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SUPERDESTRUCTIBILITY: A DUAL TO LAVER'S  
INDESTRUCTIBILITY

JOEL DAVID HAMKINS AND SAHARON SHELAH

**Abstract.** After small forcing, any  $<\kappa$ -closed forcing will destroy the supercompactness and even the strong compactness of  $\kappa$ .

In a delightful argument, Laver [3] proved that any supercompact cardinal  $\kappa$  can be made indestructible by  $<\kappa$ -directed closed forcing. This indestructibility, however, is evidently not itself indestructible, for it is always ruined by small forcing: in [1] the first author recently proved that small forcing makes any cardinal superdestructible; that is, any further  $<\kappa$ -closed forcing which adds a subset to  $\kappa$  will destroy the measurability, even the weak compactness, of  $\kappa$ . What is more, this property holds higher up: after small forcing, any further  $<\kappa$ -closed forcing which adds a subset to  $\lambda$  will destroy the  $\lambda$ -supercompactness of  $\kappa$ , provided  $\lambda$  is not too large (his proof needed that  $\lambda < \aleph_{\kappa+\delta}$ , where the small forcing is  $<\delta$ -distributive). In this paper, we happily remove this limitation on  $\lambda$ , and show that after small forcing, the supercompactness of  $\kappa$  is destroyed by any  $<\kappa$ -closed forcing. Indeed, we will show that even the strong compactness of  $\kappa$  is destroyed. By doing so we answer the questions asked at the conclusion of [1], and obtain the following attractive complement to Laver indestructibility:

**MAIN THEOREM.** *After small forcing, any  $<\kappa$ -closed forcing will destroy the supercompactness and even the strong compactness of  $\kappa$ .*

We will provide two arguments. The first, similar to but generalizing the Superdestruction Theorem of [1], will show that supercompactness is destroyed; the second, by a different technique, will show fully that strong compactness is destroyed. Both arguments will rely fundamentally on the Key Lemma, below, which was proved in [1]. Define that a set or sequence is *fresh* over  $V$  when it is not in  $V$  but every initial segment of it is in  $V$ .

**KEY LEMMA.** *Assume that  $|\mathbb{P}| = \beta$ , that  $\Vdash_{\mathbb{P}} \dot{Q}$  is  $\leq\beta$ -closed, and that  $\text{cof}(\lambda) > \beta$ . Then  $\mathbb{P} * \dot{Q}$  adds no fresh subsets of  $\lambda$ , and no fresh  $\lambda$ -sequences.*

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While in [1] it is proved only that no fresh sets are added, the following simple argument shows from this that no fresh sequences can be added: given a sequence in  $\delta^\lambda$ , code it in the natural way with a binary sequence of length  $\delta\lambda$ , by using  $\lambda$  many blocks of length  $\delta$ , each with one 1. The binary sequence corresponds to a subset of the ordinal  $\delta\lambda$ , which, since  $\text{cof}(\delta\lambda) = \text{cof}(\lambda)$ , cannot be fresh. Thus, the original  $\lambda$ -sequence cannot be fresh.

Let us give now the first argument. We will use the notion of a  $\theta$ -club to extend the inductive proof of the Superdestruction Theorem [1] to all values of  $\lambda$ .

**THEOREM.** *After small forcing, any  $<\kappa$ -closed forcing which adds a subset to  $\lambda$  will destroy the  $\lambda$ -supercompactness of  $\kappa$ .*

**PROOF.** Suppose that  $|\mathbb{P}| < \kappa$  and  $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$  is  $<\kappa$ -closed. Suppose that  $g * G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$  is  $V$ -generic, and that  $\mathbb{Q} = \dot{\mathbb{Q}}_g$  adds a new subset  $A \subseteq \lambda$ , with  $\lambda$  minimal, so that  $A \in V[g][G]$  but  $A \notin V[g]$ . By the closure of  $\mathbb{Q}$ , we know that  $\text{cof}(\lambda) \geq \kappa$ . Suppose, towards a contradiction, that  $\kappa$  is  $\lambda$ -supercompact in  $V[g][G]$ . Let  $P_\kappa\lambda$  denote  $(P_\kappa\lambda)^{V[g][G]}$ , which is also  $(P_\kappa\lambda)^{V[g]}$ .

