Large κ-Preserving Sets in Infinite Graphs

Andreas Huck
Frank Niedermeyer
INSTITUTE FÜR MATHEMATIK
UNIVERSITÄT HANNOVER
HANNOVER, FEDERAL REPUBLIC OF GERMANY
Saharon Shelah*
INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY
JERUSALEM, ISRAEL

ABSTRACT

Let κ be a cardinal. If $\kappa \geq \aleph_0$, define $\kappa' := \kappa$. Otherwise, let $\kappa' := \kappa + 1$. We prove a conjecture of Mader: Every infinite κ'-connected graph $G = (V, E)$ contains a set $S \subseteq V$ with $|S| = |V|$ such that $G \backslash S'$ is κ-connected for all $S' \subseteq S$.

Throughout this paper we let κ be a finite or infinite cardinal and $G = (V, E)$ be a graph without loops and multiple edges. V denotes the set of vertices and $E \subseteq \{e \subseteq V : |e| = 2\}$ the set of edges of G. For each $D \subseteq V$, let $G \backslash D$ denote the graph obtained from G by deleting the vertices of D. Define κ' to be κ or $\kappa + 1$ if κ is infinite or finite respectively. G is called *κ-connected* if $|V| \geq \kappa'$ and if for each subset D of V with $|D| < \kappa$, the graph $G \backslash D$ is connected. A subset S of V is called *κ-preserving* if for each $S' \subseteq S$, the graph $G \backslash S'$ is κ-connected.

We have the following results of Mader [2] and Thomassen [3].

Theorem A. If $\kappa \geq \aleph_0$, $|V| \geq \aleph_0$, and G is κ-connected, then there exists a κ-preserving set S in G with $|S| \geq \kappa$.

Theorem B. If $\kappa < \aleph_0$, $|V| \geq \aleph_0$, and G is κ'-connected, then there exists a κ-preserving set S in G with $|S| \geq \aleph_0$.

Our aim is to prove the following common extension.

*Publ 444.

Journal of Graph Theory, Vol. 18, No. 4, 413–426 (1994)
 CCC 0364-9024/94/040413-14

Theorem 1. If $|V| \geq \aleph_0$ and G is κ'-connected, then there exists a κ-preserving set S in G with $|S| = |V|$.

If κ is finite, then in Theorem 1, we are not allowed to replace κ' by κ. This is demonstrated by the graph $G^* = (V^*, E^*)$, where $V^* = \{v_{i,j} : i < \aleph_0, j \in \{1, \ldots, \kappa\}\}$ (the vertices $v_{i,j}$ mutually distinct) and $E^* = \{\{v_{i,j}, v_{i+1,k}\} : i < \aleph_0, j, k \in \{1, \ldots, \kappa\}\}$.

Before proving Theorem 1, we give some technical preliminaries and additional definitions. When dealing with ordinals and cardinals, we use the definitions of [1]. In particular, we consider an ordinal as the set of all preceding ordinals and cardinals are special ordinals. Moreover, as usual, we let κ^+ denote the cardinal successor of κ.

If $X \subseteq V$, then $G[X]$ denotes the subgraph of G induced by X, i.e., $G[X] = (X, \{e \in E : e \subseteq X\})$. For every $X, Y \subseteq V$, we define $E(X,Y) := \{\{x,y\} \in E : x \in X, y \in Y\}$, $d(X,Y) := |E(X,Y)|$, $N(X) := \{v \in V \backslash X : E(\{v\}, X) \neq \emptyset\}$, and $d(X) := |N(X)|$. An injective finite sequence $P = (v_i)_{0 \leq i \leq k}$ of vertices is called a *path* if $\{v_i, v_{i+1}\} \in E$ for all $i < k$. Let $V(P) := \{v_i : i \leq k\}$. If $X, Y \subseteq V$ such that $V(P) \cap X = \{v_0\}$ and $V(P) \cap Y = \{v_k\}$, then we call P an X $-$ Y-*path*. Two paths $P_1 = (v_i)_{0 \leq i \leq k}$, $P_2 = (w_i)_{0 \leq i \leq l}$ are called *openly disjoint* if $V(P_1) \cap V(P_2) \subseteq \{v_0, v_k\} \cap \{w_0, w_l\}$. Moreover, they are called *nearly disjoint* or *disjoint* if $V(P_1) \cap V(P_2) \subseteq \{v_0\} \cap \{w_0\}$ or $V(P_1) \cap V(P_2) = \emptyset$, respectively. If $X = \{x\}$ or $Y = \{y\}$, then we also write x instead of X or y instead of Y, respectively, in these definitions. By a *component* of G we always mean the vertex set of a connectivity component of G.

We tacitly use the following well-known facts about connectivity.

Lemma 1.

(a) If $|V| \geq 2$, then G is κ-connected iff for any two distinct $u, v \in V$, there exists a set M of pairwise openly disjoint $u - v$-paths in G with $|M| \geq \kappa$.
(b) If G is κ-connected, then for each $X \subseteq V$ with $|X| \geq \kappa$ and for each $x \in V \backslash X$, there exist κ pairwise nearly disjoint $x - X$-paths in G.
(c) If G is κ-connected, then for each $X, Y \subseteq V$ with $|X|, |Y| \geq \kappa$, there exist κ pairwise disjoint $X - Y$-paths in G.

The next two lemmas are easily seen to be true.

Lemma 2. Let $A \subseteq V$ such that $G[A]$ is κ-connected and for each $x \in V \backslash A$, there are κ pairwise nearly disjoint $x - A$-paths in G. Then G is κ-connected.

Lemma 3. $S \subseteq V$ is κ-preserving in G iff $G \backslash S$ is κ-connected and $d(s, V \backslash S) \geq \kappa$ for every $s \in S$.

If $A \subseteq V$, then a subset B of V is called a *κ-closure* of A if $A \subseteq B$ and $G[B]$ is κ-connected.

Lemma 4. Let $A \subseteq V$ and G be κ-connected. Then there exists a κ-closure B of A with $|B| \leq \max\{|A|, \kappa, \aleph_0\}$.

Proof. Clearly, we may assume that $\kappa \geq 1$ and thus $|V| \geq 2$. For every $x, y \in V$ $(x \neq y)$, let $\mathcal{P}_{x,y}$ be a set of κ pairwise openly disjoint $x-y$-paths and define $M_{x,y} := \bigcup\{V(P) : P \in \mathcal{P}_{x,y}\}$. Take any two distinct vertices a and b of G. Now for $n < \omega$, we inductively define B_n by $B_0 := A \cup \{a, b\}$ and $B_{n+1} := B_n \cup \bigcup\{M_{x,y} : x, y \in B_n, x \neq y\}$. Then $B := \bigcup_{n<\omega} B_n$ is as required. ∎

The proof of the next two lemmas are due to Mader [2].

Lemma 5. Let G be κ'-connected and let γ be a cardinal. Assume that there exist systems $(F_i)_{i<\gamma}$ and $(B_i)_{i<\gamma}$ of subsets of V such that for each $i < \gamma$, F_i is not empty and B_i is a κ'-closure of $N(F_i)$. Moreover, assume that for every $i, j < \gamma$ with $j < i$, we have $F_i \cap (F_j \cup B_j) = \emptyset$. Then there exists a κ-preserving set S in G with $|S| = \gamma$.

