ISRAEL JOURNAL OF MATHEMATICS, Vol. 62, No. 3, 1988

WAS SIERPINSKI RIGHT? I

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ABSTRACT

Aroused by Todorcevic's breakthrough we prove here some complementary consistency results, mainly $2^{\aleph_0} \to [\aleph_1]^2_3$. We also get some generalization of his theorem to, e.g., $\lambda \not\to [\lambda]^2_{\aleph_0}$ for λ regular not ω-Mahlo.

Introduction

Todorcevic had stated that the important open partition relations are $\aleph_1 \to [\aleph_1]^2_{\aleph_1}$ or $\aleph_1 \to [(\aleph_1, \aleph_1)]^2_2$, $2^{\aleph_0} \to [\aleph_1]^2_3$ and $2^{\aleph_0} \to [2^{\aleph_0}, [2^{\aleph_0}; 2^{\aleph_0}]]$. Certainly the first got more attention (maybe because of its relation to many other problems on $\aleph_1$, see e.g. [KV]). Lately he made a breakthrough proving in ZFC $\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$; Todorcevic had an older result in the direction of the consistency of $2^{\aleph_0} \to [2^{\aleph_0}, [2^{\aleph_0}; 2^{\aleph_0}]]^2$: if we add to V any number of Sacks reals with countable support (product, not iteration) then (if for simplicity V satisfies G.C.H.) $\aleph_n \to (\aleph_n, [\aleph_1, \aleph_1])^2$.

We prove here (in 1.1) the following: let V satisfy G.C.H. (for simplicity), $\aleph_0 < \kappa < \lambda \leqq \chi$, $\lambda = \kappa^{+3}$, κ successor of regular, we can blow up $2^{\aleph_0}$ to χ without collapsing cardinals by a forcing so that still $\lambda \to (\lambda, [\kappa; \kappa])^2$. So the restriction to $\aleph_1$ is removed. In fact we can replace $\aleph_0$ by any regular μ (using μ-complete forcing). The proof relies on Todorcevic's and is influenced by order used by Gitik in [G] (for an iteration).

† The author thanks the NSF and N.S.E.R.C. for partially supporting the research. Preliminary versions of §3 and §2 were circulated in November '84 and February '85, respectively.
Received March 18, 1987 and in revised form January 12, 1988

We could have still thought that Sierpinski's result $2^{\aleph_0}\nrightarrow[\aleph_1]^2_2$, Galvin and Shelah's [GS] result $2^{\aleph_0}\nrightarrow[2^{\aleph_0}]^2_{\aleph_0}$ and Todorcevic's result $\aleph_1\nrightarrow[\aleph_1]^2_{\aleph_1}$ can be strengthened to $2^{\aleph_0}\nrightarrow[\aleph_1]^2_3$. This ($2^{\aleph_0}\rightarrow[\aleph_1]^2_3$?) is quite an old problem of Erdös and Hajnal [EH]; for a discussion of its importance see e.g. Erdös [E] and III 21 of [MU]. However, our main result is (in §2) the consistency with ZFC of $2^{\aleph_0}\rightarrow[\aleph_1]^2_3$. More elaborately, if λ is a strongly inaccessible Erdös, when $\mu=\aleph_0$, measurable otherwise; and $\lambda>\mu=\mu^{<\mu}$, then for some μ-complete forcing not collapsing any cardinal, in V^P, $2^\mu=\lambda$ and $\lambda\rightarrow[\mu]^2_3$ (in fact $\lambda\rightarrow[\mu]^2_{\sigma,3}$ for $\sigma<\mu$) (see 2.1). In fact we can make 2^μ larger. Though settling the original problem a host has arisen: minimal cases are:

(1) $\aleph_2\rightarrow[\aleph_1]^2_3$?

(2) $2^{\aleph_0}\rightarrow[\aleph_1]^3_{\aleph_0}$?

(3) $2^{\aleph_0}\rightarrow[\aleph_2]^2_3$?

(4) $\lambda\rightarrow[\lambda]^2_{\aleph_0}$ not weakly compact?

Galvin had conjectured the consistency of $\{\aleph_2\rightarrow[\aleph_1]^n_{h(n)}: n<\omega\}$ for a suitable $h:\omega\rightarrow\omega$.[†]

Lately Todorcevic made a breakthrough in partition relations proving $\aleph_1\nrightarrow[\aleph_1]^2_{\aleph_1}$. He presented the proof in the MAMLS conference, Nov. '84. He told me then that he has another proof and he is working on the "family of uncountable linear ordered has no finite bases". He knew $\lambda^+\rightarrow[\lambda^+]^2_{\lambda^+}$ for λ regular.

Our proof for (A), (B), (C) below (i.e. §3) continues the work of Todorcevic [T]. We use simpler coloring, as he used coloring on ω_1 which uses more information which was relevant e.g. to a new construction of uncountable linear order I whose square is the union of $\aleph_0$ chains (this was his starting point). Such orders were first constructed in [Sh].

We prove, e.g.,

(A) If λ is regular $>\aleph_0$, $S\subseteq\lambda$ stationary with no initial segment stationary, then $\lambda\nrightarrow[\lambda]^2_\lambda$ (e.g. λ Mahlo, not 2-Mahlo or succesor of regular) (see 3.1).

(B) If $\forall n<\omega\ \exists m, k(\forall m'>m)\aleph_{m'}\nrightarrow[\aleph_k]^{<\omega}_{\aleph_n}$ (i.e. various instances of the Chang conjecture fail) [equivalently $\bigwedge_n\bigvee_k\aleph_\omega\rightarrow[\aleph_k]^{<\omega}_{\aleph_n}$] *then* $\aleph_{\omega+1}\nrightarrow[\aleph_{\omega+1}]^2_{\aleph_{\omega+1}}$.

Todorcevic had proved $\lambda^+\nrightarrow[\lambda^+]^2_{\mathrm{cf}\lambda}$ if $(\forall\mu<\lambda)[\mu^{\mathrm{cf}\lambda}<\lambda]$.

(C) Suppose λ is regular $>\aleph_0$, $\lambda\nrightarrow[\lambda]^2_{\aleph_0}$ (hence λ is ω-Mahlo). Then

[†] For further results, solving some of the problems, from Spring '86, see [Sh 2], [Sh 3] and, better, [Sh 4], [Sh 5], [Sh 6], and more applications of §3 in [Sh 7].

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(*) If $\langle C_\delta : \delta < \lambda,\ \delta \text{ inaccessible}\rangle$ is such that C_δ is a closed unbounded subset of δ and $C^+ \subseteq \lambda$ is closed unbounded, *then* there is a closed unbounded set $C^* \subseteq C^+ \subseteq \lambda$ of limit ordinals such that for some $\delta_i < \lambda$, $\alpha_i \in C_{\delta_i}$ for $i < \lambda$ we have that $\bigcap_{i<\lambda}(C_{\delta_i} \cup [\alpha_i, \lambda))$ contains a club of λ [using instances of the Chang conjecture we can weaken the hypothesis to $\lambda \to [\lambda]^2_\mu$ for suitable μ].

REMARKS. (1) On the hypothesis of (C) see 3.7, 3.11.
(2) In fact, in the cases we get $\lambda \not\to [\lambda]^2_\sigma$ we get also $\lambda \not\to [\lambda, \lambda, \lambda]^{1,1,1}_\sigma$.

Consequences of (C) are:

(D) (1) if $\lambda > \aleph_0$ is Mahlo but not ω-Mahlo, *then* $\lambda \not\to [\lambda]^2_\lambda$.
(2) If $\lambda > \aleph_0$ is regular, $S_i \subseteq \lambda$ stationary for $i < \lambda$ but for no inaccessible $\lambda' < \lambda$, $(\forall i < \lambda')$ ($S_i \cap \lambda'$ is stationary), *then* $\lambda \not\to [\lambda]^2_{\aleph_0}$.
(3) If $\lambda \to [\lambda]^2_{\aleph_0}$ ($\lambda > \aleph_0$ regular), *then* λ is weakly compact in L.
(4) If λ is successor or singular, *then* $\lambda \not\to [\lambda]^2_{\aleph_0}$.
(5) $\aleph_{\omega_1+1} \not\to [\aleph_{\omega_1+1}]^2_{\aleph_1}$.

§1. On the consistency of $\lambda \to (\lambda, [\kappa; \kappa])$

1.1. THEOREM. *Suppose $\mu < \kappa < \lambda$ are regular cardinals, $\mu = \mu^{<\mu}$, $\kappa = \kappa^{<\kappa}$, $\lambda = \lambda^{<\kappa}$, $\lambda \geqq \beth_2(\kappa)^+$ and $(\forall \theta < \kappa)[\theta^{<\mu} < \kappa]$. Then for some forcing notion P*:

(1) $|P| = \lambda$.
(2) $\Vdash_P$ "$2^\mu = \lambda$".
(3) $\Vdash_P$ "$\lambda \to (\lambda, [\kappa; \kappa])$" (*see Definition* 1.2 *below*) (*hence for* $\kappa_1 < \kappa$: $\Vdash_P$ "$\lambda \to (\lambda, [\kappa_1, \kappa_1])$").
(4) *Forcing by P does not collapse any cardinal nor change a cofinality and P is μ-complete.*

1.2. DEFINITION. (1) $\lambda \to (\mu_1, [\mu_2; \mu_2]_\theta)$ holds *iff* for every 2-place function c from λ to $\theta + 1$, at least one of the following hold:

(i) there is $A \subseteq \lambda$, $|A| = \mu_1$ such that, on A, c is constantly zero;
(ii) there are $\alpha_i, \beta_i < \lambda$ for $i < \mu_2$, pairwise distinct, and ζ, $0 < \zeta \leqq \theta$ such that for $i < j < \mu_2$, $c(\alpha_i, \beta_j) = \zeta$.

If $\theta = 1$ we omit it.

(2) $\lambda \to (\mu_1, [\mu_2, \mu_3]_\theta)$ holds *iff*, for every 2-place function c from λ to $\theta + 1$, at least one of the following holds:

(i) there is $A \subseteq \lambda$, $|A| = \mu_1$ such that, on the set A, the function c is constantly zero;

(ii) there are $\alpha_i < \lambda$ (for $i < \mu_2$) and $\beta_j < \lambda$ ($j < \mu_3$), all pairwise distinct, and ζ, $0 < \zeta \leqq \theta$ such that for $i < \mu_2$, $j < \mu_3$ we have $c(\alpha_i, \beta_j) = \zeta$.

PROOF. Let

$Q = \{g : g$ a function from some $\alpha < \mu$ to $\{0, 1\}\}$
order: inclusion
$P = \{f : f$ a function with domain a subset of λ of power $< \kappa, f(i) \in Q\}$

stipulating that when $i \notin \mathrm{Dom}_f$, $f(i) = \emptyset \in Q$ the order on P is:

$$P \models f_1 \leqq f_2 \textit{ iff } \text{for each } i \in \mathrm{Dom}\, f_1,\ f_1(i) \leqq f_2(i) \text{ (in } Q) \text{ and } \{i \in \mathrm{Dom}\, f_1 : f_1(i) \neq f_2(i)\} \text{ has power } < \mu.$$

We say $f_1 \leqq_{\mathrm{pr}} f_2$ (f_2 a pure extension of f_1) if

$$[i \in \mathrm{Dom}\, f_1 \Rightarrow f_1(i) = f_2(i)].$$

EXPLANATION. Note that $(P, \leqq_{\mathrm{pr}})$ is really adding λ Cohen subsets to κ; and $(\{f \in P : |\mathrm{Dom}\, f| < \mu\}, \leqq)$ is really adding λ Cohen subsets to μ. The point is that q extends p *if*:

(a) q gives more information,
(b) outside Dom p it gives $< \kappa$ new pieces of information,
(c) inside Dom p it gives $< \mu$ additional pieces of information.

A. FACT. P is μ-complete.

B. FACT. P satisfies the κ^+-c.c.

By the Δ-system argument

C. FACT. $|P| = \lambda^{<\kappa}$.

D. FACT. $\Vdash_P$ "$2^\mu = \lambda$".

Standard:

E. FACT. If θ is regular cardinal, $\mu^+ \leqq \theta \leqq \kappa$ *then* $\Vdash$ "θ is a regular cardinal".

PROOF. Suppose $p \in P$, $\chi < \theta$, and $p \Vdash_P$ "cf $\theta = \chi$". So there are P-names $\underset{\sim}{\zeta}_i$ (for $i < \chi$) such that:

$$p \Vdash_P \text{"each } \underset{\sim}{\zeta}_i \text{ is an ordinal } < \theta \text{ and } \theta = \sup_{i<\chi} (\underset{\sim}{\zeta}_i)\text{"}.$$

We define by induction on $\alpha \leqq \mu$, $p_\alpha \in P$ such that:

(a) for $\beta < \alpha$, $p_\beta \leqq_{\mathrm{pr}} p_\alpha$, and $p_0 = p$;

(b) if α is limit, $\mathrm{Dom}\, p_\alpha = \bigcup_{\beta<\alpha} \mathrm{Dom}\, p_\beta$,

$$p_\alpha(i) = p_\beta(i) \text{ for every } \beta < \alpha \text{ large enough;}$$

(c) if $i < \chi$, $\xi < \theta$, $q \in P$, $p_{\beta+1} \leqq q$,

$$\{j \in \mathrm{Dom}\, p_{\beta+1} : p_{\beta+1}(j) \neq q(j)\} \subseteq \mathrm{Dom}\, p_\beta \quad \text{and} \quad q \Vdash_P \text{“}\underset{\sim}{\zeta}_i = \xi\text{”}$$

then $q \restriction (\mathrm{Dom}\, p_{\beta+1}) \Vdash_P$ “$\underset{\sim}{\zeta}_i = \xi$”.

