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The Structure of Ext$(A, \mathbb{Z})$ and $V = L$

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Introduction

The original Problem 54 proposed by Fuchs [4] asks to determine the structure of $\operatorname{Ext}(A, \mathbb{Z})$ for an arbitrary abelian group A. Recall that, for any A,

$$\operatorname{Ext}(A, \mathbb{Z}) \cong \operatorname{Ext}(A/tA, \mathbb{Z}) \oplus \operatorname{Ext}(tA, \mathbb{Z}),$$

where tA denotes the torsion subgroup of A. Thus Fuchs' problem breaks up into two distinct cases, torsion and torsion-free groups. In case $A = T$, a torsion group, we have

$$\operatorname{Ext}(T, \mathbb{Z}) \cong \operatorname{Hom}(T, \mathbb{R}/\mathbb{Z}).$$

Therefore it is compact and reduced, so its structure is known explicitly [11].

It remains to study the case of a torsion-free group A. Since $\operatorname{Ext}(A, \mathbb{Z})$ is then divisible it can be written uniquely as

$$\operatorname{Ext}(A, \mathbb{Z}) \cong \bigoplus \mathbb{Q} \times \bigoplus_p (\bigoplus \mathbb{Z}(p^\infty))$$

and hence is characterized by a collection of cardinal numbers. We denote them by $\nu_0(A)$ and $\nu_p(A)$, where $\nu_0(A)$ is the (torsion-free) rank and $\nu_p(A)$ the p-rank of $\operatorname{Ext}(A, \mathbb{Z})$, respectively. For countable A the possible values of $\nu_0(A)$ and $\nu_p(A)$ have been determined by C. Jensen [12].

In this paper we consider the case where A is torsion-free and uncountable, assuming Gödel's Axiom of Constructibility $V = L$. We essentially work with the same tools as have been used successfully by the third-named author in order to solve Whitehead's Problem in L ([14], [15]). Before stating our main results we recall the following definitions: For an infinite cardinal κ, an abelian group is said to be *κ-generated* if it has a set of generators of cardinality $< \kappa$. An abelian group is called *κ-free* if every κ-generated subgroup is free. As usual $|B|$ denotes the cardinality of B. The results are the following:

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Theorem 1 $(V=L)$. *Let A be a torsion-free abelian group of uncountable cardinality κ such that, for every κ-generated subgroup B of A, A/B is not free. Then we have $\nu_0(A)=2^\kappa$.*

Corollary 1 $(V=L)$. *Let A be torsion-free but not free. Suppose that B is a subgroup of A such that A/B is free. If B is of minimal cardinality with respect to this property, then $\nu_0(A)=2^{|B|}$.*

Corollary 2 $(V=L)$. *Let A be torsion-free but not free. Then*
(a) *$\nu_0(A)$ is of the form 2^μ for some infinite cardinal μ.*
(b) *$\nu_p(A)\leqq\nu_0(A)$ for every prime p.*

Corollary 3 $(V=L)$. *Let A be κ-free but not free for some infinite cardinal κ. Then $\nu_0(A)$ is of the form 2^μ for some $\mu\geqq\kappa$.*

Corollary 4 $(V=L)$. *Let A be any abelian group such that* $\operatorname{Ext}(A,\mathbb{Z})$ *is divisible. Then either A is free, hence* $\operatorname{Ext}(A,\mathbb{Z})=0$, *or $\nu_0(A)$ is of the form 2^μ for some infinite μ.*

Corollary 1 provides us with an explicit description of the function ν_0 which is surprisingly simple. Statement (a) of Corollary 2 contains Théorème 1 of [8]. Corollary 3 extends the result of [9], whereas Corollary 4 generalizes Theorem 2 of [7] considerably. We remark that by [14, Theorem 3.5] neither Theorem 1 nor any of the corollaries can be proved on the basis of ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice) alone.

We recall some terminology of set theory. As usual an ordinal is identified with the set of its predecessors, and cardinals are understood to be initial ordinals. The *cofinality* $\operatorname{cf}(\kappa)$ of an infinite cardinal κ is the smallest cardinal σ such that there is a strictly increasing function $f\colon\sigma\to\kappa$ such that $\sup\{f(\nu)\mid\nu\in\sigma\}=\kappa$. A cardinal κ is called *regular* if $\operatorname{cf}(\kappa)=\kappa$, otherwise κ is *singular*.

The first part of the paper is devoted to the proof of Theorem 1. Thereby we use the description of $\operatorname{Ext}(A,\mathbb{Z})$ in terms of factor sets (see I.1). In Section I.2 we establish the regular case. We prove in fact a slightly more general result on extending factor sets (Proposition 1). The proof of this proposition is based on R. Jensen's combinatorial principle $\diamondsuit$. To prove Theorem 1 for κ singular we use Proposition 1 again as well as a modified version (Theorem 2) of the third-named author's theorem stating that every κ-free abelian group of cardinality κ is free [15]. This is done in Section I.3

In the first section of Part II we determine the possible values of $\nu_p(A)$ in case A is a torsion-free group such that $\operatorname{Hom}(A,\mathbb{Z})$ is zero (Proposition 2). A nice consequence of this is the following: $(V=L)$. Let A be any abelian group such that $\operatorname{Hom}(A,\mathbb{Z})$ is zero. Then $\operatorname{Ext}(A,\mathbb{Z})$ admits a compact topology (Theorem 3(a)). In II.2 we consider topological applications (assuming $V=L$). We conclude that, given a topological space X such that, for some $n\geqq 2$, $H^n(X,\mathbb{Z})$ is non-zero and divisible, then the rank of $H^n(X,\mathbb{Z})$ is of the form 2^μ for some infinite μ (Theorem 4). Finally, we characterize those abelian groups to which there exist co-Moore spaces (Proposition 3, Theorem 5).