Proof. For $i < \gamma$, take $s_i \in F_i$ and define $S := \{s_i : i < \gamma\}$. By the premises of Lemma 5, the sets F_i $(i < \gamma)$ are pairwise disjoint. Thus $|S| = \gamma$. We show that S is κ-preserving. By construction, no two vertices in S are connected by an edge and hence by Lemma 3, it suffices to prove that $G \backslash S$ is κ-connected. Take $D \subseteq V \backslash S$ with $|D| < \kappa$ and suppose that there are two vertices y_1 and y_2 separated by D in $G \backslash S$. By a standard application of Zorn's Lemma, we find an $X \subseteq S \cup D$ separating y_1 and y_2 in G that is minimal by inclusion. Let C_k be the component of $G \backslash X$ containing y_k $(k \in \{1, 2\})$. Define $S' := S \cap X$ and $D' := D \cap X$.

By the minimality of X, for each $x \in X$, we have $d(x, C_1) > 0$ and $d(x, C_2) > 0$. Moreover, $|D'| < \kappa$ and since G is κ'-connected, $|S'| \geq 2$. Let i be minimal such that $s_i \in S'$ and choose $j > i$ with $s_j \in S'$. Then since $G[C_k]$ is connected, we obtain an s_i, s_j-path P_k whose interior vertices lie in C_k $(k \in \{1, 2\})$. By construction, there is a $v_k \in V(P_k) \cap N(F_i)$ $(k \in \{1, 2\})$ and since $|D' \cup \{s_i\}| < \kappa'$ and $G[B_i]$ is κ'-connected, there exist a v_1, v_2-path P in $G[B_i]$ not containing vertices of $D' \cup \{s_i\}$. By the premises of Lemma 5 and by the minimality of i, $V(P) \cap S' = \emptyset$. Hence C_1 and C_2 are not components of $G \backslash (S' \cup D')$ and we have a contradiction. ∎

Lemma 6. Let G be κ'-connected and let γ, ρ be cardinals such that $\max\{\kappa, \aleph_0\} < \rho \leq \gamma$ and ρ is regular. Assume that there exists a set $\mathcal{F}$ of

γ pairwise disjoint nonempty subsets of V with $d(F) < \rho$ for each $F \in \mathcal{F}$. Then there exists a κ-preserving set S in G with $|S| = \gamma$.

Proof. We inductively define a sequence $(F_i)_{i<\gamma}$ of pairwise disjoint elements of $\mathcal{F}$ and a sequence $(B_i)_{i<\gamma}$ such that for each $i < \gamma$, B_i is a κ'-closure of $N(F_i)$ with $|B_i| < \rho$. Assume that F_j and B_j have been defined for each $j < i$. By construction, $|\bigcup_{j<i} B_j| < \gamma = |\mathcal{F}|$. Hence we obtain an $F_i \in \mathcal{F}\backslash\{F_j : j < i\}$ such that $F_i \cap \bigcup_{j<i} B_j = \emptyset$. Moreover, by Lemma 4, there is a κ'-closure B_i of $N(F_i)$ with $|B_i| < \rho$.

The sequences $(F_i)_{i<\gamma}$ and $(B_i)_{i<\gamma}$ satisfy the premise of Lemma 5 and hence we find an S as required. ∎

In the following we assume that G is κ'-connected and that $|V|$ is infinite. Define $\lambda := |V|$. Now we prove that G has a κ-preserving set of cardinality λ. To do this, we have to consider two cases.

Case 1. λ is regular

By Theorem A and Theorem B, we may assume that $\lambda > \max\{\kappa, \aleph_0\}$.

Let us recall some set-theoretical definitions. A subset C of λ is called a *club* (in λ) if C is closed (i.e., for each $A \subseteq C$ with $|A| < \lambda$, sup $A \in C$) and unbounded (i.e., for each $\alpha < \lambda$, there exists a $\beta \in C$ with $\beta > \alpha$). A function $\mu : \lambda \to \lambda$ is called *continuous* if $\mu(\delta) = \sup_{i<\delta} \mu(i)$ for each limit ordinal $\delta < \lambda$. It is well known that $C \subseteq \lambda$ is a club iff there exists a (unique) strictly increasing continuous function $\mu_C : \lambda \to \lambda$ with $C = \{\mu_C(\delta) : \delta < \gamma\}$. A (well-founded) *tree* is a pair $(\mathcal{T}, \prec)$, where $\mathcal{T}$ is a set and $\prec$ is a partial order on $\mathcal{T}$ (i.e., an irreflexive and transitive relation on $\mathcal{T}$) such that for every $X \in \mathcal{T}$, $\{Y \in \mathcal{T} : Y \prec X\}$ is well ordered by $\prec$. We call $A \subseteq \mathcal{T}$ an *antichain* if any two elements of A are incomparable by $\prec$. Note that for each $M \subseteq \mathcal{T}$, there exists an antichain $M' \subseteq M$ such that for each $X \in M\backslash M'$, we have a $Y \in M'$ with $Y \prec X$. M' is the set of the minimal elements of M.

Choose a numbering $(v_i)_{i<\lambda}$ of V. For each $\delta < \lambda$, we define $V^\delta := \{v_i : i < \delta\}$. If $C \subseteq \lambda$ is a club, then we let $V_C^\delta := V^{\mu_C(\delta)}$ for each $\delta < \lambda$. Moreover, we define a tree $(\mathcal{T}_C, \prec)$ as follows. For each $\delta < \lambda$, let

$$\mathcal{T}_C^\delta := \{X \subseteq V : \ X \text{ is a component of } G\backslash V_C^\delta\},$$

$$\mathcal{T}_C := \bigcup_{\delta<\lambda} \mathcal{T}_C^\delta .$$

For each $X, Y \in \mathcal{T}_C$, let

$$X \preceq Y :\Longleftrightarrow Y \subseteq X,$$

$$X \prec Y :\Longleftrightarrow (X \prec Y \text{ and } X \neq Y).$$

It is straightforward to check that $(\mathcal{T}_C, \prec)$ is indeed a tree. Obviously, for every club $C \subseteq \lambda$, we have

(1) $X, Y \in \mathcal{T}_C$ are incomparable iff $X \cap Y = \emptyset$.

Note that for each $\delta < \lambda$ and $X \in \mathcal{T}_C^\delta$, we have $N(X) \subseteq V_C^\delta$ and thus $d(X) < \lambda$. Hence by (1) and Lemma 6, we may assume that for each club $C \subseteq \lambda$, the following is true.

(2) $\mathcal{T}_C$ does not contain an antichain of cardinality λ.

Now we construct some clubs with special properties. Without loss of generality, we may assume that there exists a club $C \subseteq \lambda$ such that

(3) $|V_C^0| \geq \kappa'$ and $d(v) = \lambda$ for each $v \in V \setminus V_C^0$.

