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Q-SETS DO NOT NECESSARILY HAVE STRONG MEASURE ZERO

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ABSTRACT. The purpose of this paper is to give a negative answer to the following question (see Miller [4]): Do all Q-sets have strong measure zero?

1. Definitions and standard facts.

1.1 *Q-set.* A set of reals X is a Q-set iff every subset of X is a relative F_σ. The history of Q-sets can be found in Fleissner's paper [2]. We recall the following facts

(i) If X is a Q-set then $|X| < 2^{\aleph_0}$ and $2^{|X|} = 2^{\aleph_0} = c$.

(ii) Every Q-set has universal measure zero.

(iii) Martin's axiom implies that if $X \subseteq \mathbf{R}$ and $|X| < 2^{\aleph_0}$, then X is a Q-set.

1.2 *Strong measure zero set.* A set of reals X has strong measure zero iff given any sequence $\varepsilon_n > 0$ for $n < \omega$, X can be covered by a sequence of open sets X_n each having diameter less than ε_n.

1.3 *Ramsey ultrafilters.* An ultrafilter $U \subseteq P(\omega)$ is a Ramsey ultrafilter iff U contains the filter of cofinite sets and for any $\pi\colon [\omega]^2 \to 2$ there is an $A \in U$ with π constant on $[A]^2$. For A, B subsets of ω, we say that $A \subseteq^* B$ iff there exists $n \in \omega$ such that $A - n \subseteq B$.

We say that a family $\langle A_\alpha \colon \alpha < \kappa\rangle$, κ a cardinal, is a tower iff $A_\beta \subseteq^* A_\alpha$ and $A_\alpha \not\subseteq^* A_\beta$ for every $\alpha < \beta$, and for every $A \subseteq \omega$, it is not the case that $\forall \alpha < \kappa$ $A \subseteq^* A_\alpha$.

The following facts are well known.

(i) Martin's axiom implies $\kappa = 2^{\aleph_0}$.

(ii) Martin's axiom implies that there exists a Ramsey ultrafilter which is generated by a tower.

Let U be a Ramsey ultrafilter over ω. We define the following poset P_U: the elements of P_U are ordered pairs (s, A) such that $s \in \omega^{<\omega}$, $A \in U$, $\sup s < \inf A$, and the order is given by: $(s, A) \le (t, B)$ iff

$$s \subseteq t, \quad B \subseteq A \quad \text{and} \quad t - s \subseteq A.$$

It is clear that P_U satisfies the countable chain condition and the generic object can be regarded as a subset of ω characterized by being almost contained in every member of the filter U (see Mathias [5]).

2. THEOREM. *Let V be a model for ZFC+Martin's axiom, let $U \in V$ be a Ramsey ultrafilter generated by a tower $\langle A_\alpha \colon \alpha < c\rangle$, let P_U be the forcing notion defined above this U, and let $G \subseteq P_U$ be a generic object over V. Then*

(i) *V and $V[G]$ have the same cardinals.*

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(ii) $V[G] \vDash$ "$c = c^V$".
(iii) *If* $X \in V \cap P(\mathbf{R})$ *and* $|X| < c$, *then*

$$V[G] \vDash \text{“}X \text{ is a } Q\text{-set”}.$$

(iv) *If* $X \in V \cap P(\mathbf{R})$ *and* $|X| > \aleph_0$, *then*

$$V[G] \vDash \text{“}X \text{ does not have strong measure zero”}.$$

2.1 REMARK. In $V[G]$, the old uncountable subsets of reals, of cardinality less than c, are Q-sets but not of strong measure zero.

PROOF. Clear by (iii) and (iv).

2.2 *Proof of the theorem.* (i) By countable chain condition of P_U.

(ii) By countable chain condition every real in $V[G]$ is obtained by a name which is encodable in V by a real.

(iii) Let $X \in V \cap P(\mathbf{R})$ and $|X| < c$. Let $\mathbf{h}: X \to \{0,1\}$ be a P_U-name for a subset of X. By Mathias [**5**], for every $i \in X$ there exists $A_i \in U$ such that if $n \in A_i$ and $s \subseteq n$, then

$$(s, A_i - n) \Vdash \mathbf{h}(i) = 0 \quad \text{or} \quad (s, A_i - n) \Vdash \mathbf{h}(i) = 1.$$

Since U is generated by a tower, and $|X| < c$, there exists $A \in U$ such that for every $i \in X$, $A \subseteq^* A_i$. Therefore, for every $i \in X$ there exists $n_i \in \omega$ such that $A - n_i \subseteq A_i$ and $n_i \in A_i$.

So if $(\phi, A) \in G$, and r ($\subseteq \omega$) is the real number defined by G, we have that $\mathbf{h}(i)$ is computable from $r \restriction n_i$.

Now we define the following equivalence relation on X:

$$i \sim j \quad \text{iff} \quad n_i = n_j \quad \text{and}$$
$$(\forall s \subseteq n_i)((s, A_i - n_i) \Vdash \mathbf{h}(i) = 0 \quad \text{iff} \quad (s, A_j - n_j) \Vdash \mathbf{h}(j) = 0).$$

It is clear that $\sim$ is an equivalence relation with countably many classes, say $X = \bigcup_{l \in \omega} X_l$ where each X_l is an equivalence class and the following holds:

$$\text{if } i, j \text{ belong to } X_l \text{ for } l \in \omega, \text{ then}$$
$$(\phi, A) \Vdash \mathbf{h}(i) = \mathbf{h}(j).$$

Each X_l for $l \in \omega$ belongs to V and also $\langle X_l : l \in \omega \rangle$ is a number of V. Since $V \vDash \text{MA}$ for every $l \in \omega$, there exists Y_l, an F_σ set of reals, such that

$$V \vDash X_l = Y_l \cap X.$$

Therefore, by an absoluteness argument,

$$V[G] \vDash X_l = Y_l \cap X$$

(remember that Y_l is a definition of a set), and thus in $V[G]$

$$\{i : h(i) = 0\} = X \cap \left(\bigcup \{Y_l : (\forall i \in X_l)(h(i) = 0)\}\right),$$

and this says that $\{i : h(i) = 0\}$ is a F_σ set relative to X. This completes the proof of (iii).

(iv) This fact is well known and the proof is obtained following the argument given by Baumgartner [**1**, §9] in which it is possible to replace Mathias' forcing by P_U and to use the results proven by Mathias [**5**].

This concludes the proof of the theorem, and the following question arises: Is "ZFC+ Borel conjecture + there exists Q-set" consistent?

REFERENCES

1. J. Baumgartner, *Iterated forcing*, Surveys in Set Theory (A. R. D. Mathias, ed.), London Math. Soc. Lecture Notes Series 87, Cambridge Univ. Press, Cambridge, 1983, pp. 1–50.
2. W. Fleissner, *Current research on Q-sets*, Topology, vol. I, Colloq. Math. Soc. Janós Bolyai, 23, North-holland, 1980, pp. 413–431.
3. W. Fleissner and A. Miller, *On Q-set*, Proc. Amer. Math. Soc. **78** (1980), 280–284.
4. A. Miller, *Special subsets of the real line*, Handbook of Set-Theoretic Topology, Chapter 5 (K. Kunnen and J. Vaughan, eds.), North-Holland, 1984, pp. 201–233.
5. A. Mathias, *Happy families*, Ann. Math. Logic **12** (1977), 59–111.
6. F. Rothberger, *On some problems of Hausdorff and Sierpiński*, Fund. Math. **35** (1948), 29–46.

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