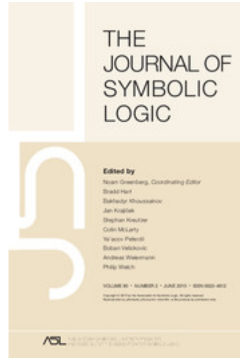

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Matthew Foreman, Menachem Magidor and Saharon Shelah

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$0^{\#}$ AND SOME FORCING PRINCIPLES

MATTHEW FOREMAN, MENACHEM MAGIDOR AND SAHARON SHELAH

§0. Introduction. It has been considered desirable by many set theorists to find maximality properties which state that the universe has in some sense "many sets". The properties isolated thus far have tended to be consistent with each other (as far as we know). For example it is a widely held view that the existence of a supercompact cardinal is consistent with the axiom of determinacy holding in $L(\mathbf{R})$. This consistency has been held to be evidence for the truth of these properties. It is with this in mind that the first author suggested the following:

Maximality Principle If $\mathbf{P}$ is a partial ordering and $G \subseteq \mathbf{P}$ is a V-generic ultrafilter then either

a) there is a real number $r \in V[G]$ with $r \notin V$, or

b) there is an ordinal α such that α is a cardinal in V but not in $V[G]$.

This maximality principle applied to garden variety partial orderings has startling results for the structure of V.

For example, if for some $\kappa \geq \aleph_1$, $\kappa^{\underline{\kappa}} = \kappa$, then $\mathbf{P} = \langle\{p: p \subseteq \kappa, |p| < \kappa\}, \subseteq\rangle$ neither adds a real nor collapses a cardinal. Thus from the maximality principle we can deduce that the G. C. H. fails everywhere and there are no inaccessible cardinals. (Hence this principle contradicts large cardinals.) Similarly one can show that there are no Suslin trees on any cardinal κ. These consequences help justify the title "maximality principle".

Since the maximality principle implies that the G. C. H. fails at strong singular limit cardinals it has consistency strength at least that of "many large cardinals". (See [M].) On the other hand it is not known to be consistent, relative to any assumptions. Many of its consequences however have been shown to be consistent (see e.g. [FW]).

It is also not clear exactly what the consequences of this principle are. For example the first author has made two (contradictory) conjectures:

Conjecture 1. If α is an ordinal then the maximality principle is consistent with $2^{\aleph_0} \geq \aleph_\alpha$.

Conjecture 2. The maximality principle implies $2^{\aleph_0} = \aleph_{71}$.

The first conjecture seems rather more likely. The point, however, is that it is not known how to add reals and preserve the maximality principle.

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There have been many partial results in the direction of showing that the maximality principle is consistent. In this paper we prove the following theorem:

THEOREM. *If* $0^{\#}$ *exists and* $\mathbf{P}$ *is a nonatomic partial ordering in* L *then forcing with* $\mathbf{P}$ *over* V *adds a real.*

Thus if we have enough large cardinals then forcing over V with a set of conditions from a sufficiently structured inner-model adds a real. (Note here we do not need clause b) of the maximality principle.)

The theorem in its present form is due to the third author. The first two authors originally proved: "If for all n, $0^{\#\#\#\#\cdots\#}$ (n sharps) exists then every partial ordering in L adds a real." The third author then improved the result to the one given.

Let the weak maximality principle be the following statement: If $\mathbf{P}$ is a partial order that does not add a real and does not collapse $\aleph_2$ and $\langle D_\alpha : \alpha < \aleph_1 \rangle$ is a collection of $\aleph_1$ dense sets in $\mathbf{P}$, then there is a filter $G \subseteq \mathbf{P}$ such that $G \cap D_\alpha \neq \varnothing$ for all $\alpha < \aleph_1$.

The following theorem occurs in a forthcoming paper.

THEOREM. *If* ZFC + *"there is a supercompact cardinal" is consistent then so is* ZFC + *"weak maximality principle".*

We now make some miscellaneous remarks establishing some conventions for the paper. If $0^{\#}$ exists then there is a closed unbounded class I of indiscernibles for L such that every $x \in L$ is the result of applying some term τ in the language of L to some indiscernibles $i_1, \ldots, i_n$. Since we have two kinds of "terms" to worry about, namely terms for elements of L and terms for elements of a Boolean extension, we will try to call terms for elements of L "terms" and forcing terms "names". So every element of L is denoted by a *term* applied to some indiscernibles and every element of a forcing extension of V by $\mathbf{P}$ is denoted by a *name* in $V^{\mathbf{P}}$.