Proof. By Lemma 6, we may assume that $|\{v \in V : d(v) < \lambda\}| < \lambda$. Thus we obtain an $\alpha \geq \kappa'$ such that $\{v \in V : d(v) < \lambda\} \subseteq V^\alpha$. The club $C := \{\delta : \alpha \leq \delta \leq \lambda\}$ satisfies (3). ∎

Such a club C also satisfies

(4) $|X| = \lambda$ for every $X \in \mathcal{T}_C$.

Proof. Let $X \in \mathcal{T}_C$. Then $d(v) = \lambda$ for each $v \in X$. Thus by $d(X) < \lambda$, we have $|X| = \lambda$. ∎

There exists a club $C \subseteq \lambda$ with the following property.

(5) For each $\delta < \lambda$, the graph $G[V_C^\delta]$ is κ-connected.

Proof. Define C by μ_C as follows. Assume that $\mu_C(\gamma)$ has been defined for each $\gamma < \delta$. If $\delta > 0$ is a limit ordinal, let $\mu_C(\delta) := \sup_{\gamma<\delta} \mu_C(\gamma)$. If $\delta = 0$ or $\delta = \gamma + 1$, then we define ordinals $\varepsilon_n < \lambda$ $(n < \omega)$ as follows. Let $\varepsilon_0 := 0$ or $\varepsilon_0 := \mu_C(\gamma) + 1$, respectively. Now assume that ε_n has been defined. Take a κ'-closure B of V^{ε_n} with $|B| < \lambda$ and let $\varepsilon_{n+1} < \lambda$ be such that $B \subseteq V^{\varepsilon_{n+1}}$. Finally, let $\mu_C(\delta) := \sup_{n<\omega} \varepsilon_n$.
By construction, C satisfies (5). ∎

There exists a club $C \subseteq \lambda$ with the following property.

(6) For each $X, Y \in \mathcal{T}_C$ with $X \preceq Y$, $N(X) \subseteq N(Y)$.

Proof. For each $i < \lambda$, define

$$I_i := \{X \in \mathcal{T}_\lambda : \text{ for each } Y \in \mathcal{T}_\lambda \text{ with } X \preceq Y, E(v_i, Y) \neq \emptyset\},$$

$$J_i := \{X \in \mathcal{T}_\lambda : E(v_i, X) = \varnothing\}.$$

Then for each $i < \lambda$, we obtain a $\beta_i < \lambda$ with $\bigcup_{\gamma \geq \beta_i} \mathcal{T}_\lambda^\gamma \subseteq I_i \cup J_i$ as follows. Let J_i' be the set of the minimal elements of J_i. Since J_i' is an antichain, we have $|J_i'| < \lambda$ by (2). So there exists $\beta_i < \lambda$ with $J_i' \subseteq \bigcup_{\gamma < \beta_i} \mathcal{T}_\lambda^\gamma$. Now assume that $Z \in \bigcup_{\gamma \geq \beta_i} \mathcal{T}_\lambda^\gamma \setminus I_i$. Then there exists $X \in J_i$ with $Z \preceq X$. Let $Y \in J_i'$ with $Y \preceq X$. Since $\mathcal{T}_\lambda$ is a tree, Y and Z are comparable and therefore, by the choice of β_i, $Y \preceq Z$. Thus $Z \in J_i$.

Now define C by μ_C as follows. Let $\mu_C(0) := 0$. Assume that $\delta > 0$ and that for each $\gamma < \delta$, $\mu_C(\gamma)$ is already defined. If δ is a limit ordinal, then let $\mu_C(\delta) := \sup_{\gamma < \delta} \mu_C(\gamma)$. If $\delta = \gamma + 1$, then we define ordinals $\varepsilon_n < \lambda$ $(n < \omega)$ by $\varepsilon_0 := \mu_C(\gamma) + 1$ and $\varepsilon_{n+1} := \sup_{i < \varepsilon_n} \beta_i$. Finally, let $\mu_C(\delta) := \sup_{n < \omega} \varepsilon_n$. Obviously, for each $\delta < \lambda$ and $i < \mu_C(\delta)$, we have $\beta_i \leq \mu_C(\delta)$. To prove that C satisfies (6), let $X \in \mathcal{T}_C$ and $i < \lambda$ with $v_i \in N(X)$. It suffices to show that $X \in I_i$. Choose $\delta < \lambda$ such that $X \in \mathcal{T}_C^\delta = \mathcal{T}_\lambda^{\mu_C(\delta)}$. Then $i < \mu_C(\delta)$ and thus $\beta_i \leq \mu_C(\delta)$. Hence $X \in \bigcup_{\gamma \geq \beta_i} \mathcal{T}_\lambda^\gamma$. Since $X \notin J_i$, we have $X \in I_i$. ∎

Now we choose clubs C_3, C_5, and C_6 satisfying (3), (5), and (6) respectively. It is well known that the intersection of any two clubs is also a club. Hence $C := C_3 \cap C_5 \cap C_6$ is a club satisfying (3), (4), (5), and (6). Moreover, we have

(7) Let $X \in \mathcal{T}_C$ and $x, y \in N(X)$ be distinct. Then there exists a system $(P_i)_{i<\lambda}$ of pairwise openly disjoint $x - y$-paths in G.

Proof. The $x - y$-paths P_i $(i < \lambda)$ will be defined inductively. Assume that for each $j < i$, P_j has been defined. Since $M := \bigcup_{j<i} V(P_i)$ has cardinality less than λ, we find $\gamma < \lambda$ with $M \subseteq V_C^\gamma$. Since $|X| = \lambda$ by (4), we have $X \setminus V_C^\gamma \neq \varnothing$. Thus there exists a $Y \in \mathcal{T}_C^\gamma$ with $X \preceq Y$. By (6), we have $x, y \in N(Y)$. Since $G[Y]$ is connected, there exists an $x - y$-path P_i with $V(P_i) \setminus \{x, y\} \subseteq Y$. By construction, the paths P_i $(i < \lambda)$ are pairwise openly disjoint. ∎

Now we obtain the following.

(8) If $v \in V \setminus V_C^0$ and if $A \subseteq V \setminus (V_C^0 \cup \{v\})$ with $|A| < \lambda$, then there exist κ pairwise nearly disjoint $v - V_C^0$-paths in $G \setminus A$.