I. The Torsion-Free Rank of Ext$(A, \mathbb{Z})$

I.1 Given two abelian groups A and G, the group Ext(A, G) can be defined in various ways. In our context its description in terms of factor sets is the most appropriate.

Recall that a *factor set* on A to G is a function $f: A \times A \to G$ satisfying certain identities (see e.g. [5], pp. 209–211). In particular, to each function $g: A \to G$ with $g(0)=0$ there is a factor set δg, given by

$$\delta g(a, b) = g(a) - g(a+b) + g(b).$$

A factor set of this form is called a *transformation set*. By termwise addition the set Fact(A, G) of all factor sets on A to G becomes an abelian group, and the transformation sets form a subgroup Trans(A, G). Then Ext(A, G) can be defined as

$$\mathrm{Ext}(A, G) = \mathrm{Fact}(A, G)/\mathrm{Trans}(A, G).$$

Moreover, a homomorphism $\varphi: A \to B$ induces a homomorphism φ_2^*: Fact$(B, G) \to$ Fact(A, G), given by

$$(\varphi_2^* f)(a, b) = f(\varphi a, \varphi b).$$

Clearly, $\varphi_1^* = \varphi_2^*|_{\mathrm{Trans}(B, G)}$ factors through Trans(A, G). Hence φ induces a homomorphism φ^*: Ext$(B, G) \to$ Ext(A, G). Finally recall (cf. e.g. [5], Theorem 51.3) that every short exact sequence $0 \to B \xrightarrow{\mu} A \xrightarrow{\varepsilon} C \to 0$ of abelian groups induces an exact sequence

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(A, G) \to \mathrm{Hom}(B, G) \xrightarrow{\omega} \mathrm{Ext}(C, G) \to$$

$$\xrightarrow{\varepsilon^*} \mathrm{Ext}(A, G) \xrightarrow{\mu^*} \mathrm{Ext}(B, G) \to 0.$$

The following lemmata are crucial for the proof of the regular case of Theorem 1.

Lemma 1. *Let B be a subgroup of an abelian group A. Then each element $f \in$ Fact(B, G) can be extended to an element $f_1 \in$ Fact(A, G).*

Proof. We consider the following commutative diagram with exact rows

$$\text{(E)} \qquad \begin{array}{ccccccccc} 0 \to & \mathrm{Trans}(A, G) & \to & \mathrm{Fact}(A, G) & \to & \mathrm{Ext}(A, G) & \to 0 \\ & \downarrow \iota_1^* & & \downarrow \iota_2^* & & \downarrow \iota^* & \\ 0 \to & \mathrm{Trans}(B, G) & \to & \mathrm{Fact}(B, G) & \to & \mathrm{Ext}(B, G) & \to 0, \end{array}$$

where the vertical maps are induced by $\iota: B \hookrightarrow A$. We have to show that ι_2^* is surjective. But this is easily seen from the diagram, using the fact that ι_1^* and ι^* are surjective.

Lemma 2. *Let B be a pure subgroup of a torsion-free abelian group A such that* $\operatorname{Ext}(A/B, G) \neq 0$. *Let $f \in \operatorname{Fact}(B, G)$ such that $nf = \delta g$ for some $n > 0$ and some function $g\colon B \to G$. Then there exists $\tilde{f} \in \operatorname{Fact}(A, G)$ extending f such that there is no function $\tilde{g}\colon A \to G$ which both extends g and satisfies $n\tilde{f} = \delta\tilde{g}$.*

Proof. In the first part we prove the following special case:
(*) Given $n > 0$, there exists $f_1 \in \operatorname{Fact}(A, G)$ extending $0 \in \operatorname{Fact}(B, G)$ such that there is no function $g_1\colon A \to G$ which both satisfies $g_1|_B = 0$ and $nf_1 = \delta g_1$.

For this purpose we consider the exact sequence

$$\operatorname{Hom}(A, G) \xrightarrow{\iota^*} \operatorname{Hom}(B, G) \xrightarrow{\omega} \operatorname{Ext}(A/B, G) \xrightarrow{\pi^*} \operatorname{Ext}(A, G) \xrightarrow{\iota^*} \operatorname{Ext}(B, G) \to 0$$

which is induced by $0 \to B \xhookrightarrow{\iota} A \xrightarrow{\pi} A/B \to 0$, π denoting the projection map. We distinguish two cases according to whether the image I of π^* is trivial or not. First assume that I is non-trivial. Since A/B is torsion-free, $\operatorname{Ext}(A/B, G)$ is divisible; hence I is likewise divisible. Thus we can pick $\eta \in I$ such that $n\eta \neq 0$. As $\eta \in \ker \iota^*$, we see from diagram (E) that we can choose a representative $f_1 \in \operatorname{Fact}(A, G)$ of η which is in the kernel of $\iota_2^*\colon \operatorname{Fact}(A, G) \to \operatorname{Fact}(B, G)$. So f_1 extends the zero factor set on B. On the other hand, by the choice of η, nf_1 is not a transformation set. Hence f_1 is as required.