Let $I = \langle i_\alpha : \alpha \in \mathrm{OR} \rangle$ be the canonical closed unbounded set of indiscernibles for L. When we speak of elements of $i_\alpha \in I$ we will be more concerned about their ordinal subscripts than the indiscernible itself. Accordingly we will use the notation $\underline{\alpha}$ for the αth indiscernible i_α. Also $\underline{\vec{\alpha}}$, where $\vec{\alpha} = (\alpha_0, \ldots, \alpha_n)$, will be the n-tuple $(i_{\alpha_0}, \ldots, i_{\alpha_n})$.

To avoid east-west conflict about the notation $p \leq q$ for p, q elements of a partial ordering Q we will use the notation $p \Vdash q$ to mean that p is stronger than q or p extends q. Similarly we will use Boolean algebra and partial ordering notation interchangeably, $\|\phi(\tau)\| \geq p$ being equivalent to $p \Vdash \phi(\tau)$. Whenever $\| \ \|$ occurs it is meant to be the Boolean value taken in V. If we want to take the Boolean value in a particular partial ordering $\mathbf{P}$ we will use $\| \ \|_{\mathbf{P}}$. Occasionally we will use the Boolean operation $\tilde{p} = 1 \sim p$, but this operation is absolute so it will not cause ambiguities. If $W \in V^{\mathbf{P}}$ is a name for an element of V then we will use $p \| W$ to mean p determines W. ($p \not\| W$ means p does not determine W.) If ϕ is a formula and τ is a forcing name then $p \| \phi(\tau)$ means either $p \Vdash \phi(\tau)$ or $p \Vdash \neg \phi(\tau)$. All of our partial orderings will be separative. We will denote the trivial Boolean algebra corresponding to the trivial partial ordering by $[\varnothing]$.

If $q \in Q$, let $\hat{q} = \{p \in Q : q \Vdash p\}$. We will use $|X|$ for the cardinality of X. For the cardinality of X in the extension $V^{\mathbf{P}}$ we will use $|X|^{V^{\mathbf{P}}}$, or $|X|^{\mathbf{P}}$ if V is clear from the context.

We will use standard iteration notation, $\mathbf{P} * Q$ having the usual meaning. If f is a 1-1 order preserving function from $\mathbf{P}$ into Q, $f : \mathbf{P} \hookrightarrow Q$ means that whenever $A \subseteq \mathbf{P}$ is a maximal antichain then $f''A \subseteq Q$ is a maximal antichain. If τ is a forcing name for an element of $V^{\mathbf{P}}$, we will use $\tau^{\mathbf{P}}$, $\tau^{V^{\mathbf{P}}}$ and $\tau^{V[G]}$ variously for its realization in the generic extension.

As is standard practice, $H(\lambda)$ will be the collection of sets of hereditary cardinality $<\lambda$. If X is a set and λ is a cardinal then $[X]^{\lambda}$ and $[X]^{<\lambda}$ will denote the collection of subsets of X of cardinality λ and cardinality $<\lambda$ respectively.

If $\alpha > \beta$ are ordinals, $\alpha - \beta$ is the ordinal γ such that $\beta + \gamma = \alpha$. The residue class of α modulo β, written $\alpha(\operatorname{mod}\beta)$ is the $\gamma < \beta$ such that for some δ, $\alpha = \beta \cdot \delta + \gamma$. If $\alpha < \beta$ then $\alpha(\operatorname{mod}\beta) = \alpha$. Similarly if $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ is a sequence of ordinals then $\vec{\alpha}(\operatorname{mod}\beta) = \langle \alpha_0(\operatorname{mod}\beta), \alpha_1(\operatorname{mod}\beta), \ldots, \alpha_{n-1}(\operatorname{mod}\beta)\rangle$. If $\vec{\alpha}$ is a sequence, $l(\vec{\alpha}) =$ length of $\vec{\alpha}$. If $\vec{\alpha}$ is a sequence of ordinals the β neighborhood of $\vec{\alpha}$ is all ordinals γ such that for some α_i, $\alpha_i - \gamma < \beta$ or $\gamma - \alpha_i < \beta$. We use $\vec{\alpha}$, $\vec{\beta}$, etc. for finite sequences of ordinals.

If $\mathfrak{A}$ and $\mathfrak{L}$ are structures in the same language, then $\mathfrak{A} \prec \mathfrak{L}$ will mean that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{L}$. If $\vec{\alpha}$, $\vec{\beta}$ are sequences of ordinals, $l(\vec{\beta}) = n$, then the type of $\vec{\beta}$ over $\vec{\alpha}$ or $\text{type}_{\vec{\alpha}}\,\vec{\beta}$ will mean the collection of atomic formulas in n variables and constants α_i completely describing the order relationship of the ordinals $\vec{\alpha} \cup \vec{\beta}$.

§1. Let $\mathbf{P} \in L$ and assume $0^{\#}$ exists.