This is enough: for each $\xi < \theta$, necessarily, as $p \Vdash$ “$\theta = \sup_{i<\chi}(\underset{\sim}{\zeta}_i)$” (and $p = p_0 \leqq p_\mu$) there are $q^\xi \in P$, satisfying $p_\mu \leqq q^\xi$, an ordinal $\zeta[\xi] < \theta$ and $i(\xi) < \chi$ such that

$$q^\xi \Vdash_P \text{“}\theta > \underset{\sim}{\zeta}_{i(\xi)} = \zeta[\xi] > \xi\text{”}.$$

As $\{i \in \mathrm{Dom}(p_\mu) : p_\mu(i) \neq q^\xi(i)\}$ has power $< \mu$ it is included in $\mathrm{Dom}\, p_{\beta(\xi)}$ for some $\beta(\xi) < \mu$. By (c) above

$$q^\xi \restriction \mathrm{Dom}(p_{\beta(\xi)+1}) \Vdash_P \text{“}\underset{\sim}{\zeta}_{i(\xi)} = \zeta[\xi]\text{”}$$

hence $q^\xi \restriction \mathrm{Dom}(p_\mu) \Vdash_P$ “$\underset{\sim}{\zeta}_{i(\xi)} = \zeta[\xi]$”. As the number of $i(\xi)$ is $\chi < \theta$, θ regular (in V) there is a set $S \subseteq \theta$, $|S| = \theta$ such that $i(\xi) = i(*)$ for $\xi \in S$ and $\zeta[\xi_1] < \zeta[\xi_2]$ when $\xi_1 \in S$, $\xi_2 \in S$, $\xi_1 < \xi_2$. Let

$$u_\xi = \{j \in \mathrm{Dom}(p_\mu) : q^\xi(i) \neq p_\mu(i)\},$$

so $|u_\xi| < \mu$. As $|\{q^\xi(j) : \xi \in S\}| \leqq \mu$ for each $j \in \mathrm{Dom}(p_\mu)$ and as

$$|u_\xi| < \mu = \mu^{<\mu} < |\{\xi : \xi < \theta\}|$$

for some $\xi < \zeta$ in S, $q^\xi \restriction (u_\xi \cap u_\zeta) = q^\zeta \restriction (u_\xi \cap u_\zeta)$ hence $q^\xi \restriction \mathrm{Dom}(p_\mu)$, $q^\zeta \restriction \mathrm{Dom}(p_\mu)$ are compatible and $(q^\xi \restriction u_\xi) \cup (q^\zeta \restriction (\mathrm{Dom}\, p_\mu - u_\xi))$ is a common upper bound; but they force different values on $\underset{\sim}{\zeta}_{i(*)}$, contradiction.

We still have to carry the definition of the p_α's. For $\alpha = 0$, α limit no problem. For $\alpha = \beta + 1$, let $\{(i_\xi, r_\xi) : \xi < \xi(*)\}$ list all pairs (i, r), i an ordinal $< \chi$, $r \in P$, $\mathrm{Dom}\, r$ a subset of $\mathrm{Dom}\, p_\beta$ of power $< \mu$. The number of $\mathrm{Dom}\, r_\xi$ is $< \kappa$ as $(\forall \theta < \kappa)(\theta^{<\mu} < \kappa)$ and $|\mathrm{Dom}\, p_\beta| < \kappa$. For each such domain the number of conditions is $\leqq \mu^{<\mu} = \mu < \kappa$. Lastly the number of values of i is $\chi < \theta \leqq \kappa$. So $\xi(*) < \kappa$. We now define by induction on $\xi \leqq \xi(*)$ a condition $p_{\beta,\xi} \in P$ such that: $p_{\beta,0} = p_\beta$, $(\forall \zeta < \xi)\, p_{\beta,\zeta} \leqq_{\mathrm{pr}} p_{\beta,\xi}$, for ξ limit $p_{\beta,\xi} = \bigcup_{\zeta<\xi} p_{\beta,\zeta}$ and for each $\xi < \xi(*)$ *if* there is q, $p_{\beta,\xi} \leqq q \in P$, q forces a value for $\underset{\sim}{\zeta}_{i_\xi}$,

$q \restriction (\mathrm{Dom}\, r_\xi) = r_\xi$ and $[\forall i \in \mathrm{Dom}(p_{\beta,\xi}) - \mathrm{Dom}(r_\xi)]\, [p_{\beta,\xi}(i) = q(i)]$ *then* $p_{\beta,\xi+1}$ satisfies this.

Now let $p_\alpha = p_{\beta+1} \stackrel{\text{def}}{=} p_{\beta,\xi(*)}$. It is as required.

F. Fact. Suppose λ_2 is regular, $\lambda_2 \to (\kappa^+)^2_\kappa$, $\lambda_2 > \theta$, and $\lambda_1 = [2^{<\lambda_2}]^+$ (or just λ_1 is regular and $(\forall \sigma < \lambda_1)[\sigma^{<\lambda_2} < \lambda_1]$. Then $\Vdash_P \lambda_1 \to (\lambda_1, [\kappa; \kappa]_\theta)$.

Proof. Let $\underset{\sim}{d}$ be a P-name of a 2-place function from λ_1 to θ, $p_0 \in P$. For $\alpha < \beta < \lambda_1$ choose $p_{\alpha,\beta}$, $p_0 \leqq p_{\alpha,\beta} \in P$ such that for some $\psi_{\alpha,\beta} \in \theta$, $p_{\alpha,\beta} \Vdash_P$ "$\underset{\sim}{d}(\alpha,\beta) = \psi_{\alpha,\beta}$", and if possible, $\psi_{\alpha,\beta} \neq 0$. So $\psi_{\alpha,\beta} = 0$ implies $p_0 \Vdash_P$ "$\underset{\sim}{d}(\alpha,\beta) = \psi_{\alpha,\beta}$".

Let $\mathrm{Dom}\, p_{\alpha,\beta} = \{i_{\alpha,\beta}(\zeta) : \zeta < \zeta_{\alpha,\beta} < \kappa\}$ where $i_{\alpha,\beta}(\zeta)$ increases with ζ.

We define a 3-place function H with domain $\lambda_1 : H(\alpha,\beta,\gamma)$ is a sequence consisting of

(i) $\zeta_{\alpha,\beta}$,

(ii) $\{\langle \zeta_1, \zeta_2 \rangle : i_{\alpha,\beta}(\zeta_1) = i_{\alpha,\gamma}(\zeta_2)\}$,

(iii) $\{\langle \zeta, p_{\alpha,\beta}(i_{\alpha,\beta}(\zeta))\rangle : \zeta < \zeta_{\alpha,\beta}\}$,

(iv) $\{\langle \zeta_1, \zeta_2 \rangle : i_{\alpha,\gamma}(\zeta_1) = i_{\beta,\gamma}(\zeta_2)\}$,

(v) $\{\langle \zeta, p_{\alpha,\gamma}(i_{\alpha,\beta}(\zeta))\rangle : \zeta < \zeta_{\alpha,\beta}\}$,

(vi) $\{\langle \zeta_1, \zeta_2 \rangle : i_{\alpha,\beta}(\zeta_1) = i_{\beta,\gamma}(\zeta_2)\}$.

So we have two colorings: $\psi_{\alpha,\beta}$ (two place with θ colors) and H (three place with κ colors as $\kappa = \kappa^{<\kappa}$).

As $\lambda_1 = [2^{<\lambda_2}]^+$, there is a subset A of λ_1, such that: either

(I) $\psi_{\alpha,\beta} = 0$ for every $\alpha < \beta$ from A, and $|A| = \lambda_1$

or

(II) $|A| = \lambda_2$, A has order type λ_2, and such that:

(1) $\psi_{\alpha,\beta} \neq 0$ for $\alpha < \beta$ from A,

(2) for $\alpha < \beta < \gamma$ from A, $\psi_{\alpha,\beta} = \psi_{\alpha,\gamma}$,

(3) for $\alpha_1 < \alpha_2 < \beta < \gamma$ from A, $H(\alpha_1, \alpha_2, \beta) = H(\alpha_1, \alpha_2, \gamma)$.

So on A we can define a 2-place function H',

$$H'(\alpha,\beta) = H(\alpha,\beta,\gamma) \qquad \text{for every } \gamma \in A - (\alpha+\beta+1).$$

If (I) holds, $p_0 \Vdash_P$ "$\underset{\sim}{d}$ is constantly 0 on A" and we finish. So we shall assume (II). Note that $\psi_{\alpha,\beta}(\alpha < \beta,\ \alpha \in A,\ \beta \in A)$ depends on α only. So as $\lambda_2 > \theta$ is regular w.l.o.g. for some ψ, $(0 < \psi \leqq \theta)$, $\psi_{\alpha,\beta} = \psi$ for every $\alpha < \beta$ from A. As $\lambda_2 \to (\kappa^+)^2_\kappa$, there is a subset B of A of cardinality (and order type) κ^+, on which H' is constant.

So, the function H is constant on B. Hence for every $\alpha \in B$ (by (ii))

$\langle \mathrm{Dom}\, p_{\alpha,\beta} : \alpha < \beta \in B \rangle$ form a Δ-system, and let its "heart" be b_α, and let $r_\alpha = p_{\alpha,\beta} \restriction b_\alpha$ for $\alpha < \beta \in B$ (the choice of β is immaterial). So for each $\alpha \in B$: $\langle \mathrm{Dom}\, p_{\alpha,\beta} - b_\alpha : \alpha < \beta \in B \rangle$ are pairwise disjoint.

As $|B| = \kappa^+$, for some $C \subseteq B$, C has cardinality and order types κ^+, and $\langle r_\alpha : \alpha \in C \rangle$ form a Δ-system, i.e. for some r^*,

$$r^* = r_\alpha \restriction (\mathrm{Dom}\, r^*),$$
$$\langle \mathrm{Dom}\, r_\alpha - \mathrm{Dom}\, r^* : \alpha \in C \rangle \text{ are pairwise disjoint.}$$

We now define in V^P by induction on $i < \kappa^+$ ordinals α_i, β_i (pairwise distinct) from C as follows:

(i) $\alpha_i \in C$ is minimal such that $r_{\alpha_i} \in \underset{\sim}{G}_P$ and $\alpha_i > \bigcup_{j<i} (\alpha_j \cup \beta_j)$,

(ii) $\beta_i \in C$ is minimal such that $p_{\alpha_j,\beta_i} \in \underset{\sim}{G}_P$ and $\beta_i > \alpha_j$ for every $j \leqq i$.

If α_i, β_i are defined for every $i < \kappa$, then as clearly in V^P $d(\alpha_j, \beta_i) = \psi$ for $j \leqq i$ (as p_{α_j,β_i} force this) we have finished. So it suffices to show

$$r^* \Vdash_P \text{“}\alpha_i, \beta_i \text{ are defined for every } i < \kappa\text{”}.$$

We have two cases (according to whether the first to be undefined is an α_i or β_i).

Suppose first $r^* \leqq r^+ \in P$, and $r^+ \Vdash_P$ "α_i is not defined (but α_j, β_j are defined for $j < i$)"; w.l.o.g. for $j < i$, $r_{\beta_j} \leqq r^+$ and for $j_1 < j_2 < i$, $p_{\alpha_{j_1},\beta_{j_2}} \leqq r^+$.

But $\mathrm{Dom}\, r_\alpha - \mathrm{Dom}\, r^*$ $(\alpha \in C - \bigcup_{j<i} (\alpha_j \cup \beta_j))$ are pairwise disjoint and their number is κ^+ (really κ suffices).

So for some α, $\bigcup_{j<i} (\alpha_j \cup \beta_j) < \alpha \in C$, $\mathrm{Dom}\, r_\alpha - \mathrm{Dom}\, r^*$ is disjoint to $\mathrm{Dom}\, r^+$. As $r_\alpha \restriction (\mathrm{Dom}\, r^*) = r^* \subseteq r^+$, clearly r^+, r_α are compatible:

$$r^{++} \overset{\mathrm{def}}{=} r^+ \cup r_\alpha \restriction (\mathrm{Dom}\, r_\alpha - \mathrm{Dom}\, r^*)$$

is an upper bound but $r^{++} \Vdash$ "α is a good candiate for α_i". Hence α_i is defined.

Contradiction. Suppose secondly $r^* \leqq r^+ \in P$ and $r^+ \Vdash_P$ "β_i is not defined but α_j $(j \leqq i)$ β_j $(j < i)$ are defined". W.l.o.g. for $j \leqq i$ we have $r_{\alpha_j} \leqq r^+$ and for $j_1 < j_2 < i$ we have $p_{\alpha_{j_1},\beta_{j_2}} \leqq r^+$.

For each $j \leqq i$, $\langle \mathrm{Dom}\, p_{\alpha_j,\beta} - \mathrm{Dom}\, r_{\alpha_j} : \alpha_i < \beta \in C \rangle$ are pairwise disjoint, hence for all except $< \kappa$ of the ordinals $\beta \in C - (\alpha_i + 1)$ we have: $\mathrm{Dom}\, p_{\alpha_j,\beta} - \mathrm{Dom}\, r_{\alpha_j}$ is disjoint to $\mathrm{Dom}\, r^+$. As $|C - (\alpha_i + 1)| \geqq \kappa$, for some $\beta \in C$, $\beta > \alpha_i$, and for every $j \leqq i$, $\mathrm{Dom}\, p_{\alpha_j,\beta} - \mathrm{Dom}\, r_{\alpha_j}$ is disjoint to $\mathrm{Dom}\, r^+$. As $p_{\alpha_j,\beta} \restriction \mathrm{Dom}\, r_{\alpha_j} = r_{\alpha_j} = r_{\alpha_j} \leqq r^+$, similarly to the first case r^+, $p_{\alpha_j,\beta}$ are compatible.