**LEMMA.** *Every normal fine measure on  $P_\kappa\lambda$  in  $V[g][G]$  concentrates on  $(P_\kappa\lambda)^V$ .*

**PROOF.** Let us begin with some definitions. Fix a regular cardinal  $\theta$  such that  $|\mathbb{P}| < \theta < \kappa$ . A set  $C \subseteq P_\kappa\lambda$  is *unbounded* if and only if for every  $\sigma \in P_\kappa\lambda$  there is  $\tau \in C$  such that  $\sigma \subseteq \tau$ . A set  $D \subseteq P_\kappa\lambda$  is  $\theta$ -*directed* if and only if whenever  $B \subseteq D$  and  $|B| < \theta$  then there is some  $\tau \in D$  such that  $\sigma \subseteq \tau$  for every  $\sigma \in B$ . The set  $C$  is  $\theta$ -*closed* if and only if every  $\theta$ -directed  $D \subseteq C$  with  $|D| < \kappa$  has  $\bigcup D \in C$ . Finally,  $C$  is a  $\theta$ -*club* if and only if  $C$  is both  $\theta$ -closed and unbounded.

**CLAIM.** *A normal fine measure on  $P_\kappa\lambda$  contains every  $\theta$ -club.*

**PROOF.** Work in any model  $\vec{V}$ . Suppose that  $C$  is a  $\theta$ -club in  $P_\kappa\lambda$  and that  $\mu$  is a normal fine measure on  $P_\kappa\lambda$ . Let  $j: \vec{V} \rightarrow M$  be the ultrapower by  $\mu$ . It is well known that  $j''\lambda$  is a seed for  $\mu$  in the sense that  $X \in \mu$  if and only if  $j''\lambda \in j(X)$  for  $X \subseteq P_\kappa\lambda$ . By elementarity  $j(C)$  is a  $\theta$ -club in  $M$  and  $j''C \subseteq j(C)$ . (We know  $j''C \in M$  because  $M$  is closed under  $\lambda^{<\kappa}$  sequences in  $\vec{V}$ .) Also, it is easy to check that  $j''C$  is  $\theta$ -directed. Thus, by the definition of  $\theta$ -club, we know  $\bigcup(j''C) \in j(C)$ . But

$$\bigcup(j''C) = \bigcup_{\sigma \in C} j(\sigma) = \bigcup_{\sigma \in C} (j''\sigma) = j''\lambda.$$

Thus,  $j''\lambda \in j(C)$  and so  $C \in \mu$ . +

Now let  $C = (P_\kappa\lambda)^V$ . We will show that  $C$  is a  $\theta$ -club in  $V[g][G]$ . First, let us show that  $C$  is unbounded. If  $\sigma \in P_\kappa\lambda$  in  $V[g][G]$ , then actually  $\sigma \in V[g]$ , and so  $\sigma = \dot{\sigma}_g$  for some  $\mathbb{P}$ -name  $\dot{\sigma} \in V$ . We may assume that  $\Vdash |\dot{\sigma}| < \kappa = 1$  and consequently  $\sigma \subseteq \{\alpha \mid \Vdash \check{\alpha} \in \dot{\sigma}\} \in C$ ; so  $\sigma$  is covered as desired. To show that  $C$  is  $\theta$ -closed, suppose in  $V[g][G]$  that  $D \subseteq C$  has size less than  $\kappa$  and is  $\theta$ -directed. We have to show that  $\bigcup D \in C$ . It suffices to show that  $\bigcup D \in V$  since  $C = P_\kappa\lambda \cap V$ . Since  $\mathbb{Q}$  is  $<\kappa$ -closed, we know that  $D \in V[g]$ , and thus  $D = \dot{D}_g$  for some name  $\dot{D} \in V$ . In  $V$  let  $D_p = \{\sigma \in C \mid p \Vdash \dot{\sigma} \in \dot{D}\}$ . It follows that  $D = \bigcup_{p \in g} D_p$ . There must be some  $p \in g$  such that  $D_p$  is  $\subseteq$ -cofinal in  $D$ ; for if not, then for each  $p \in g$  we may choose  $\sigma_p \in D$  such that  $D_p$  contains no supersets

of  $\sigma_p$ . Since  $D$  is  $\theta$ -directed and  $|g| < \theta$  there is some  $\sigma \in D$  such that  $\sigma_p \subseteq \sigma$  for all  $p \in g$ . But  $\sigma$  must be forced into  $D$  by some condition  $p \in g$ , so  $\sigma \in D_p$  for some  $p \in g$ , contradicting the choice of  $\sigma_p$ . So we may fix some  $p \in g$  such that  $D_p$  is  $\subseteq$ -cofinal in  $D$ . But in this case  $\bigcup D_p = \bigcup D$  and since  $D_p \in V$  we conclude  $\bigcup D \in V$ . Thus  $C$  is a  $\theta$ -club in  $V[g][G]$ , and the lemma is proved.  $\dashv$

Let us now continue with the theorem. Since  $\kappa$  is  $\lambda$ -supercompact in  $V[g][G]$  there must be an embedding  $j: V[g][G] \rightarrow M[g][j(G)]$  which is the ultrapower by a normal fine measure  $\mu$  on  $P_\kappa\lambda$ .