Our strategy will be to produce a real in steps. We will have two major cases. In each case we will break $\mathbf{P}$ into subsets $\langle \mathbf{P}_n : n \in \omega \rangle$, each defined by a single term τ_n. The behavior of our generic object on $\mathbf{P}_n$ will be determined by a term σ_n and a finite set of ordinals W_n. Hence we get either a new real or a new ω-sequence of ordinals. If the new ω-sequence of ordinals is a subset of a countable set in V, then again we get a new real. Otherwise we show that the residues modulo the ordinal ω^{ω} of this new countable sequence form a new real.

We start with some elementary combinatorial lemmas.

DEFINITION. Say that $\vec{\alpha}\, E^n_{\vec{\gamma}}\, \vec{\beta}$ iff

a) $l(\vec{\alpha}) = l(\vec{\beta})$,
b) $\text{type}_{\vec{\gamma}}\,\vec{\alpha} = \text{type}_{\vec{\gamma}}\,\vec{\beta}$,
c) $\alpha_i - \alpha_j < \omega^n$ iff $\beta_i - \beta_j < \omega^n$,
d) $\alpha_i - \alpha_j = \beta_i - \beta_j \,(\operatorname{mod} \omega^n)$,
e) $\alpha_i - \gamma_l < \omega^n$ iff $\beta_i - \gamma_l < \omega^n$,
f) $\alpha_i - \gamma_l = \beta_i - \gamma_l \,(\operatorname{mod} \omega^n)$,
g) $\alpha_i - 0 < \omega^n$ iff $\beta_i - 0 < \omega^n$, and
h) $\alpha_i = \beta_i \,(\operatorname{mod} \omega^n)$.

The following lemma is easy to see and well known:

LEMMA 1. *Suppose $\vec{\alpha}\, E^n_{\vec{\gamma}}\, \vec{\beta}$. Then for every finite sequence of ordinals $\vec{\delta}$ there is a sequence $\vec{\rho}$ of ordinals such that $\vec{\alpha} \cup \vec{\delta}\, E^{n-1}_{\vec{\gamma}}\, \vec{\beta} \cup \vec{\rho}$.*

We now need some definitions:

If $n \in \omega$, a *block* in n is a sequence of consecutive integers in n. Iff B_1, B_2 are blocks, then $B_1 < B_2$ iff $\sup B_1 < \inf B_2$.

Let $\langle W_i : i < \delta \rangle$ be a collection of increasing sequences of ordinals of fixed length n. The $\langle W_i : i < \delta \rangle$ form a *generalized Δ-system* iff there is a partition of n into blocks $B_1 < B_2 < \cdots < B_k$ such that for each block B_j either

a) $W_i \upharpoonright B_j = W_{i'} \upharpoonright B_j$ for all $i, i' \in \delta$, or
b) if $i < i' \in \delta$ then $\sup W_i \upharpoonright B_j < \inf W_{i'} \upharpoonright B_j$.

Note that an ordinary Δ-system is a generalized Δ-system with two blocks.

If $\langle W_i : i < \delta \rangle$ form a generalized Δ-system, the *kernel* of the $\langle W_i : i < \delta \rangle$ is $\bigcup \{W_i \restriction B_j$: for all $i < \delta$, $W_i \restriction B_j$ is constant$\}$.

The following is a folk lemma and corollary.

LEMMA 2. *Let $\langle W_i : i < \delta \rangle$ be a collection of increasing sequences of ordinals of fixed length n. Then there are a cofinal set $S \subseteq \delta$ and a $1 \leq k \leq n$ such that either*

a) *for all $i, i' \in S$*

$$W_i \restriction k = W_{i'} \restriction k,$$

or

b) *for all $i < i' \in S$*

$$\sup W_i \restriction k < \inf W_{i'} \restriction k \quad \textit{and} \quad \sup W_{i'} \restriction k < \inf W_i \restriction (n - k).$$

PROOF. It is enough to handle the case where δ is a regular cardinal. Suppose a) does not hold. Then we can assume that the first elements of $\langle W_i : i < \delta \rangle$ form an increasing sequence of ordinals.

For each $i < \delta$, let k_i be the largest integer such that there is an i', $i < i' < \delta$, with the first element of $W_{i'} >$ the k_ith element of W_i.