In the second case, we assume that I is trivial. Then ω is surjective and so, as $\operatorname{Ext}(A/B, G)$ is non-zero and divisible, we can choose $\varphi \in \operatorname{Hom}(B, G)$ such that $n\varphi$ is not in the image of ι^*. Now define $h\colon A \to G$ by

$$h(a) = \begin{cases} \varphi(a) & \text{if } a \in B, \\ 0 & \text{if } a \notin B, \end{cases}$$

and let $f_1 = \delta h \in \operatorname{Fact}(A, G)$. We claim that f_1 satisfies condition (*). Clearly f_1 extends the zero factor set on B. Suppose that there is a function $g_1\colon A \to G$ such that $g_1|_B = 0$ and $nf_1 = \delta g_1$. But then the map $\psi = nh - g_1\colon A \to G$ is a homomorphism and satisfies, for all $b \in B$,

$$\psi(b) = nh(b) - g_1(b) = n\varphi(b).$$

Thus $n\varphi$ is in the image of ι^*, contradicting the choice of φ. This completes the proof of (*).

Now we consider the general case. So let $f \in \operatorname{Fact}(B, G)$ such that $nf = \delta g$ for some $g\colon B \to G$. By Lemma 1, f can be extended to some $f_2 \in \operatorname{Fact}(A, G)$. Suppose that there is a $g_2\colon A \to G$ both extending g and satisfying $nf_2 = \delta g_2$ (otherwise let $\tilde{f} = f_2$). In this case let $\tilde{f} = f_1 + f_2$, where f_1 satisfies condition (*). Now assume that $\tilde{g}\colon A \to G$ extends g. Then $\tilde{g} - g_2$ extends the zero function on B. Thus by the choice of f_1 we have

$$\begin{aligned} \delta\tilde{g} &= \delta(\tilde{g} - g_2) + \delta g_2 \\ &\neq nf_1 + nf_2 = n\tilde{f}. \end{aligned}$$

Hence $\tilde{f}$ satisfies the assertion, and the lemma is proved.

I.2 This section is devoted to the proof of the following proposition which implies the regular case of Theorem 1. Moreover it will be used again in the proof of the singular case.

Proposition 1 ($V=L$). *Let κ be a regular uncountable cardinal. Suppose that A is a torsion-free abelian group of cardinality κ such that, for every κ-generated subgroup C, A/C is not free. Let B be any κ-generated pure subgroup of A. Then each $f\in\text{Fact}(B,\mathbb{Z})$ can be extended to $f^\alpha\in\text{Fact}(A,\mathbb{Z})$, $\alpha\in 2^\kappa$, such that*

(*) *for every pair $\alpha\neq\beta$, $f^\alpha-f^\beta$ represents an element of infinite order of* $\text{Ext}(A,\mathbb{Z})$.

First we recall some terminology. Given any infinite cardinals κ and σ, a function $f:\sigma\to\kappa$ is called *normal* if it is strictly increasing and satisfies $f(\lambda)=\sup\{f(\alpha)|\alpha<\lambda\}$ for every limit ordinal $\lambda\in\sigma$. A subset S of κ is called *stationary* in κ if S meets the range of every normal function $f:\text{cf}(\kappa)\to\kappa$. An ascending chain of sets (or abelian groups)

$$A_0\subseteq A_1\subseteq\cdots\subseteq A_\alpha\subseteq\cdots,\qquad \alpha\in\kappa,$$

indexed by a cardinal κ, is called a *smooth chain* if, for every limit ordinal $\lambda\in\kappa$, $A_\lambda=\bigcup_{\alpha<\lambda}A_\alpha$.

The following lemma is easily derived from the combinatorial property $\diamondsuit$ of L which was discovered by R. Jensen [13, Lemma 6.5].

Lemma 3 ($V=L$). *Let $\sigma\leqq\kappa$ be cardinals, κ regular uncountable. Let X be a set of cardinality κ which is the union of a smooth chain of sets $\{X_\nu|\nu\in\kappa\}$ of cardinality $<\kappa$, and let Y be any set of cardinality $\leqq\kappa$. If S is a stationary subset of κ, then there is a sequence $\{(n_\nu,\alpha_\nu,g_\nu)|\nu\in S\}$, $n_\nu\in\omega$, $\alpha_\nu\in\sigma$, $g_\nu:X_\nu\to Y$ for every $\nu\in S$, such that for any triple (n,α,g), $n\in\omega$, $\alpha\in\sigma$, $g:X\to Y$, there exists $\nu\in S$ such that $n=n_\nu$, $\alpha=\alpha_\nu$ and $g|_{X_\nu}=g_\nu$.*

Proof of Proposition 1. As $V=L$ implies the General Continuum Hypothesis, it suffices to show that each $f\in\text{Fact}(B,\mathbb{Z})$ can be extended to $f^\alpha\in\text{Fact}(A,\mathbb{Z})$, $\alpha\in\kappa^+$, such that condition (*) is satisfied. So suppose that there is an $f_0\in\text{Fact}(B,\mathbb{Z})$ which can be extended only to $\sigma\leqq\kappa$ many $f^\alpha\in\text{Fact}(A,\mathbb{Z})$ which satisfy (*). We can assume that the set $E=\{f^\alpha|\alpha\in\sigma\}$ is maximal with respect to (*).