We want to show that the set $\{r^+\} \cup \{p_{\alpha_j,\beta} : j \leqq i\}$ has an upper bound in P. By the definition of P it suffices to show that any two are compatible. As we have shown that r^+, $p_{\alpha_i,\beta}$ are compatible when $j \leqq i$, it is enough to show that $p_{\alpha_{j(1)},\beta}$, $p_{\alpha_{j(2)},\beta}$ are compatible when $j(1) < j(2) \leqq i$. This follows as the function H is constant on the set $C \subseteq \lambda_2$, using the definition of H.

By the definition of P, there is $r^{++} \in P$ such that $r^+ \leqq r^{++}$, $p_{\alpha_j,\beta} \leqq r^{++}$ for $j \leqq i$. Clearly $r^{++} \Vdash_P$ "β is a good candidate for β_i hence β_i is defined". Contradiction.

§2. On the consistency of $2^{\aleph_0} \to [\aleph_1]^2_3$

2.1. THEOREM. *Suppose $\mu = \mu^{<\mu} < \lambda = \chi$ and λ is a strongly inaccessible measurable cardinal $> \mu$ (or $\lambda \to (\omega_1)_2^{<\omega}$, λ minimal).*

Then there is a forcing notion P such that:

(α) *P is μ-complete,*

(β) $|P| = \chi$,

(γ) $\Vdash_P$ "$\lambda \to [\mu^+]^2_3$",

(δ) *P collapses no cardinal $\leqq \lambda$, changes no cofinality, adds no sequence of ordinals of length $< \mu$ and $\Vdash_P$ "$2^\mu = \chi$".*

2.1A. REMARK. At the urging of the referee we concentrate here on the case $\mu = \aleph_0$, $\lambda = \chi$ the first measurable.

2.1B. REMARK. (1) See 2.7 for the improvement in the hypothesis on λ.

(2) In (γ) we can get $\lambda \to [\mu^+]^2_{\theta,3}$ for $\theta < \mu$. For this in (6) below $\underset{\sim}{d}$ is a function from λ to $\underset{\sim}{\theta}_\alpha$, $\underset{\sim}{\theta}_\alpha < \mu$ and $\underset{\sim}{e}^l_d < \theta_\alpha$.

PROOF. We try to define by induction on $\alpha \leqq \chi$:

$$\bar{Q} = \langle P_j, \underset{\sim}{Q}_i : i < \alpha, j \leqq \alpha \rangle \quad \text{and} \quad \mathbf{e}^*_\alpha \in \{0, 1\}$$

as follows:

(1) P_j is a forcing notion and satisfying the $\aleph_1$-c.c.

(2) $\underset{\sim}{Q}_i$ is a P_i-name of a forcing notion of power $\aleph_1$ (with minimal element $\underset{\sim}{\varnothing}$ or $\varnothing_i$).

(3) $\bar{Q}$ is a finite support iteration, i.e.

$P_j = \{f : f$ is a function with domain a finite subset of j and for $i \in \mathrm{Dom}(f)$, $f(i)$ is a P_i-name, $(f \restriction i) \Vdash_{P_i}$ "$f(i) \in \underset{\sim}{Q}_i$" and $f(i) \in H(2^\chi)^+)$ (to avoid classes)$\}$

and

$$P_j \models \text{“}f \leqq g\text{” } \textit{iff} \text{ for each } i \in \mathrm{Dom}\, f,\ g \restriction i \Vdash_{P_i} \text{“}f(i) \leqq g(i)\text{”}.$$

We let for $f \in P_j$, $i < j$, $i \notin \mathrm{Dom}(f)$: $f(i) = \varnothing$ or $f(i) = \varnothing_i$. Note that for the

$\underset{\sim}{Q}_i$ we are using, the set $P'_j = \{ f \in P_j : f(i) \in V$ (i.e. not just forced to be in V but is specific element)$\}$ is a dense subset of P_j.

(4) $\mathbf{e}^*_\alpha$ is an ordinal < 2 such that $[\mathbf{e}^*_\alpha = 1 \Rightarrow \mathrm{cf}(\alpha) = \aleph_1]$ (it just tells us what we are doing in $\underset{\sim}{Q}_\alpha$).

(5) If $\mathbf{e}^*_\alpha = 0$ then

$$Q_\alpha = \{ f : f \text{ a function from some } \xi < \aleph_1 \text{ to } \{0, 1\}\}$$

ordered by being an end-extension.

(6) If $\mathbf{e}^*_\alpha = 1$ then for some $\underset{\sim}{d}_\alpha$, e^1_α, e^2_α, I and N^α_s, $h^\alpha_{s,t}$ $(s, t \in I \overset{\mathrm{def}}{=} \{ t \subseteq \aleph_1 : |t| \leqq 2\}$, $|s| = |t|)$ and r^α_ζ, $\theta^\alpha_\zeta (\zeta < \aleph_1)$ the following holds:

(i) α is an ordinal of cofinality $\aleph_1$, $\underset{\sim}{d}_\alpha$ is a P_α-name of a partial function from λ to $\{0, 1, 2\}$, C_α a closed unbounded subset of α, and for $\beta \in C_\alpha$, $\underset{\sim}{d}_\alpha \restriction \beta$ is a P_β-name and e^1_α, e^2_α are ordinals < 3.

(ii) If $s \in I$ *then* $N^\alpha_s \prec (H(2^\chi)^+, \in)$, $N^\alpha_s \cap \lambda \subseteq \alpha$, $\aleph_1 \subseteq N^\alpha_s$, $\| N^\alpha_s \| = \aleph_1$, $\aleph_1 \in N^\alpha_s$ (remember that $|N|$ is the universe of the model N, so $\| N \|$ is its cardinality) and $C_\alpha \cap N^\alpha_s$ is unbounded in $\alpha \cap N^\alpha_s$, $\bigcup_{s \in I} (\lambda \cap N^\alpha_s)$ is in Dom $\underset{\sim}{d}_\alpha$ (i.e. on all pairs from each $\lambda \cap N^\alpha_s$),

$$\Big[\beta \in N^\alpha_s \wedge \mathbf{e}^*_\beta = 1 \Rightarrow \bigcup_{t \in I} N^\beta_t \subseteq N^\alpha_s \Big],$$

$$\{(\beta, \underset{\sim}{d}_\alpha \restriction \beta) : \beta \in C_\alpha \cap N^\alpha_s\} \subseteq N^\alpha_s$$

and

$$\{\langle P_j, \underset{\sim}{Q}_i : j \leqq \beta, i < \beta \rangle : \beta \in C_\alpha \cap N^\alpha_s\} \subseteq N^\alpha_s.$$

(iii) If $s, t \in I$ then $N^\alpha_s \cap N^\alpha_s = N^\alpha_{s \cap t}$.

(iv) If $|s| = |t|$, then $h^\alpha_{s,t}$ is an isomorphism from N^α_s onto N^α_t, mapping $\{(\beta, \underset{\sim}{d}_\alpha \restriction \beta) : \beta \in C_\alpha \cap N^\alpha_s\}$ onto $\{(\beta, \underset{\sim}{d}_\alpha \restriction \beta) : \beta \in C_\alpha \cap N^\alpha_t\}$ and $\{\langle P_j, \underset{\sim}{Q}_i : j \leqq \beta, \ i < \beta \rangle : \beta \in N^\alpha_s \cap C_\alpha\}$ onto $\{\langle P_j, \underset{\sim}{Q}_i : j \leqq \beta, \ i < \beta \rangle : \beta \in N^\alpha_t \cap C_\alpha\}$, $h_{s,t}$ is the identity on $N_\varnothing$, it extends $h_{\{\max(s)\},\{\max(t)\}}$ and $h_{\{\min(s)\},\{\min(t)\}}$ and $h_{s,t}$ is the identity when $s = t$ and $h_{s,t} = h^{-1}_{t,s}$.

(v) For $\zeta < \aleph_1$, $\theta^\alpha_\zeta \in N^\alpha_{\{\zeta\}} \cap \lambda$ is an ordinal, $[\xi < \zeta < \aleph_1 \Rightarrow \theta^\alpha_\xi < \theta^\alpha_\zeta]$, $[\zeta \neq \xi \Rightarrow \theta^\alpha_\zeta \notin N^\alpha_{\{\xi\}}]$ and $r^\alpha_\zeta \in P_\alpha \cap N^\alpha_{\{\zeta\}}$, $h^\alpha_{\{\zeta\},\{\xi\}}(r^\alpha_\zeta) = r^\alpha_\xi$, $h^\alpha_{\{\zeta\},\{\xi\}}(\theta^\alpha_\zeta) = \theta^\alpha_\xi$.

(vi) If $r^\alpha_\zeta \leqq p \in N^\alpha_{\{\zeta\}} \cap P_\alpha$ then there are p_1, p_2 such that: $p \leqq p_1 \in N^\alpha_{\{\zeta\}} \cap P_\alpha$, $p \leqq p_2 \in N^\alpha_{\{\zeta\}} \cap P_\alpha$, and *if* $\zeta < \xi < \mu^+$ *then* for some $q_1, q_2 \in N^\alpha_{\{\zeta,\xi\}} \cap P_\alpha$, for $l = 1, 2$,

$$q_l \Vdash \text{“} \underset{\sim}{d}_\alpha(\theta^\alpha_\zeta, \theta^\alpha_\xi) = e^\alpha_l \text{”}, \quad q_l \restriction (N^\alpha_{\{\zeta\}} \cap \alpha) = p_l, \quad q_l \restriction (N^\alpha_{\{\xi\}} \cap \alpha) = h_{\{\zeta\},\{\xi\}}(p_{3-l})$$

$(e^1_\alpha, e^2_\alpha$ are ordinals $< 3)$.

(vii) For each α for which $\mathrm{e}^*_\alpha = 1$

(a) $\mathrm{Min}(N^\alpha_{\{\zeta,\xi\}} - N^\alpha_\emptyset) = \mathrm{Min}(N^\alpha_{\{\zeta\}} - N^\alpha_0)$ for $\zeta < \xi < \aleph_1$,

(b) $\langle \mathrm{Min}(N^\alpha_{\{\zeta\}} - N^\alpha_\emptyset) : \zeta < \mu^+ \rangle$ is increasing and converges to α,

(c) for each ζ, $\langle \mathrm{Min}(N^\alpha_{\{\zeta,\xi\}} - N^\alpha_{\{\zeta\}}) : \zeta < \xi < \aleph_1 \rangle$ is increasing and converges to α, hence

(d) if $\beta < \alpha$, $\mathrm{e}^*_\alpha = 1 = \mathrm{e}^*_\beta$ *then* for some $\zeta(*) < \aleph_1$ $\bigcup\{(N^\alpha_t - N^\alpha_s) \cap \lambda : t \in I, t \neq s = t \cap \zeta(*)\}$, is disjoint to

$$\bigcup\{(N^\beta_t - N^\beta_s) \cap \lambda : t \in I, t \neq s = t \cap \zeta(*)\}.$$

(viii) $\underset{\sim}{Q}_\alpha = \{w \subseteq \aleph_1 : |w| < \mu$, and for every $\zeta \in w$, $r^\alpha_\zeta \in G_{P_\alpha}$ and for every $\zeta < \xi$ from w there is $q \in N^\alpha_{\{\zeta,\xi\}} \cap P_\alpha \cap \underset{\sim}{G}_{P_\alpha}$ such that $q \Vdash_{P_\alpha}$ "$\underset{\sim}{d}_\alpha(\theta^\alpha_\zeta, \theta^\alpha_\xi) \in \{e^1_\alpha, e^2_\alpha\}$" and there is $q' \in N^\alpha_{\{\zeta,\xi\}} \cap P_\alpha$, $q' \Vdash_{P_\alpha}$ "$\underset{\sim}{d}_\alpha(\theta^\alpha_\zeta, \theta^\alpha_\xi) \in \{e^1_\alpha, e^2_\alpha\}$" and $q' \restriction (\lambda \cap N^\alpha_{\{\xi\}}) = h_{\{\zeta\},\{\xi\}}(q \restriction (N^\alpha_{\{\zeta\}} \cap \lambda)$, and $q \restriction (\lambda \cap N^\alpha_{\{\xi\}}) = h_{\{\zeta\},\{\xi\}}(q' \restriction (\lambda \cap N^\alpha_{\{\zeta\}}))$ and these elements are in $P_\alpha\}$.

$\underset{\sim}{Q}_\alpha$ is ordered by inclusion.

2.2. Notation. If $\Gamma \subseteq P_\alpha$, $|\Gamma| < \aleph_0$ we define $q = \bigcup\Gamma$; it is a function with domain $a \overset{\text{def}}{=} \bigcup_{p\in\Gamma} \mathrm{Dom}\, p$ and for each $\gamma \in \alpha$, $g(\gamma) = \bigcup_{p\in\Gamma} p(\gamma)$.

In general q need not be in P_α (e.g. maybe for some $p_1, p_2 \in P$ and γ, $p_1(\gamma) \cup p_2(\gamma) \notin \underset{\sim}{Q}_\gamma$).

2.2A. Fact. Suppose:

(1) $\Gamma \subseteq P_\alpha$, $|\Gamma| < \mu$ and for every $p_1, p_2 \in \Gamma$ and $\gamma \in \mathrm{Dom}\, p_1 \cap \mathrm{Dom}\, p_2$ the following holds:

(i) $\bigcup_{r\in\Gamma}(r \restriction \gamma) \Vdash_{P_\gamma}$ "$p_1(\gamma) \leqq p_2(\gamma)$ in $\underset{\sim}{Q}_\gamma$" *or*

(ii) $\bigcup_{r\in\Gamma}(r \restriction \gamma) \Vdash$ "$p_2(\gamma) \leqq p_1(\gamma)$ in $\underset{\sim}{Q}_\gamma$"

then $\bigcup\Gamma \in P_\alpha$ is the least upper bound of Γ.

