cap (a^* - x) = \tau_n \cap (h(x) \cap a^* - x)$$

It is $\ne 0$ [as $\underline{d}(\tau_n) \cap (\underline{d}(x) \cup \underline{d}(h(x)) \cup \underline{d}(a^*)) = \phi$) and $h(x) \cap a^* - x \ne 0$, $\tau_n \ne 0$], and for different n we get different values. So

$\{h(y \cap x) \cap (a^*-x) : x \in B_{\alpha^*}\}$, is infinite. Hence $\{h(y \cap x)-x ; y \in B_{\alpha^*}\}$ is infinite, leading to the assumption of Subcase III 2 (with x here for a^* there).

Subcase III. 4. For some $\rho^* \in \mathrm{T}$, and $a^* \in B_{\alpha^*} - Ex\ Ker^*(h)$ for every $\tau \in \langle x_\nu : \rho^* \leq \nu \in \mathrm{T} \rangle_{B_0^c}$ $h(\tau \cap a^*) \cap a^*$ is based on $\{\nu : \rho^* \not\leq \nu \in \mathrm{T}\}$.

W.l.o.g. the hypothesis of subcase III 1 fail, hence $\{h(\tau \cap a^*) : \tau \in \langle x_\nu : \rho^* \leq \nu \in \mathrm{T} \rangle_{B_0^c}\}$ is infinite. As also w.l.o.g. the hypothesis of subcase III 2 fail we get $\{h(\tau \cap a^*) \cap a^* : \tau \in \langle x_\nu : \rho^* \leq \nu \in \mathrm{T} \rangle_{B_0^c}\}$ is infinite. So by 3.9 we we can find $d_n \in \langle x_\nu : \rho^* \leq \nu \in \mathrm{T} \rangle_{B_0^c}$ such that $\langle d_n : n < \omega \rangle$ is a maximal antichains in B_0^c, and there is no $x \in B_{\alpha^*}$, $x \cap h(d_{2n}) = h(d_{2n})$, $x \cap h(d_{2n+1}) = 0$, and $d_0 = 1-a^*$.

As before we can assume $\rho^* \in \mathrm{Rang}\ (f^*)$ and $d_n \in N_0^\alpha$ for $n < \omega$. We suppose $\eta_\alpha \notin \{\eta_\beta : \beta < \alpha\}$ is an ω-branch of $f^\alpha, \rho^* < \eta_\alpha$.

For any suitable $\bar{\tau}$, if $y[\bar{\tau}, \bar{d}] \in \langle B_\alpha, a_\alpha[\bar{\tau}, \bar{d}] \rangle_{B_0^c}$ satisfies $\tau_n \in \langle x_\nu : \rho^* \leq \nu \in T \rangle_{B_0^c}$ and $y[\bar{\tau}, \bar{d}] \cap h(d_n) = h(\tau_n \cap d_n)$, (for every n) then by 3.3 we easily get $y[\bar{\tau}, \bar{d}] \in B_\alpha$, and then get contradiction by trying four $\bar{\tau}$'s, as in subcase III2.

Subcase III. 5. There are $\rho^* \in \mathrm{T}$ and atomless countable subalgebra $Y \subseteq B_{\alpha^*}$ and pairwise disjoint $c_\ell \in Y (\ell < \omega)$ such that for every ℓ and $\rho_\ell \in \{\rho : \rho^* \leq \rho \in \mathrm{T}\}$ for some $\tau_\ell \in \langle x_\nu : \rho_\ell \leq \nu \in \mathrm{T} \rangle_{B_0^c}$, the following holds: for no $x \in B_0^c$ is $\underline{d}(x) \subseteq \{\nu : \rho_\ell \not\leq \nu \in \mathrm{T}\}$ and $x \cap h(c_\ell) \cap c_\ell - \tau_\ell = h(c_\ell \cap \tau_\ell) \cap c_\ell - \tau_\ell$.

Let $\langle d_n : n < \omega \rangle$ be a maximal antichain of B_{α^*} such that $d_{2n} = c_{2n}$.