There is a cofinal set $T \subseteq \delta$ such that for all $i, i' \in T$, $k_i = k_{i'}$. Since δ is regular, and the first elements of $\langle W_i : i \in \gamma \rangle$ are increasing we can choose a cofinal set $S \subseteq T$ such that if $i < i' \in S$, then $\sup W_i \restriction k_i < \inf W_{i'}$. We claim that for all $i < i' \in S$

$$\sup W_{i'} \restriction k_i < \inf W_i \restriction n - k_i.$$

Otherwise for some $i < i'$ the $(k_i + 1)$st element of W_i satisfies $w_{k_i+1} \leq \sup W_{i'} \restriction k_i$. Take $j > i'$, $j \in S$. Then $\inf W_j > \sup W_{i'} \restriction k_i \geq w_{k_i+1}$. This contradicts the definition of k_i as the largest integer such that there is a $j > i$ with $\inf W_j > \sup W_i \restriction k_i$. □

COROLLARY 3. *If $\langle W_i : i < \delta \rangle$ is a collection of increasing sequences of ordinals of fixed length, then there is an $S \subseteq \delta$, S cofinal in δ, such that the $\langle W_i : i \in S \rangle$ form a generalized Δ-system.*

PROOF. Repeatedly apply Lemma 2 to get the blocks for the Δ-system.

We return to the proof of Theorem 1. Let $\mathbf{P}$ be a partial ordering in L. Then $\mathbf{P} = \tau(\vec{\underset{\sim}{\alpha}}^*)$ for a term τ and some indiscernibles $\vec{\underset{\sim}{\alpha}}^*$. Well-order the terms over L in order type ω. Look at all possible k-types Γ of finite sequences of ordinals over α^*. Enumerate all of the possibilities for terms and types, $\langle (\tau_n, \Gamma_n) : n \in \omega \rangle$. Let $\mathbf{P}_n = \{p \in \mathbf{P} : p = \tau_n(\vec{\underset{\sim}{\alpha}})$ where $\vec{\alpha}$ has type Γ_n over $\alpha^*\}$. Then $\mathbf{P} = \bigcup_{n \in \omega} \mathbf{P}_n$. Let G be a V-generic object for $\mathbf{P}$. Then $G = \bigcup_{n \in \omega} G \cap \mathbf{P}_n$. Hence the sequence $\langle G \cap \mathbf{P}_n : n \in \omega \rangle \notin V$.

LEMMA 4. *Let $S \subseteq \mathbf{P}$, $S \in V$. Let $p \in \mathbf{P}$ and $X \in V$ be such that $p \Vdash X \cap S = G \cap S$. Then $p \Vdash G \cap S = \hat{p} \cap S$.*

PROOF. Recall $\hat{p} = \{q \in \mathbf{P} : p \Vdash q\}$. Since $G \supseteq \hat{p}$, $p \Vdash G \cap S \supseteq \hat{p} \cap S$. On the other hand if $p \Vdash q \in G \cap S$, then $p \Vdash q \in G$, so $p \Vdash q$. □

COROLLARY 5. *Suppose $S \subseteq \mathbf{P}$, $S \in V$ and $p \Vdash$ for no set $X \in L$ is $G \cap S = X \cap S$. Then $p \Vdash$ for no set $X \in V$ is $G \cap S = X \cap S$.*

We break our proof of Theorem 1 into two cases:

Case I. For each n, $\| \exists X \in L, G \cap \mathbf{P}_n = X \cap \mathbf{P}_n \| = 1$.

Case II. Otherwise.

For each $q \in \mathbf{P}$, let σ_q be the least term of minimal length, such that there is a sequence $\underline{\vec{\alpha}} \in I$ with $q = \sigma_q(\underline{\vec{\alpha}})$. Let $\underline{\vec{\alpha}}_q$ be the lexicographically least such sequence.

We do Case I first:

For each n, choose $q \in G$ such that $\underline{\vec{\alpha}}_q$ has minimal length and $\vec{\alpha}_q$ is lexicographically minimal and $\overline{\alpha_q}$ is least with the property that $q \Vdash \hat{q} \cap \mathbf{P}_n = G \cap \mathbf{P}_n$. Let $\sigma_n = \sigma_q$ and $W_n = \underline{\vec{\alpha}}_q$. If the sequence $\langle \sigma_n : n \in \omega \rangle \notin V$ then we have added a new real to V, so we can assume that $\langle \sigma_n : n \in \omega \rangle \in V$. Further, if there are a countable set $A \in V$ and a $p \in \mathbf{P}$ such that $p \Vdash \bigcup W \subseteq A$, then we also have added a new real. Hence we can also assume that there is no such A. (Note that at this point we have already shown that there is a new ω-sequence of ordinals in $V[G]$.)

Claim (*). a) *Suppose* $r = \sigma_r(\underline{\vec{\alpha}}_r)$ *and* $s = \sigma_r(\underline{\vec{\alpha}}_s)$ *and* $\vec{\alpha}_r E^2_{\vec{\alpha}^*} \vec{\alpha}_s$. *Then* $r \| G \cap \mathbf{P}_k$ *iff* $s \| G \cap \mathbf{P}_k$.

b) *Suppose* q *and* q' *are conditions such that* $\sigma_q = \sigma_{q'}$ *and* $q \Vdash W_k = \vec{\alpha}$. *If* $\vec{\alpha}_q \cup \vec{\alpha} E^4_{\vec{\alpha}^*} \vec{\alpha}_{q'} \cup \vec{\alpha}'$ *then* $q' \Vdash W_k = \vec{\alpha}'$.