(2) We can of course omit in (i), (ii) above "$\bigcup_{r\in\Gamma} r \restriction \gamma$": this is particularly useful when $\Gamma \subseteq P'_\alpha$ (P'_α — defined above).

(3) We can add in (1):

or

(iii) $\bigcup_{r\in\Gamma} r \restriction \gamma \Vdash_{P_\gamma}$ "$p_1(\gamma) \cup p_2(\gamma) \in \underset{\sim}{Q}_\gamma$".

2.3. Notation. $P''_\alpha = \{p \in P_\alpha$: for $\beta \in \mathrm{Dom}\, p$, $p(\beta)$ is an actual subset of $\aleph_1$ (or function from $\aleph_0$ to 2), not just a P_β-name, and if $\mathrm{e}^*_\beta = 1$, $\zeta < \xi$, $\zeta \in p(\beta)$, $\xi \in p(\beta)$, *then* for some $r \in N^\beta_{\{\zeta,\xi\}} \cap P''_\beta$, $r \leqq p \restriction \beta$ (so p forces that r will belong to the generic subset of p_β) and $r \Vdash_{P_\beta}$ "$\underset{\sim}{d}_\beta(\zeta, \xi) \in \{e^1_\beta, e^2_\beta\}$" and there is $r' \in P''_\beta \cap N^\beta_{\{\zeta,\xi\}}$ (note that generally r' is incompatible with $p(\beta)$!) such that:

"$r' \Vdash_{P_\beta} \underset{\sim}{d}_\beta(\zeta, \xi) \in \{\underset{\sim}{e}^1_\beta, \underset{\sim}{e}^2_\beta\}$", $h^\beta_{\{\zeta\},\{\xi\}}(r' \upharpoonright N^\beta_{\{\zeta\}}) = r \upharpoonright N^\beta_{\{\xi\}}$, $h^\beta_{\{\zeta\},\{\xi\}}(r \upharpoonright N^\beta_{\{\zeta\}}) = r' \upharpoonright N^\beta_{\{\xi\}}\}$. Note that

(i) $\{(\beta, P''_\beta) : \beta \in C_\alpha \cap N^\alpha_s\} \subseteq N^\alpha_s$ when $\mathrm{e}^*_\alpha = 1$, $s \in I$,

(ii) $r \cup (p \upharpoonright (N^\beta_{\{\zeta\}})) \cup (p \upharpoonright N^\beta_{\{\zeta\}})$ can serve instead r above.

2.4. FACT. (1) If $\mathrm{e}^*_\alpha = 1$, $p \in P''_\alpha$, $t \in I$, then $p \upharpoonright N^\alpha_t \in N^\alpha_t \cap P''_\alpha$.

(2) P''_α is a dense subset of P_α.

PROOF. Note that if $\beta \in N^\alpha_t$, then $(\beta \cap \bigcup_{s \in I} N^\beta_s) \subseteq N^\alpha_t$.

2.5. FACT. *P_α satisfies the $\aleph_1$-c.c.*

By well-known theorems, the only problematic case is $\alpha + 1$, $\mathrm{e}^*_\alpha = 1$. Let $\alpha = \bigcup_{\zeta < \mu^+} \psi_{\alpha,\zeta}$, $\langle \psi_{\alpha,\zeta} : \zeta < \aleph_1 \rangle$ be increasing continuous, $\psi_{\alpha,\zeta} < \alpha$. So suppose $\langle p_\zeta : \zeta < \aleph_1 \rangle$ is given, $p_\zeta \in P_{\alpha+1}$. By 2.4(2) w.l.o.g. $p_\zeta \in P''_{\alpha+1}$. Let

$$w_\zeta = \{i < \aleph_1 : i \in p_\zeta(\alpha) \textit{ or } \operatorname{dom}(p_\zeta) \text{ is not disjoint to } N^\alpha_{\{i\}} - N^\alpha_\varnothing, \textit{ or} \\ \text{for some } \xi < \aleph_1, \operatorname{dom}(p_\zeta) \text{ is not disjoint to } N^\alpha_{\{i,\xi\}} - N^\alpha_{\{\xi\}} \cup N^\alpha_{\{i\}}\}.$$

Clearly w_ζ is a subset of $\aleph_1$ of cardinality $< \aleph_0$, (dom p_ζ) $\cap\, \alpha$ a subset of α of cardinality $< \aleph_0$. Hence by the Fodor lemma, for some stationary $S \subseteq \{\delta < \aleph_1 : \operatorname{cf} \delta = \aleph_0\}$ the following holds:

$$(\forall \zeta, \xi \in S)(\zeta \neq \xi \Rightarrow w_\zeta \cap w_\xi = w^*).$$

$\operatorname{Min}(w_\zeta - w^*) \geqq \zeta$.

As (dom p_ζ) $\cap\, \alpha$ is a finite subset of α, by the Fodor lemma w.l.o.g. for some $\beta(*) < \alpha$ for every $\zeta \in S$: $(\operatorname{dom} p_\zeta) \cap \psi_{\alpha,\zeta} \subseteq \beta(*)$, and for $i < \zeta$, (dom p_i) $\cap\, \alpha \subseteq \psi_{\alpha,\zeta}$ and $(N^\alpha_{\{i,\zeta\}} - N^\alpha_{\{i\}}) \cap \psi_{\alpha,\zeta} = \varnothing$ for $i < \zeta$.

Let $w_\zeta - w^* = \{\varepsilon_\sigma(\zeta) : \sigma < \sigma^\zeta\}$ (increasing with σ), so σ^ζ is finite and w.l.o.g. for $\zeta \in S$, $\sigma^\zeta = \sigma^*$. Let $M_\zeta = \bigcup\{N^\alpha_{\{i,j\}} : i, j \in w_\zeta\}$ (so M_ζ is normally not an elementary submodel of $(H(\chi), \in)$).

Let $\zeta(*)$ be the minimal element of S.

Let us define for $\zeta \in S$, p^a_ζ as $p_\zeta \upharpoonright (M_\zeta \cap \alpha)$. ($a$, b and c below serve just to denote a variant of p_ζ.) Now $p^a_\zeta \in P''_\alpha$, as: it is a function, with domain a finite subset of α, and for each $i \in \operatorname{Dom} p^a_\zeta$, $p^a_\zeta(i)$ is a set or function of the right kind. But why is $i \in \operatorname{Dom} p^a_\zeta \wedge \mathrm{e}^*_i = 1 \Rightarrow p^a_\zeta \upharpoonright i \Vdash p^a_\zeta(i) \in Q_i$? By (viii) of (6) above and "$[\beta \in N^\alpha_s \wedge \mathrm{e}^*_\beta = 1 \Rightarrow \bigcup_{t \in I} N^\beta_t \subseteq N^\alpha_s]$" from (ii) of (6) above.

Next we define a condition $p^b_\zeta \in P''_\alpha$; we define it by demanding Dom p^b_ζ is a subset of $\alpha \cap M_{\zeta(*)}$ and

(*) *if*

(a) $i(1), i(2) \in w_{\zeta(*)}$, $j(1), j(2) \in w_\zeta$, and for $l = 1, 2$

$$[i(l)\in w(*) \wedge i(l)=j(l)]$$

$$\vee\, [i(l)\in(w_{\zeta(*)}-w^*) \wedge (\exists\,\sigma)(i(l)=\varepsilon_\sigma(\zeta(*))) \wedge j(l)=\alpha_\sigma(\zeta)]$$

then

(b) $h^\alpha_{\{i(1),i(2)\},\{j(1),j(2)\}}(p^b_\zeta \restriction N^\alpha_{\{i(1),i(2)\}}) = p^a_\zeta \restriction N^\alpha_{\{j(1),j(2)\}}$.

Why is $p^b_\zeta \in P''_\alpha$? By 2.2A. [Explanation: p^b_ζ is p^a_ζ mapped to a condition with domain $\subseteq M_{\zeta(*)}$, as far as is feasible.]

Clearly for some $\beta(1)<\alpha$, $\beta(1)>\beta(*)$, $\{p^b_\zeta : \zeta\in S\}\subseteq P''_\beta$, hence by the induction hypothesis, for some $\zeta_1<\zeta_2$ from S, for some $q\in P''_{\beta(1)}$, $p^b_{\zeta_1}, p^b_{\zeta_2}\leqq q$. Again we can show that $q\restriction M_{\zeta(*)}\in P''_{\beta(1)}$. (Note that we are strongly using "each Q_γ has power $\leqq \aleph_1$".)

Let for $\zeta\in S$, $p^c_\zeta \in P''_\alpha$ be defined by the following: $\mathrm{dom}(p^c_\zeta)\subseteq \alpha\cap M_\zeta$ and

(**) *if* $i(1), i(2), j(1), j(2)$ satisfies (a) above *then*

$$h^\alpha_{\{i(1),i(2)\},\{j(1),j(2)\}}(q\restriction N^\alpha_{\{i(1),i(2)\}}) = p^c_\zeta \restriction N^\alpha_{\{j(1),j(2)\}}.$$

To get the desired upper bound of p_{ζ_1}, p_{ζ_2} we shall apply 2.2A to

$$\Gamma \overset{\text{def}}{=} \Gamma_0\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3$$

where the Γ_l are defined below.

Let $\Gamma_0=\{p_{\zeta_1}, p_{\zeta_2}, q, p^c_{\zeta_1}, p^c_{\zeta_2}\}$.

[Explanation: Note that $(\bigcup\Gamma_0)\restriction\alpha\in P''_\alpha$, so the rest are designed to force that $p_{\zeta_1}(\alpha)\cup p_{\zeta_2}(\alpha)$ is a condition in Q_α, mainly: for $\sigma(1), \sigma(2)<\sigma^*$ we want that $\underset{\sim}{d}_\alpha[\theta^\alpha_{\varepsilon_{\sigma(1)}(\zeta_1)}, \theta^\alpha_{\varepsilon_{\sigma(2)}(\zeta_2)}]$ is e^α_1 or e^α_2. Now $\Gamma_1, \Gamma_2, \Gamma_3$ will deal respectively with the cases $\sigma(1)<\sigma(2)$, $\sigma(1)>\sigma(2)$ and $\sigma(1)=\sigma(2)$.]

Let $\Gamma_1=\{h_{\{\varepsilon_{\sigma(1)}(\zeta_1),\varepsilon_{\sigma(2)}(\zeta_1)\},\{\varepsilon_{\sigma(1)}(\zeta_1),\varepsilon_{\sigma(2)}(\zeta_2)\}}(p^c_{\zeta_1}\restriction{}_{\{\varepsilon_{\sigma(1)}(\zeta_1),\varepsilon_{\sigma(2)}(\zeta_1)\}}) : \sigma(1)<\sigma(2)<\sigma^*\}$.

Let for $\sigma(2)<\sigma(1)<\sigma^*$, $q_{\sigma(2),\sigma(1)}\in N^\alpha_{\{\varepsilon_{\sigma(2)}(\zeta_1),\varepsilon_{\sigma(1)}(\zeta_1)\}}\cap P''_\alpha$ be such that:

(A) (a) $h^\alpha_{\{\varepsilon_{\sigma(2)}(\zeta_1)\},\{\varepsilon_{\sigma(1)}(\zeta_1)\}}(q_{\sigma(2),\sigma(1)}\restriction N^\alpha_{\{\varepsilon_{\sigma(2)}(\zeta_1)\}})\leqq p^c_{\zeta_1}\restriction N^\alpha_{\{\varepsilon_{\sigma(1)}(\zeta_1)\}}$,

(b) $h^\alpha_{\{\varepsilon_{\sigma(1)}(\zeta_1)\},\{\varepsilon_{\sigma(2),\sigma(1)}\}}(q_{\sigma(2),\sigma(1)}\restriction N^\alpha_{\{\varepsilon_{\sigma(1)}(\zeta_1)\}})\leqq p^c_{\zeta_1}\restriction N^\alpha_{\{\varepsilon_{\sigma(2)}(\zeta_1)\}}$,

(c) $q_{\sigma(2),\sigma(1)} \Vdash_{P_\alpha} \underset{\sim}{d}_\alpha(\theta^\alpha_{\varepsilon_{\sigma(2)}(\zeta_1)}, \theta^\alpha_{\varepsilon_{\sigma(1)}(\zeta_1)})\in\{e^1_\alpha, e^2_\alpha\}$

(exist; see 2.3, in particular (ii)).

We let

$$\Gamma_2=\{h^\alpha_{\{\varepsilon_{\sigma(2)}(\zeta_1),\varepsilon_{\sigma(1)}(\zeta_1)\},\{\varepsilon_{\sigma(1)}(\zeta_1),\varepsilon_{\sigma(2)}(\zeta_2)\}}(q_{\sigma(2),\sigma(1)}) : \sigma(2)<\sigma(1)<\sigma^*\}.$$

Lastly, for each $\sigma<\sigma^*$, there is $q_\sigma\in N^\alpha_{\{\varepsilon_\sigma(\zeta_1),\varepsilon_\sigma(\zeta_2)\}}$, such that (it exists by the demands on the r^α_i's — see (6)(vi)):

(B) (a) $q_\sigma\in P''_\alpha\cap N^\alpha_{\{\varepsilon_\sigma(\zeta_1),\varepsilon_\sigma(\zeta_2)\}}$,

(b) $p^c_{\zeta_1}\restriction N^\alpha_{\{\varepsilon_\sigma(\zeta_1)\}}\geqq q_\sigma$,

(c) $p^{\zeta}_{\zeta_2} \restriction N^{\alpha}_{\{\varepsilon_\sigma(\zeta_2)\}} \leqq q_\sigma$,

(d) $q_\sigma \Vdash \underset{\sim}{d}_\alpha(\theta^{\alpha}_{\varepsilon_\sigma(\zeta_1)}, \theta^{\alpha}_{\varepsilon_\sigma(\zeta_2)}) \in \{e^1_\alpha, e^2_\alpha\}$.