LEMMA.  $P(\lambda)^M = P(\lambda)^V$ .

PROOF. ( $\supseteq$ ). By the previous lemma we know that  $(P_\kappa\lambda)^V \in \mu$  and so  $j^{\text{``}}\lambda \in j((P_\kappa\lambda)^V) = (P_\kappa\lambda)^M$ . Since  $M$  is transitive, it follows that  $j^{\text{``}}\lambda \in M$ . And obtaining this fact was the only reason for proving the previous lemma. Now if  $B \subseteq \lambda$  and  $B \in V$  then  $j(B) \in M$ , and since  $B$  is constructible from  $j(B)$  and  $j^{\text{``}}\lambda$  it follows that  $B \in M$  as well.

( $\subseteq$ ). Now we prove the converse. By induction we will show that  $P(\delta)^M \subseteq V$  for all  $\delta \leq \lambda$ . Suppose that  $B \subseteq \delta$  and  $B \in M$  and every initial segment of  $B$  is in  $V$ . By the Key Lemma it follows that  $B \in V$  unless  $\text{cof}(\delta) < \kappa$ . So suppose  $\text{cof}(\delta) < \kappa$ . By the closure of  $\mathbb{Q}$  we know in this case that  $B \in V[g]$  and so  $B = \dot{B}_g$  for some name  $\dot{B} \in V$ . We may view  $\dot{B}$  as a function from  $\delta$  to the set of antichains of  $\mathbb{P}$ . Since  $\dot{B}$  may be coded with a subset of  $\delta$ , we know  $\dot{B} \in M$  by the previous direction of this lemma. Thus, both  $B$  and  $\dot{B}$  are in  $M$  and  $g$  is  $M$ -generic. Since  $B = \dot{B}_g$  in  $M[g]$  there is in  $M$  a condition  $p \in g$  such that  $p \Vdash \dot{B} = \dot{B}$ . That is,  $p$  decides every antichain of  $\dot{B}$  in a way that makes it agree with  $B$ . Use  $p$  to decide  $\dot{B}$  in  $V$  and conclude that  $B \in V$ . This completes the induction.  $\dashv$

Now we are nearly done. Consider again the new set  $A \subseteq \lambda$  such that  $A \in V[g][G]$  but  $A \notin V[g]$ . Since  $j$  is a  $\lambda$ -supercompact embedding, we know  $A \in M[g][j(G)]$ . Since the  $j(G)$  forcing is  $< j(\kappa)$ -closed, we know  $A \in M[g]$ . Therefore  $A = \dot{A}_g$  for some name  $\dot{A} \in M$ . Viewing  $\dot{A}$  as a function from  $\lambda$  to the set of antichains in  $\mathbb{P}$ , we can code  $\dot{A}$  with a subset of  $\lambda$ , and so by the last lemma we know  $\dot{A} \in V$ . Thus,  $A = \dot{A}_g \in V[g]$ , contradicting the choice of  $A$ .  $\dashv$

COROLLARY. *By first adding in the usual way a generic subset to  $\beta$  and then to  $\lambda$ , where  $\text{cof}(\lambda) > \beta$ , one destroys all supercompact cardinals between  $\beta$  and  $\lambda$ .*

In fact, one does not even need to add them in the usual way. This is because the proof of the theorem does not really use the full  $< \kappa$ -closure of  $\mathbb{Q}$ . Rather, if  $\mathbb{P}$  has size  $\beta$ , then we only need that  $\mathbb{Q}$  is  $\leq \beta$ -closed and adds no new elements of  $P_\kappa\lambda$ . Thus, we have actually proved the following theorem.

THEOREM. *After any forcing of size  $\beta < \kappa$ , any further  $\leq \beta$ -closed forcing which adds a subset to  $\lambda$  but no elements to  $P_\kappa\lambda$  will destroy the  $\lambda$ -supercompactness of  $\kappa$ .*

This improvement is striking when  $\beta$  is small, having the consequence that after adding a Cohen real, any countably-closed forcing which adds a subset to some minimal  $\lambda$  destroys all supercompact cardinals up to  $\lambda$ .