Let A be represented as the union of a smooth chain of κ-generated pure subgroups $\{A_\nu|\nu\in\kappa\}$ such that $A_0=B$. It is easily seen that we can assume $\{A_\nu\}$ to satisfy the following condition: For all $\nu\in\kappa$, if A_ρ/A_ν is not free for some $\rho>\nu$, then $A_{\nu+1}/A_\nu$ is not free. We claim that

$$S=\{\nu\in\kappa|A_{\nu+1}/A_\nu\ \text{is not free}\}$$

is a stationary subset of κ. Suppose that there is a normal function $f:\kappa\to\kappa$ such that $S\cap f(\kappa)=\emptyset$. Then the definition $B_\nu:=A_{f(\nu)}$ yields a smooth chain of κ-

generated subgroups $\{B_\nu | \nu \in \kappa\}$ with union A such that $B_{\nu+1}/B_\nu$ is free for all $\nu \in \kappa$. Therefore, by [3, Theorem 2.6] A/B_0 must be free, contradicting the hypothesis. Hence S is stationary in κ.

Now by Lemma 3 there is a sequence $\{(n_\nu, \alpha_\nu, g_\nu) | \nu \in S\}$, $n_\nu \in \omega$, $\alpha_\nu \in \sigma$, $g_\nu : A_\nu \to \mathbb{Z}$ for every $\nu \in S$, such that for any triple (n, α, g), $n \in \omega$, $\alpha \in \sigma$, $g : A \to \mathbb{Z}$, there exists $\nu \in S$ such that $n = n_\nu$, $\alpha = \alpha_\nu$ and $g|_{A_\nu} = g_\nu$. This enables us to define inductively a sequence of factor sets $\{f_\nu^* : A_\nu \times A_\nu \to \mathbb{Z} | \nu \in \kappa\}$ such that (i), (ii) and (iii) are satisfied:

(i) $f_0^* = f_0$;

(ii) $f_\nu^*|_{A_\mu \times A_\mu} = f_\mu^*$ for all $\mu < \nu$;

(iii) for all $\alpha \in \sigma$, $f^* - f^\alpha$ represents an element of infinite order of $\mathrm{Ext}(A, \mathbb{Z})$ (f^* denoting the union of $\{f_\nu^*\}$).

Suppose that f_μ^* has been defined for every $\mu < \nu$. If ν is a limit ordinal, let $f_\nu^* = \bigcup_{\mu < \nu} f_\mu^*$. If ν is a successor ordinal, say $\nu = \mu + 1$, we distinguish two cases. First assume that $\mu \in S$, $n_\mu > 0$, and the condition

$$(**) \quad n_\mu (f_\mu^* - f_\mu^{\alpha_\mu}) = \delta g_\mu$$

holds, where $f_\mu^{\alpha_\mu}$ denotes the restriction $f^{\alpha_\mu}|_{A_\mu \times A_\mu}$. Since $\mu \in S$, the result of [15] implies that $\mathrm{Ext}(A_{\mu+1}/A_\mu, \mathbb{Z})$ is non-zero. Therefore, by Lemma 2, we can extend $f_\mu^* - f_\mu^{\alpha_\mu}$ to an $\tilde{f} \in \mathrm{Fact}(A_{\mu+1}, \mathbb{Z})$ such that there is no function $\tilde{g} : A_{\mu+1} \to \mathbb{Z}$ both extending g_μ and satisfying $n\tilde{f} = \delta \tilde{g}$. Now let $f_{\mu+1}^* = \tilde{f} + f_{\mu+1}^{\alpha_\mu} \in \mathrm{Fact}(A_{\mu+1}, \mathbb{Z})$, so $f_{\mu+1}^*$ extends f_μ^*. In the second case, if $\mu \notin S$ or if $\mu \in S$ but $n_\mu = 0$ or $(**)$ does not hold, let $f_{\mu+1}^*$ be any factor set on $A_{\mu+1}$ to $\mathbb{Z}$ which extends f_μ^*. By Lemma 1 such an $f_{\mu+1}^*$ always exists.

Finally let $f^* \in \mathrm{Fact}(A, \mathbb{Z})$ be the union of the sequence $\{f_\nu^* | \nu \in \kappa\}$. We claim that condition (iii) is satisfied. So suppose that, for some $\alpha \in \sigma$, $f^* - f^\alpha$ represents an element of $\mathrm{Ext}(A, \mathbb{Z})$ of finite order, say $n (> 0)$; i.e., there is a function $g : A \to \mathbb{Z}$ such that

$$n(f^* - f^\alpha) = \delta g.$$

Now there is a $\mu \in S$ such that $n = n_\mu$, $\alpha = \alpha_\mu$ and $g|_{A_\mu} = g_\mu$; hence $(**)$ holds for this μ. But in this case $f_{\mu+1}^*$ has been defined such that there is no function $\tilde{g} : A_{\mu+1} \to \mathbb{Z}$ both extending g_μ and satisfying

$$n(f_{\mu+1}^* - f_{\mu+1}^\alpha) = \delta \tilde{g}.$$

This is a contradiction, hence the claim is proved.