Let

$$\Gamma_3 = \{q_\sigma : \sigma < \sigma^*\}.$$

Now $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ satisfies the assumptions of 2.2A (the point is that $N^{\alpha}_s \cap N^{\alpha}_t = N^{\alpha}_{s \cap t}$ for $s, t \in I$), so, as said above, we finish.

To finish the proof of 2.1 we need (note that $\Vdash_{P_\lambda}$ "$2^{\aleph_0} = \lambda$" is trivial)

2.6. Claim. $\Vdash_{P_\lambda}$ "$\lambda \to [\aleph_1]^2_3$".

Proof. For this is suffices that:

(***) for every P_λ-name $\underset{\sim}{d}$ of a function from λ to $\{0, 1, 2\}$ and $p_0 \in P$, for some α, and p_1, $\underset{\sim}{d} \restriction \alpha = \underset{\sim}{d}_\alpha$, $p_0 \leqq p_1 \in P_{\alpha+1}$ and $e^*_\alpha = 1$ and $p_1 \Vdash_{P_{\alpha+1}}$ "$\underset{\sim}{G}_{Q_\alpha}$ is unbounded in $\aleph_1$".

A way to guarantee this is to use a preliminary forcing R, the conditions are sequences $\langle P_j, \underset{\sim}{Q}_i : i < \alpha, j \leqq \alpha \rangle$ as required above, the order being an initial segment. This is a λ-complete forcing of power $\lambda^{<\lambda}$.

By the following Claim 2.8 the generic $\langle P_j, \underset{\sim}{Q}_i : i < \lambda, j \leqq \lambda \rangle$ is as required, i.e. $\Vdash_{P_\lambda}$ "$\lambda \to [\aleph_1]^2_3$".

Why? Suppose $\underset{\approx}{d}$ is an R-name of a P_α-name, $r_0 \in R$, r_0 forces: $p_0 \in \underset{\sim}{P}_\lambda$ forces ($\Vdash_{P_\lambda}$) $\underset{\approx}{d}$ forms a counter example. We can choose by induction on $\beta < \lambda$, $r_\beta \in R$, such that $\wedge_{\gamma<\beta} r_\gamma \leqq r_\beta$ and r_β forces a value $\underset{\sim}{d}^\beta$ to $\underset{\approx}{d} \restriction \beta$. Let

$$r_\beta = \langle P_i, \underset{\sim}{Q}_j : i \leqq \alpha_\beta, j < \alpha_\beta \rangle.$$

So the iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leqq \lambda, j < \lambda \rangle$ is uniquely defined and is as required in (1)–(6). Let $\underset{\sim}{d}$ be $\bigcup_{\alpha<\lambda} \underset{\sim}{d}^\alpha$ and apply 2.7 on $(H(2^\lambda)^+), \in, \lambda, \bar{Q}, \underset{\sim}{d})$ (more exactly — expand by Skolem functions and find an elementary submodel of power λ which includes $\{i : i < \lambda\}$). So we can find δ such that $\mathrm{cf}\,\delta = \aleph_1$, $\wedge_{\beta<\delta} \alpha_\beta < \delta$, for a club of $\alpha < \delta$, $\underset{\sim}{d} \restriction \alpha$ is a P_α-name, and there are $\langle N_s : s \subseteq \mathrm{cf}(\delta) \text{ finite} \rangle$, $h_{s,t}$ as in 2.7. Then we can easily find the r^α_ζ (i.e. r^α_0) above p_0 which is wlog in $N_\varnothing$.

Let $\langle \psi_\alpha : \alpha < \aleph_1 \rangle$ be increasingly continuous, $\bigcup_{\alpha<\aleph_1} \psi_\alpha = \alpha$ and for $s \in I$

$$N^\alpha_s \overset{\text{def}}{=} N_{\{\psi_\zeta : \zeta \in s\}},$$

$$\theta^\alpha_\zeta = \mathrm{Min}[(N_{\{\psi_\alpha\}} - N_\varnothing) \cap \alpha],$$

$$h^\alpha_{s,t} = h_{\{\psi_\zeta : \zeta \in s\}, \{\psi_\zeta : \zeta \in t\}}.$$

This choice defines a forcing notion Q in V^{P_α}. Now

$$\bar{Q}_\alpha = \langle P_i, \underset{\sim}{Q}_j : j \leqq \alpha, j < \alpha \rangle$$

can be continued by choosing $\underset{\sim}{Q}_\alpha$ as above and we get r^*. But if $r^* \in \underset{\sim}{G}_R$, then the iteration in $V[\underset{\sim}{G}_R]$ satisfies (***) above. So we finish.

2.7. Claim. Suppose (a) λ is measurable $> \mu$ or (b) $\mu = \aleph_0$, λ the first cardinal satisfying $\lambda \rightarrow (\omega_1)^{<\omega}_{\aleph_0}$.

If M is an algebra with μ (finitary) operations and universe λ, *then* the set of ordinals $\delta < \lambda$, satisfying the following, is closed unbounded or stationary $\subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mu^+\}$:

(*) there are $N_s (s \in I \stackrel{\mathrm{def}}{=} \{s \subseteq \mathrm{cf}(\delta) : |s| < \aleph_0\}$, $\theta_\zeta(\zeta < \delta))$ such that:

(1) For $s \in I$, N_s is a bounded subset of δ, $\| N_s \| = \mu^+$ including $\{i : i < \mu^+\}$.

(2) For $s, t \in I$, $N_{s \cap t} = N_s \cap N_t$.

(3) For $s, t \in I$, $|s| = |t|$ there is an order preserving isomorphism $h_{s,t}$ from N_s onto N_t.

(4) If $s = t \cap \alpha$, $s \in I$, $t \in I$, then N_s is an initial segment of N_t.

(5) $\langle \mathrm{Min}(N_{\{\zeta\}} - N_\varnothing) : \zeta < \mathrm{cf}(\delta) \rangle$ increases and converges to δ, and even for $s \subseteq \mathrm{cf}(\delta)$, $0 \leqq |s| < \aleph_0$, $\langle \mathrm{Min}(N_{s \cup \{\zeta\}} - N_s) : \max(s) < \zeta < \mathrm{cf}(\delta) \rangle$ increases and converges to δ.

(6) If $|s_1| = |s_2| = |s_3|$, $|s_l| = m$ then $h_{s_1,s_3} = h_{s_2,s_3} \circ h_{s_1,s_2}$.

(7) $h_{s,t} = h^{-1}_{t,s}$.

(8) $h_{s,t} \restriction N_\varnothing = \mathrm{id}$.

(9) If g is an order preserving function from s onto t, $s \in I$, $t \in I$, $s_1 \subseteq s$, $t_2 = g''(s_1)$, then $h_{s_1,t_1} \subseteq h_{s,t}$.

(10) $\theta_\zeta = \mathrm{Min}(N_{\{\zeta\}} - N_\varphi)$.

(11) We can allow the functions to have $< \mu$ places if $\mu^{<\mu} \leqq \mu^+$.

Remark. For λ measurable we really can have $\delta = \lambda$.

Proof. Easy (or see [Sh 3]).

2.8. Theorem. *Assume $\mu = \mu^{<\mu} < \lambda \leqq \chi$, λ is the first strongly inaccessible Erdös when $\mu = \aleph_0$, measurable otherwise $\lambda > \mu$ and $\chi = \chi^\mu > \lambda$.*

Then we can get the conclusion of 2.1.

We delay this to part II.

2.9. Theorem. *In* 2.1 *we can add* (ε) *if* $\mu = \aleph_0$: $\Vdash_P$ "$\mathrm{MA}_{\aleph_1}$" *and if* $\mu > \aleph_0$:

$\Vdash_P$ "*if Q is a forcing notion of cardinality μ^+, satisfying* $*[\mu]$, *and $D_i \subseteq Q$ is dense for $i < i(*) < \mathrm{cf}\,\chi$, then there is a directed $G \subseteq Q$ not disjoint to any D_i".*

PROOF. Same for $\mu = \aleph_0$; for $\mu > \aleph_0$ see [Sh 2].

2.10. DISCUSSION. We can replace, in 2.9, $\aleph_1$ by $\mu^* > \aleph_1$ (except in 2.1 (γ)) but then we need few changes — $\| N^\alpha_s \| = \mu^*$, $\{i : i \leqq \mu^*\} \subseteq N^\alpha_s$, and so in 2.7 we also consider μ^* instead of μ.

§3. Generalizations of the Todorcevic Theorem

3.1. THEOREM. *Suppose λ is regular $> \aleph_0$, $S \subseteq \lambda$ a stationary set, not reflected. Then $\lambda \not\to [\lambda]^2_\lambda$.*

EXAMPLES. $\aleph_1$, successor of regular, $(\alpha + 1)$-Mahlo not $(\alpha + 2)$-Mahlo are such cardinals. If $0^\#$ does not exist there are lots of cardinals with such S (e.g., any successor of singular cardinals).

PROOF. For each $i < \lambda$, $i \neq 0$ we choose a set $C_i \subseteq i$ such that:

(1) if i is a successor then $C_i = \{i - 1, 0\}$,

(2) if i is limit, let C_i be a closed unbounded subset of i, disjoint to S, $0 \in C_i$, successors in C_i are successors in λ.

Note: if $\delta \in S$, $0 < i < \lambda$ then $\delta \in C_i \Leftrightarrow i = \delta + 1$.

We can partition S to λ pairwise disjoint stationary subsets (of λ) S_ξ ($\xi < \lambda$) so $S = \bigcup_{\xi<\lambda} S_\xi$.

Now we define the coloring: a 2-place function d from λ to λ:

For any $\alpha < \beta$ define a $\gamma_l^+(\beta, \alpha)$, $\gamma_l^-(\beta, \alpha)$ by induction on l:

(a) $\gamma_0^+(\beta, \alpha) = \beta$, $\gamma_0^-(\beta, \alpha) = 0$,

(b) if $\gamma_l^+(\beta, \alpha)$ is defined and $> \alpha$ let $\gamma_{l+1}^+(\beta, \alpha)$ be the first member of $C_{\gamma_l^+(\beta,\alpha)}$ which is $\geqq \alpha$, and $\gamma_{l+1}^-(\beta, \alpha)$ be the last member of the closure of

$$[C_{\gamma_l^+(\beta,\alpha)} \cap \alpha],$$

[i.e. last member of $C_{\gamma_l^+(\beta,\alpha)}$ which is $< \alpha$, if there is one and α otherwise]. Next let $k = k(\beta, \alpha)$ be the first k such that $\gamma_k^+(\beta, \alpha) = \alpha$.

Note that

(*) if $\lambda > \beta > \alpha > 0$, for $m < k(\beta, \alpha)$, $\gamma_m^-(\beta, \alpha) < \alpha < \gamma_m^+(\beta, \alpha)$ and for $m = k(\beta,\alpha)$, $\gamma_m^-(\beta, \alpha) \leqq \alpha = \gamma_m^+(\beta, \alpha)$, and [$\gamma_m^-(\beta, \alpha) = \alpha$ iff α is an accumulation point of $C_{\gamma_m^+(\beta,\alpha)}$].

Suppose $\alpha < \beta$, $m \leqq k(\beta, \alpha)$; let

$$\varepsilon = \varepsilon_m(\beta, \alpha) = \mathrm{Max}\{\gamma_l^-(\beta, \alpha) + 1 : l \leqq m\},$$

then $\varepsilon \leqq \alpha + 1$ and clearly

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(**) $\gamma_l^+(\beta,\alpha)=\gamma_l^+(\beta,\xi)$, $\gamma_l^-(\beta,\alpha)=\gamma_l^-(\beta,\xi)$ when $\varepsilon \leqq \xi \leqq \alpha$ for $l \leqq m$.
We define d:

> suppose $\alpha<\beta$, let $m \leqq k(\beta,\alpha)$ be maximal such that: $\varepsilon \overset{\text{def}}{=} \varepsilon_m(\beta,\alpha)<\alpha$, $\gamma_l^-(\alpha,\varepsilon)=\gamma_l^-(\beta,\varepsilon)$ for $l \leqq m$ and $\gamma_m^+(\beta,\alpha)\in S$; now let $d(\beta,\alpha)$ be the unique ξ such that $\gamma_m^+(\beta,\alpha)\in S_\xi$.

If this does not define $d(\beta,\alpha)$ then let $d(\beta,\alpha)=0$.

Suppose $Y \subseteq \lambda$ has cardinality λ, and $\xi<\lambda$. We shall show $\xi \in \mathrm{Rang}(d \restriction Y)$.

Let M be a model with universe λ and the following three relations: $x<y$, $x \in Y$, $i \in C_j$.