Let us now give the second argument, which will improve the previous results with a different technique and establish fully that strong compactness is destroyed.

**THEOREM.** *After small forcing, any  $<\kappa$ -closed forcing which adds a  $\lambda$ -sequence will destroy the  $\lambda$ -strong compactness of  $\kappa$ .*

**PROOF.** Define that a cardinal  $\kappa$  is  $\lambda$ -measurable if and only if there is a  $\kappa$ -complete (non  $\kappa^+$ -complete) uniform measure on  $\lambda$ . Necessarily  $\kappa \leq \text{cof}(\lambda)$ . This notion is studied in [2].

**LEMMA.** *Assume that  $|\mathbb{P}| < \kappa \leq \lambda$ , that  $\dot{\mathbb{Q}}$  adds a new  $\lambda$ -sequence over  $V^{\mathbb{P}}$ ,  $\lambda$  minimal, and that  $\kappa$  is  $\lambda$ -measurable in  $V^{\mathbb{P} * \dot{\mathbb{Q}}}$ . Then  $\mathbb{P} * \dot{\mathbb{Q}}$  must add a fresh  $\lambda$ -sequence over  $V$ .*

**PROOF.** This lemma is the heart of the proof. Assume the hypotheses of the lemma. So, for suitable names,  $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}} \dot{s}$  is a  $\lambda$ -sequence of ordinals not in  $V^{\mathbb{P}}$ , and  $\dot{\mu}$  is a  $\kappa$ -complete uniform measure on  $\lambda$ . Without loss of generality, we may assume that  $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$  is a complete boolean algebra on an ordinal. Suppose now that  $g * G$  is  $V$ -generic for  $\mathbb{P} * \dot{\mathbb{Q}}$ . Let  $\mathbb{Q} = \dot{\mathbb{Q}}_g$ , and  $s = \dot{s}_{g * G}$ .

In  $V[g]$ , let  $T = \{u \in \text{ORD}^{<\lambda} \mid \llbracket \dot{u} \subseteq \dot{s} \rrbracket^{\mathbb{Q}} \neq 0\}$ . Thus, under inclusion,  $T$  is a tree with  $\lambda$  many levels, and  $\mathbb{Q}$  adds the  $\lambda$ -branch  $s$ . For  $u \in T$ , let  $b_u = \llbracket \dot{u} \subseteq \dot{s} \rrbracket^{\mathbb{Q}}$ . Thus,  $b_u$  is an ordinal. Let  $I = \{\langle \ell(u), b_u \rangle \mid u \in T\}$ , where  $\ell(u)$  denotes the length of  $u$ , and define  $\langle \alpha, b_\alpha \rangle \triangleleft \langle \alpha', b_{\alpha'} \rangle$  when  $\alpha' < \alpha$  and  $b_\alpha \leq_{\mathbb{Q}} b_{\alpha'}$ . Since  $u \supset v$  if and only if  $\langle \ell(u), b_u \rangle \triangleleft \langle \ell(v), b_v \rangle$  it follows that  $\langle T, \supset \rangle \cong \langle I, \triangleleft \rangle$ , and consequently  $I$  is also a tree, under the relation  $\triangleleft$ , with  $\lambda$  many levels. Furthermore, the  $\alpha$ th level of  $I$  consists of pairs of the form  $\langle \alpha, \beta \rangle$ . For  $p \in \mathbb{P}$  let us define that  $a \triangleleft_p b$  when  $p \Vdash a \triangleleft b$ . Thus,  $\triangleleft = \bigcup_{p \in \mathbb{P}} \triangleleft_p$ .

