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# THE KAPLANSKY TEST PROBLEMS FOR $\aleph_1$-SEPARABLE GROUPS

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ABSTRACT. We answer a long-standing open question by proving in ordinary set theory, ZFC, that the Kaplansky test problems have negative answers for $\aleph_1$-separable abelian groups of cardinality $\aleph_1$. In fact, there is an $\aleph_1$-separable abelian group $M$ such that $M$ is isomorphic to $M \oplus M \oplus M$ but not to $M \oplus M$. We also derive some relevant information about the endomorphism ring of $M$.

## INTRODUCTION

Kaplansky [15, pp. 12f.] posed two test problems in order to "know when we have a satisfactory [structure] theorem. ... We suggest that a tangible criterion be employed: the success of the alleged structure theorem in solving an explicit problem." The two problems were:

(I) If $A$ is isomorphic to a direct summand of $B$ and conversely, are $A$ and $B$ isomorphic?

(II) If $A \oplus A$ and $B \oplus B$ are isomorphic, are $A$ and $B$ isomorphic?

In fact, he says ([15, p. 75]) that he invented the problems "to show that Ulm's theorem [a structure theory for countable abelian $p$-groups] could really be used". For some other classes of abelian groups, such as finitely-generated groups, free groups, divisible groups, or completely decomposable torsion-free groups, the existence of a structure theory leads to an affirmative answer to the test problems. On the other hand, negative answers are taken as evidence of the absence of a useful classification theorem for a given class; Kaplansky says "I believe their defeat is convincing evidence that no reasonable invariants exist" [15, p. 75]. Negative answers to both questions have been proven, for example, for the class of uncountable abelian $p$-groups and for the class of countable torsion-free abelian groups.

Of particular interest is the method developed by Corner (cf. [1], [2], [4]) which, by realizing certain rings as endomorphism rings of groups, provides negative answers to both test problems (for a given class) as special cases of an even more extreme pathology. More precisely, Corner's method — where applicable — yields, for any positive integer $r$, an abelian group $G_r$ (in the class) such that for any

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positive integers $m$ and $k$, the direct sum of $m$ copies of $G_r$ is isomorphic to the direct sum of $k$ copies of $G_r$ if and only if $m$ is congruent to $k$ mod $r$. (See, for example, [2] or [11, Thm. 91.6, p. 145].) Then we obtain negative answers to both test problems by letting $A = G_2$ ($\cong G_2 \oplus G_2 \oplus G_2$) and $B = G_2 \oplus G_2$.

Our focus here is on the class of $\aleph_1$-separable abelian groups (of cardinality $\aleph_1$). We will prove, in ordinary set theory (ZFC), that both test problems have negative answers by deriving the Corner pathology:

**Theorem 0.1.** *For any positive integer $r$ there is an $\aleph_1$-separable group $M = M_r$ of cardinality $\aleph_1$ such that for any positive integers $m$ and $k$, $M^m$ is isomorphic to $M^k$ if and only if $m$ is congruent to $k$ mod $r$. (Here $M^m$ denotes the direct sum of $m$ copies of $M$.)*

We do not determine the endomorphism ring of $M$, even modulo an ideal. However, we can derive a property of the endomorphism ring of $M$ which is sufficient to imply the Corner pathology: see section 3.

A group $M$ is called $\aleph_1$-separable [10, p. 184] (respectively, strongly $\aleph_1$-free) if it is abelian and every countable subset is contained in a countable free direct summand of $M$ (resp., contained in a countable free subgroup $H$ which is a direct summand of every countable subgroup of $M$ containing $H$). Obviously, an $\aleph_1$-separable group is strongly $\aleph_1$-free, so a negative answer to one of the test problems for the class of $\aleph_1$-separable groups implies a negative answer to the problem for the class of strongly $\aleph_1$-free groups. (It is independent of ZFC whether these classes are different for groups of cardinality $\aleph_1$: the weak Continuum Hypothesis ($2^{\aleph_0} < 2^{\aleph_1}$) implies that there are strongly $\aleph_1$-free groups of cardinality $\aleph_1$ which are not $\aleph_1$-separable; on the other hand, Martin's Axiom (MA) plus the negation of the Continuum Hypothesis ($\neg$CH) implies that every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is $\aleph_1$-separable; cf. [16].)

