Sh:341

Topology and its Applications 32 (1989) 289-294
North-Holland

$\pi(X)=\delta(X)$ FOR COMPACT X

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Received 11 December 1987

We prove that if X is a compact T_2 space (and $x\in X$) and $\pi(X)=\kappa$ ($\pi\chi(x, X)=\kappa$), then there is a dense subset $Y\subset X$ (resp. a set $Y\subset X$ with $x\in \bar{Y}$) such that $d(Y)=\kappa$ (resp. $x\notin \bar{Z}$ for any $Z\subset Y$ with $|Z|<\kappa$). Previously this only has been proven for κ regular. A consequence is that the point-picking game $G_\alpha^D(X)$ is always determined if X is compact T_2.

AMS (MOS) Subj. Class.: 54A25, 54D30

Boolean algebra of regular open subsets
singular cardinals   point-picking game
compact π-weight   dense set

In [5] the authors introduced the cardinal function

$$\delta(X)=\sup\{d(Y)\colon\ Y\subset X \text{ dense in } X\},$$

and raised the following interesting problem: is $\pi(X)=\delta(X)$ for a compact T_2 space X? It was shown in [2 and 4] that every compact T_2 space X has a dense subspace Y left separated in type $\pi(X)$, hence if $\pi(X)$ is regular, then the answer to the above question is "yes", and in fact we have a dense set Y with $\delta(X)=d(Y)=\pi(X)$, i.e. sup = max. It also follows then easily that under GCH we have $\pi(X)=\delta(X)$ always. But the problem then remained whether the extra set-theoretic assumption is necessary here for singular values of $\pi(X)$? We are going to show below that in fact it is not, though the proof of this is definitely more difficult than that of the case in which $\pi(X)$ is regular.

Theorem. *If X is any compact T_2 space, then X has a dense subspace Y with $d(Y)=\pi(X)$. Consequently, $\pi(X)=\delta(X)$.*

Proof. Since this has been known if $\pi(X)$ is a regular cardinal, let us assume now that $\pi(X)=\kappa$ is singular.

* Research partially supported by OTKA grant no. 1805.

Let RO(X) be the boolean algebra of regular open subsets of X and let us put $\mathcal{R}=\mathrm{RO}(X)\setminus\{0\}$. Given a regular cardinal λ and a function $p\in{}^{\lambda}\mathcal{R}$ (i.e. $p:\lambda\to\mathcal{R}$) we put, for any $b\in\mathcal{R}$,

$$A_b(p)=\left\{\alpha\in\lambda:\exists a\in[\alpha]^{<\omega}\left(0\neq b-\bigvee_{i\in a}p(i)\leqslant p(\alpha)\right)\right\},$$

where, of course, $-$, $\vee$ and $\leqslant$ are taken in RO(X). Now, we let I_p denote the λ-complete ideal on λ generated by the family

$$\{A_b(p):\ b\in\mathcal{R}\},$$

i.e. we have

$$I_p=\{A\subset\lambda:\exists\mathcal{B}\in[\mathcal{R}]^{<\lambda}\ (A\subset\bigcup\{A_b(p):\ b\in\mathcal{B}\})\}.$$

Note that for any $\alpha\in\lambda$ we have $\alpha\in A_{p(\alpha)}(p)$, hence I_p contains all singletons and thus all subsets of λ of size $<\lambda$.

For any collection $\mathcal{B}\subset\mathrm{RO}(X)$ we shall denote by $\langle\mathcal{B}\rangle$ the (not necessarily complete) subalgebra of RO(X) generated by $\mathcal{B}$.

We now formulate a lemma that is perhaps the crux of the whole proof.