In  $V[g][G]$  let  $b_\gamma = \langle \gamma, b_{s \upharpoonright \gamma} \rangle$ . Thus,  $b_\gamma \in I$ , and if  $\gamma < \zeta$  then  $b_\zeta \triangleleft b_\gamma$  and so there is some  $r \in g$  such that  $b_\zeta \triangleleft_r b_\gamma$ . Since there are fewer than  $\kappa$  many such  $r$ , for each  $\gamma$  there must be an  $r$  which works for  $\mu$ -almost every  $\zeta$ . But then again, since there are relatively few  $r$ , it must be that there is some  $r^* \in g$  which has this property for  $\mu$ -almost every  $\gamma$ . So, fix  $r^* \in g$  such that for  $\mu$ -almost every  $\gamma$ , for  $\mu$ -almost every  $\zeta$ , we have  $b_\zeta \triangleleft_{r^*} b_\gamma$ . Fix also a condition  $\langle p_0, q_0 \rangle \in g * G$  forcing  $r^*$  to have this property. Let  $t = \langle b_\gamma \mid \gamma < \lambda \text{ \& for } \mu\text{-a.e. } \zeta, b_\zeta \triangleleft_{r^*} b_\gamma \rangle$ . Thus,  $t$  is a partial function from  $\lambda$  to pairs of ordinals, and  $\text{dom}(t) \in \mu$ . In particular,  $\text{dom}(t)$  is unbounded in  $\lambda$ .

We will argue that  $t$  is fresh over  $V$ . First, notice that  $t \notin V[g]$  since in  $V[g]$  knowing  $t$  we could read off the branch  $s$ . Thus,  $t \notin V$ .

Nevertheless, we will argue that every initial segment of  $t$  is in  $V$ . Suppose  $\delta < \lambda$ , and let  $t_\delta = t \upharpoonright \delta$ . By the minimality of  $\lambda$  it follows that  $t_\delta \in V[g]$ , and so there is a  $\mathbb{P}$ -name  $\dot{t}_\delta$  and a condition  $\langle p_1, q_1 \rangle \in g * G$ , stronger than  $\langle p_0, q_0 \rangle$ , forcing this name to work. Assume towards a contradiction that  $t_\delta \notin V$ , and that this is forced by  $p_1$ . Then, for each  $r \in \mathbb{P}$  below  $p_1$  we may choose  $\gamma_r < \delta$  such that  $r$  does not decide  $t(\gamma_r)$  (or whether  $\gamma_r$  is in the domain of  $t$ ). But, nevertheless, for each  $r$  either for  $\mu$ -almost every  $\zeta$ ,  $b_\zeta \triangleleft_{r^*} b_{\gamma_r}$ , or else for  $\mu$ -almost every  $\zeta$ ,  $b_\zeta \not\triangleleft_{r^*} b_{\gamma_r}$  (but not both). In the first case it follows that  $t(\gamma_r) = b_{\gamma_r}$ , and in the second it follows that  $\gamma_r \notin \text{dom}(t)$ . Since there are relatively few  $r$ , by intersecting these sets of  $\zeta$  we can find a single  $\zeta$  which acts, with respect to the  $\gamma_r$ , exactly the way  $\mu$ -almost every  $\zeta$  acts. Fix such a  $\zeta$ . Thus, for each  $r$  we have either  $b_\zeta \triangleleft_{r^*} b_{\gamma_r}$ , and consequently  $t(\gamma_r) = b_{\gamma_r}$ , or else  $\gamma_r \notin \text{dom}(t)$  (but not both). Notice that  $\zeta$  and  $b_\zeta$  are just some particular ordinals. Fix some condition  $\langle p^*, q^* \rangle$  below  $\langle p_1, q_1 \rangle$  forcing  $\zeta$  and  $b_\zeta$  to have the property we

mention in the sentence before last. Now we will argue that this is a contradiction. Let  $\gamma = \gamma_{p^*}$ . There are two cases. First, it might happen that  $b_\zeta \triangleleft_{r^*} \langle \gamma, \beta \rangle$  for some ordinal  $\beta$ . Such a situation can be observed in  $V$ . In this case,  $\langle p^*, q^* \rangle$  forces  $\beta = b_{s|\gamma}$  and therefore, by the assumption on  $\zeta$ , it also forces  $t(\gamma) = \langle \gamma, \beta \rangle$ . Since  $i_\delta$  is a  $\mathbb{P}$ -name, it follows that  $p^* \Vdash i_\delta(\check{\gamma}) = \langle \check{\gamma}, \check{\beta} \rangle$ , contrary to the choice of  $\gamma = \gamma_{p^*}$ . Alternatively, in the second case, it may happen that  $b_\zeta \not\triangleleft_{r^*} \langle \gamma, \beta \rangle$  for every  $\beta$ . In this case, by the assumption on  $\zeta$ , it must be that  $\langle p^*, q^* \rangle$  forces that  $\gamma \notin \text{dom}(t)$ . Again, since  $i_\delta$  is a  $\mathbb{P}$ -name, it follows that  $p^* \Vdash \gamma \notin \text{dom}(i_\delta)$ , contrary again to the choice of  $\gamma = \gamma_{p^*}$ . Thus, in either case we reach a contradiction, and so we have proven that  $\mathbb{P} * \mathbb{Q}$  must add a fresh  $\lambda$ -sequence.  $\dashv$