Dugas and Göbel [5] proved that ZFC + $2^{\aleph_0} < 2^{\aleph_1}$ implies that the Corner pathology exists for the class of strongly $\aleph_1$-free groups of cardinality $\aleph_1$; in fact, they showed that there is a strongly $\aleph_1$-free group $G$ whose endomorphism ring is an appropriate ring (the ring $A = A_r$ of the next section). (See also [12].) This group $G$ cannot be $\aleph_1$-separable since the endomorphism ring of an $\aleph_1$-separable group has too many idempotents. However, Thomé ([20] and [21]) showed that ZFC plus V = L (Gödel's Axiom of Constructibility) implies the Corner pathology for $\aleph_1$-separable groups of cardinality $\aleph_1$; he did this by constructing an $\aleph_1$-separable $G$ such that End($G$) is a split extension of $A$ by $I$ (in the sense of [3, p. 277]), where $I$ is the ideal of endomorphisms with a countable image.

It follows from known structure theorems for the class of $\aleph_1$-separable groups of cardinality $\aleph_1$ under the hypothesis MA + $\neg$CH that the Dugas-Göbel and Thomé realization results are *not* theorems of ZFC (cf. [7] or [17]). The fact that there *are* positive structure theorems for the class of $\aleph_1$-separable groups assuming MA + $\neg$CH or the stronger Proper Forcing Axiom (PFA) — see, for example, [8] or [18] — led to the question of whether the Kaplansky test problems could have affirmative answers for this class assuming, say, PFA. Thomé [21] gave a negative answer to the second test problem in ZFC, using a result of Jónsson [14] for countable torsion-free groups; however, till now, the first test problem as well as the Corner pathology were open (in ZFC).

Our construction of the Corner pathology involves a direct construction of the pathological group $M$ using a tree-like ladder system and a "countable template" which comes from the Corner example for countable torsion-free groups. A key role is played by a paper of Göbel and Goldsmith [13] which — while it does not itself prove any new results about the Kaplansky test problems for strongly $\aleph_1$-free or $\aleph_1$-separable groups — provides the tools for creating a suitable template from the Corner example.

## 1. The countable template

Fix a positive integer $r$. For this $r$, let $A = A_r$ be the countable ring constructed by Corner in [2]. (See also [11, p. 146].) Specifically, $A$ is the ring freely generated by symbols $\rho_i$ and $\sigma_i$ $(i = 0, 1, ..., r)$ subject to the relations

$$\rho_j \sigma_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{i=0}^{r} \sigma_i \rho_i = 1.$$

Then $A$ is free as an abelian group, and $\sigma_0\rho_0, ..., \sigma_r\rho_r$ are pairwise orthogonal idempotents. Moreover, if $M$ is a right $A$-module, then $M = M\sigma_0\rho_0 \oplus M\sigma_1\rho_1 \oplus ... \oplus M\sigma_r\rho_r$, and $M\sigma_i\rho_i \cong M$ because $\sigma_i\rho_i\sigma_i : M \to M\sigma_i\rho_i$ and $\rho_i\sigma_i\rho_i : M\sigma_i\rho_i \to M$ are inverses; therefore $M \cong M^{r+1}$.

Our construction will work for any countable torsion-free ring $A$ whose additive subgroup is free; but hereafter $A$ will denote the ring $A_r$ just defined.