Proof. Take $X \in \mathcal{T}_C^0$ with $v \in X$. Of course, $N(X) \subseteq V_C^0$ and since G is κ-connected and $|V_C^0| \geq \kappa$ (by (3)), $d(X) \geq \kappa$. Choose $\delta < \lambda$ with $v \in V_C^\delta$. Then since by (3), $d(v) = \lambda > |V_C^\delta|$, we obtain a $Z \in T_C^\delta$ with $v \in N(Z)$. By construction, $X \prec Z$ and thus $N(X) \subseteq N(Z)$ by (6). Hence, by (7) and $v \in N(Z) \setminus N(X)$, for each $y \in N(X)$, we find a system of λ pairwise openly disjoint $v - y$-paths. Now define inductively $v-$

V_C^0-paths P_i $(i < \kappa)$ as follows. If for each $j < i$, P_j has been defined, then $|\bigcup_{j<i} V(P_j) \cap V_C^0| \leq |i| < \kappa \leq d(X)$ and $|A \cup \bigcup_{j<i} V(P_j)| < \lambda$. Thus we find a $y \in N(X) \setminus \bigcup_{j<i} V(P_j)$ and a $v - y$-path P with $V(P) \cap (A \cup \bigcup_{j<i} V(P_j)) = \{v\}$. Since $y \in N(X) \subseteq V_C^0$, P contains a $v - V_C^0$-path P_i. By construction, the paths P_i $(i < \kappa)$ are pairwise nearly disjoint. ∎

Now we construct a κ-preserving set of cardinality λ. For all i with $\mu_C(0) \leq i < \lambda$, we define an ordinal δ_i with $\mu_C(0) \leq \delta_i < \lambda$ as follows. Assume that δ_j has been defined for each j with $\mu_C(0) \leq j < i$. Let $A := \{v_{\delta_j} : \mu_C(0) \leq j < i\}$. Then by (8), there is a set $\mathcal{P}$ of κ nearly disjoint $v_i - V_C^0$-paths in $G \setminus (A \setminus \{v_i\})$. Choose $\delta_i \geq \mu_C(0)$ such that $A \cup \bigcup \{V(P) : P \in \mathcal{P}\} \subseteq V^{\delta_i}$.

By construction, the sequence $(\delta_i)_{\mu_C(0) \leq i < \lambda}$ is strictly increasing and thus $S := \{v_{\delta_i} : \mu_C(0) \leq i < \lambda\}$ has cardinality λ. Moreover, by construction, for every i with $\mu_C(0) \leq i < \lambda$, there exist κ pairwise nearly disjoint $v_i - V_C^0$-paths in $G \setminus (S \setminus \{v_i\})$. Thus since $G[V_C^0]$ is κ-connected, also $G \setminus S$ is κ-connected by Lemma 2. Moreover, $d(v, V \setminus S) \geq \kappa$ for each $v \in S$. Hence S is κ-preserving in G by Lemma 3.

This completes the proof of Theorem 1 if λ is regular. Now we obtain the following result for arbitrary, not necessarily regular λ.

Proposition 1. Let $\gamma \leq \lambda$ be a regular cardinal. Then there exists a κ-preserving set S in G with $|S| \geq \gamma$.

Proof. Again, by Theorem A and Theorem B, we may assume that $\lambda > \max\{\kappa, \aleph_0\}$. Moreover, by Case 1, we may let λ be a singular cardinal. Thus there exists a regular cardinal $\xi < \lambda$ with $\xi > \max\{\gamma, \kappa, \aleph_0\}$. By Lemma 4, we find a $W \subseteq V$ with $|W| = \xi$ such that $G[W]$ is κ'-connected. By Case 1, there exists a κ-preserving set $T \subseteq W$ in $G[W]$ with $|T| = \xi$. Take a partition of T into pairwise disjoint sets S_α $(\alpha < \xi)$ with $|S_\alpha| = \gamma$. Note that each S_α is again a κ-preserving set in $G[W]$. If there is an $\alpha < \xi$ such that $G \setminus S_\alpha$ is κ-connected, then by Lemma 3, S_α is also κ-preserving in G. So we may assume that for each $\alpha < \xi$, there exists $D_\alpha \subseteq V \setminus S_\alpha$ such that $|D_\alpha| < \kappa$ and $G \setminus (S_\alpha \cup D_\alpha)$ is not connected. Since $G[W] \setminus (S_\alpha \cup D_\alpha)$ is connected for each $\alpha < \xi$, we find a component C_α of $G \setminus (S_\alpha \cup D_\alpha)$ with $C_\alpha \cap W = \emptyset$.

We define inductively a strictly increasing sequence $(\alpha_i)_{i<\xi}$ in ξ with

$$(*) \qquad (S_{\alpha_i} \cup C_{\alpha_i}) \cap \bigcup_{j<i} D_{\alpha_j} = \emptyset \text{ for each } i < \xi,$$

as follows. Assume that α_j has been defined for every $j < i$, and let $\alpha^* := \sup_{j<i} \alpha_j$ and $D^* := \bigcup_{j<i} D_{\alpha_j}$. Suppose for a contradiction that for every $\alpha > \alpha^*$, there exists a $y_\alpha \in (S_\alpha \cup C_\alpha) \cap D^*$ (otherwise we could find an $\alpha_i > \alpha^*$ as required). Since $|D^*| < \xi$, there exists $y \in D^*$ such that $|\{\alpha > \alpha^* : y = y_\alpha\}| = \xi$ Thus we may assume for simplicity that $y \in (S_\alpha \cup C_\alpha) \cap D^*$ for each $\alpha > \alpha^*$. Since the sets S_α $(\alpha < \xi)$ are pairwise disjoint and $C_\alpha \cap W = \emptyset$, we have $y \in C_\alpha$ for all $\alpha > \alpha^*$. G is κ'-connected and thus for each $\alpha > \alpha^*$, since $|D_\alpha| < \kappa$, we find a $y - S_\alpha$-path P_α with $V(P_\alpha)\backslash S_\alpha \subseteq C_\alpha$. Since $\xi > \aleph_0$, there exists $m < \omega$ such that $|\{\alpha > \alpha^* : \text{the length of } P_\alpha \text{ is } m\}| = \xi$. So we may assume for simplicity that the length of each path P_α $(\alpha > \alpha^*)$ is m.

Let $X_{-1} := \{\alpha : \alpha^* < \alpha < \xi\}$. Now we define sequences $(X_n)_{n<\omega}$ and $(d_n)_{n<\omega}$ such that for each $n < \omega$, the following conditions are true.

(i) $|X_n| = \xi$ and $X_k \supseteq X_n$ for each k with $-1 \leq k < n$.
(ii) $\{d_0, \ldots, d_n\} \subseteq V(P_\alpha)$ for each $\alpha \in X_n$.
(iii) $d_k \neq d_n$ for each $k < n$.

Then by (ii) and (iii), we obtain a contradiction to the fact that each path P_α has length m and the existence of a sequence $(\alpha_i)_{i<\xi}$ satisfying $(*)$ is proved. Assume that $n < \omega$ and that X_k and d_k are defined for each $k < n$. Let $\mu := \min X_{n-1}$. For every $\alpha \in X_{n-1}\backslash\{\mu\}$, $D_\mu \cup S_\mu$ separates y and S_α since $y \in C_\mu$ and $S_\alpha \subseteq W$. Thus, since $V(P_\alpha) \cap S_\mu = \emptyset$, we obtain $V(P_\alpha) \cap D_\mu \neq \emptyset$ and hence, since $|D_\mu| < \kappa$, there exists $d_n \in D_\mu$ such that $X_n := \{\alpha \in X_{n-1} : d_n \in V(P_\alpha)\}$ has cardinality ξ. By construction, (i) and (ii) are satisfied. Moreover, for each $k < n$, we have $d_k \in V(P_\mu)$ by $\mu \in X_k$. Thus, since $V(P_\mu) \cap D_\mu = \emptyset$, also (iii) is satisfied.