LEMMA. *If  $\kappa \leq \text{cof}(\lambda)$  and  $\kappa$  is  $\lambda$ -strongly compact, then  $\kappa$  is  $\lambda$ -measurable.*

PROOF. Let  $j: V \rightarrow M$  be the ultrapower map witnessing that  $\kappa$  is  $\lambda$ -strongly compact. By our assumption on  $\text{cof}(\lambda)$ , it follows that  $\sup j^{<\lambda} < j(\lambda)$ . Let  $\alpha = (\sup j^{<\lambda}) + \kappa$ , and let  $\mu$  be the measure germinated by the seed  $\alpha$ . That is,  $X \in \mu$  if and only if  $\alpha \in j(X)$ . Since  $\alpha < j(\lambda)$  it follows that  $\mu$  is a measure on  $\lambda$ . Since  $j(\beta) < \alpha$  for all  $\beta < \lambda$  it follows that  $\mu$  is uniform. Since  $\text{cp}(j) = \kappa$  it follows that  $\mu$  is  $\kappa$ -complete. For  $\gamma < \kappa$ , let  $B_\gamma = \{ \beta \mid \gamma < \text{cof}(\beta) < \kappa \}$ . Since  $\text{cof}(\alpha) = \kappa$  in  $M$ , it follows that  $\alpha \in j(B_\gamma)$  and consequently  $B_\gamma \in \mu$  for every  $\gamma < \kappa$ . Since  $\bigcap_\gamma B_\gamma = \emptyset$ , it follows that  $\mu$  is not  $\kappa^+$ -complete, as desired.  $\dashv$

REMARK. Ketonen [2] has proved that if  $\kappa$  is  $\lambda$ -measurable for every regular  $\lambda$  above  $\kappa$ , then  $\kappa$  is strongly compact. This cannot, however, be true level-by-level, since if  $\kappa < \lambda$  are both measurable, with measures  $\mu$  and  $\nu$ , then  $\mu \times \nu$  is a  $\kappa$ -complete, non- $\kappa^+$ -complete, uniform measure on  $\kappa \times \lambda$ . Thus, in this situation,  $\kappa$  will be  $\lambda$ -measurable, even when it may not be even  $\kappa^+$ -strongly compact. But the previous lemma establishes that the direction we need does indeed hold level-by-level.

Let us now finish the proof of the theorem. Suppose that  $V[g][G]$  is a forcing extension by  $\mathbb{P} * \mathbb{Q}$ , where  $|\mathbb{P}| < \kappa$  and  $\mathbb{Q}$  is  $<\kappa$ -closed. Let  $\lambda$  be least such that  $\mathbb{Q}$  adds a new  $\lambda$ -sequence not in  $V[g]$ . Necessarily,  $\kappa \leq \lambda$  and  $\lambda$  is regular. By the Key Lemma  $V[g][G]$  has no  $\lambda$ -sequences which are fresh over  $V$ . Thus, by the first lemma  $\kappa$  is not  $\lambda$ -measurable in  $V[g][G]$ . Therefore, by the second lemma,  $\kappa$  is not  $\lambda$ -strongly compact in  $V[g][G]$ .  $\dashv$

So the proof actually establishes that after small forcing of size  $\beta < \kappa$ , any  $\leq \beta$ -closed forcing which adds a new  $\lambda$ -sequence for some minimal  $\lambda$ , with  $\lambda \geq \kappa$ , will destroy the  $\lambda$ -measurability of  $\kappa$ . This subtlety about adding a  $\lambda$ -sequence as opposed to a *subset* of  $\lambda$  has the following intriguing consequence, which is connected with the possibilities of changing the cofinalities of very large cardinals.

COROLLARY. *Suppose that  $\kappa$  is  $\lambda$ -measurable. Then after forcing with  $\mathbb{P}$  of size  $\beta < \kappa$ , any  $\leq \beta$ -closed  $\mathbb{Q}$  which adds a  $\lambda$ -sequence, but no shorter sequences, must necessarily add subsets to  $\lambda$ .*

PROOF. Such forcing will destroy the  $\lambda$ -measurability of  $\kappa$ . Hence, it must add subsets to  $\lambda$ .  $\dashv$

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