Corner shows that there is a torsion-free countable abelian group $G$ whose endomorphism ring is $A$; thus $G$ is an $A$-module and hence $G \cong G^{r+1}$. Furthermore, he shows that $G^\ell$ is not isomorphic to $G^n$ if $1 \leq \ell < n \leq r$, and hence $G^m$ is not isomorphic to $G^k$ if $m$ is not congruent to $k$ mod $r$. We shall require these and further properties of $G$, which we summarize in the following:

**Proposition 1.1.** *There are countable free $A$-modules $B \subseteq H$ such that $G \cong H/B$ and $B$ is the union of a chain of free $A$-modules, $B = \bigcup_{n\in\omega} B_n$, such that $B_0 = 0$ and for all $n \in \omega$, $H/B_n$ and $B_{n+1}/B_n$ are free $A$-modules of rank $\omega$. Moreover for any positive integers $m$ and $k$, if $m$ is not congruent to $k$ mod $r$, then $G^m \oplus \mathbf{Z}^{(\omega)}$ is not isomorphic to $G^k \oplus \mathbf{Z}^{(\omega)}$.*

The main work in proving Proposition 1.1 will be done in two lemmas from [13]. For the first one, we give a revised proof (cf. [13, p. 343]). We maintain the above notation.

**Lemma 1.2.** *The group $G$ is the union, $G = \bigcup_{n\geq 1} G_n$, of an increasing chain of free $A$-modules.*

*Proof.* By [1, p. 699] $G$ is the pure closure $\langle G_1 \rangle_*$ in $\hat{A}$ of a free $A$-module $G_1 = \bigoplus_{i\in I} e_i A \oplus A$ containing $A$. Here $\hat{A}$ is the natural, or $\mathbf{Z}$-adic, completion of $A$ (cf. [1, p. 692]). We will define inductively $G_n = \bigoplus_{i\in I} e_{i,n} A \oplus A$ such that $G_n \supseteq G_{n-1}$ and for all $i \in I$, $ne_{i,n} + A = e_{i,n-1} + A$. Let $e_{i,1} = e_i$ for all $i \in I$. If $G_{n-1} \subseteq G$ has been defined for some $n > 1$, then since $A$ is dense in $\hat{A}$, there exists $e_{i,n} \in \hat{A}$ such that $ne_{i,n} + A = e_{i,n-1} + A$; say $ne_{i,n} = e_{i,n-1} + a_i$. By the definition of $G$, $e_{i,n} \in G$. We need to show that $\{e_{i,n} : i \in I\} \cup \{1\}$ is $A$-linearly independent. Suppose that $\sum_{i\in I} e_{i,n} c_i + 1 \cdot c_0 = 0$ for some $c_0, c_i \in A$. Then $\sum_{i\in I} ne_{i,n} c_i + nc_0 = 0$, so $\sum_{i\in I} e_{i,n-1} c_i + 1 \cdot (\sum_{i\in I} a_i c_i + nc_0) = 0$. By the $A$-linear independence of $\{e_{i,n-1} : i \in I\} \cup \{1\}$, we can conclude that each $c_i$ equals 0 and hence also $c_0$ equals 0. This completes the definition of $G_n$.

It remains to prove that $G \subseteq \bigcup_{n\geq 1} G_n$. Let $g \in G \setminus G_1$. For some $n > 1$, $ng \in G_1$. We claim that $g \in G_n$. Since $ng \in G_{n-1}$, $ng = \sum_{i\in I} e_{i,n-1}c_i + c_0$ for some $c_i, c_0 \in A$. Then

$$ng = \sum_{i\in I}(ne_{i,n} - a_i)c_i + c_0 = n\sum_{i\in I} e_{i,n}c_i + a'$$

for some $a' \in A$. Since $A$ is pure in $\hat{A}$, $a' = na''$ for some $a'' \in A$. Thus $g = \sum_{i\in I} e_{i,n}c_i + a'' \in G_n$. □

The second lemma is proved in [13, Lemma 2.5], generalizing [9, Lemma XII.1.4]. We state it here for the sake of completeness.