For simplicity we may assume that $(S_i \cup C_i) \cap \bigcup_{j<i} D_j = \emptyset$ for every $i < \xi$. Then we obtain $C_i \cap C_j = \emptyset$ for all $j < i$ as follows. Suppose that there exists an $x \in C_i \cap C_j$. Since G is κ'-connected and $|D_i| < \kappa$, there is an $x - S_i$-path P with $V(P)\backslash S_i \subseteq C_i$. Let y be the endvertex of P. Since $(S_i \cup C_i) \cap (S_j \cup D_j) = \emptyset$ and $V(P) \subseteq S_i \cup C_i$, we have $y \in C_j$. This contradicts $y \in W$ and $C_j \cap W = \emptyset$.

For each $i < \xi$, we have $N(C_i) \subseteq S_i \cup D_i$ and thus $d(C_i) < \xi$. Hence by Lemma 6, there exists a κ-preserving set $S \subseteq V$ in G with $|S| = \xi \geq \gamma$. ∎

Now we are able to proof Theorem 1 if λ is singular.

Case 2. λ is singular.

Again, using Theorem A and Theorem B, we may assume that $\lambda > \max\{\kappa, \aleph_0\}$. Suppose for contradiction that G does not contain a κ-preserving set of cardinality λ.

Let ϑ be the cofinality of λ and for each cardinal γ, let $\Gamma_\gamma := \{x \in V : d(x) \leq \gamma\}$. If for each $i < \vartheta$, $\lambda_i \leq \lambda$ is a cardinal, then the sequence $(\lambda_i)_{i<\vartheta}$ is called *λ-convergent* if for each $\gamma < \lambda$, there exists an $i_0 < \vartheta$ with $\lambda_i \geq \gamma$ for all $i \geq i_0$. Moreover, a strictly increasing sequence $(\lambda_i)_{i<\vartheta}$ of regular cardinals less then λ is called *regular* if it is λ-convergent and if $\lambda_0 > \max\{\vartheta, \kappa\}$. Finally, we call a sequence $(A_i)_{i<\vartheta}$ of subsets of V *exhaustive*, if $A_i \subseteq A_j$ for each i, j with $i < j$ and $\bigcup_{i<\vartheta} A_i = V$.

(9) There exists a regular sequence $(\lambda_i)_{i<\vartheta}$ with $|\Gamma_{\lambda_i}| \leq \lambda_i$ for each $i < \vartheta$.

Proof. Suppose that there is no such sequence. Then it is easy to construct a regular sequence $(\lambda_i)_{i<\vartheta}$ with $|\Gamma_{\lambda_i}| > \lambda_i$ for each $i < \vartheta$. For each $j < \lambda$, let $\gamma_j := \min\{\lambda_i: i < \vartheta, j \leq \lambda_i\}$. Now for each $j < \lambda$, define a vertex x_j and a subset B_j of V with $|B_j| \leq \gamma_j$ as follows. Assume that x_k and B_k have been defined for each $k < j$. Since $|\{x_k: k < j\} \cup \bigcup_{k<j} B_k| \leq \gamma_j$, there is an $x_j \in \Gamma_{\gamma_j}$ such that $x_j \notin B_k \cup \{x_k\}$ for all $k < j$. By Lemma 4, there exists a κ'-closure B_j of $N(x_j)$ with $|B_j| \leq \gamma_j$. Now by Lemma 5, we obtain a κ-preserving set in G of cardinality λ. This is a contradiction. ∎

For each $A \subseteq V$, let C_A be the set of the components of $G\backslash A$. Moreover, for each cardinal γ, define $C_A^{\gamma} := \{C \in C_A: |C| = \gamma\}$ and $C_A^{<\gamma} := \{C \in C_A: |C| < \gamma\}$. Finally, let $R_A := \bigcup C_A^{<\gamma}$.

(10) Let $A \subseteq V$ with $|A| < \lambda$, and let $(C_i)_{i<\vartheta}$ be a sequence of pairwise disjoint elements of C_A. Then $(|C_i|)_{i<\vartheta}$ is not λ-convergent.

Proof. Suppose that $(|C_i|)_{i<\vartheta}$ is λ-convergent. Clearly, there exists a regular sequence $(\lambda_i)_{i<\vartheta}$ with $\lambda_0 > |A|$. Without loss of generality, we may assume that for each $i < \vartheta$, $\lambda_i \leq |C_i|$ (otherwise we consider a subsequence of $(C_i)_{i<\vartheta}$ satisfying this). Let B be a κ'-closure of A with $\kappa' \leq |B| < \lambda_0$. Then for each $C \in C_A$, since $N(C\backslash B) \subseteq B$, $G[C \cup B]$ is κ'-connected by Lemma 2. Hence by Proposition 1, for each $i < \vartheta$, there is a κ-preserving S_i in $G[C_i \cup B]$ with $|S_i| \geq \lambda_i$. Since $|B| < \lambda$, we may assume that $S_i \subseteq C_i\backslash B$. Let $S := \bigcup_{i<\vartheta} S_i$. Then $|S| = \lambda$. We show that S is κ-preserving in G, which is a contradiction. By construction, $E(S_i, S_j) = \varnothing$ for each i, j with $i \neq j$ and thus by Lemma 3, $d(v, V\backslash S) \geq \kappa$ for each $v \in S$. Hence by Lemma 3, it suffices to show that $G\backslash S$ is κ-connected. Of course, since $S \cap B = \varnothing$, $G[B]\backslash S = G[B]$ is κ-connected. Furthermore, for each $i < \vartheta$, $G[C_i \cup B]\backslash S = G[C_i \cup B]\backslash S_i$ is κ-connected and for every $C \in C_A\backslash\{C_i: i < \vartheta\}$, $G[C \cup B]\backslash S = G[C \cup B]$ is κ-connected. Therefore $G\backslash S$ is κ-connected by Lemma 2. ∎

(11) Let $A \subseteq V$ with $|A| < \lambda$. Then $|C_A^{\gamma}| < \vartheta$ and $|R_A| < \lambda$.

Proof. $|C_A^{\gamma}| < \vartheta$ immediately follows by (10). Now suppose that $|R_A| = \lambda$. Take a λ-convergent sequence $(\lambda_i)_{i<\vartheta}$ with $\lambda_i < \lambda$ for each $i < \vartheta$ and define inductively a sequence $(C_i)_{i<\vartheta}$ in $C_A^{<\gamma}$ as follows. Assume that C_j has been defined for each $j < i$. Since $i < \vartheta$, we have $|\bigcup\{C_j: j < i\}| < \lambda$.

Define $C^* := C_A \backslash \{C_j\colon j < i\}$. Then, since $|R_A| = \lambda$, also $|\bigcup C^*| = \lambda$. Moreover, by Lemma 6 and $N(C) \subseteq A$ for each $C \in C^*$, we have $|C^*| < \lambda$. Hence there exists a $C_i \in C^*$ with $|C_i| \geq \lambda_i$. $(|C_i|)_{i<\vartheta}$ is λ-convergent, contradicting (10). ∎

There exists a regular sequence $(\lambda_i)_{i<\vartheta}$ and an exhaustive sequence $(A_i)_{i<\vartheta}$ satisfying the following conditions.