Let N_i $(i<\lambda)$ be increasing continuous sequence of elementary submodels of M, $\| N_i \| <\lambda$ and $i \in N_{i+1}$. We can find a limit $\delta \in S_\xi$, such that N_δ has universe δ. Choose $\beta \in Y$, $\beta \notin N_{\delta+1}$. So $k(\beta,\delta)$ is well defined and >0. Let

$$\varepsilon \overset{\text{def}}{=} \varepsilon_{k(\beta,\delta)}(\beta,\delta).$$

We claim that ε is $<\delta$. Why? If $l<k(\beta,\delta)$ then by (*) $\gamma_l^-(\beta,\delta)<\delta$, and as δ is a limit, $\gamma_l^-(\beta,\delta)+1<\delta$. Suppose $l=k(\beta,\delta)$, $\gamma_l^-(\beta,\delta)$ is $\leqq \delta$, if equality holds then by (*) (as $\gamma_{l-1}^+(\beta,\delta)>\delta$) δ is a point of $C_{\gamma_{l-1}^+(\beta,\delta)}$, but then (as $\delta \in S$) $\gamma_{l-1}^+(\beta,\delta)$ (which is $>\delta$) cannot be a limit ordinal. Hence $\gamma_{l-1}^+(\beta,\delta)$ is a successor ordinal, so it can be only $\delta+1$. But then easily $C_{\gamma_{l-1}^+(\beta,\delta)}=\{\delta,0\}$, hence $\gamma_l^-(\beta,\delta)=0<\delta$. So even if $l=k(\beta,\delta)$, $\gamma_l^-(\beta,\delta)<\delta$ so again as δ is a limit ordinal, $\gamma_l^-(\beta,\delta)+1<\delta$. By the definition of $\varepsilon_{k(\beta,\delta)}(\beta,\delta)$ we can conclude that it is $<\delta$.

Remember $\varepsilon=\varepsilon_{k(\beta,\delta)}(\beta,\delta)$.

Let the formula $\varphi(x,y)=\varphi(x,y,\varepsilon,\gamma_l^-(\beta,\delta))_{l \leqq k(\beta,\delta)}$ say that: y is limit, $x \in Y$, $\varepsilon<y<x$, $\gamma_l^-(x,y)=\gamma_l^-(\beta,\delta)$ for $l \leqq k(\beta,\delta)$ and $\gamma_{k(\beta,\delta)}^+(x,y)=y$. This is a first order formula with parameters from N_δ and $M \models \varphi(\beta,\delta)$. As $\delta \notin N_\delta$, $\delta \in N_{\delta+1}$, $\beta \notin N_{\delta+1}$ clearly

$$M \Vdash \forall y\, \exists y'>y\, \forall x\, \exists x'>x \varphi(x',y').$$

Hence for some $\delta'<\beta'$ in N_δ, $M \models \varphi(\beta',\delta')$, $\delta'>\xi$, ε and the interval (δ',β') is not disjoint to C_δ.

By (**), we can prove by induction on $l \leqq k(\beta,\delta)$ that $\gamma_l^-(\beta,\beta')=\gamma_l^-(\beta,\delta)=\gamma_l^-(\beta,\varepsilon)$, $\varepsilon_l(\beta,\beta')=\varepsilon_l(\beta,\delta)\leqq\varepsilon$, and $\gamma_l^+(\beta,\beta')=\gamma_l^+(\beta,\delta)=\gamma_l^+(\beta,\varepsilon)$.

So $\gamma_{k(\beta,\delta)}^+(\beta,\beta')=\delta$. By the choice of (β',δ'), e.g., for $l \leqq$

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$k(\beta,\delta)$: $\gamma_l^-(\beta',\delta')=\gamma_l^-(\beta',\varepsilon)=\gamma_l^-(\beta,\delta)$, $\gamma^+_{k(\beta,\delta)}(\beta',\delta')=\delta'$. We note that $k(\delta,\beta)$ satisfies the requirement on m in the definition of d.

Now for $l=k(\beta,\delta)+1$, $\gamma_l^-(\beta,\beta')$ is the last member of the closure of $\beta'\cap C_\delta$, so as $(\delta',\beta')\cap C_\delta\neq\emptyset$, it is $>\delta'$; hence $\gamma_l^-(\beta,\beta')$ cannot be equal to $\gamma_l(\beta',\varepsilon_l(\beta,\beta'))$ as the latter is $\leqq\gamma_{l-1}(\beta',\varepsilon_{l-1}(\beta,\beta'))=\delta'$. So easily every $m'\geqq k(\beta,\delta)+1$ does not satisfy the requirement on m in the definition of d.

So in the definition of $d(\beta',\beta)$, m is $k(\alpha,\beta)$ and $\gamma_m^+(\beta',\beta)$ is δ, and as $\delta\in S_\xi$ we finish.

3.2. Observation. If λ is regular $>\aleph_0$, $S\subseteq\lambda$ stationary not reflected *then* $\lambda\not\rightarrow[\lambda;\lambda,\lambda]^{1,1,1}$.

3.2A. Explanation. Remember $\lambda\rightarrow[\lambda;\lambda;\lambda]^{1,1,1}_\mu$ means that for some 3-place function d from λ to μ, there are $\zeta<\mu$ and pairwise distinct ordinals α_i, β_i, γ_i $(i<\lambda)$ such that

$$i_1<i_2<i_3<\lambda\Rightarrow d(\alpha_i,\beta_i,\gamma_i)\neq\zeta.$$

Proof. Let $C_\alpha(\alpha<\lambda)$, $S_\xi(\xi<\lambda)$, $\gamma_l^+(\beta,\alpha)$, $\gamma_l^-(\beta,\alpha)$, $k(\beta,\alpha)$, $\varepsilon_m(\beta,\alpha)$ be as in the proof of 3.1.

We define a 3-place function from λ to λ: if $\alpha<\beta<\gamma$, and $m\leqq k(\gamma,\beta)$ is maximal such that: $\gamma_l^-(\gamma,\alpha)=\gamma_l^-(\gamma,\beta)$ for $l\leqq m$ and $\gamma_m^+(\gamma,\alpha)\in S_\xi$ *then* $d(\beta,\alpha)=\xi$, otherwise it is zero.

Let for $l=1,2,3$, $Y_l=\{y^l_\alpha:\alpha<\lambda\}\subseteq\lambda$, y^l_α increasing in α and let $\xi<\lambda$. We should find $\alpha<\beta<\gamma<\lambda$ such that $d(y^1_\alpha,y^2_\beta,y^3_\gamma)=\xi$.

Define M as in 3.1 but with the predicates $x\in Y_l$ for $l=1,2,3$ and also N_i $(i<\lambda)$ and δ will be chosen as in the proof of 3.1.

Choose $\gamma\in Y_3$, $\gamma\notin N_{\delta+1}$. Let $k=k(\gamma,\delta)$ and $\varepsilon=\varepsilon_{k(\gamma,\delta)}(\gamma,\delta)$; now as in the proof of 3.1 ε is $<\delta$. Now choose $\alpha\in N_\delta\cap Y_1$, $\alpha>\varepsilon$ and then choose $\beta\in N_\delta\cap Y_2$, such that not only $\beta>\alpha$ but $(\alpha,\beta)\cap C_\delta=\emptyset$. Now $d(\alpha,\beta,\gamma)=\xi$.

3.3. Theorem. (1) *Suppose λ is regular $>\aleph_0$, $\theta<\lambda$ regular, $S\subseteq\{\delta:\delta<\lambda,\operatorname{cf}\delta=\theta\}$ is stationary not reflecting in any inaccessible, $\sigma\leqq\theta$, and for every regular cardinal κ in the (open) interval (θ,λ), $\kappa\not\rightarrow[\theta]^{<\omega}_\sigma$,* then $\lambda\not\rightarrow[\lambda]^2_\sigma$.

(2) *Suppose $\langle\theta_i:i<i(*)\rangle$ is an increasing sequence of regular cardinals $<\lambda$, λ regular $(>\aleph_0)$ and for each i, $S_i\subseteq\{\delta:\delta<\lambda,\operatorname{cf}\delta=\theta_i\}$ is stationary not reflecting in inaccessibles $(<\lambda)$, $S_i\cap S_j=\emptyset$ for $i\neq j$ and*

$$(\forall\kappa)\,(\kappa\text{ regular}\wedge\kappa<\lambda\Rightarrow\kappa\not\rightarrow\{[\theta_j]^{<\omega}_{\sigma_j}:\theta_j<\kappa\})$$

(*see below Definition* 3.4).

Then $\lambda \nrightarrow [\lambda]^2_\sigma$ *where* $\sigma = \Sigma_{i<i(*)} \sigma_i$.

(3) *In part* (2) *if* $\lambda = \sigma^+$ *we can conclude*

$$\text{even } \lambda \nrightarrow [\lambda]^2_\lambda.$$

(4) *Suppose in* (2) *we replace*

$$(\forall \kappa)\,(\kappa \text{ regular} \wedge \kappa < \lambda \Rightarrow \kappa \nrightarrow \{[\theta_j]^{<\omega}_{\sigma_j} : \theta_j < \kappa\})$$

by

$$(\forall \kappa)\,[\kappa \text{ regular} \wedge \kappa < \lambda \Rightarrow \kappa \nrightarrow \{[\theta_j]^{<\omega}_{\sigma_j} : j \in A_\kappa\}].$$

The conclusion still holds if

(*) *for every* $j < \sigma$ *there is* $\kappa_0 < \lambda$ *regular such that*:

$$[\kappa_0 \leqq \kappa < \lambda \wedge \kappa = \mathrm{cf}\,\kappa \Rightarrow j \in A_\kappa].$$

3.4. Definition. $\kappa \nrightarrow \{[\theta_j]^{<\omega}_{\sigma_j} : j < j(*)\}$ means that there is a function F from $[\kappa]^{<\omega}[\overset{\text{def}}{=} \{w \subseteq \kappa : |w| < \aleph_0\}]$ to κ such that for every $j < j(*)$ and $A \subseteq \kappa$ of cardinality θ_j, $\{F(w) : w \in [A]^{<\omega}\}$ includes σ_j.

3.4A. Remark. Note that in 3.3(2), the condition $\kappa \nrightarrow \{[\theta_j]^{<\omega}_{\sigma_j} : \theta_j < \kappa\}$ is trivially satisfied when $\sigma_j \leqq \aleph_0$ for $j < i(*)$.

Proof of 3.3. (1) Follows by (2).

(2) For each regular $\kappa < \lambda$ there is a function g_κ from $[\kappa]^{<\omega}$ to κ exemplifying $\kappa \nrightarrow [\theta_j]^{<\omega}_{\sigma_j}$ whenever $\theta_j < \kappa$ (or, for 3.3(4): $j \in A_\kappa$). For each i, $0 < i < \lambda$ choose C_i, such that:

(α) if i is a successor ordinal, $C_i = \{i - 1, 0\}$;

(β) if i is a limit ordinal, $\mathrm{cf}\, i < i$, let C_i be a closed unbounded subset of i of order type $\mathrm{cf}(i)$, $0 \in C_i$ and $\mathrm{cf}(i) < \mathrm{Min}(C_i - \{0\})$ and an ordinal which is a successor in C_i is a successor in λ;

(γ) if i is an inaccessible cardinal C_i is a closed unbounded subset of i disjoint to

$$S \overset{\text{def}}{=} \bigcup \{S_j : j < i(*), j < i\}.$$

(δ) if i is a regular cardinal but not inaccessible, it is a successor cardinal so we can find a closed unbounded $C_i \subseteq i$ such that

$$\alpha \in C_i \wedge \alpha > 0 \Rightarrow |\alpha|^+ = i.$$

W.l.o.g. $S_i \cap (i + 1) = \varnothing$ for each i, hence S does not reflect in any inaccessible cardinal.

Now for $\alpha<\beta$, $\alpha>0$ we define by induction on l, $\gamma_l^+(\beta,\alpha)$, $\gamma_l^-(\beta,\alpha)$, and then $k(\beta,\alpha)$, $\varepsilon(\beta,\alpha)$ (note that in addition to the use of g_κ we have some minor differences from the proof of 3.1).

(A) $\gamma_0^+(\beta,\alpha)=\beta$, $\gamma_0^-(\beta,\alpha)=0$.

(B) If $\gamma_l^+(\beta,\alpha)$ is defined and $>\alpha$ and α is not a limit point of $C_{\gamma_l^+(\beta,\alpha)}$ then we let $\gamma_{l+1}^+(\beta,\alpha)$ be the minimal member of $C_{\gamma_l^+(\beta,\alpha)}$ which is $\geqq\alpha$ and let $\gamma_{l+1}^-(\beta,\alpha)$ be the maximal member of $C_{\gamma_l^+(\beta,\alpha)}$ which is $<\alpha$ (by the choice of $C_{\gamma_l^+(\beta,\alpha)}$ and the demands on $\gamma_l^+(\beta,\alpha)$ they are well defined).

Otherwise $\gamma_{l+1}^+(\beta,\alpha)$, $\gamma_{l+1}^-(\beta,\alpha)$ is undefined.

So

(B_1) (a) $\gamma_l^-(\beta,\alpha)<\alpha\leqq\gamma_l^+(\beta,\alpha)$,

(b) $\gamma_{l+1}^+(\beta,\alpha)<\gamma_l^+(\beta,\alpha)$ when both are defined.

(C) Let $k=k(\beta,\alpha)$ be the maximal number k such that $\gamma_k^+(\beta,\alpha)$ is defined (it is well defined as $\langle\gamma_l^+(\beta,\alpha):l\leqq k\rangle$ is strictly decreasing). So

(C_1) $\gamma_{k(\beta,\alpha)}^+(\beta,\alpha)=\alpha$

or $\gamma_{k(\beta,\alpha)}^+>\alpha$, $\gamma_{k(\beta,\alpha)}^+$ is a limit ordinal and α is a limit point of $C_{\gamma^+_{k(\beta,\alpha)}(\beta,\alpha)}$.