Lemma. *There are in κ cofinally many regular cardinals $\lambda<\kappa$ for which there is some $p\in{}^{\lambda}\mathcal{R}$ such that the ideal I_p on λ is proper, i.e. $I_p\neq P(\lambda)$.*

Proof. Assume, indirectly, that there is a cardinal $\nu<\kappa$ such that whenever λ is a regular cardinal with $\nu<\lambda<\kappa$, one has for every $p\in{}^{\lambda}\mathcal{R}$, $I_p=P(\lambda)$. We are going to show that then for every cardinal λ with $\nu\leqslant\lambda\leqslant\kappa$ we have

$$\text{for every } \mathcal{B}\in[\mathcal{R}]^{\lambda} \text{ there is } \mathcal{C}\in[\mathcal{R}]^{\nu} \text{ such that } \mathcal{C}<_{\pi}\mathcal{B}, \tag{$*$}$$

where $\mathcal{C}<_{\pi}\mathcal{B}$ means that for every $b\in\mathcal{B}$ there is a $c\in\mathcal{C}$ with $c\leqslant b$. Of course, then $(*)_{\kappa}$ implies $\pi(X)\leqslant\nu$, a contradiction.

Now, $(*)_{\lambda}$ is proven by induction on λ. Of course, $(*)_{\nu}$ is trivial and thus assume next that $\nu<\lambda\leqslant\kappa$ and that $(*)_{\mu}$ has been shown to hold whenever $\nu\leqslant\mu<\lambda$. The case in which λ is singular is easy:

Let $\mu=\mathrm{cf}(\lambda)<\lambda$ and $\lambda=\sum\{\lambda_i:\ i\in\mu\}$ with $\lambda_i<\lambda$ for each $i\in\mu$, and let $\mathcal{B}\in[\mathcal{R}]^{\lambda}$ where $\mathcal{B}=\bigcup\{\mathcal{B}_i:\ i\in\mu\}$ with $|\mathcal{B}_i|=\lambda_i$ for $i\in\mu$. By induction we have a $\mathcal{C}_i\in[\mathcal{R}]^{\nu}$ such that $\mathcal{C}_i<_{\pi}\mathcal{B}_i$ for each $i\in\mu$. Then

$$\left|\bigcup\{\mathcal{C}_i:\ i\in\mu\}\right|\leqslant\nu\cdot\mu<\lambda,$$

hence now by $(*)_{\nu\cdot\mu}$ there is a set $\mathcal{C}\in[\mathcal{R}]^{\nu}$ such that

$$\mathcal{C}<_{\pi}\bigcup\{\mathcal{C}_i:\ i\in\mu\}<_{\pi}\mathcal{B},$$

i.e. $(*)_{\lambda}$ holds.

Next we assume that $\nu<\lambda<\kappa$, and λ is regular, and consider

$$\mathcal{B}=\{b_{\alpha}:\ \alpha\in\lambda\}\in[\mathcal{R}]^{\lambda}.$$

Let us put, for any $\alpha \in \lambda$,

$$\mathcal{B}_\alpha = \{b_\beta : \beta \in \nu + \alpha\},$$

then we get a sequence $\langle \mathcal{B}_\alpha : \alpha \in \lambda \rangle$ of subsets of $\mathcal{R}$ which is increasing, continuous (i.e. $\mathcal{B}_\alpha = \bigcup\{\mathcal{B}_\beta : \beta \in \alpha\}$ if α is limit) and satisfies $|\mathcal{B}_\alpha| \leq \nu + |\alpha| < \lambda$ for each $\alpha \in \lambda$. In what follows we are going to say that a sequence with all of these properties is nice.

Claim. For every nice sequence $\langle \mathcal{B}_\alpha : \alpha \in \lambda \rangle$ there is a nice sequence $\langle \mathcal{B}'_\alpha : \alpha \in \lambda \rangle$ such that $\mathcal{B}'_\alpha <_\pi \mathcal{B}_{\alpha+1}$ for every $\alpha \in \lambda$.

Proof. Let us write, for $\alpha \in \lambda$,

$$\mathcal{B}_{\alpha+1} = \{q_\beta^{(\alpha)} : \beta \in \nu + \alpha\}.$$

For fixed $\beta \in \lambda$ let γ_β be the smallest ordinal α such that $q_\beta^{(\alpha)}$ is defined. (Clearly, $\gamma_\beta = 0$ if $\beta < \nu$ and $\beta + 1 = \nu + \gamma_\beta$ otherwise.)