PROOF. a) If $r \nparallel G \cap \mathbf{P}_k$ then there is a $p = \tau_k(\underline{\vec{\alpha}})$ such that $r \not\Vdash p$ and $r \not\Vdash \tilde{p}$. Since $\vec{\alpha}_r E^2_{\vec{\alpha}^*} \vec{\alpha}_s$ there is a $\vec{\beta}$ such that $\text{type}_{\vec{\alpha}^*} \vec{\alpha}_r \cup \vec{\alpha} = \text{type}_{\vec{\alpha}^*} \vec{\alpha}_s \cup \vec{\beta}$. But then by indiscernibility $s \not\Vdash \tau_k(\vec{\beta})$ and $s \not\Vdash \tau_k(\tilde{\vec{\beta}})$. Clause a) follows by symmetry.

b) Suppose $q \Vdash W_k = \vec{\alpha}$ and $\vec{\alpha}_q \cup \vec{\alpha} E^4_{\vec{\alpha}^*} \vec{\alpha}_{q'} \cup \vec{\alpha}'$. If $q' \not\Vdash W_k = \vec{\alpha}'$ then there are a term σ and a sequence of ordinals $\vec{\beta}$ such that either $l(\vec{\beta}) < l(\vec{\alpha}')$ or $\vec{\beta}$ is lexicographically less than $\vec{\alpha}'$ and

i) $\sigma(\vec{\beta})$ is compatible with q', and

ii) $\sigma(\vec{\beta}) \| G \cap \mathbf{P}_k$.

By Lemma 1, there is a $\vec{\gamma}$ such that $\vec{\alpha}_q \cup \vec{\alpha} \cup \vec{\gamma} E^3_{\vec{\alpha}^*} \vec{\alpha}_{q'} \cup \vec{\alpha}' \cup \vec{\beta}$. Hence by a) and indiscernibility,

i) $\sigma(\vec{\gamma})$ is compatible with q, and

ii) $\sigma(\vec{\gamma}) \| G \cap \mathbf{P}_k$.

But this is a contradiction since $\vec{\gamma}$ stands in the same relation to $\vec{\alpha}$ as $\vec{\beta}$ does to $\vec{\alpha}'$. □

Define a $\mathbf{P}$-name for an ω-sequence of ordinals less than the ordinal $\omega^\omega = \sup_{n \in \omega} \omega^n$ by $\dot{x} = \langle W_k (\text{mod}\, \omega^\omega) : k \in \omega \rangle$. We claim that this yields a new real.

It is enough to see that for each $p \in \mathbf{P}$ there are $q, q' \Vdash p$ such that q and q' give conflicting information about $\dot{x}$.

Let $N_p = \{\beta : \text{there is an } \alpha \in \vec{\alpha}_p \cup \vec{\alpha}^* \;\; \alpha - \beta < \omega^6 \text{ or } \beta - \alpha < \omega^6\}$ be the ω^6 neighborhood of $\vec{\alpha}_p \cup \vec{\alpha}^*$. N_p is a countable set of ordinals in V, hence there are a $q \Vdash p$ and a $k \in \omega$ and an ordinal $j \notin N_p$ such that $q \Vdash j \in W_k$. We can assume that $q \| W_k$.

Suppose $q = \sigma_q(\underline{\vec{\alpha}}_q)$ and $q \Vdash W_k = \vec{\rho}$. Consider $\underline{\vec{\alpha}}_q \cup \vec{\rho}$. Let $\beta_1, \ldots, \beta_l$ be a list of the ordinals in $\underline{\vec{\alpha}}_q \cup \vec{\rho}$ that are not in N_p. Let $\vec{\alpha}'$ and $\vec{\rho}'$ be the result of replacing $\beta_1, \ldots, \beta_l$ by $\beta_1 + \omega^5, \beta_2 + \omega^5, \ldots, \beta_l + \omega^5$. Let $q' = \sigma_q(\vec{\alpha}')$. Then, since $\vec{\alpha}' \cup \vec{\rho}' \; E^4_{\vec{\alpha}^* \cup \vec{\alpha}_p} \vec{\alpha}_q \cup \vec{\rho}$, $q' \Vdash W_k = \vec{\rho}'$ and $q' \Vdash p$. But $\rho' \neq \rho \,\text{mod}(\omega^\omega)$, so q and q' give different information about x. This completes the proof in Case I.