(E) Let for $m\leqq k(\beta,\alpha)$:

$$\varepsilon_m(\beta,\alpha)=\mathrm{Max}\{\gamma_l^-(\beta,\alpha)+1:l\leqq m\}.$$

Note

(E_1) (a) $\varepsilon_m(\beta,\alpha)\leqq\alpha$ (if defined) and

(b) If α is limit then $\varepsilon_m(\beta,\alpha)<\alpha$ (if defined).

(c) If $\varepsilon_m(\beta,\alpha)\leqq\xi\leqq\alpha$ *then* for every $l\leqq m$

$$\gamma_l^+(\beta,\alpha)=\gamma_l^+(\beta,\xi),\quad \gamma_l^-(\beta,\alpha)=\gamma_l^-(\beta,\xi),\quad \varepsilon_l(\beta,\alpha)=\varepsilon_l(\beta,\xi).$$

[Explanation for (c): if $\varepsilon_m(\beta,\alpha)<\alpha$ this is easy (check the definition) and if $\varepsilon_m(\beta,\alpha)=\alpha$, necessarily $\xi=\alpha$ and it is trivial.]

(d) If $l\leqq n$ then $\varepsilon_l(\beta,\alpha)\leqq\varepsilon_n(\beta,\alpha)$.

(F) Let $n(\beta,\alpha)$ be the maximal $n\leqq k(\beta,\alpha)$ such that for $l\leqq n$, $\gamma_l^-(\alpha,\varepsilon_l(\beta,\alpha))=\gamma_l^-(\beta,\alpha)$.

(G) Let $\varepsilon(\beta,\alpha)=\varepsilon_{n(\beta,\alpha)}(\beta,\alpha)$.

(G_1) For $0<\alpha<\beta<\lambda$, clearly (a) $n(\beta,\alpha)\geqq 0$ is well defined, and (b) when α is a limit $\varepsilon(\beta,\alpha)<\alpha$.

Let us partition S_i to σ_i pairwise disjoint stationary sets, $S_{i,j}$ $(j<\sigma_i)$. Now we define the function

$$d:[\lambda]^2\to\sigma=\sum_{i<i(*)}\sigma_i.$$

3.4B. Definition. We define $d(\beta,\alpha)$, $\alpha<\beta$, by cases, letting $n=n(\beta,\alpha)$.

Case 1. There are ordinals ξ, ζ, i and j such that:

(i) $\xi < \alpha < \zeta < \gamma_n^+(\beta, \alpha)$,

(ii) $\mathrm{Sup}[C_{\gamma_n^+(\beta,\alpha)} \cap \xi] = \mathrm{Sup}[C_{\gamma_n^+(\alpha,\varepsilon_n(\beta,\alpha))} \cap \xi]$,

(iii) $C_{\gamma_n^+(\beta,\alpha)} \cap [\xi, \zeta] = \varnothing$,

(iv) $\gamma_n^+(\beta, \alpha) \in S_{i,j}$.

Then let $d(\beta, \alpha) = j$.

Case 2. Not case 1, $0 < \alpha < \beta$ but $\gamma_n^+(\beta, \alpha)$, $\gamma_n^+(\alpha, \varepsilon_n(\beta, \alpha))$ are limit and the set w defined below is finite. Let

$$i(*) = \sup[C_{\gamma_n^+(\beta,\alpha)} \cap C_{\gamma_n^+(\alpha,\varepsilon_n(\beta,\alpha))}].$$

Let E be the following equivalence relation on $C_{\gamma_n^+(\beta,\alpha)} - i(*)$:

$$\gamma_1 E \gamma_2 \Leftrightarrow (\forall \gamma \in C_{\gamma_n^+(\alpha,\varepsilon_n(\beta,\alpha))})[\gamma_1 < \gamma \equiv \gamma_2 < \gamma].$$

We assume that the set

$$w \overset{\text{def}}{=} \{\gamma \in C_{\gamma_n^+(\beta,\alpha)} : \gamma > i(*), \gamma = \mathrm{Max}(\gamma/E)\}$$

is finite (really, if it is infinite, its accumulation points are in the closures of $C_{\gamma_n^+(\beta,\alpha)}$ and of $C_{\gamma_n^+(\alpha,\alpha_n(\beta,\alpha))}$).

We let $d(\beta, \alpha) = g_\kappa(w')$ if $g_\kappa(w') < \sigma$, zero otherwise, where

$$\kappa = \mathrm{cf}(\gamma_n^+(\beta, \alpha)) = |C_{\gamma_n^+(\beta,\alpha)}|$$

and w' is the image of w under the Mostowski collapse $\mathrm{Col}_{C_{\gamma_n^+(\beta,\alpha)}}$ (of $C_{\gamma_n^+(\beta,\alpha)}$).

Case 3. Not cases 1, 2.

Let $d(\beta, \alpha) = 0$.

Now suppose that $Y \subseteq \lambda$, $|Y| = \lambda$, and $d < \sigma$. We shall find $\alpha < \beta$ in Y such that $d(\beta, \alpha) = d$. Suppose $d < \sigma_i$, let M be a model with universe λ and all relevant relations and functions (countable many). Let $\langle N_i : i < \lambda \rangle$ be a sequence of elementary submodels of M, strictly increasing and continuous, $\| N_i \| < \lambda$, the universe of N_i is an ordinal, and not a successor cardinal.

Choose $\delta \in S_{i,d}$ such that $|N_\delta| = \delta$. Choose $\beta \in Y$, $\beta \notin N_{\delta+1}$. Let $n = k(\beta, \delta)$. Let $\varepsilon = \varepsilon(\beta, \delta)$ (which is $< \delta$, see (G_1)(b)).

Case A: $\gamma_n^+(\beta, \delta) = \delta$.

Now δ is singular (as it $\in S_i$) hence C_δ has order type $< \delta$, so we can easily (as in the proof of Theorem 3.1) find $\beta' < \delta$ in Y such that case 1 of the definition of $d(\beta, \beta')$ applies and $d(\beta, \beta') = d$ as required.

Case B. Not case A.

Then necessarily $\delta \in C_{\delta(*)}$ where $\delta(*) \stackrel{\text{def}}{=} \gamma_n^+(\beta, \delta)$. As $\delta \in C_{\delta(*)}$, $\mathrm{cf}\,\delta(*) > |C_\delta| \geqq \mathrm{cf}\,\delta = \theta_i$ ($\mathrm{cf}\,\delta = \theta_i$ as $\delta \in S_{i,d}$). Hence $\delta(*)$ has cofinality $> \theta_i$. So

$$C_{\delta(*)} \cap S \supseteq C_{\delta(*)} \cap S_{i,d} \neq \varnothing$$

hence (by (γ) above) $\mathrm{cf}\,\delta(*) < \delta(*)$ hence (by (β) above) $C_{\delta(*)}$ has order type $< \mathrm{Min}[C_{\delta(*)} - \{0\}] < \delta$. Hence (as $|N_\delta| = \delta$):

$$\begin{aligned} D = \{\xi \in C_{\delta(*)} : \xi < \delta, &\text{ for some } \zeta < \delta \\ &\xi = \mathrm{Sup}(C_{\delta(*)} \cap N_\zeta) = \mathrm{Max}(C_{\delta(*)} \cap N_\zeta) \\ &\text{and } \varnothing = C_{\delta(*)} \cap (|N_{\zeta+1}| - |N_\zeta|)\} \end{aligned}$$

is unbounded below δ hence has power $\geqq \mathrm{cf}\,\delta = \theta_i$. By the choice of g_κ (where $\kappa \stackrel{\text{def}}{=} \mathrm{cf}\,\delta(*)$) it is enough to show:

$\oplus$ for any $\xi_0 < \xi_1 < \cdots < \xi_p$ from D, ζ_l a witness for $\xi_l \in D$, if $\varepsilon(\beta, \alpha) < \xi_0$, then for some $\beta' \in N_{\zeta_p+1}$

$$n(\beta, \beta') = n = k(\beta, \delta), \quad \varepsilon_n(\beta, \beta') = \varepsilon_n(\beta, \alpha) = \varepsilon,$$

$C_{\gamma^+_{n(\beta',\varepsilon)}(\beta',\varepsilon)}$ satisfies: ξ_0 belongs to it, it is included in

$$\xi_0 \cup [|N_{\zeta_1+1}| - |N_{\zeta_1}|] \cup \cdots \cup [|N_{\zeta_p+1}| - |N_{\zeta_p}|]$$

and is not disjoint to any

$$|N_{\zeta_q+1}| - |N_{\zeta_q}| \qquad \text{for } q = 1, \ldots, p.$$

Now $\oplus$ is quite easy by definition of elementary submodel.

(3) Let $\langle h_\beta : \beta < \lambda \rangle$ be such that: h_β is a function from σ onto β. We now define a coloring d' (where d comes from the proof of part (2)): for $\alpha < \beta < \lambda$, $d'(\beta, \alpha) = h_\beta(d(\beta, \alpha))$.

Why is d' as required? So let $Y \subseteq \lambda$, $|Y| = \lambda$, $d' < \lambda$ and we shall find $\alpha < \beta$ in Y such that $d'(\beta, \alpha) = d'$. Let M be a model with universe λ and all relevant relations and functions. Let $\langle N_i : i < \lambda \rangle$ be a strictly increasing continuous chain of elementary submodels of M such that $|N_i|$ is an ordinal. For every pair (i, d), $i < i(*)$, $d < \sigma_i$ choose $\delta_{i,d} \in S_{i,d}$ such that $N_{\delta_{i,d}}$ has universe $\delta_{i,d}$, clearly $\gamma = \bigcup\{\delta_{i,d} : i < i(*), d < \sigma_i\}$ is $< \lambda$ so there is $\beta \in Y$ such that ($\beta > \gamma$ and) $\beta \notin N_{\gamma+1}$. Choose $d < \sigma$ such that $h_\beta(d) = d'$ (h_β is from σ onto β) and choose $i < i(*)$ such that $d < \sigma_i$. Let $\delta = \delta_{i,d}$ and continue as in the proof of part (2).

(4) Same proof as part (2).

3.5. Conclusion. (1) E.g. if $n_i < \omega$, $\bigwedge_{i<\omega} \exists m(\forall j > m)\aleph_j \nrightarrow [\aleph_{n_i}]^{<\omega}_{\aleph_i}$ then $\aleph_{\omega+1} \nrightarrow [\aleph_{\omega+1}]^2_{\aleph_{\omega+1}}$.

(2) $\lambda^+ \nrightarrow [\lambda^+]^2_{\aleph_0}$.

(3) *If* λ is an inaccessible not Mahlo then $\lambda \nrightarrow [\lambda]^2_{\aleph_0}$.

(4) $\aleph^+_{\omega_1} \nrightarrow [\aleph^+_{\omega_1}]^2_{\aleph_1}$.

Proof. (1) By 3.3(2) $\aleph_{\omega+1} \nrightarrow [\aleph_{\omega+1}]^2_{\aleph_\omega}$ (just let $m < \omega$, $g_m : {}^{\omega>}[\aleph_m] \rightarrow \aleph_m$ be such that for every $l < k < \omega$, if $\aleph_m \nrightarrow [\aleph_k]^{<\omega}_{\aleph_l}$ then for every $A \subseteq \aleph_m$ of cardinality $\aleph_k$, $\aleph_l \subseteq \{g_m(w) : w$ a finite subset of $A\}$).

The stronger version $\aleph_{\omega+1} \nrightarrow [\aleph_{\omega+1}]^2_{\aleph\omega+1}$ follows by 3.3(3).

(2) Follows by 3.3(1) applied to $S = \{\delta : \delta < \lambda^+$ is limit $> \lambda\}$.

(3) Follows by 3.3(1) applied to S, a club of λ consisting of singular ordinals.

(4) Follows by 3.3(4) if $\kappa < \aleph_{\omega_1}$ is regular; let $\kappa = \aleph_{j+1}$ and, e.g., $g_k(w) = h_j(|w|)$ where h_j is a one-to-one map from ω onto $j+1$.

3.6. Observation. Under the assumption of 3.3(1) $\lambda \nrightarrow [\lambda; \lambda; \lambda]^2_\sigma$. Similarly for 3.3(2), (3), (4).

Proof. Combine the proof of 3.2, 3.3.

3.7. Claim. Let λ be a Mahlo cardinal, S_{in} be $\mathrm{inac}(\lambda) \stackrel{\mathrm{def}}{=} \{\kappa < \lambda : \kappa$ inaccessible$\}$. For $C \subseteq \lambda$ let $\lim(C) = \{\delta \in C : \delta = \sup(\delta \cap C)\}$.

Let C_κ denote a club of κ. Then the following statements are equivalent:

(1) For every $\langle C_\kappa : \kappa \in S_{\mathrm{in}}\rangle$ for some club C^* of λ, $(\forall \delta < \lambda)$ $(\exists \kappa \in S_{\mathrm{in}})$ $[C^* \cap \delta \subseteq C_\kappa \cap \delta]$.

(2)$^-$ For some stationary $A \subseteq \lambda$ for every $\langle C_\kappa : \kappa \in S_{\mathrm{in}}\rangle$ there is a club C^* of λ such that:

$$(\forall \delta \in \lim C^* \cap A)(\exists \kappa \in S_{\mathrm{in}})[\delta \in \lim C_\kappa \wedge \sup(C^* \cap \delta - C_\kappa) < \delta].$$

(2)$^+$ Like (2)$^-$, for every stationary A.

(3)$^-$ For some stationary $A \subseteq \lambda$ for every $\langle C_\kappa : \kappa \in S_{\mathrm{in}}\rangle$ there is a club C^* of λ such that:

$$(\forall \delta \in \lim C^* \cap A)(\exists \kappa \in S_{\mathrm{in}})[\delta \in \lim C_\kappa \text{ and for every large enough } i \in C^* \cap \delta, \mathrm{Min}[C_\kappa - (i+1)] < \mathrm{Min}[C^* - (i+1)]].$$

(3)$^+$ Take (3)$^-$ for every stationary A.