**Lemma 1.3.** *Let $G$ be a countable $A$-module which is the union, $G = \bigcup_{n\geq 1} G_n$, of an increasing chain of free $A$-modules. Then there exist countable free $A$-modules $B \subseteq H$ such that $G \cong H/B$ and $B$ is the union of a chain of free $A$-modules, $B = \bigcup_{n\geq 1} B_n$, such that for all $n \geq 1$, $H/B_n$ and $B_{n+1}/B_n$ are free $A$-modules.* □

*Proof of Proposition 1.1.* The existence of $H$, $B$, and the $B_n$ is now an immediate consequence of Lemmas 1.2 and 1.3. All that is left to show is that if $m$ is not congruent to $k$ mod $r$, then $G^m \oplus \mathbf{Z}^{(\omega)}$ is not isomorphic to $G^k \oplus \mathbf{Z}^{(\omega)}$. Since $G^m$ is not isomorphic to $G^k$, it is enough to show that $R_{\mathbf{Z}}(G^l \oplus \mathbf{Z}^{(\omega)}) = G^l$ for any $l \in \omega$. Here $R_{\mathbf{Z}}(N)$ is the $\mathbf{Z}$-radical of $N$, that is, $R_{\mathbf{Z}}(N) = \bigcap\{\ker(\varphi) : \varphi : N \to \mathbf{Z}\}$. (See, for example, [9, pp. 289f.].) To show that $R_{\mathbf{Z}}(G^l \oplus \mathbf{Z}^{(\omega)}) = G^l$ it is enough to show that $\mathrm{Hom}(G^l, \mathbf{Z}) = 0$, or, equivalently, $\mathrm{Hom}(G, \mathbf{Z}) = 0$. This follows from Observation 2.7 of [13], but we give here a self-contained argument based on the notation of Lemma 1.2. Suppose $\psi \in \mathrm{Hom}(G, \mathbf{Z})$; we can regard $\psi$ as an endomorphism of $G$ by identifying $\mathbf{Z}$ with the subgroup $\langle 1\rangle$ of $A \subseteq G$ which is generated by the unit 1 of $A$. Since the endomorphism ring of $G$ is $A$, there is $a \in A$ such that $\psi(g) = ga$ for all $g \in G$. By considering $\psi(1) = 1a = a$, we see that $a \in \langle 1\rangle$. Now consider $\psi(e_i)$ for any $e_i$; since $\psi(e_i) = e_ia$ and since $e_iA \cap \langle 1\rangle = \{0\}$, we see that $a = 0$. □

## 2. The main construction

Fix a positive integer $r$ and let $A, H, B, B_n$ and $G$ be as in Proposition 1.1. For each $n \in \omega$, fix a basis $\{b_{n,i} + B_n : i \in \omega\}$ of $B_{n+1}/B_n$ (as $A$-module). Also, fix a set of representatives $\{h_i : i \in \omega\}$ for $H/B$ where $h_0 = 0$; thus each coset $h + B$ equals $h_i + B$ for a unique $i \in \omega$.

Fix a stationary subset $E$ of $\omega_1$ consisting of limit ordinals and a ladder system $\{\eta_\delta : \delta \in E\}$. That is, for every $\delta$ in $E$, $\eta_\delta : \omega \to \delta$ is a strictly increasing function whose range is cofinal in $\delta$; we shall also choose $\eta_\delta$ so that its range is disjoint from $E$. Furthermore, we choose a ladder system which is *tree-like*, that is, for all $\delta, \gamma \in E$ and $n, m \in \omega$, $\eta_\delta(n) = \eta_\gamma(m)$ implies that $m = n$ and $\eta_\delta(l) = \eta_\gamma(l)$ for all $l < n$ (cf. [9, pp. 368, 386]).