Case II. For some p and n, $p \Vdash$ for no $X \in L$, $X \cap \mathbf{P}_n = G \cap \mathbf{P}_n$.

PROOF. Well-order $\mathbf{P}_n$ lexicographically by the ordinals appearing in the canonical representation of each condition. So $\mathbf{P}_n = \langle p_\beta : \beta < \mu \rangle$.

Let $q^* \Vdash p$ and $\delta^* \in \text{OR}$ be such that

a) $q^* \Vdash$ for no $X \in V$, $X \cap \langle p_\beta : \beta < \delta^* \rangle = G \cap \langle p_\beta : \beta < \delta^* \rangle$, and

b) for each $\gamma < \delta^*$, $q^* \Vdash$ there is an $X \in L$ such that

$$X \cap \langle p_\beta : \beta < \gamma \rangle = G \cap \langle p_\beta : \beta < \gamma \rangle.$$

There is such a q^* by Corollary 5.

By Lemma 4, in $V[G]$, to each $\gamma < \delta^*$ we can associate a condition $q_\gamma = \sigma(\vec{\alpha}) \in G$ such that

$$q_\gamma \Vdash \hat{q}_\gamma \cap \langle p_\beta : \beta < \gamma \rangle = G \cap \langle p_\beta : \beta < \gamma \rangle.$$

Choose the condition q_γ to minimize $l(\vec{\alpha})$ and $\vec{\alpha}$ lexicographically. Let $n_\gamma = l(\vec{\alpha})$ and $W_\gamma = \vec{\alpha}$. It is clear that $\gamma \le \gamma'$ implies $n_\gamma \le n_{\gamma'}$.

Claim. For each $l \in \omega$, there is a γ such that $n_\gamma \ge l$.

PROOF. Suppose that the n_γ's are eventually constant. In $V[G]$, using Corollary 3 (the one on generalized Δ-systems) we can choose a cofinal set of γ's such that the W_γ's form a generalized Δ-system with blocks $B_1, \dots, B_k$. We can assume without loss of generality that all of the W_γ's are in this generalized Δ-system. Let H be the union of the kernel of this generalized Δ-system and $\{\eta$: for some block B_i, $\eta = \sup_{\gamma < \delta^*} W_\gamma \restriction B_i\}$.

Suppose that $p_\alpha = \tau_n(\underset{\sim}{\vec{\alpha}})$ and $p_{\alpha'} = \tau_n(\underset{\sim}{\vec{\alpha}}')$ are elements of $\mathbf{P}_n$ with $\alpha, \alpha' < \delta^*$ and $\text{type}_H \vec{\alpha} = \text{type}_H \vec{\alpha}'$. We claim that $p_\alpha \in G \cap \langle p_\beta : \beta < \delta^* \rangle$ iff $p_{\alpha'} \in G \cap \langle p_\beta : \beta < \delta^* \rangle$. This is because whenever $\text{type}_H \vec{\alpha} = \text{type}_H \vec{\alpha}'$ there is a γ_0 such that for all $\gamma < \gamma_0$, $\text{type}_{W_\gamma} \vec{\alpha} = \text{type}_{W_\gamma} \vec{\alpha}'$. Hence, by indiscernibility, $q_\gamma \Vdash p_\alpha$ iff $q_\alpha \Vdash p_{\alpha'}$. Thus, taking $\gamma < \sup\{\alpha, \alpha'\}$, $p_\alpha \in G$ iff $p_{\alpha'} \in G$.

Hence for an $\alpha < \delta^*$, being in or out of G depends only on the type of the ordinals in the representation of p_α over H. There are only finitely many such types over H and finitely many combinations of these types. Hence the table of which types give rise to elements of $G \cap \langle p_\gamma : \gamma < \delta^* \rangle$ lies in V. Thus $G \cap \langle p_\gamma : \gamma < \delta^* \rangle \in G$, a contradiction. □

In $V[G]$, let γ_n = least γ such that $|W_\gamma| \ge n$. Then $\langle \gamma_n : n \in \omega \rangle$ is an ω-sequence of ordinals cofinal in δ^*. Thus we know at this stage that we are adding a new ω-sequence of ordinals. We now need a coarser measure of $\langle G \cap \langle p_\gamma : \gamma < \gamma_n \rangle : n \in \omega \rangle$ than these ordinals. Again we turn to residue classes mod ω^ω. Our "real" will be $\dot{x} = \langle W_{\gamma_n} (\text{mod}\, \omega^\omega) : n \in \omega \rangle$.