(12.1) $|A_i| \leq \lambda_i$ for each $i < \vartheta$.

(12.2) $\Gamma_{\lambda_i} \subseteq A_i$ for each $i < \vartheta$.

(12.3) For each $i < \vartheta$ and $B \subseteq V$ with $A_i \subseteq B$ and $|B| \leq \lambda_i$, we have $R_B = \emptyset$.

Proof. Suppose not. By (9), we find a regular sequence $(\lambda_i)_{i<\vartheta}$ and an exhaustive sequence $(X_i)_{i<\vartheta}$ with $\Gamma_{\lambda_i} \subseteq X_i$ and $|X_i| \leq \lambda_i$ for each $i < \vartheta$. There exists a subsequence $(\lambda_{j_i})_{i<\vartheta}$ of $(\lambda_i)_{i<\vartheta}$ and a sequence $(Y_i)_{i<\vartheta}$ of subsets of V such that for each $i < \vartheta$, we have $X_{j_i} \subseteq Y_i$, $|Y_i| \leq \lambda_{j_i}$, and $R_{Y_i} \neq \emptyset$ (otherwise we find a sequence $(A_i)_{i<\vartheta}$ as required). Without loss of generality, we may assume that $j_i = i$ for each $i < \vartheta$. By (11), $|R_{Y_i}| < \lambda$ for each $i < \vartheta$. Thus we obtain a subsequence $(\lambda_{j_i})_{i<\vartheta}$ of $(\lambda_i)_{i<\vartheta}$ with $|R_{Y_{j_i}}| \leq \lambda_{j_{i+1}}$ for each $i < \vartheta$. Without loss of generality, let $j_i = i$ for each $i < \vartheta$.

Define inductively and exhaustive sequence $(Z_i)_{i<\vartheta}$ with $|Z_i| = \lambda_i$ for each $i < \vartheta$ as follows. Assume that Z_j has been defined for each $j < i$. Then $U := \bigcup_{j<i}(Z_j \cup R_{Y_j}) \cup Y_i$ has cardinality at most λ_i. Let Z_i be a κ'-closure of U with $|Z_i| = \lambda_i$.

For each $i < \vartheta$, let $R_i := R_{Y_i} \backslash Z_i$. Note that the sets R_i $(i < \vartheta)$ are pairwise disjoint. Since $N(R_{Y_i}) \subseteq Y_i$, $d(R_{Y_i}) \leq \lambda_i$. Now by $\Gamma_{\lambda_i} \subseteq Y_i$ and $R_{Y_i} \neq \emptyset$, we have $|R_{Y_i}| > \lambda_i$ and thus $|R_i| > \lambda_i$. Since $N(R_i) \subseteq Z_i$ and G and $G[Z_i]$ are κ'-connected, also $G[Z_i \cup R_i]$ is κ'-connected by Lemma 2. Hence by Proposition 1, there exists a κ-preserving S_i in $G[Z_i \cup R_i]$ with $|S_i| \geq \lambda_i^+$ and $S_i \subseteq R_i$. Let $S := \bigcup_{i<\vartheta} S_i$. Then $|S| = \lambda$.

We show that S is κ-preserving in G, which is a contradiction. By construction, for every i, j with $i < j$, we have $S_i \cup N(S_i) \subseteq Z_i$. $S_i \cap S_j = \emptyset$ and $E(S_i, S_j) = \emptyset$ follow. Now for each $i < \vartheta$ and $v \in S$, we obtain $N(v) \backslash S_i = N(v) \backslash S$, and thus, since S_i is κ-preserving in $G[Z_i \cup R_i]$, we have $d(v, V \backslash S) \geq \kappa$ by Lemma 3. Therefore by Lemma 3, it remains to show that $G \backslash S$ is κ-connected. We do this by proving by induction on i that for each $i < \vartheta$, $G[Z_i \cup R_i] \backslash S$ is κ-connected (then since $V = \bigcup_{i<\vartheta} Z_i \cup R_i$ and since this union is nested, we obtain the required result). Clearly, $G[Z_0 \cup R_0] \backslash S = G[Z_0 \cup R_0] \backslash S_0$ is κ-connected. Now assume that $i > 0$ and $G[Z_j \cup R_j] \backslash S$ is κ-connected for each $j < i$. Let $B := \bigcup_{j<i}(Z_j \cup R_j) \subseteq Z_i$. Since the union is nested, also $G[B] \backslash S$ is κ-connected. Suppose

that $G[Z_i \cup R_i]\backslash(S \cup D)$ is not connected for some $D \subseteq (Z_i \cup R_i)\backslash S$ with $|D| < \kappa$. Then there exists a component C of $G[Z_i \cup R_i]\backslash(S \cup D)$ such that $C \cap B = \emptyset$. Choose $x \in C$. Since S_i is κ-preserving in $G[Z_i \cup R_i]$, there exists an $x - B$-path P in $G[Z_i \cup R_i]\backslash(S_i \cup D)$. Since $N(\bigcup_{j<i} S_j) \subseteq B$, P terminates in a vertex of $B\backslash\bigcup_{j<i} S_j$. Thus P is a path in $G[Z_i \cup R_i]\backslash(S \cup D)$ contradicting $C \cap B = \emptyset$. ∎

(13) Let $\gamma < \lambda$ be a cardinal and $A \subseteq V$ with $|A| \leq \gamma$. Moreover, assume that $R_A = \emptyset$ and for each $C \in C_A$, $G[C]$ is γ^+-connected. Then also for each $B \subseteq V$ with $A \subseteq B$ and $|B| \leq \gamma$, we have $R_B = \emptyset$ and $G[C]$ is γ^+-connected for each $C \in C_B$.

Proof. Let $C \in C_B$. Then there exists a $C' \in C_A$ with $C \subseteq C'$. Thus C is a component of $G[C']\backslash B$. Since $G[C']$ is γ^+-connected and $|B| \leq \gamma$, we obtain $C = C'\backslash B$ and $G[C]$ is also γ^+-connected. Moreover, $|C| = \lambda$ since $|C'| = \lambda$. ∎

Now we obtain a regular sequence $(\lambda_i)_{i<\vartheta}$ and an exhaustive sequence $(A_i)_{i<\vartheta}$ with the following properties.

(14.1) $|A_i| \leq \lambda_i$ for each $i < \vartheta$.

(14.2) $R_{A_i} = \emptyset$ for each $i < \vartheta$.

(14.3) For each $i < \vartheta$ and $C \in C_{A_i}$, $G[C]$ is λ_i^+-connected.

Proof. Take a regular sequence $(\lambda_i)_{i<\vartheta}$ and an exhaustive sequence $(A_i')_{i<\vartheta}$ satisfying (12.1), (12.2), and (12.3). It suffices to prove that for each $i < \vartheta$, there exists a $B_i \subseteq V$ such that $A_i' \subseteq B_i$, $|B_i| \leq \lambda_i$, $R_{B_i} = \emptyset$, and for each $C \in C_{B_i}$, $G[C]$ is λ_i^+-connected, because if for each $i < \vartheta$, we have such a B_i, define $A_i := \bigcup_{j<i} B_j$. Then $(A_i)_{i<\vartheta}$ is an exhaustive sequence satisfying (14.1). Moreover, by (13), also (14.2) and (14.3) are satisfied.