It follows that the elements of $E^* = E \cup \{f^*\}$ extend f_0 and satisfy $(*)$, contradicting the maximality of E. This completes the proof of Proposition 1.

I.3 For the proof of Theorem 1 in case κ is singular we need the following

Theorem 2. *Let κ and λ be infinite cardinals, κ singular and $\lambda < \kappa$. Suppose that A is an abelian group of cardinality κ such that every κ-generated subgroup B of A*

contains a λ^+-generated subgroup C such that B/C is free. Then A contains a λ^+-generated subgroup C such that A/C is free.

Proof. This follows from the third-named author's compactness theorem [15] with $F=F_1 \cup F_2$, where

$$F_1=\{(B,0) \mid B \text{ is a subgroup of } A \text{ containing a } \lambda^+\text{-generated subgroup } C \text{ such that } B/C \text{ is free}\}$$

$$F_2=\{(B, C) \mid 0 \neq C \subseteq B,\ B \text{ and } C \text{ are subgroups of } A \text{ such that } B/C \text{ is free}\}.$$

The language of the model has power λ, so every M includes each $\alpha \leqq \lambda+1$. The axioms are checked as in the case of free abelian groups (see [15], pp. 337–338).

Note that Theorem 2 is proved on the basis of ZFC.

Proof of Theorem 1. First observe that 2^κ is an upper bound for the rank of Ext($A, \mathbb{Z}$). In case κ is regular, the fact that 2^κ is also a lower bound is an easy consequence of Proposition 1. Indeed, for $B=0$ this proposition just says that there is a family of factor sets $\{f^\alpha: A \times A \to \mathbb{Z} \mid \alpha \in 2^\kappa\}$ such that, for every pair $\alpha \neq \beta$, $f^\alpha - f^\beta$ represents an element of infinite order of Ext($A, \mathbb{Z}$). We conclude that the cardinality of the quotient group of Ext($A, \mathbb{Z}$) modulo torsion is $\geqq 2^\kappa$; hence we have rank(Ext($A, \mathbb{Z}$)) $\geqq 2^\kappa$.

Now suppose that κ is singular. In this case, let A be represented as the union of any ascending chain of κ-generated subgroups $\{A_\alpha \mid \alpha \in \mathrm{cf}(\kappa)\}$. We define by induction on $\alpha \in \mathrm{cf}(\kappa)$ a chain of pure subgroups $\{B_\alpha \mid \alpha \in \mathrm{cf}(\kappa)\}$ of A such that, for all $\alpha \in \mathrm{cf}(\kappa)$,

(i) B_α contains A_α;
(ii) $|B_\alpha|$ is a regular cardinal $>|\bigcup_{\beta<\alpha} B_\beta|$;
(iii) if C is a $|B_\alpha|$-generated subgroup of B_α, then B_α/C is not free.

Suppose that B_β has been defined for all $\beta<\alpha$. Let B' be the pure closure of $\bigcup_{\beta<\alpha} B_\beta + A_\alpha$. Applying Theorem 2 to the group A/B' and the cardinal $\lambda=|B'|$, we find a κ-generated subgroup B_α of A containing B' such that for every λ^+-generated subgroup C with $B' \subseteq C \subseteq B_\alpha$, B_α/C is not free. We can assume that B_α is of minimal cardinality and pure in A. Clearly we have $|B_\alpha| > \lambda \geqq |\bigcup_{\beta<\alpha} B_\beta|$, and $|B_\alpha|$ must be regular by Theorem 2. Hence (i) and (ii) are satisfied. We claim that condition (iii) holds too. So suppose that there is a $|B_\alpha|$-generated subgroup C of B_α such that B_α/C is free. Then $(B'+C)/C$ is contained in a $|B_\alpha|$-generated direct summand of B_α/C, say D/C. So D contains B', and B_α/D is free. It follows by the definition of B_α that for every λ^+-generated subgroup C with $B' \subseteq C \subseteq D$, D/C is not free. But this contradicts the minimality of $|B_\alpha|$; hence B_α is as required.

Let B_α^0 denote the union of $\{B_\nu \mid \nu<\alpha\}$, $\alpha \in \mathrm{cf}(\kappa)$. Now we assign, by induction on $\alpha \in \mathrm{cf}(\kappa)$, to each sequence η of ordinals of length α with $\eta(\nu) \in 2^{|B_\nu|}$, $\nu<\alpha$, a factor set $f^\eta \in \mathrm{Fact}(B_\alpha^0, \mathbb{Z})$, such that

(iv) if ξ is an initial segment of η, then f^η extends f^ξ;
(v) if $\xi \neq \eta$ are of the same length α, then $f^\xi - f^\eta$ represents an element of infinite order of Ext($B_\alpha^0, \mathbb{Z}$).