Now we may apply our indirect assumption to the function $q_\beta \in {}^\lambda\mathcal{R}$ defined by

$$q_\beta(\alpha) = q_\beta^{(\gamma_\beta + \alpha)}$$

and conclude that I_{q_β} is not proper, i.e. there is some $\mathcal{C}_\beta \in [\mathcal{R}]^{<\lambda}$ such that

$$\bigcup\{A_b(q_\beta) : b \in \mathcal{C}_\beta\} = \lambda.$$

In particular this means that for every $\alpha \in \lambda \setminus \gamma_\beta$ there is a non-zero element c of

$$\langle \mathcal{C}_\beta \cup \{q_\beta^{(i)} : \gamma_\beta \leq i < \alpha\}\rangle$$

with $c \leq q_\beta^{(\alpha)}$. Thus we put, for any $\alpha \in \lambda$,

$$\mathcal{D}_\alpha = \bigcup\{\langle \mathcal{C}_\beta \cup \{q_\beta^{(i)} : \gamma_\beta \leq i < \alpha\}\rangle : \beta \in \nu + \alpha\} \setminus \{0\}.$$

Clearly, the sequence $\mathcal{D}_\alpha$ is continuous, and, according to our above remark, we have

$$\mathcal{D}_\alpha <_\pi \mathcal{B}_{\alpha+1}.$$

But $|\mathcal{D}_\alpha| < \lambda$, hence by induction we may find $\mathcal{E}_\alpha \in [\mathcal{R}]^\nu$ such that $\mathcal{E}_\alpha <_\pi \mathcal{D}_\alpha$.

Let us now define the sequence $\langle \mathcal{B}'_\alpha : \alpha \in \lambda \rangle$ as follows:

$$\mathcal{B}'_\alpha = \begin{cases} \bigcup\{\mathcal{E}_\beta : \beta \leq \alpha\}, & \text{if } \alpha \text{ is not limit;} \\ \bigcup\{\mathcal{E}_\beta : \beta \in \alpha\}, & \text{if } \alpha \text{ is limit.} \end{cases}$$

It is clear that $\langle \mathcal{B}'_\alpha : \alpha \in \lambda \rangle$ is a nice sequence. If α is not limit, then $\mathcal{E}_\alpha \subset \mathcal{B}'_\alpha$, hence $\mathcal{B}'_\alpha <_\pi \mathcal{D}_\alpha <_\pi \mathcal{B}_{\alpha+1}$. If, on the other hand, α is limit, then we have $\mathcal{D}_\alpha = \bigcup\{\mathcal{D}_\beta : \beta \in \alpha\}$ and $\mathcal{E}_\beta <_\pi \mathcal{D}_\beta$ for each $\beta \in \alpha$, hence

$$\mathcal{B}'_\alpha = \bigcup\{\mathcal{E}_\beta : \beta \in \alpha\} <_\pi \mathcal{D}_\alpha <_\pi \mathcal{B}_{\alpha+1},$$

i.e. the sequence $\langle \mathcal{B}'_\alpha : \alpha \in \lambda \rangle$ is as required.

Now, starting with our original nice sequence $\langle \mathcal{B}_\alpha = \{b_\beta : \beta \in \nu + \alpha\} : \alpha \in \lambda \rangle$ we repeatedly apply our claim to define nice sequences $\langle \mathcal{B}^n_\alpha : \alpha \in \lambda \rangle$ by induction on

$n\in\omega$ as follows. We put $\mathcal{B}^0_\alpha=\mathcal{B}_\alpha$ and if $\langle\mathcal{B}^n_\alpha\colon \alpha\in\lambda\rangle$ is defined we choose a nice sequence $\langle\mathcal{B}^{n+1}_\alpha\colon \alpha\in\lambda\rangle$ such that $\mathcal{B}^{n+1}_\alpha <_\pi \mathcal{B}^n_{\alpha+1}$ for $\alpha\in\lambda$.