Proof. (1)$\Rightarrow$(2)$^-$, (2)$^+$, (3)$^-$, (3)$^+$. Trivial (use the set of limit point of the C^* given for $\langle C_\kappa : \kappa \in S_{\mathrm{in}}\rangle$ by (1)).

$(3)^- \Rightarrow (2)^-$. Let $A \subseteq \lambda$ be a stationary set exemplifying $(3)^-$ and we shall prove that it exemplifies $(2)^-$. So let $\langle C_\kappa : \kappa \in S_{\text{in}} \rangle$ be given, and we should find a club as required in $(2)^-$.

Let C^* be a club of λ as guaranteed in $(3)^-$: i.e.

$$(\forall \delta \in C^* \cap A)(\exists \kappa \in S_{\text{in}})[\delta \in \lim C_\kappa \text{ and for every large enough } i \in C^* \cap \delta,\ \mathrm{Min}[C_\kappa - (i+1)] < \mathrm{Min}[C^* - (i+1)]];$$

let

$$C^{**} = \{\delta \in C^* : \delta \text{ a limit ordinal and } \delta = \sup(\delta \cap C^*)\}.$$

We shall prove that C^{**} satisfies the requirements in $(2)^-$.

So let $\delta \in \lim(C^{**}) \cap A$ be given. Clearly $\delta \in C^* \cap A$. So by the choice of C^*

$$(\exists \kappa \in S_{\text{in}})[\delta \in \lim C_\kappa \wedge \text{for every large enough } i \in C^* \cap \delta:\ \mathrm{Min}[C_\kappa \cap \delta - (i+1)] < \mathrm{Min}[C^* - (i+1)]]$$

and let κ exemplify it and let "for every large enough i" means $i > i(\delta)$. It suffices to prove

$$\delta \in \lim C_\kappa \wedge \mathrm{Sup}[C^{**} \cap \delta - C_\kappa] < \delta.$$

The first conjunct we already know. For the second we prove $\mathrm{Sup}(C^{**} \cap \delta - C_\kappa) \leqq i(\delta)$. So suppose $\varepsilon \in C^{**} \cap \delta$, $\varepsilon > i(\delta)$. As $\varepsilon \in C^{**}$, $\varepsilon = \bigcup_{\zeta < \mathrm{cf}\,\varepsilon} \varepsilon_\zeta$, $\varepsilon_\zeta > i(\delta)$ strictly increasing, $\varepsilon_\zeta \in C^*$, and so clearly $\varepsilon_\zeta < \delta$. By the choice of κ [using ε_ζ, as i] $[\varepsilon_\zeta, \varepsilon_{\zeta+1}] \cap C_\kappa \neq \varnothing$ for each $\zeta < \mathrm{cf}\,\varepsilon$ hence $\varepsilon \in C_\kappa$ hence $\varepsilon \notin C^{**} \cap \delta - C_\kappa$. We have proved $(C^{**} \cap \delta - C_\kappa) \subseteq i(\delta)$ as required.

$(2)^- \Rightarrow (1)$. Let $\langle C_\kappa : \kappa \in S_{\text{in}} \rangle$ be given. Choose $A \subseteq \lambda$, a stationary set exemplifying $(2)^-$. Applying $(2)^-$ to $\langle C_\kappa : \kappa \in S_{\text{in}} \rangle$, we get a club C^* of λ such that

$$(\forall \delta \in C^* \cap A)(\exists \kappa \in S_{\text{in}})[\delta \in \lim C_\kappa \wedge \mathrm{Sup}(C^* \cap \delta - C_\kappa) < \delta].$$

For a (limit) $\delta \in C^* \cap A$ let $\kappa_\delta \in S_{\text{in}}$ and $h(\delta) < \delta$ be such that $\delta \in C_{\kappa_\delta}$ and $C^* \cap \delta - C_{\kappa_\delta} \subseteq h(\delta)$. By Fodor's lemma for some stationary $B \subseteq A \cap C^*$ and $\gamma < \lambda$, $(\forall \delta \in B)[h(\delta) = \gamma]$. Let $C^{**} = C^* - \gamma$. So C^{**} is a club of λ, and for every $\delta < \lambda$ there is $\delta_1 \in B$, $\delta_1 > \delta$, so (letting $\kappa = \kappa_{\delta_1}$)

$$C^{**} \cap \delta \subseteq C^* \cap \delta - \gamma \subseteq C^* \cap \delta_1 - \gamma$$

$$= C^* \cap \delta_1 - h(\delta_1) \subseteq C_\kappa \cap \delta_1 - h(\delta_1) \subseteq C_\kappa \cap \delta_1$$

hence $C^{**} \cap \delta \subseteq C_\kappa \cap \delta$ as required.

$(2)^+ \Rightarrow (2)^-$. Trivial.

$(3)^+ \Rightarrow (3)^-$. Trivial.

3.8. REMARK. (1) If λ is weakly compact, 3.7(1) holds.

(2) If $\mu < \lambda$, λ satisfies (1) of 3.7 and P is a forcing notion satisfying the μ^+-c.c. *then* in V^P 3.6(1) still holds for λ.

3.9. CLAIM. If λ is Mahlo, $S_i \subseteq \lambda$ is stationary for $i < \lambda$, and for no inaccessible $\kappa < \lambda$ $(\forall i < \kappa)[\kappa \cap S_i$ is stationary], then 3.6(1) fail.

3.10. REMARKS. (1) Saying that the set of such κ is not stationary makes no change, as we could have shrunk the S_i's.

(2) By a result of Magidor [Mg], 3.9 implies: if λ satisfies 3.6(1) then λ is weakly compact in L.

PROOF. For $\kappa \in S_{\text{in}} = \{\kappa < \lambda : \kappa \text{ inaccessible}\}$ let $h(\kappa) < \kappa$ be minimal such that $\kappa \cap S_{h(\kappa)}$ is not stationary, and C^*_κ be a club of κ disjoint to $S_{h(\kappa)}$, and to $(h(\kappa)+1)$. Suppose $C^* \subseteq \lambda$ is a club as guaranteed for $\langle C_\kappa : \kappa \in S_{\text{in}} \rangle$ by 3.7(1). As κ is Mahlo and $S_i \cap C^*$ is unbounded in C^* for each i (being stationary) clearly $C^- = \{\delta < \lambda : \delta > 0 \text{ and for } i < \delta, S_i \cap C^* \text{ has order type } \delta\}$ is a club of λ.

Choose $\delta \in C^-$ so for some $\kappa \in S_{\text{in}}$, $C^* \cap \delta \subseteq C_\kappa \cap \delta$.

Now $C^* \cap \delta \neq \varnothing$ (as $\delta \in C^-$) hence $C_\kappa \cap \delta \neq \varnothing$, hence $h(\kappa) < \text{Min } C_\kappa < \delta$. This implies $C^* \cap S_{h(\kappa)} \cap \delta \neq \varnothing$ (as $\delta \in C^-$). However $C_\kappa \cap S_{h(\kappa)} = \varnothing$, contradiction.

3.11. THEOREM. *Suppose, for a Mahlo cardinal λ, that* 3.7(1) *fails and*:

(a) $\langle S_i : i < \sigma \rangle$ *are pairwise disjoint stationary subsets of* λ, $\sigma \leqq \lambda$;

(b) $C^+ \subseteq \lambda$ *is a club consisting of limit cardinals*;

(c) *for each inaccessible* $\kappa < \lambda$, *there is a function* $g_\kappa : [\kappa]^{<\omega} \to \sigma$ *and club* $C^\kappa \subseteq \kappa$ *such that*: if $i < \sigma$, $i < \delta < \kappa$, $\delta \in C^\kappa \cap C^+ \cap S_i$, *and* $Y \subseteq \delta$ *of cardinality* δ then $i \in \{g_\kappa(w) : w \in [Y]^{<\omega}\}$.

Then $\lambda \not\to [\lambda]^2_\sigma$.

3.12. REMARK. We can get also $\lambda \not\to [\lambda; \lambda; \lambda]^2_\sigma$. Remember $S_{\text{in}} = \{\kappa < \lambda : \kappa$ inaccessible$\}$.

PROOF OF 3.11. Like 3.3.

W.l.o.g. $[\sigma < \lambda \Rightarrow \sigma < \text{Min } C^+]$, $S_i \cap (i+1) = \varnothing$, $S_i \subseteq C^+$. As 3.6(1) fails also 3.6(3) fails for $A \subseteq S_i$ (which is stationary), so there are $\langle C^i_\kappa : \kappa \in S_{\text{in}} \rangle$ which exemplify the failure of 3.6(3) for S_i.

We now choose C_α for $\alpha < \lambda$ as follows:

(α) $C_0=\emptyset$, $C_{i+1}=\{0,i\}$;

(β) if $\alpha=\delta$ is singular ordinal (i.e. $\mathrm{cf}\,\delta<\delta$) C_δ will be a closed unbounded subset of δ of order type $\mathrm{cf}\,\delta$, $0\in C_\delta$, $\mathrm{cf}\,\delta<\mathrm{Min}(C_\delta-\{0\})$ and $(\forall i\in C_\delta)[i\neq \mathrm{Sup}(C_\delta\cap i)\Rightarrow(\exists j)(i=j+1)]$;

(γ) suppose $\alpha=\kappa\in S_{\mathrm{in}}\cap\lim(C^+)$, let C^a_κ be $(\kappa\cap C^+)\cap\bigcap_{i<\sigma}C^i_\kappa$ if $\sigma<\kappa$ and let C^a_κ be $\kappa\cap C^+\cap\{\delta<\kappa:\delta\in\bigcap_{i<\delta}C^i_\kappa\}$ if $\sigma\geqq\kappa$

[equivalently if $\sigma=\lambda$].

Let

$$C_\kappa=\{i: i=0\ or\ i=\mathrm{Sup}(i\cap C^a_\kappa)\in C^a_\kappa\ or\ (\exists j\in C^a_\kappa)[(i=j+1\wedge j>\mathrm{Sup}(j\cap C^a_\kappa)]\}.$$

[The last part in order that every limit ordinal in C_κ will be an accumulation point of C_κ].

(δ) If $\alpha=\kappa\in S_{\mathrm{in}}-C^+$, let $C_\kappa\subseteq\kappa$ be a club, $0\in C_\kappa$, $\mathrm{Min}(C_\kappa-\{0\})>\mathrm{Sup}(\kappa\cap C^+)$ and $(\forall i\in C_\kappa)[i\neq\mathrm{Sup}(C_\kappa\cap i)\Rightarrow(\exists j)(i=j+1)]$.

For the rest of the proof see the proof of 3.3.

3.13. Remark. In 3.7 the equivalence holds for each $\langle C_\kappa:\kappa\in S_{\mathrm{in}}\rangle$ separately.

3.14. Lemma. *Suppose*

$\oplus$ *$S\subseteq\lambda$ is stationary*, $[\delta\in S\Rightarrow\mathrm{cf}\,\delta=\theta]$.

S does not reflect in any inaccessible $\lambda'<\lambda$, and for every regular $\kappa\in(\theta,\lambda)$

$(*)_{\kappa,\theta}$ *there is $g_\kappa:[\kappa]^{<\omega}\to\kappa$ such that: <u>if</u> $A\subseteq\kappa$, $|A|=\theta$, A closed under g_κ,* $\mathrm{cf}(\sup A)=\theta$ <u>then</u> *A includes a club of* $(\sup A)$.

Then $\lambda\not\to[\lambda]^2_\lambda$.

3.14A. Remark. The condition $(*)_{\kappa,\theta}$ holds if there is no inner model with large enough Erdös cardinals, by Magidor covering theorem [Mg2] (i.e. if $\kappa>\theta>\aleph_0$, and in the inner model K, $\kappa\not\to(\theta)^{<\omega}_2$ then $(*)_{\kappa,\theta}$ holds).

Proof. Like the proof of 3.3; let $\bar S=\langle S_\zeta:\zeta<\lambda\rangle$ be a partition of S to pairwise disjoint stationary subsets. Now note that w.l.o.g. for each ζ,

$$F(S_\zeta)=\{\delta:\delta<\lambda,\ S_\zeta\cap\delta \text{ is a stationary subset of } \delta\}$$

is a stationary subset of λ (otherwise apply 3.1). So as $F(S_\zeta)$ has no inaccessible member we have for some θ_ζ,

$$S^1_\zeta=\{\delta:\delta<\lambda,\ \mathrm{cf}\,\delta=\theta_\zeta,\ \delta\in F(S_\zeta)]$$

is stationary.

In the definition of the coloring d, is case 1 we replace (iv) by

(iv)′ $\gamma_n^+(\beta, \alpha) \in S_j$

(and then let $d(\beta, \alpha) = j$).

In case 2, we let $\kappa = \mathrm{cf}(\gamma_n^+(\beta, \alpha))$ which is equal to $|C_{\gamma_n^+(\beta,\alpha)}|$, and we let g' be a function from the family of finite subsets of $C_{\gamma_n^+(\beta,\alpha)}$ into $C_{\gamma_n^+(\beta,\alpha)}$ such that $[C_{\gamma_n^+(\beta,\alpha)}, g'] \cong (\kappa, g_\kappa)$, and let $d(\beta, \alpha)$ be the unique ζ such that $g'(w)$ belongs to S_ζ. The rest is similar (but for the color d we use ordinals in S_d^1; provided we arrange g_κ such that

(∗) for every $A \subseteq \kappa$, $\{g_\kappa(w) : w \subseteq A \text{ finite}\}$ includes A and is closed under g_κ.

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