So w.l.o.g. $Y \cup \{d_n : n < \omega\} \subseteq N_0^\alpha, \rho^* \in \mathrm{Rang}\ (f^\alpha)$ (using $Gm'(W)$), and even $\rho^* < \eta_\alpha$, and each N_m^α is closed under the functions h and $\rho_\ell \to \tau_\ell$ (implicit in the assumption of the subcase).

We can now choose by induction on n, $\tau_n \in N_n^\alpha$,

$$\tau_n = \left\langle x_\nu : \eta_\alpha \upharpoonright n \trianglelefteq \nu \in \mathrm{T} \right\rangle_{B_0^c}$$

such that

(*) (a) for even n, for no $x \in B_0^c$ based on $\{\nu : \eta_\alpha \upharpoonright n \ntrianglelefteq \nu \in \mathrm{T}\}$ is $x \cap h(d_n) \cap d_n - \tau_n = h(d_n \cap \tau_n) \cap d_n - \tau_n$.

Why is this sufficient? We let $\bar{d} = \langle d_n : n < \omega \rangle$, and $\bar{\tau} = \langle \tau_n : n < \omega \rangle$. So assume some $\bar{y}[\bar{\tau},\bar{d}] \in \langle B_\alpha, a_\alpha[\bar{\tau},\bar{d}] \rangle_{B_0^c}$ realizes $p_\alpha[\bar{\tau},\bar{d}]$, i.e. satisfies $y[\bar{\tau},\bar{d}] \cap h(d_n) = h(d_n \cap \tau_n)$ for every n. As $y[\bar{\tau},\bar{d}] \in \langle B_\alpha, a_\alpha[\bar{\tau},\bar{d}] \rangle_{B_0^c}$ for some pairwise disjoint $e_0[\bar{\tau},\bar{d}], e_1[\bar{\tau},\bar{d}], e_2[\bar{\tau},\bar{d}] \in B_\alpha$,
$y[\bar{\tau},\bar{d}] = e_0[\bar{\tau},\bar{d}] \cup (e_1[\bar{\tau},\bar{d}] \cap a_\alpha[\bar{\tau},\bar{d}]) \cup (e_2[\bar{\tau},\bar{d}] - a_\alpha[\bar{\tau},\bar{d}])$.
For some $m(*) < \omega$, $\underline{d}(e_0[\bar{\tau},\bar{d}]) \cup \underline{d}(e_1[\bar{\tau},\bar{d}]) \cup \underline{d}(e_2[\bar{\tau},\bar{d}])$ is disjoint to $\{\nu : \eta_\alpha \upharpoonright m(*) \trianglelefteq \nu \in \mathrm{T}\}$ (see 3.3(2)).

Now we compute for n even $> m(*)$:

$$z \overset{\mathrm{def}}{=} h(d_n \cap \tau_n) \cap d_n - \tau_n =$$

$$= y[\bar{\tau},\bar{d}] \cap h(d_n) \cap d_n - \tau_n \text{ (by the choice of } y[\bar{\tau},\bar{d}])$$

$$= (e_0[\bar{\tau},\bar{d}] \cup (e_1[\bar{\tau},\bar{d}] \cap a_\alpha[\bar{\tau},\bar{d}]) \cup (e_2[\bar{\tau},\bar{d}] - a[\bar{\tau},\bar{d}])) \cap h(d_n) \cap d_n - \tau_n =$$
$$= (e_0[\bar{\tau},\bar{d}] \cap h(d_n) \cap d_n - \tau_n) \cup ((e_1[\bar{\tau},\bar{d}] \cap a_\alpha[\bar{\tau},\bar{d}]) \cap h(d_n) \cap d_n - \tau_n) \cup$$
$$\cup ((e_2[\bar{\tau},\bar{d}] - a_\alpha[\bar{\tau},\bar{d}]) \cap h(d_n) \cap d_n - \tau_n)$$