Suppose that the f^{ξ}'s have been defined for all sequences ξ of length $<\alpha$. If α is a limit ordinal, then every sequence η of length α determines a unique factor set f^{η} on B^0_{α}. In case α is a successor ordinal, say $\alpha=\nu+1$, we proceed as follows: By definition, $B^0_{\alpha}(=B_{\nu})$ satisfies the hypothesis of Proposition 1 and B^0_{ν} is a $|B^0_{\alpha}|$-generated pure subgroup of B^0_{α}. Therefore, given $f^{\xi}\in\mathrm{Fact}(B^0_{\nu},\mathbb{Z})$, we can find $f^{\eta}\in\mathrm{Fact}(B^0_{\alpha},\mathbb{Z})$ for each sequence η of length α satisfying $\eta(\mu)=\xi(\mu)$ for all $\mu<\nu$ and $\eta(\nu)\in 2^{|B_{\nu}|}$, such that the following holds: For all such η's, f^{η} extends f^{ξ}, and for every pair $\eta\neq\zeta$, $f^{\eta}-f^{\zeta}$ represents an element of infinite order of $\mathrm{Ext}(B^0_{\alpha},\mathbb{Z})$.

In the limit we obtain $\prod\limits_{\nu\in\mathrm{cf}(\kappa)} 2^{|B_{\nu}|}=2^{\kappa}$ factor sets on A to $\mathbb{Z}$ which represent pairwise different elements of $\mathrm{Ext}(A,\mathbb{Z})$ modulo torsion. Hence, also in the singular case, the rank of $\mathrm{Ext}(A,\mathbb{Z})$ is $\geqq 2^{\kappa}$. This completes the proof of Theorem 1.

Remark. We can prove Theorem 1 assuming the following axiom of set theory (which is much weaker than $V=L$).

(Hyp) *Every stationary subset S of any regular uncountable cardinal κ satisfies $\Phi_{\kappa}(S)$.*

Hereby $\Phi_{\kappa}(S)$ denotes the assertion that for any $F\colon \bigcup\limits_{\alpha<\kappa} 2^{\alpha}\to 2$ there is a $g\in 2^{\kappa}$ such that for any $f\in 2^{\kappa}$, the set $\{\alpha\in S\,|\,F(f|\alpha)=g(\alpha)\}$ is stationary in κ (cf. [2]).

(Hyp) is consistent with any function $\aleph_{\alpha}\to 2^{\aleph_{\alpha}}$ which is strictly increasing.

Proof of Corollary 1. For B uncountable Theorem 1 applies, whereas the case B countable is settled by Théorème 2.7 of [12].

Note that Corollaries 2 and 3 are immediate consequences of Corollary 1. For the proof of Corollary 4 one uses in addition that $\mathrm{Ext}(A,\mathbb{Z})$ divisible implies A torsion-free.

II. The p-rank of $\mathrm{Ext}(A,\mathbb{Z})$ and Topological Applications

II.1 Recall that for a torsion-free abelian group A, $\nu_0(A)$ $[\nu_p(A)]$ denotes the torsion-free rank [the p-rank] of $\mathrm{Ext}(A,\mathbb{Z})$. Our Corollary 2 says that, if A is not free, then

(a) $\nu_0(A)$ is of the form 2^{μ} for some infinite μ.

(b) $\nu_p(A)\leqq\nu_0(A)$ for every prime p.

It remains to determine the possible values of $\nu_p(A)$. For A countable torsion-free but not free, C. Jensen proved [12, Théorème 2.7] that $\nu_p(A)$ is finite or $2^{\aleph_0}$. This led us to expect that for any torsion-free non-free group A condition (c) holds:

(c) $\nu_p(A)$ is finite or of the form 2^{μ_p}, μ_p infinite.

We remark that by a result of Hulanicki [10] the triple of conditions (a), (b), (c) characterizes exactly the divisible abelian groups which admit a compact topology.

However, the third-named author has constructed an (uncountable) torsion-free group such that (c) is not satisfied [16]. But we can prove that (c) holds in the following special case. Note that hereby we do not need the hypothesis $V=L$.

Proposition 2. *If A is a torsion-free abelian group such that* Hom$(A,\mathbb{Z})$ *is zero, then for every prime p, $v_p(A)$ is finite or of the form 2^{μ_p}, μ_p infinite.*

By the way, as such a group A has no free quotients, we have $v_0(A)=2^{|A|}$ by Corollary 1.

Proof. We consider the exact sequence

$$\mathrm{Hom}(A,\mathbb{Z})\to \mathrm{Ext}(A/pA,\mathbb{Z}) \xrightarrow{\varepsilon^*} \mathrm{Ext}(A,\mathbb{Z}) \xrightarrow{p^*} \mathrm{Ext}(A,\mathbb{Z})\to 0$$

which is induced by the exact sequence $0\to A\xrightarrow{p} A\xrightarrow{\varepsilon} A/pA\to 0$, p denoting multiplication by p. By hypothesis Hom$(A,\mathbb{Z})$ is zero, hence ε^* is injective. Since p^* coincides with the multiplication by p, it follows that $v_p(A)$ is the dimension of the $\mathbb{Z}/p$-vector space Ext$(A/pA,\mathbb{Z})$. But Ext$(A/pA,\mathbb{Z})$ is a direct product of copies of $\mathbb{Z}/p$, hence $v_p(A)$ is as required.