Inductively define free $A$-modules $M_\beta$ ($\beta < \omega_1$) as follows: if $\beta$ is a limit ordinal, $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$; if $\beta = \alpha + 1$ where $\alpha \notin E$, let

$$M_\beta = M_\alpha \oplus \bigoplus_{i\in\omega} x_{\alpha,i}A.$$

If $\beta = \delta + 1$ where $\delta \in E$, define an embedding $\iota_\delta : B \to M_\delta$ by sending the basis element $b_{n,i}$ to $x_{\eta_\delta(n),i}$. Essentially $M_{\delta+1}$ will be defined to be the pushout of

$$\begin{array}{lcl} M_\delta & & \\ \uparrow \iota_\delta & & \\ B & \hookrightarrow & H \end{array}$$

but we will be more explicit in order to avoid the necessity of identifying isomorphic copies. Let $y_{\delta,0} = 0$, and let $\{y_{\delta,i} : i \in \omega \setminus \{0\}\}$ be a new set of distinct elements (not in $M_\delta$). Then define $M_{\delta+1}$ to be $\{y_{\delta,i} + u : u \in M_\delta,\ i \in \omega\}$, where the operations on $M_{\delta+1}$ extend those on $M_\delta$ and are otherwise determined by the rules

$$\begin{array}{lll} y_{\delta,i} + y_{\delta,j} = y_{\delta,k} + \iota_\delta(b) & \text{if} & h_i + h_j = h_k + b, \\ y_{\delta,i}a = y_{\delta,\ell} + \iota_\delta(b) & \text{if} & h_i a = h_\ell + b, \end{array}$$

where $b \in B$ and $a \in A$. Then there is an embedding $\theta_\delta : H \to M_{\delta+1}$ extending $\iota_\delta$ which takes $h_i$ to $y_{\delta,i}$ and induces an isomorphism of $H/B$ with $M_{\delta+1}/M_\delta$.

This completes the inductive definition of the $M_\beta$. Let $M = \bigcup_{\beta<\omega_1} M_\beta$. Note that it follows from the construction that every element of $M$ has a unique representation in the form

$$\sum_{j=1}^{s} y_{\delta_j,n_j} + \sum_{\ell=1}^{t} x_{\alpha_\ell,i_\ell} a_\ell,$$

where $\delta_1 < \delta_2 < \ldots < \delta_s$ are elements of $E$, $n_j \in \omega \setminus \{0\}$, $\alpha_\ell \in \omega_1 \setminus E$, $i_\ell \in \omega$, $a_\ell \in A$, and the pairs $(\alpha_\ell, i_\ell)$ $(\ell = 1, \ldots, t)$ are distinct.

Since $M$ is constructed to be an $A$-module, $M$ is isomorphic to $M^{r+1}$. We claim that

(†) $M$ is $\aleph_1$-separable; in fact for all $\alpha < \omega_1$, $M_{\alpha+1}$ is a free direct summand of $M$.

Assuming this for the moment, we can show that

(††) $M^m$ is not isomorphic to $M^k$ if $m$ is not congruent to $k$ mod $r$.

In brief, this is because $M^m$ and $M^k$ are not quotient-equivalent (cf. [9, pp. 251f.]), since for all $\delta \in E$, $(M_{\delta+1}/M_\delta)^m \oplus \mathbf{Z}^{(\omega)}$ is not isomorphic to $(M_{\delta+1}/M_\delta)^k \oplus \mathbf{Z}^{(\omega)}$ by Proposition 1.1. In more detail, if there is an isomorphism $\varphi : M^m \to M^k$, then there is a closed unbounded subset $C$ of $\omega_1$ such that for $\beta \in C$, $\varphi[M^m_\beta] = M^k_\beta$. Since $E$ is stationary in $\omega_1$, there exist $\delta \in C \cap E$; choose $\beta > \delta$ such that $\beta \in C$. Then $\varphi$ induces an isomorphism of $M^m_\beta/M^m_\delta$ with $M^k_\beta/M^k_\delta$. Since $M_\beta/M_{\delta+1}$ is free (of infinite rank) by (†), we can conclude that

$$\begin{aligned} (M_{\delta+1}/M_\delta)^m \oplus \mathbf{Z}^{(\omega)} &\cong (M^m_{\delta+1}/M^m_\delta) \oplus (M^m_\beta/M^m_{\delta+1}) \cong M^m_\beta/M^m_\delta \cong M^k_\beta/M^k_\delta \\ &\cong (M^k_{\delta+1}/M^k_\delta) \oplus (M^k_\beta/M^k_{\delta+1}) \cong (M_{\delta+1}/M_\delta)^k \oplus \mathbf{Z}^{(\omega)}, \end{aligned}$$

which contradicts Proposition 1.1.