But $a_\alpha[\bar{\tau},\bar{d}] \cap d_n = \tau_n \cap d_n$ hence

$$(e_1[\bar{\tau},\bar{d}] \cap a_\alpha[\bar{\tau},\bar{d}]) \cap d_n = (e_1[\bar{\tau},\bar{d}] \cap \tau_n) \cap d_n$$

$$(e_2[\bar{\tau},\bar{d}] - a_\alpha[\bar{\tau},\bar{d}]) \cap d_n = (e_2[\bar{\tau},\bar{d}] - \tau_n) \cap d_n$$

Hence

$$z = (e_0[\bar{\tau},\bar{d}] \cap h(d_n) \cap d_n - \tau_n) \cup ((e_1[\bar{\tau},\bar{d}] \cap \tau_n) \cap h(d_n) \cap d_n - \tau_n) \cup$$
$$((e_2[\bar{\tau},\bar{d}] - \tau_n) \cap h(d_n) \cap d_n - \tau_n)$$

But the second term is zero and in the third the first $-\tau_n$ is redundant, so

$$z = (e_0[\bar{\tau},\bar{d}] \cap h(d_n) \cap d_n - \tau_n) \cup (e_2 \cap h(d_n) \cap d_n - \tau_n) =$$

$$= (e_0[\bar{\tau},\bar{d}] \cup e_2[\bar{\tau},\bar{d}]) \cap h(d_n) \cap d_n - \tau_n$$

We can conclude

$$(e_0[\bar{\tau},\bar{d}] \cup e_2[\bar{\tau},\bar{d}]) \cap h(d_n) \cap d_n - \tau_n = h(d_n \cap \tau_n) \cap d_n - \tau_n$$

contradicting the choice of τ_n.

To finish Case III (hence the proof of 3(10) we need only

Why the five subcases exhaust all possibilities?

Suppose none of III 1-5 occurs. By not subcase III 1 for some $\rho^0 \in \mathrm{T}$,

a) $h(\tau) \neq 0$ for every $\tau \in \langle x_\eta : \rho^0 \trianglelefteq \eta \in \mathrm{T} \rangle_{B_0^c}$.

Let Y be the $\langle x_{\rho^{0\frown}<i>} : i < \omega \rangle_{B_0^c}$. As Y is countable, for some $i(*) < \lambda$, $\{\nu : \rho^0 \frown <i(*)> \trianglelefteq \nu \in \mathrm{T}\}$ is disjoint to $\bigcup \{\underline{d}(y) \cup \underline{d}(h(y)) : y \in Y\}$. As "not subcase III 5" for some ρ^1, $\rho^0 \frown \langle i(*) \rangle \trianglelefteq \rho^1 \in \mathrm{T}$, and

(b) there are no pairwise disjoint non zero $c_\ell \in Y(\ell < \omega)$, such that for every $\rho_\ell^1, \rho^1 \triangleleft \rho_\ell^1 \in \mathrm{T}$ for some $\tau_\ell \in \langle x_\nu : \rho_\ell^1 \trianglelefteq \nu \in \mathrm{T} \rangle_{B_0^c}$, the following holds:

(*) for no $x \in B_0^c$, $\underline{d}(x) \subseteq \{\nu : \rho_\ell^1 \ntrianglelefteq \nu \in \mathrm{T}\}$ and

$$x \cap h(c_\ell) \cap c_\ell - \tau_\ell = h(c_\ell \cap \tau_\ell) \cap c_\ell - \tau_\ell$$

Clearly

c) $\bigcup \{\underline{d}(y) \cup \underline{d}(h(y)) : y \in Y\}$ is disjoint to $\{\nu : \rho^1 \trianglelefteq \nu \in \mathrm{T}\}$.

Let $Z = \{c \in Y$: for some $\rho_c^1, \rho^1 \trianglelefteq \rho_c^1 \in \mathrm{T}$ for no $\tau \in \langle x_\nu : \rho^1 \trianglelefteq \nu \in \mathrm{T} \rangle_{B_0^c}$ does (*) of (b) hold (with c, τ instead c_ℓ, τ_ℓ)$\}$.