Let us now construct sets B_i as described above. Let $i < \vartheta$ and suppose that there is no B_i are required. We define inductively sequences $(D_j)_{j<\vartheta}$ and $(C_j)_{j<\vartheta}$ such that for each $j < \vartheta$, the following conditions are satisfied.

(i) $|D_j| \leq \lambda_i$.
(ii) $D_j \subseteq C_j \in C_{F_j}$ for $F_j := \bigcup_{k<j} D_k \cup A_i'$.
(iii) $G[C_j]\backslash D_j$ is not connected.

Assume that D_k and C_k have been defined for each $k < j$. Then also F_j is defined and $|F_j| \leq \lambda_i$. Thus, by (12.3), $R_{F_j} = \emptyset$. Since there is no B_i as required, there exists a $C_j \in C_{F_j}$ such that $G[C_j]$ is not λ_i^+-connected. By (12.3), $|C_j| = \lambda > \lambda_i^+$. Thus we find $D_j \subseteq C_j$ with $|D_j| \leq \lambda_i$ such that $G[C_j]\backslash D_j$ is not connected.

Let $F := \bigcup_{j<\vartheta} D_j \cup A_i'$. Then $F = \bigcup_{j<\vartheta} F_j$ and $|F| \leq \lambda_i$. Now for each $j < \vartheta$, we define inductively $x_j \in V\backslash F$ such that for each $k < j$

and $l > j$, the vertices x_k and x_j lie in different components of $G - F_l$. Assume that x_k has been defined for each $k < j$. By construction and $C_j \in \mathcal{C}_{F_j}$ we have $x_k \in C_j$ for at most one $k < j$. Thus by (iii), there exists a component C' of $G[C_j]\backslash D_j$ with $C' \cap \{x_k: k < j\} = \varnothing$. By (12.3), $|C'| = \lambda$. Thus there is an $x_j \in C'\backslash F$. Now let $k < j$ and $l > j$. Then since $F_{j+1} = F_j \cup D_j$, x_k and x_j lie in different components of $G - F_{j+1}$. Moreover, since $F_{j+1} \subseteq F_l$, x_k and x_j also lie in different components of $G - F_l$.

For each $j < \vartheta$, let C_j^* be the component of $G\backslash F$ containing x_j. By (12.3), $|C_j^*| = \lambda$ for each $j < \vartheta$. Thus by (11), there are $k, j < \vartheta$ with $k < j$ and $C_j^* = C_k^*$. Let $l := j + 1$ and let $C \in \mathcal{C}_{F_l}$ with $C_j^* \subseteq C$. Then $x_k, x_j \in C$ contradicts our construction. ∎

Take a regular sequence $(\lambda_i)_{i<\vartheta}$ and an exhaustive sequence $(A_i')_{i<\vartheta}$ satisfying (14.1), (14.2), and (14.3). For each $i < \vartheta$ and $C \in \mathcal{C}_{A_i'}$, choose $M_{i,C} \subseteq C$ with $|M_{i,C}| = \kappa'$. Define $M := \bigcup\{M_{i,C}: i < \vartheta, C \in \mathcal{C}_{A_i'}\}$. Then by (11), $|M| \leq \max\{\vartheta, \kappa'\} < \lambda_0$. Now we obtain an exhaustive sequence $(A_i)_{i<\vartheta}$ and a sequence $(Y_i)_{i<\vartheta}$ satisfying the following conditions.

(15.1) $|A_i| = \lambda_i$ for each $i < \vartheta$.

(15.2) $M \subseteq A_0$.

(15.3) $Y_0 = \varnothing$ and for each $i < \vartheta$, $Y_i \subseteq A_i\backslash \bigcup_{j<i} A_j$ and $N(Y_i) \subseteq A_i$.

(15.4) $G[A_0]$ is κ'-connected.

(15.5) For each $i < \vartheta$ and $x \in A_i\backslash(Y_i \cup \bigcup_{j<i} A_j)$, there exist κ' pairwise nearly disjoint $x - M$-paths in $G[A_i]$ not containing vertices of $Y_i \cup (\bigcup_{j<i} A_j\backslash M)$.

(15.6) For each $i < \vartheta$ and $y \in Y_i$, there exist $\sup_{j<i}\lambda_j$ pairwise nearly disjoint $y - V\backslash Y_i$-paths in G.

Proof. For each $i < \vartheta$, define inductively A_i and Y_i satisfying (15.1), ..., (15.6) as follows. Let $B := A_0' \cup M$. Then by (13), (14.2), and (14.3), for each $C \in \mathcal{C}_B$, we have $|C| = \lambda$ and $G[C]$ is λ_0^+-connected. Moreover, $|\mathcal{C}_B| < \vartheta$ by (11). For each $C \in \mathcal{C}_B$, take $L_C \subseteq C$ with $|L_C| = \lambda_0$. Let A_0 be a κ'-closure of $B \cup \bigcup\{L_C: C \in \mathcal{C}_B\}$ with $|A_0| = \lambda_0$ and let $Y_0 = \varnothing$.

Now assume that $i > 0$ and that A_j and Y_j have been defined for each $j < i$. Let $A^* := \bigcup_{j<i} A_j$, $B := A^* \cup A_i'$ and $Y_i := \{y \in B\backslash A^*: d(y) \leq \lambda_i\}$. Then by construction, $|Y_i| \leq |B| \leq \lambda_i$ and thus $d(Y_i) \leq \lambda_i$. Moreover, by (13), (14.2), and (14.3), for each $C \in \mathcal{C}_B$, we have $|C| = \lambda$ and $G[C]$ is λ_i^+-connected. Furthermore, for each $x \in B\backslash(Y_i \cup A^*)$, $|N(x)\backslash B| \geq \lambda_i^+$. Thus by (11), for each $x \in B\backslash(Y_i \cup A^*)$, there is a

Sh:444

$C_x \in \mathcal{C}_B$ with $d(x, C_x) \geq \kappa'$. Choose $N_x \subseteq N(x) \cap C_x$ with $|N_x| = \kappa'$. Let $C \in \mathcal{C}_B$. Then by (14.3) and $|B| \leq \lambda_i$, there exists a $D \in \mathcal{C}_{A_i'}$ with $C = D \backslash B$. Moreover, for each $y \in M_{i,D}$, we have $d(y, C) \geq \lambda_i^+ > \kappa'$ since $d(y) \geq \lambda_i^+$ (by (14.3)) and $N(y) \subseteq C \cup B$. Thus by $|M_{i,D}| = \kappa'$, it is easy to find pairwise disjoint edges $\{x_{k,C}, y_{k,C}\}$ ($k < \kappa'$) with $x_{k,C} \in C$ and $y_{k,C} \in M_{i,D}$ for each $k < \kappa'$. Choose an $L_C \subseteq C$ with $|L_C| = \lambda_i$ and define $U_C := \{x_{k,C}: k < \kappa'\} \cup \bigcup\{N_x: x \in B\backslash(Y_i \cup A^*)$ and $C_x = C\} \cup (N(Y_i) \cap C) \cup L_C$. Then $|U_C| = \lambda_i$. Take a κ'-closure $\overline{U_C}$ of U_C in $G[C]$ of cardinality λ_i. Finally, define $A_i := B \cup \bigcup\{\overline{U_C}: C \in \mathcal{C}_B\}$.