LEMMA 6. *Suppose we have conditions $q = \tau_i(\underset{\sim}{\vec{\alpha}})$ and $q' = \tau_i(\underset{\sim}{\vec{\alpha}}')$ and ordinals $\eta = \phi(\vec{\eta})$ and $\eta' = \phi(\vec{\eta}')$ for some terms τ_i, ϕ and indiscernibles $\underset{\sim}{\vec{\alpha}}$, $\underset{\sim}{\vec{\alpha}}'$, $\vec{\eta}$, $\vec{\eta}'$. Suppose that $\vec{\alpha}_q \cup \vec{\eta}\, E^4_{\vec{\alpha}^*_q \cup \vec{\alpha}^*} \vec{\alpha}' \cup \vec{\eta}'$ and $r = \sigma_r(\vec{\alpha}_r)$ is compatible with q (respectively $r \Vdash q$) and*

$$r \Vdash G \cap \langle p_\beta : \beta < \eta \rangle = \hat{r} \cap \langle p_\beta : \beta < \eta \rangle.$$

Then for any sequence of ordinals $\vec{\xi}$ such that

$$\vec{\alpha} \cup \vec{\eta} \cup \vec{\alpha}_r\, E^3_{\vec{\alpha}^*_q \cup \vec{\alpha}^*} \vec{\alpha}' \cup \vec{\eta}' \cup \vec{\xi},$$

if $r' = \sigma_r(\underset{\sim}{\vec{\xi}})$ then r' is compatible with q' (respectively $r' \Vdash q'$) and

$$r' \Vdash G \cap \langle p_\beta : \beta < \eta' \rangle = \hat{r}' \cap \langle p_\beta : \beta < \eta' \rangle.$$

PROOF. The proof is essentially a repetition of Claim (*) in Case I. Since $\text{type}_{\vec{\alpha}'} \vec{\xi} = \text{type}_{\vec{\alpha}} \vec{\alpha}_r$, by indiscernibility we get that r' is compatible with q' (respectively $r' \Vdash q'$). Suppose r' does not determine $G \cap \langle p_\beta : \beta < \eta' \rangle$. Then we can choose some condition $p'_\beta = \tau_n(\vec{\beta}')$ such that $r' \Vdash p'_\beta$ and $r' \Vdash \bar{p}'_\beta$. By our condition on ordinals we can choose a $\vec{\beta}$ such that

$$\vec{\alpha} \cup \vec{\eta} \cup \vec{\alpha}_r \cup \vec{\beta}\, E^2_{\vec{\alpha}^*_q \cup \vec{\alpha}^*} \vec{\alpha}' \cup \vec{\eta}' \cup \vec{\xi}' \cup \vec{\beta}'.$$

It is now easy to check that

a) r does not decide $\tau_n(\vec{\beta})$, and

b) $\tau_n(\vec{\beta}) = p_\beta$ for some $\beta < \eta$. □

Let $p \in \mathbf{P}$, $p \Vdash q^*$. We want to show that p does not determine the real $\dot{x}$. Recall $p = \sigma_p(\underline{\vec{\alpha}}_p)$ for some term σ_p and indiscernibles $\underline{\vec{\alpha}}_p$.

Let $N_p = \{\alpha'$: There is an $\alpha \in \vec{\alpha}_p \cup \vec{\alpha}^* \cup \vec{\alpha}^*_q$ with $\alpha' - \alpha < \omega^6$ or $\alpha - \alpha' < \omega^6\}$ be the ω^6 neighborhood of $\vec{\alpha}_p \cup \vec{\alpha}^* \cup \vec{\alpha}^*$. Then N_p is countable. As we argued earlier, if there is a $q \Vdash p$ such that $q \Vdash \bigcup W_{\gamma_n} \subseteq N_p$, then we add a new real to V by forcing below q. Thus we can assume there is a $q \Vdash p$ such that for some n and a finite sequence of ordinals $\vec{\rho}$, $q \,\|\, \dot{\gamma}_n$ and $q \Vdash W_{\gamma_n} = \vec{\rho}$ and there is an ordinal $\rho_i \notin N_p$. Express $\gamma_n = \phi(\underline{\vec{\gamma}})$ for some term ϕ and indiscernibles $\vec{\gamma}$ and $q = \sigma_q(\underline{\vec{\alpha}}_q)$. Let $\beta_1, \ldots, \beta_k$ be the ordinals in $\vec{\alpha}_q \cup \vec{\rho} \cup \vec{\gamma}$ that are not in N_p. Let $\vec{\alpha}'$, $\vec{\rho}'$, $\vec{\gamma}'$ be the result of replacing $\beta_1, \ldots, \beta_k$ in $\vec{\alpha}_q \cup \vec{\rho} \cup \vec{\gamma}$ by $\beta_1 + \omega^5, \beta_2 + \omega^5, \ldots \beta_k + \omega^5$. Then $\vec{\alpha}_q \cup \vec{\rho} \cup \vec{\gamma}\, E^5_{\vec{\alpha}_p \cup \vec{\alpha}^*_q} \vec{\alpha}' \cup \vec{\rho}' \cup \vec{\gamma}'$.