By (b) among any $\aleph_0$ pairwise disjoint members of Y, at least one belong to Z.

It is quite easy to define $y_n \in Z((n < \omega)$ such that $[y_n \in Ex\ Ker^*(h) \Longrightarrow y_n \in Ex\ Ker(h)]$, $[m < n \Longrightarrow y_n \cap y_m = 0]$, and for every $y \in Y - \{0\}$ for some n, $y \cap (\bigcup_{\ell<n} y_\ell) \neq 0$ or $y_n \leq y$. So (by the choice of

$Y)$ $\langle y_n : n < \omega \rangle$ is a maximal antichain of B_0^c. We shall show $y_n \in Ex\ Ker(h)$; fix n for a while, and suppose $y_n \notin Ex\ Ker(h)$, and let $\rho_n^1, \rho^1 \leq \rho_n^1 \in \mathbf{T}$ be such that for no $\tau \in \langle x_\nu : \rho_n^1 \leq \nu \in \mathbf{T} \rangle_{B_0^c}$ does (*) of (b) hold.

Now for each $\tau \in \langle x_\nu : \rho_n^1 \leq \nu \in \mathbf{T} \rangle_{B_0^c}$ as $y_n \in Z$, clearly [as (*) of (b) fail for y_n, τ (and ρ_n^1)] for some $x_1 \in B_0^c$, $\underline{d}(x_1) \subseteq \{\nu : \rho_n^1 \not\leq \nu \in \mathbf{T}\}$ and $x_1 \cap h(y_n) \cap y_n - \tau = h(y_n \cap \tau) \cap y_n - \tau$. Applying the failure of (*) of (b) for $y_n, 1-\tau, \rho_n^1$ we get $x_2 \in B_c^0$, $\underline{d}(x_2) \subseteq \{\nu : \rho_n^1 \not\leq \nu \in \mathrm{t}\}$ and $x_2 \cap h(y_n) \cap y_n - (1-\tau) = h(y_n \cap (1-\tau)) \cap y_n - (1-\tau)$; note that $h(y_n \cap \tau) \leq h(y_n)$, and $h(y_n \cap (1-\tau)) = h(y_n) - h(y_n \cap \tau)$. By these equations and as y_n $h(y_n), x_1, x_2$ are based on $\{\nu : \rho_n^1 \not\leq \nu \in \mathbf{T}\}$ (by (c) and their choice resp.) clearly for some partition of 1, $e_0^\tau, e_1^\tau, e_2^\tau, e_3^\tau \in B_0^c$, based on $\{\nu : \rho^1 \not\leq \nu \in \mathbf{T}\}$:

(i) $h(\tau \cap y_n) \cap y_n = e_0^\tau \cup (e_1^\tau \cap \tau) \cup (e_2^\tau - \tau)$.

Now for any $\tau, \sigma \in \langle x_\nu : \rho_n^1 \leq \nu \in \mathbf{T} \rangle$, easily (as h is an endomorphism):

(ii) $h((\tau \cup \sigma) \cap y_n) \cap y_n = (h(\tau \cap y_n) \cap y_n) \cap (h(\sigma \cap y_n) \cap y_n)$.

(iii) $h((\tau \cup \sigma) \cap y_n) \cap y_n = (h(\tau \cap y_n) \cap y_n) \cup (h(\sigma \cap y_n) \cap y_n)$.

We can apply (i) to τ, σ and also to $\tau \cap \sigma, \tau \cup \sigma$, and substitute in (ii) (iii).