We are left with the task of proving (†). First we shall show that each $M_{\alpha+1}$ is a direct summand of $M$ by defining a projection $\pi_\alpha$ of $M$ onto $M_{\alpha+1}$ (that is, $\pi_\alpha \restriction M_{\alpha+1}$ is the identity). For every integer $k$ there is a projection $\rho_k : H \to B_{k+1}$, since $H/B_{k+1}$ is free. Given $\alpha$, for each $\delta \in E$ with $\delta > \alpha$, let $k_\delta$ be the maximal integer $k$ such that $\eta_\delta(k) \leq \alpha$. For each $\delta \in E$, we let $\pi_\alpha$ act like $\rho_{k_\delta}$ on the isomorphic copy, $\theta_\delta[H]$, of $H$. More precisely, for each element $z$ of $\theta_\delta[H]$, define $\pi_\alpha(z)$ to be $\theta_\delta(\rho_{k_\delta}(\theta_\delta^{-1}(z)))$; if $\nu \notin \bigcup\{\mathrm{ran}(\eta_\delta) : \delta \in E\}$ and $\nu > \alpha$, define $\pi_\alpha(x_{\nu,i}) = 0$. Extend to an arbitrary element of $M$ by additivity; this will define a homomorphism on $M$ provided that $\pi_\alpha$ is well-defined. It is easy to see, using the unique representation of elements, that the question of well-definition reduces to showing that the definition of $\pi_\alpha(x_{\beta,i})$ for $x_{\beta,i} \in \theta_\delta[H]$ is independent of $\delta$. If $\beta \leq \alpha$, then $\pi_\alpha(x_{\beta,i}) = x_{\beta,i}$. Say $\beta > \alpha$ and $\beta = \eta_\delta(n) = \eta_\gamma(n)$; by the tree-like property, $\eta_\delta(m) = \eta_\gamma(m)$ for all $m \leq n$, and hence $k_\delta = k_\gamma$. Hence $\pi_\alpha(x_{\beta,i})$ is well-defined because $\rho_{k_\delta} = \rho_{k_\gamma}$ and thus $\theta_\delta(\rho_{k_\delta}(\theta_\delta^{-1}(x_{\beta,i}))) = \theta_\gamma(\rho_{k_\gamma}(\theta_\gamma^{-1}(x_{\beta,i})))$.

It remains to prove that each $M_\beta$ is $\aleph_1$-free (as an abelian group). Since $A$ is free as an abelian group, it suffices to show that $M_{\delta+1}$ is a free $A$-module for every $\delta \in E$. We will inductively define $S_n$ so that

$$B = \bigcup_{n\in\omega} S_n \cup \{x_{\nu,i} : \nu \in \delta \setminus (E \cup \bigcup\{\operatorname{ran}(\eta_\mu) : \mu \in E \cap (\delta+1)\}), i \in \omega\}$$

is an $A$-basis of $M_{\delta+1}$. Let $S_0$ be the image under $\theta_\delta$ of a basis of $H$. Fix a bijection $\psi : \omega \to E \cap \delta$; also, for convenience, let $\psi(-1) = \delta$. Suppose that $S_m$ has been defined for $m \leq n$ so that $\bigcup_{m\leq n} S_m$ is $A$-linearly independent and generates $\bigcup\{\theta_{\psi(m)}[H] : -1 \leq m < n\}$. Let $\gamma = \psi(n)$, and let $k = k_n$ be maximal such that $\eta_\gamma(k) = \eta_{\psi(m)}(k)$ for some $-1 \leq m < n$. Notice that $\{x_{\eta_\gamma(\ell),i} : \ell \leq k, i \in \omega\}$ is contained in the $A$-submodule generated by $\bigcup_{m\leq n} S_m$. Since $H/B_{k+1}$ is $A$-free, we can write $H = B_{k+1} \oplus C_k$ for some $A$-free module $C_k$ ($= \ker(\rho_k)$); let $S_{n+1}$ be the image under $\theta_\gamma$ of a basis of $C_k$. This completes the inductive construction. One can then easily verify that $B$ is an $A$-basis of $M_{\delta+1}$; indeed, the fact that $\bigcup_{m\leq n} S_m$ is $A$-linearly independent can be proved by induction on $n$, using the unique representation of elements of $M$ to show that if $\sum_{i=1}^r z_i a_i \in \left\langle \bigcup_{m\leq n} S_m \right\rangle$, where $z_1, ..., z_r$ are distinct elements of $S_{n+1}$, then $a_i = 0$ for all $i = 1, ..., r$.