Let $q' = \sigma_q(\vec{\alpha}')$. Then $q' \Vdash p$ by indiscernibility.

Claim. $q' \Vdash \gamma_n = \phi(\vec{\gamma}')$ *and* $W_{\gamma_n} = \vec{\rho}'$.

Proof. We need to see that $\phi(\vec{\gamma}')$ is the least ordinal needing at least n indiscernibles to decide $G \cap \langle p_\beta : \beta < \phi(\vec{\gamma}')$.

First we show that you need at least $l(\vec{\rho}')$ indiscernibles to decide $G \cap \langle p_\beta : \beta < \phi(\underline{\vec{\gamma}}')\rangle$. Otherwise there would be an r' compatible with q' such that

$$r' \Vdash \hat{r}' \cap \langle p_\beta : \beta < \phi(\underline{\vec{\gamma}}')\rangle = G \cap \langle p_\beta : \beta < \phi(\underline{\vec{\gamma}}')\rangle$$

and $r' = \sigma_{r'}(\vec{\alpha}_{r'})$ for some $\vec{\alpha}_{r'}$ with $l(\vec{\alpha}_{r'}) < l(\rho')$. But now we can choose $\vec{\alpha}_r$ such that

$$\vec{\alpha}_q \cup \vec{\rho} \cup \vec{\gamma} \cup \vec{\alpha}_r\, E^4_{\vec{\alpha}_p \cup \vec{\alpha} \cup \vec{\alpha}^*_q} \vec{\alpha}' \cup \vec{\rho}' \cup \vec{\gamma}' \cup \vec{\alpha}_{r'}.$$

Applying Lemma 6 to $r = \sigma_{r'}(\underline{\vec{\alpha}}_r)$, we contradict $q \Vdash \gamma_n = \phi(\gamma)$.

Similarly if $\delta < \phi(\gamma')$ we can represent δ as $\psi(\eta')$ for some term ψ and indiscernibles $\vec{\eta}'$. Choose an $s \Vdash q$, $s = \sigma_s(\underline{\vec{\alpha}}_s)$. Choose η, β_s such that

$$\vec{\alpha}_q \cup \vec{\rho} \cup \vec{\alpha} \cup \vec{\eta} \cup \vec{\beta}_s\, E^4_{\vec{\alpha}_p \cup \vec{\alpha}^* \cup \vec{\alpha}^*_p} \vec{\alpha}' \cup \vec{\rho}' \cup \vec{\gamma}' \cup \vec{\eta}' \cup \vec{\alpha}_s.$$

Then there is an $r \Vdash \sigma_s(\vec{\beta}_s)$ with $l(\vec{\alpha}_r) < n$ compatible with q that determines $G \cap \langle p_\beta : \beta < \psi(\vec{\eta})\rangle$. If we choose $\vec{\alpha}'$ approximately, Lemma 6 implies that $\sigma_r(\underline{\vec{\alpha}}')$ determines $G \cap \langle p_\beta : \beta < \psi(\vec{\eta}')\rangle$ and $\sigma_r(\vec{\alpha}')$ is compatible with q' and $\sigma_r(\underline{\vec{\alpha}}') \Vdash \sigma_s(\underline{\vec{\alpha}}_s)$. Hence on a dense set below $q \langle p_\beta : \beta < \psi(\vec{\eta})\rangle$ is decidable by a condition with $< n$ ordinals mentioned. Hence $\gamma_n > \psi(\vec{\eta})$, as desired.

This completes the proof of Theorem 2.

The proof is more general than this particular theorem. For example, it shows that if O^+ exists and L_μ is the model consisting of sets constructible relative to a normal measure, then every partial ordering in L_μ adds a real to V. Similarly if $x^\#$ exists then any partial ordering in $L[x]$ adds a real over V.

Further if $\mathbf{P} \in V$ is a partial ordering that is generated by a countable set of terms and some indiscernibles then forcing over V and $\mathbf{P}$ adds a real. Careful reading of the proof above shows that we did not use indiscernibility over L, only over the partial ordering. Hence if our partial ordering is an Ehrenfeucht-Mostowski model generated by some well-ordered indiscernibles then it adds a real. Also if the collection of terms has cardinality λ then we add a new subset to λ.

P. Welch has pointed out a number of other situations in which the theorem applies.

We finish with a question: Suppose every partial order in L adds a real. Does $O^{\#}$ exist? This is the converse of our theorem. J. Steel has proved that if this question is answered positively and if $V \neq L$ then the reals of L are not an uncountable Σ^1_1 set.

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DEPARTMENT OF MATHEMATICS
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91125

INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY
JERUSALEM, ISRAEL