We get that

(α) $e_2^\tau \cap e_2^\sigma = 0$ if $\underline{d}(\tau) \cap \underline{d}(\sigma) = 0$, $\tau, \sigma \in \langle x_\nu : \rho^1 \leq \nu \in \mathbf{T} \rangle_{B_0^c}$ (otherwise substitute (i) in (ii) and intersect with $e_2^\tau \cap e_2^\sigma$) and get $(h((\tau \cap \sigma) \cap y_m) \cap (e_2^\tau \cap e_2^\sigma) = (e_2^\tau - \tau) \cap (e_2^\sigma - \sigma) = e_2^\tau \cap \tau_2^\sigma \cap (\tau \cup \sigma)$, and by the assumptions on the $\underline{d}(e_2^\tau)$, $\underline{d}(e_2^\sigma)$, $\underline{d}(\tau)$, $\underline{d}(\sigma)$ we get $(h((\tau \cap \sigma) \cap y_n) \cap y_n) \cap (e_2^\tau \cap e_2^\sigma) \notin \langle \{x : \underline{d}(x) \subseteq \{\nu : \rho_m^1 \leq \nu \in \mathbf{T}\} \cup (\tau \cap \sigma) \rangle_{B_0^c}$ contradiction to (i) for $\sigma \cap \tau$.

So let $\{\tau^i : i < \alpha\}$ be maximal such that $\underline{d}(\tau_i)$ are pairwise disjoint $e_2^{\tau^i} \neq 0$, and $\tau^i \in \langle x_\nu : \rho_n^1 \underline{\lessdot} \nu \in \mathrm{T}\rangle_{B_0^c}$, then $\alpha < \omega_1$, and we can choose ρ_n^2 such that:
$\rho_n^1 \underline{\lessdot} \rho_n^2 \in \mathrm{T}$, and $[\tau \in \langle x_\nu : \rho_n^2 \underline{\lessdot} \nu \in \tau\rangle_{B_0^c} \Longrightarrow e_2^\tau = 0]$.

Next we can get

(β) $e_1^\tau \cap e_0^\sigma = 0$ (if $\underline{d}(\tau) \cap \underline{d}(\sigma) = 0$, and $\tau,\sigma \in \langle x_\nu : \rho_n^2 \underline{\lessdot} \nu \in \mathrm{T}\rangle_{B_0^c}$).

The proof is similar to that of (α), using $\tau \cap \sigma$.

As B_0^c satisfies the $\aleph_1$-c.c. we can find $\{\tau^i : i < \omega\} \subseteq \langle x_\nu : \rho_n^2 \underline{\lessdot} \nu \in \mathrm{T}\rangle_{B_0^c}$, such that (in B_0^c) $e_\ell^* \overset{def}{=} \bigcup_{i<\omega} e_\ell^{\tau^i} = \bigcup\{e_\ell^\tau : \tau \in \langle x_\nu : \rho_n^2 \underline{\lessdot} \nu \in \mathrm{T}\rangle_{B_0^c}\}$ for $\ell = 0,1$. We can find $\rho_n^3, \rho_n^2 \underline{\lessdot} \rho_n^3 \in \mathrm{T}$, such that $\bigcup_{\ell<\omega} \underline{d}(\tau^i)$ is disjoint to $\{\nu : \rho_n^3 \underline{\lessdot} \nu \in \mathrm{T}\}$. So for every $\tau \in \langle x_\nu : \rho_n^3 \underline{\lessdot} \nu \in \mathrm{T}\rangle_{B_0^c}$, $e_0^\tau \le e_0^*$ (by the choice of e_0^*), and $e_0^\tau \cap e_1^{\tau^i} = 0$ for $i < \omega$ (by (β)) hence $e_0^\tau \cap e_1^* = 0$, hence

(γ) $e_0^\tau \le e_0^* - e_1^*$.

Similarly

(δ) $e_1^\tau \le e_1^* - e_0^*$.