Consequently, if A is a torsion-free abelian group such that Hom$(A,\mathbb{Z})=0$, then Ext$(A,\mathbb{Z})$ admits a compact topology. More generally we have

Theorem 3 (a) ($V=L$). *If A is any abelian group such that* Hom$(A,\mathbb{Z})=0$, *then* Ext$(A,\mathbb{Z})$ *admits a compact topology.*

(b) *Conversely, if G is an arbitrary compact abelian group, then there exists an abelian group A such that* Hom$(A,\mathbb{Z})=0$ *and* Ext$(A,\mathbb{Z})\cong G$.

Proof. (a) Recall that Ext$(A,\mathbb{Z})$ is of the form

$$\mathrm{Ext}(A,\mathbb{Z})\cong \mathrm{Ext}(A/tA,\mathbb{Z})\oplus \mathrm{Ext}(tA,\mathbb{Z}).$$

We already noticed that Ext$(tA,\mathbb{Z})$ admits a compact topology (see Introduction). So it remains to show that Ext$(A/tA,\mathbb{Z})$ can carry a compact topology. But this follows from the remark before Theorem 3, since Hom$(A/tA,\mathbb{Z})$ $\cong$ Hom$(A,\mathbb{Z})=0$.

(b) Let G be an arbitrary compact abelian group, and denote its character group by $\hat{G}$. Then it is well-known that

$$G\cong \hat{\hat{G}}\cong \mathrm{Hom}(\hat{G}/t\hat{G},\mathbb{R}/\mathbb{Z})\oplus \mathrm{Hom}(t\hat{G},\mathbb{R}/\mathbb{Z}).$$

Now Hom$(\hat{G}/t\hat{G},\mathbb{R}/\mathbb{Z})$ is divisible and compact, hence of the form

$$\mathrm{Hom}(\hat{G}/t\hat{G},\mathbb{R}/\mathbb{Z})\cong \bigoplus_{r_0}\mathbb{Q}\times \bigoplus_{p}\Big(\bigoplus_{r_p}\mathbb{Z}(p^\infty)\Big),$$

where by the main result of [10] the following conditions hold:

(i) r_0 is of the form 2^{μ} for some infinite μ.
(ii) r_p is finite or of the form 2^{μ_p}, μ_p infinite.
(iii) $r_p\leq r_0$ for every p.

Then we define

$$B=\bigoplus_{\mu}\mathbb{Q}\times\bigoplus_{p}(\bigoplus_{\mu_p}\mathbb{Z}_{(p)}),$$

where $\mu_p=r_p$ if r_p is finite; $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ with respect to p. We claim that $\mathrm{Ext}(B,\mathbb{Z})$ is isomorphic to $\mathrm{Hom}(\hat{G}/t\hat{G},\mathbb{R}/\mathbb{Z})$. First we have

$$\mathrm{Ext}(B,\mathbb{Z})\cong\prod_{\mu}\mathrm{Ext}(\mathbb{Q},\mathbb{Z})\times\prod_{p}(\prod_{\mu_p}\mathrm{Ext}(\mathbb{Z}_{(p)},\mathbb{Z})).$$

It is easily seen that $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})\cong\prod_{\aleph_0}\mathbb{Q}$ and $\mathrm{Ext}(\mathbb{Z}_{(p)},\mathbb{Z})\cong\mathbb{Z}(p^\infty)\oplus\prod_{\aleph_0}\mathbb{Q}$. Using (i), (ii) and (iii), we thus obtain

$$\begin{aligned}\mathrm{Ext}(B,\mathbb{Z})&\cong\prod_{\mu}\mathbb{Q}\times\prod_{p}[\prod_{\mu_p}(Z(p^\infty)\oplus\prod_{\aleph_0}\mathbb{Q})]\\&\cong\bigoplus_{r_0}\mathbb{Q}\times\bigoplus_{p}(\bigoplus_{r_p}\mathbb{Z}(p^\infty));\end{aligned}$$

hence the claim is proved. Now let $A=B\oplus t\hat{G}$; then we have

$$\begin{aligned}\mathrm{Hom}(A,\mathbb{Z})&\cong\mathrm{Hom}(B,\mathbb{Z})\oplus\mathrm{Hom}(t\hat{G},\mathbb{Z})\\&\cong\prod_{\mu}\mathrm{Hom}(\mathbb{Q},\mathbb{Z})\oplus\prod_{p}(\prod_{\mu p}\mathrm{Hom}(\mathbb{Z}_{(p)},\mathbb{Z}))\\&=0,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Ext}(A,\mathbb{Z})&\cong\mathrm{Ext}(B,\mathbb{Z})\oplus\mathrm{Ext}(t\hat{G},\mathbb{Z})\\&\cong\mathrm{Hom}(\hat{G}/t\hat{G},\mathbb{R}/\mathbb{Z})\oplus\mathrm{Hom}(t\hat{G},\mathbb{R}/\mathbb{Z})\\&\cong G.\end{aligned}$$

This completes the proof of Theorem 3.