## 3. The endomorphism ring of $M$

While we cannot show that $\operatorname{End}(M)$ is a split extension of $A$ by an ideal, we can obtain enough information about $\operatorname{End}(M)$ to imply the negative results on the Kaplansky test problems. (A similar idea is used in [19, p. 118].)

The ring $A$ is naturally a subring of $\operatorname{End}(M)$. We say that $A$ is *algebraically closed* in $\operatorname{End}(M)$ when every finite set of ring equations with parameters from $A$ (i.e., polynomials in several variables over $A$) which is satisfied in $\operatorname{End}(M)$ is also satisfied in $A$.

**Proposition 3.1.** *If $A = A_r$ is as in section 1, and $A$ is algebraically closed in* $\operatorname{End}(M)$*, then for any positive integers $m$ and $k$, $M^m$ is isomorphic to $M^k$ if and only if $m$ is congruent to $k$ mod $r$.*

*Proof.* Since $M$ is an $A$-module, $M \cong M^{r+1}$. If $M^\ell$ is isomorphic to $M^n$ where $1 \leq \ell < n \leq r$, then $\sum_{i=1}^{\ell} M\sigma_i\rho_i \cong \sum_{i=1}^{n} M\sigma_i\rho_i$, so by Lemma 2 of [2], there are elements $x$ and $y$ of $\operatorname{End}(M)$ such that $xy = \sum_{i=1}^{\ell} \sigma_i\rho_i$ and $yx = \sum_{i=1}^{n} \sigma_i\rho_i$. So by hypothesis, such elements $x$ and $y$ exist in $A$. We then obtain a contradiction as in [2, p. 45]. □

**Proposition 3.2.** *If $M$ is defined as in section 2, then $A$ is algebraically closed in* $\operatorname{End}(M)$.

*Proof.* For any $\sigma \in \operatorname{End}(M)$, there is a closed unbounded subset $C_\sigma$ of $\omega_1$ such that for all $\alpha \in C_\sigma$, $\sigma[M_\alpha] \subseteq M_\alpha$. For any $\sigma_1, ..., \sigma_n$ in $\operatorname{End}(M)$, choose $\alpha < \beta$ in $C_{\sigma_1} \cap ... \cap C_{\sigma_n}$ so that also $\alpha \in E$. Then each $\sigma_i$ induces an endomorphism, also denoted $\sigma_i$, of $M_\beta/M_\alpha$. The endomorphism ring of $M_\beta/M_\alpha$ is $\mathrm{End}(G \oplus \mathbf{Z}^{(\omega)})$, and restriction to $G$ defines a natural homomorphism, $\pi$, of $\mathrm{End}(G \oplus \mathbf{Z}^{(\omega)})$ onto $\mathrm{End}(G) \cong A$, because $\mathrm{Hom}(G, \mathbf{Z}^{(\omega)}) = 0$. If $\sigma_i = a \in A$ (regarded as an element of $\mathrm{End}(M)$), then $\pi(a) = a$. Hence if $\sigma_1, ..., \sigma_m$ satisfy some ring equations over $A$, then so do $\pi(\sigma_1), ..., \pi(\sigma_m)$. □

Propositions 3.1 and 3.2 provide an alternative proof of (††).

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