Now we can prove that $e_1^\tau = e_1^\sigma$ when $\underline{d}(\tau) \cap \underline{d}(\sigma) = 0$, $\tau,\sigma \in \langle x_\nu : \rho_n^3 \underline{\lessdot} \nu \in \mathrm{T}\rangle_{B_0^c}$, (repeat the proof of (α) intersecting with $e_1^\tau - e_1^\sigma$ or with $e_1^\sigma - e_1^\tau$). By the transitivity of equality $e_1^\tau = e_1^\sigma$ when $\tau,\sigma \in \langle x_\nu : \rho_n^3 \underline{\lessdot} \nu \in \mathrm{T}\rangle_{B_0^c}$. So let $e_1 \in B_{\alpha^*}$ be the common value, so

$(*)$ $h(\tau \cap y_n) \cap y_n = e_0^\tau \cup (e_1 \cap \tau)$ for $\tau \in \langle x_\nu : \rho_n^3 \underline{\lessdot} \nu \in \mathrm{T}\rangle_{B_0^c}$; and $e_0^\tau \le y_n - e_1$,

Let $e_0 = y_n - e_1$, so $y_n = e_0 \cup e_1$, $e_0 \cap e_1 = 0$.

so $e_0^\tau \le e_0$ for every $\tau \in \langle x_\nu : \rho_n^3 \underline{\lessdot} \nu \in \mathrm{T}\rangle_{B_0^c}$.

As $y_n \not\in Ex\ Ker^*(h)$, at least one of the elements, e_0, e_1 is not in $Ex\ Ker^*(h)$. As not subcase III 2, for $\ell = 1,2$ the homomorphism g_ℓ from $B_{\alpha^*} \restriction e_\ell$ to $B_{\alpha^*} \restriction (1-e_\ell)$, $g_\ell(x) = h(x) - e_\ell$ (for $x = e_1$) has a finite range. Hence for some ideal $\mathfrak{I}$ of B_0^c $y_n / \mathfrak{I}$ is a finite union of atoms and

for every $\tau \in \langle x_\nu : \rho_n^3 \underline{<} \nu \in T \rangle \cap \mathfrak{I}$

for $\ell = 0,1$ $h(\tau \cap y_n) \cap e_\ell = h(\tau \cap e_\ell) \cap e_\ell$

hence $h(\tau \cap e_\ell) \cap e_\ell = (e_0^{\mathfrak{I}} \cup (e_1^{\mathfrak{I}} \cap \tau)) \cap e_\ell$.

So (for $\tau \in \langle x_\nu : \rho_n^3 \underline{<} \nu \in T \rangle_{B_0^c} \cap \mathfrak{I}$):

$$h(\tau \cap e_0) \cap e_0 = e_0^{\mathfrak{I}}$$
$$h(\tau \cap e_1) \cap e_1 = \tau \cap e_1$$

If $e_1 \not\in Ex\ Ker^*(h)$, we get contradiction to "not subcase III 3" [use ρ_n^3 for ρ^* there, now for any ρ, $\rho_n^3 \underline{<} \rho \in T$ choose pairwise disjoint $\tau_\ell \in \langle x_\nu : \rho \underline{<} \nu \in T \rangle_{B_0^c}$ for $\ell < \omega$ now by the choice of $\mathfrak{I}$ for at least one $\ell, \tau_\ell \in \mathfrak{I}$, so τ_ℓ is as required there]. So assume $e_0 \not\in Ex\ Ker^*(h)$ and get contradiction to "not subcase III 4" [for some $\ell < m < \omega$ $x_{\rho_n^3 \frown <\ell>} - x_{\rho_n^3 \frown <n>}$ is in $\mathfrak{I}$, use $\rho_n^3 \frown <\alpha>$, $e_0 \cap (x_{\rho_n^3 \frown <\ell>} - x_{\rho_n^3 \frown <n>})$ for ρ^*, α^* with α large enough].

So for each $n, y_n \in Ex\ Ker^*(h)$ (the y_n were chosen after (b)) hence $y_n \in Ex\ Ker(h)$, (by their choice) so let $y_n = y_n^0 \cup y_n^1$ (both in B_{α^*}), $h(y_n^0) = 0$, $h(x) = x$ for $x \leq y_n^1$, $x \in B_{\alpha^*}$. Let $I \subseteq T$ be a countable set such that $\underline{d}(y_n^0), \underline{d}(y_n^1) \subseteq I$, and for $x \in B_{\alpha^*}$. $\underline{d}(h(x - y_n) \cap y_n) \subseteq I$ (by "not subcase III 2", for each n we have only finitely many elements of this form).