Next we show that

$$\bigcup\{\mathcal{B}^n_0\colon n\in\omega\}<_\pi\mathcal{B}=\bigcup\{\mathcal{B}_\alpha\colon \alpha\in\lambda\}.$$

Indeed, if $b\in\bigcup\{\mathcal{B}^n_\alpha\colon \alpha\in\lambda\}$, then the minimal α with $b\in\mathcal{B}^n_\alpha$ is either 0 or successor, say $\alpha=\beta+1$, since $\langle\mathcal{B}^n_\alpha\colon \alpha\in\lambda\rangle$ is continuous, thus in the latter case, by $\mathcal{B}^{n+1}_\beta<_\pi\mathcal{B}^n_{\beta+1}$ there is an ordinal $\alpha'<\alpha$ and a $b'\in\mathcal{B}^{n+1}_{\alpha'}$ with $b'\leq b$. Using this repeatedly, and starting with any $b\in\mathcal{B}_\alpha=\mathcal{B}^0_\alpha$ we can define a decreasing sequence of ordinals that, after finitely many steps must end up with 0 and yield some $c\in\mathcal{B}^n_0$ with $c\subset b$.

In other words, we have

$$\mathcal{C}=\bigcup\{\mathcal{B}^n_0\colon n\in\omega\}<_\pi\mathcal{B},$$

while $\mathcal{C}\in[\mathcal{R}]^\nu$, i.e. $(*)_\lambda$ holds, and the proof of the lemma is thus completed. □

Let us now return to the proof of our theorem. By the lemma we may choose an increasing sequence $\langle\lambda_i\colon i\in\mu=\mathrm{cf}(\kappa)\rangle$ of regular cardinals with $\lambda_0>\mu$ such that for each $i\in\mu$ there is a function $p_i\in{}^{\lambda_i}\mathcal{R}$ for which the ideal I_{p_i} is proper. Let us now put

$$Y=\{y\in X\colon (\forall i\in\mu)\,(\{\alpha\in\lambda_i\colon y\in p_i(\alpha)\}\in I_{p_i})\};$$

we claim that Y is the required dense subset of X, i.e. $d(Y)=\kappa$.

To show that Y is dense in X it will clearly suffice to prove that $Y\cap\bar{b}\neq\emptyset$ for each $b\in\mathcal{R}$. So let $b\in\mathcal{R}$, we will then define by induction on $i\in\mu$ sets $A_i\in I_{p_i}$ such that the collection

$$\mathcal{C}=\bigcup_{i\in\mu}\{b-p_i(\alpha)\colon \alpha\in\lambda_i\setminus A_i\}\subset\mathcal{R}$$

is centered, i.e. any finite subset of $\mathcal{C}$ has non-zero meet in $\mathrm{RO}(X)$. This in turn implies that

$$\bigcup_{i\in\mu}\{\bar{b}\setminus p_i(\alpha)\colon \alpha\in\lambda_i\setminus A_i\}$$

is a centered collection of closed sets in X, hence by compactness there is a point y in its intersection. But then, for each $i\in\mu$, we have

$$\{\alpha\in\lambda_i\colon y\in p_i(\alpha)\}\subset A_i\in I_{p_i},$$

hence $y\in Y\cap\bar{b}$, and we are done.

To start our induction we put

$$A_0=A_b(p_0).$$

Now, if $\alpha_1,\ldots,\alpha_n\in\lambda_0\setminus A_0$ with $\alpha_1<\cdots<\alpha_n$, we show by induction on $l\leq n$ that

$$\bigwedge_{k=1}^{l}[b-p_0(\alpha_k)]=b-\bigvee_{k=1}^{l}p_0(\alpha_k)\neq 0,$$

using the fact that $\alpha_l \notin A_b(p_0)$ for each $l \leq n$. Hence the set $\{b - p_0(\alpha) : \alpha \in \lambda_0 \setminus A_0\}$ is indeed centered in RO(X).