By construction, it is obvious that the defined sets A_i and Y_i satisfy (15.1),...,(15.5). To prove (15.6), note that for each $i < \vartheta$, by adding the sets L_C to A_i, we achieved that for each $y \in V\backslash A_i$, there exist λ_i pairwise nearly disjoint $y - A_i$-paths in G since $G[C]$ is always λ_i^+-connected. Now let $y \in Y_i$. For each $j < i$, be define inductively a set $\mathcal{P}_j$ of λ_j pairwise nearly disjoint $y - V\backslash Y_i$-paths in G as follows. Assume that $\mathcal{P}_k$ is defined for each $k < j$. Since $(\lambda_k)_{k<\vartheta}$ is regular, $\sup_{k<j}\lambda_k < \lambda_j$. Thus by construction, we find a set $\mathcal{P}_j$ of λ_j pairwise nearly disjoint $y - V\backslash Y_i$-paths in G not containing vertices of $\bigcup\{V(P): k < j, P \in \mathcal{P}_k\}$. $\bigcup_{j<i} \mathcal{P}_j$ is a set of $\sup_{j<i}\lambda_j$ pairwise nearly disjoint $y - V\backslash Y_i$-paths in G. ∎

Let $Y := \bigcup_{i<\vartheta} Y_i$. Moreover, for each $i > 0$, define $B_i := A_0 \cup (A_i\backslash(Y_i \cup \bigcup_{j<i} A_j))$. Then by (15.2), (15.4), (15.5), and Lemma 2. $G[B_i]$ is always κ'-connected.

Subcase 2.1. $|Y| < \lambda$.

Then there exists an $i_0 > 0$ with $\lambda_{i_0} > |Y|$. Moreover, since $(\lambda_i)_{i<\vartheta}$ is regular, $|A_i| > |\bigcup_{j<i} A_j|$ for each $i \geq i_0$, by (15.1). Thus, for each $i \geq i_0$, $|B_i| = \lambda_i$ so that by Case 1, there is a κ-preserving S_i in $G[B_i]$ with $|S_i| = \lambda_i$ and $S_i \cap A_0 = \varnothing$. For each $i < i_0$, let $S_i := \varnothing$. Define $S := \bigcup_{i<\vartheta} S_i$. Then $|S| = \lambda$. Moreover, $G[A_0]\backslash S = G[A_0]$ is κ-connected, $G[B_i]\backslash S = G[B_i]\backslash S_i$ is κ-connected for each $i < \vartheta$, and $B_i \cap B_j = A_0$ for every i, j with $i \neq j$. Thus by Lemma 2, the graph $G\backslash(Y \cup S) = G[\bigcup_{i<\vartheta} B_i]\backslash S$ is κ-connected. Furthermore, for every $i < \vartheta$ and $s \in S_i$, we have $d(s, V\backslash(Y \cup S)) \geq d(s, B_i\backslash S_i) \geq \kappa$. Hence by Lemma 3, S is κ-preserving in $G\backslash Y$.

Since G is κ-connected, for each $y \in Y$ there exist κ pairwise nearly disjoint $y - V\backslash Y$-paths in G; let T_y be the set of the endvertices of these paths and define $T := \bigcup_{y\in Y} T_y$. Then $|T| < \lambda$. Thus $S' := S\backslash T$ is also a κ-preserving set in $G\backslash Y$ of cardinality λ. Moreover, for each $y \in Y$, there are κ pairwise nearly disjoint $y - V\backslash Y$-paths in $G\backslash S'$. Hence $G\backslash S'$ is κ-connected by Lemma 2 and thus, by Lemma 3, S' is κ-preserving in G. This is a contradiction.

Subcase 2.2. $|Y| = \lambda$.

Define a subsequence $(Y_{j_i})_{i<\vartheta}$ of $(Y_i)_{i<\vartheta}$ as follows. Assume that j_k has been defined for each $k < i$. Let $\alpha := \max\{\sup_{k<i} j_k, i + 1\}$. Then, since $|Y| = \lambda$, there is a $j_i > \alpha$ with $|Y_{j_i}| > \lambda_i$.

For each $i < \vartheta$, let $Z_i := Y_{j_i}$ and $\gamma_i := \sup_{k<j_i} \lambda_k$. Then since $j_i \geq i + 2$, we have $\gamma_i > \lambda_i$. Moreover, $|Z_i| > \lambda_i$ and for each $z \in Z_i$, there are γ_i pairwise nearly disjoint $z - V\backslash Z_i$-paths in G by (15.6).

For each $i < \vartheta$, choose $S_i \subseteq Z_i$ with $|S_i| = \lambda_i$ and let $S := \bigcup_{i<\vartheta} S_i$. Then $|S| = \lambda$. Moreover, $G[A_0]$ is κ-connected, $G[B_i]$ is κ-connected for $i < \vartheta$, and $B_i \cup B_j = A_0$ for every i, j with $i \neq j$. Thus the graph $G\backslash Y = G[\bigcup_{i<\vartheta} B_i]$ is also κ-connected by Lemma 2. Now let $z \in Y\backslash S$. If $z \in Z_i$ for an $i < \vartheta$, there exist κ pairwise nearly disjoint $z - V\backslash Z_i$-paths in $G\backslash S_i$ since $\gamma_i > |S_i| = \lambda_i \geq \kappa$. Otherwise we have $z \in Y_j$ for a $j \notin \{j_i : i < \vartheta\}$, and since G is κ-connected, there are κ pairwise nearly disjoint $z - V\backslash Y_j$-paths in G. By (15.3), $E(Y_i, Y_j) = \emptyset$ for every i, j with $i \neq j$. Thus in each case the paths avoid S and end in $G\backslash Y$. Hence by Lemma 2, $G\backslash S$ is κ-connected. Moreover, for each $i < \vartheta$ and $s \in S_i$, we have $d(s) \geq \gamma_i$ by (15.6) and thus $|N(s)\backslash S_i| \geq \kappa$. Since $E(Y_i, Y_j) = \emptyset$ for every i, j with $i \neq j$, we even have $|N(s)\backslash S| \geq \kappa$. Hence by Lemma 3, S is κ-preserving in G. Again we have a contradiction.

This completes the proof of Theorem 1 if λ is singular.

ACKNOWLEDGMENT

The authors would like to thank the editor and the unknown referee for their useful hints.

This work was supported by the Volkswagen-Stiftung.

References

[1] T. Jech, *Set Theory.* Academic Press, New York (1978).

[2] W. Mader, On infinite n-connected graphs. *Cycles and Rays.* Kluwer Academic Publishers, Netherlands (1990) 151–160.

[3] C. Thomassen, Large α-preserving sets in infinite α-connected graphs. A. Baker, B. Bollobas, A. Hajnal: A Tribute to Paul Erdös. Cambridge University Press, New York (1990) 445–450.