We can easily show that for every $x \in B_{\alpha^*}$ for some $a \in B_0^c$ based on I, $h(x) - x = a - x$, [as $\langle y_n : n < \omega \rangle$ is a maximal antichain in B_{α^*}, for this it suffices to show that for every $n < \omega$ there is $a_n \in B_{\alpha^*}^c$, $a_n \leq y_n$ such that $(h(x) - x) \cap y_n = a_n - x$; But $(h(x) - x) \cap y_n$ is the union of $(h(x \cap y_n) - x) \cap y_n$ which is zero as $(\forall z \leq y_n) h(z) \leq z$ and of $(h(x - y_n) - x) \cap y_n$ which we know is based as wanted] So

$h(x) = e_0^x \cup (e_1^x \cap x) \cup (e_2^x - x)$

where each e_ℓ^* is based on I, $\langle e_\ell^* : \ell < 3 \rangle$ pairwise disjoint $e_\ell^* \in B_0^c$. As in the analysis above of $h(x \cap y_n) \cap y_n$, possibly increasing I, applied to $x \in B_{\alpha^*}$ with $\underline{d}(x) \cap I = 0$, we get $e_2^x = 0, e_1^x = e_1$. If $e_1 \notin Ex\ Ker^*(h)$ we get contradiction to "not subcase III 3.". So $1-e_1 \notin Ex\ Ker^*(h)$ and apply "not subcase III 4."

So we finish the proof of 3.11; so B_{α^*} is endo-rigid.

3.12 Lemma : B_{α^*} is indecomposable.

Proof : Suppose K_0, K_1 are disjoint ideals of B_{α^*}, each with no maximal members, which generate a maximal ideal of B_{α^*}. For $\ell = 1,2$ let $\{d_n^\ell : \ell < \omega\}$ be a maximal antichain $\subseteq K_\ell$ (they are countable as B_{α^*} satisfies the c.c.c., and may be chosen infinite as $K_\ell \neq \{0\}$, B_{α^*} is atomless). Let K be the ideal $K_0 \cup K_1$ generates.

Now, e.g. for some $\xi < \lambda$, $\{d_n^\ell : \ell < 2, n < \omega\} \subseteq B_{[\xi]}$. Clearly $a_{<\xi>} \in K$ or $1-a_{<\xi>} \in K$. For notational simplicity assume $a_{<\xi>} \in K$. So $a_{<\xi>} = b^0 \cup b^1$, $b^\ell \in K_\ell$. Now $pr_\xi(b^\ell) \in B_{[\xi]}$ and is disjoint to each $d_n^{1-\ell}$, (as b^ℓ and is, $d_n^\ell \in B_{[\xi]}$,), so by the maximality of $\{d_n^{1-\ell} : n < \omega\}$, $pr_\xi(b^\ell)$ is disjoint to every member of $K_{1-\ell}$. As $K_0 \cup K_1$ generate a maximal ideal, clearly $pr_\xi(b^\ell) \in K_\ell$ [otherwise $pr_\xi(b^\ell) = 1-c^1 \cup c^2$, for some $c^1 \in K_1$, $c^2 \in K_2$, and then $c^{1-\ell}$ is necessarily a maximal member of $K_{1-\ell}$, so $K_{1-\ell}$ is principal contradiction]. So $pr_\xi(b^0) \cup pr_\xi(b^2) < 1$ but $1 = pr_\xi(a_{<\xi>}) = \bigcup_{\ell=0}^{2} pr_\xi(b^\ell)$ contradiction.

3.13 Theorem : In 3.1 we can get $2^{\lambda^{\aleph_0}}$ such B. A. such that any homomorphism from one to the other has finite range.

Proof : Left to the reader (see [Sh 4, 3]).

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