Now assume that $i \in \mu \setminus \{0\}$ and that for every $j < i$ we have already defined $A_j \in I_{p_i}$ such that the family

$$\mathscr{C}_i = \bigcup_{j \in i} \{b - p_j(\alpha) : \alpha \in \lambda_j \setminus A_j\}$$

is centered in RO(X). Let $\mathscr{C}_i^* \subset \mathscr{R}$ be the family of all finite meets of elements of $\mathscr{C}_i$. Then it follows from our assumptions that

$$|\mathscr{C}_i| = |\mathscr{C}_i^*| < \lambda_i.$$

Consequently, using that I_{p_i} is λ_i-complete, we get that

$$A_i = \bigcup \{A_c(p_i) : c \in \mathscr{C}_i^*\} \in I_{p_i}.$$

Now let $\alpha_1, \ldots, \alpha_n \in \lambda_i \setminus A_i$ with $\alpha_1 < \cdots < \alpha_n$, moreover let c be the meet of any finite subset of $\mathscr{C}_i$, i.e. $c \in \mathscr{C}_i^*$. We want to show that

$$c \wedge \bigwedge_{l=1}^{n} b - p_i(\alpha_l) \neq 0.$$

This is shown by induction on $l \leq n$ in exactly the same way as it was shown for $i = 0$, but now using the fact that

$$\alpha_l \notin A_c(p_i) \cup A_b(p_i)$$

for every $l \leq n$. This, however, means that the inductive hypothesis is preserved and thus the induction defining the A_i's is completed.

Finally, to show $d(Y) = \kappa$, let $Z \subset Y$ with $|Z| < \kappa$. Then there is an $i \in \mu$ such that $|Z| < \lambda_i$. Now, for each $z \in Z$ we have $\{\alpha \in \lambda_i : z \in p_i(\alpha)\} \in I_{p_i}$, hence

$$\{\alpha \in \lambda_i : Z \cap p_i(\alpha) \neq \emptyset\} = \bigcup_{z \in Z} \{\alpha \in \lambda_i : z \in p_i(\alpha)\} \in I_{p_i}$$

as well. But I_{p_i} is proper, hence there is some $\alpha \in \lambda_i$ with $Z \cap p_i(\alpha) = \emptyset$ showing that Z is not dense in X, hence not dense in Y as well. This completes the proof of the theorem. □

In [1] the so-called point-picking game $G_\alpha^D(X)$ was introduced and studied. From our theorem we get the following result concerning this game.

Corollary. *If X is compact T_2, then the game $G_\alpha^D(X)$ is determined for any ordinal α.*

Proof. Indeed, if $\pi(X) \leq \alpha$, then player I has an obvious winning strategy. If, on the other hand $\kappa = \pi(X) > \alpha$, then by our theorem player II will win by restricting his choices to a dense set $Y \subset X$ with $d(Y) = \kappa > \alpha$. □

This result is, at least consistently, false for non-compact spaces (cf. [1, 3]). In fact non-determined spaces for the game G_ω^D exist under ◆ or $MA_{\aleph_1}$. However, it is still open whether undetermined spaces exist in ZFC.

Finally, we note that the proof of our theorem actually yields the following more general result, in which $\pi(\mathcal{R}')$ for some $\mathcal{R}' \subset \mathcal{R}$ is defined by

$$\pi(\mathcal{R}') = \min\{|\mathcal{P}|: \mathcal{P} \subset \mathcal{R} \;\&\; \mathcal{P} <_\pi \mathcal{R}'\}.$$

Theorem′. *If X is compact T_2 and $\mathcal{R}' \subset \mathcal{R} = \mathrm{RO}(X)\backslash\{0\}$ with $\pi(\mathcal{R}') = \kappa$, then there is some set $Y \subset X$ such that $Y \cap \bar{b} \neq \emptyset$ for all $b \in \mathcal{R}'$ while for every $Z \subset Y$ with $|Z| < \kappa$ there is some $b \in \mathcal{R}'$ with $Z \cap b = \emptyset$.*

To see that this is not an "idle" generalization, consider a point $x \in X$ and put

$$\mathcal{R}' = \{b \in \mathcal{R}: x \in b\}.$$

Then $\pi(\mathcal{R}') = \pi\chi(x, X)$ and thus the following corollary is obtained.

Corollary′. *For any point x in a compact T_2 space X there is a set Y such that $x \in \bar{Y}$ but for any $Z \subset Y$ with $|Z| < \pi\chi(x, X)$ we have $x \notin \bar{Z}$ (or, in short, $a(x, Y) = \pi\chi(x, X)$).*

Again (cf. [2] or [4]), this was known in case $\pi\chi(x, X)$ is a regular cardinal but is new if it is singular.

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