II.2 We now turn to algebraic-topological applications. The following generalizes the main result of [6] considerably.

Theorem 4 ($V=L$). *Let X be a topological space. If, for some integer $n\geqq 2$, the singular cohomology group $H^n(X,\mathbb{Z})$ is non-zero and divisible, then its rank is of the form 2^μ for some infinite cardinal μ.*

Proof. By the Universal Coefficient Formula we have

$$H^n(X,\mathbb{Z})\cong\mathrm{Hom}(H_n(X),\mathbb{Z})\oplus\mathrm{Ext}(H_{n-1}(X),\mathbb{Z}).$$

Since the Hom group is reduced, it must vanish. We are thus reduced to showing that the rank of $\mathrm{Ext}(H_{n-1}(X),\mathbb{Z})$ is of the form 2^μ, μ infinite. But this we know from Corollary 4.

Next we determine those abelian groups to which there exist co-Moore spaces. A topological space X is said to be *co-Moore of type* (G,n) ($n\geqq 1, G$ an abelian group) if

$$\tilde{H}^i(X,\mathbb{Z})\cong\begin{cases}G & \text{if } i=n\\ 0 & \text{otherwise.}\end{cases}$$

(This does not completely correspond to the usual definition of Moore space since we make no assumption about the fundamental groups. Compare [1, p. 31].)

Proposition 3 ($V=L$). *There exists a co-Moore space of type* $(G,1)$ *if and only if* G *is a direct product of copies of* $\mathbb{Z}$.

Proof. Suppose that X is a co-Moore space of type $(G,1)$. Then

$$G\cong\tilde{H}^1(X,\mathbb{Z})\cong\mathrm{Hom}(H_1(X),\mathbb{Z})$$

and

$$0=H^2(X,\mathbb{Z})\cong\mathrm{Hom}(H_2(X),\mathbb{Z})\oplus\mathrm{Ext}(H_1(X),\mathbb{Z}).$$

Hence $\mathrm{Ext}(H_1(X),\mathbb{Z})=0$ and $\mathrm{Hom}(H_1(X),\mathbb{Z})\cong G$. It follows that $H_1(X)$ is free, and thus $G\cong\mathrm{Hom}(H_1(X),\mathbb{Z})\cong\prod\mathbb{Z}$.

Conversely, suppose that $G\cong\prod_I\mathbb{Z}$. It is well-known that any countable sequence of abelian groups can be realized as the (positive) singular homology groups of an appropriately constructed space X. Choose X such that

$$\tilde{H}_i(X)=\begin{cases}\bigoplus_I\mathbb{Z} & \text{if } i=1\\ 0 & \text{otherwise.}\end{cases}$$

Then, indeed, X is co-Moore of type $(G,1)$.

Theorem 5 ($V=L$). *There exists a co-Moore space of type* (G,n), $n\geqq 2$, *if and only if* G *is of the form* $G=C\oplus D$, *where* C *is compact and* D *isomorphic to a direct product of copies of* $\mathbb{Z}$.

Proof. Suppose that X is co-Moore of type (G,n). Then by repeated application of the Universal Coefficient Theorem we deduce that $G\cong\mathrm{Hom}(A,\mathbb{Z})\oplus\mathrm{Ext}(B,\mathbb{Z})$ for groups A,B with $\mathrm{Hom}(B,\mathbb{Z})=0=\mathrm{Ext}(A,\mathbb{Z})$. Therefore $\mathrm{Ext}(B,\mathbb{Z})$ is compact by Theorem 3, and $\mathrm{Hom}(A,\mathbb{Z})$ is isomorphic to a direct product of copies of $\mathbb{Z}$. Hence G is as required.

Conversely, let $G=C\oplus D$ where C is compact and $D\cong\prod_I\mathbb{Z}$. Then by Theorem 3 there is a group A such that $\mathrm{Hom}(A,\mathbb{Z})=0$ and $\mathrm{Ext}(A,\mathbb{Z})\cong C$. Now choose X such that

$$\tilde{H}_i(X)=\begin{cases}\bigoplus_I\mathbb{Z} & \text{if } i=n\\ A & \text{if } i=n-1\\ 0 & \text{otherwise.}\end{cases}$$

Then we have

$$\tilde{H}^i(X,\mathbb{Z})\cong\begin{cases}\mathrm{Hom}(\bigoplus_I \mathbb{Z},\mathbb{Z})\oplus\mathrm{Ext}(A,\mathbb{Z})\cong C\oplus D & \text{if } i=n\\ \mathrm{Hom}(A,\mathbb{Z})=0 & \text{if } i=n-1\\ 0 & \text{otherwise.}\end{cases}$$

Hence X is co-Moore of type (G,n). This completes our proof.

Remark. The "if" part of the above theorem does not require any assumption on set theory. Moreover, by the same method we can prove that, given any sequence of abelian groups $\{G_n|n\geqq 2\}$ such that $G_n=C_n\oplus D_n$, C_n compact and $D_n\cong\prod\mathbb{Z}$, there exists a topological space X such that $H^n(X,\mathbb{Z})\cong G_n$ for every $n\geqq 2$.

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