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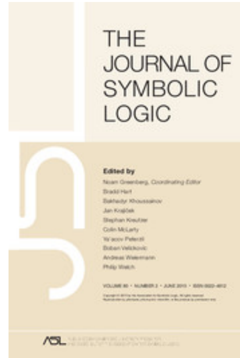

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FORCING THE FAILURE OF CH BY ADDING A REAL

SAHARON SHELAH AND HUGH WOODIN

We prove several independence results relevant to an old question in the folklore of set theory. These results complement those in [Sh, Chapter XIII, §4]. The question is the following. Suppose $V \models$ "ZFC + CH" and r is a real not in V. Must $V[r] \models$ CH? To avoid trivialities assume $\aleph_1^V = \aleph_1^{V[r]}$.

We answer this question negatively. Specifically we find pairs of models (W, V) such that $W \models$ ZFC + CH, $V = W[r]$, r a real, $\aleph_1^W = \aleph_1^V$ and $V \models \neg$CH. Actually we find a spectrum of such pairs using ZFC up to "ZFC + there exist measurable cardinals". Basically the nicer the pair is as a solution, the more we need to assume in order to construct it.

The relevant results in [Sh, Chapter XIII] state that if a pair (of inner models) (W, V) satisfies (1) and (2) then there is an inaccessible cardinal in L; if in addition $V \models 2^{\aleph_0} > \aleph_2$ then $0^{\#}$ exists; and finally if (W, V) satisfies (1), (2) and (3) with $V \models 2^{\aleph_0} > \aleph_\omega$, then there is an inner model with a measurable cardinal.

DEFINITION 1. For a pair (W, V) we shall consider the following conditions:

(1) $V = W[r]$, r a real, $\aleph_1^V = \aleph_1^W$, $W \models$ ZFC + CH but CH fails in V.

(2) $W \models$ GCH.

(3) W and V have the same cardinals.

THEOREM 1. *Assume ZFC. Then there is a pair (W, V) of generic extensions of L satisfying* (1). *In fact W has the same cardinals as L and the only cardinals that are collapsed in passing from W to V are $\aleph_2$ and $\aleph_3$. Furthermore V may be chosen with $2^{\aleph_0}$ arbitrarily large.*

PROOF. We start with $V_0 = L$. Fix λ, any cardinal of L with $\mathrm{cof}(\lambda) > \omega_2$. Let P_0 denote the Cohen type forcing conditions for adding λ subsets of ω_1, i.e. $P_0 = \{f: \mathrm{dom}\, f \to \{0,1\} \mid \mathrm{dom}\, f \subseteq \lambda, \mathrm{dom}\, f \text{ countable}\}$.

Suppose G_0 is a generic subset of P_0 over V_0 and let $V_1 = V_0[G_0]$. Since P_0 is countably closed and satisfies the $\aleph_2$ chain condition, V_1 and V_0 have the same cardinals and reals.

Let P_1 denote in V_1 the following variant of Namba forcing. Conditions are subsets $T \subseteq \omega_2^{<\omega}$ such that for each $s \in T$, $\{t \in T \mid t \text{ extends } s\}$ is of size $\aleph_2$. The ordering on P_1 is the obvious one: for $T_1, T_2 \in P_1$, T_1 is stronger than T_2 ($T_1 \geq T_2$) if $T_1 \subseteq T_2$.

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We claim that P_1 satisfies the $\aleph_4$ chain condition in V_1. This follows by a standard chain condition argument; one simply shows that the iteration $P_0 * P_1$ has the $\aleph_4$ chain condition in V_0. The situation is identical to that with P_1 replaced by Namba forcing (for more details see Claim 4.3 in [Sh, Chapter XI]).

Assume G_1 is a generic subset of P_1 over V_1 and let $V_2 = V_1[G_1]$. All of the standard fusion arguments for Namba forcing work for P_1; in particular since CH holds in V_1, V_2 and V_1 have the same reals. Further the only cardinals of V_1 that are collapsed are $\aleph_2$ and $\aleph_3$, which in V_2 have cofinalities ω and ω_1 respectively.

Note that in V_2 any countable set A of ordinals can be covered by a set $X \in V_1$ with $|X|^{V_1} \le \aleph_2^{V_1}$.

Define forcing conditions P_2 in V_2 as follows. Conditions are pairs (A, X) such that $A \subseteq X \subseteq \lambda$, A is countable, $X \in V_1$, $|X|^{V_1} \le \aleph_2^{V_1}$ and $A \cap Y$ is finite for each $Y \in V_1$, $|Y|^{V_1} = \aleph_1$. The ordering on P_2 is given by $(A_1, X_1) \le (A_2, X_2)$ if $A_1 \subseteq A_2$, $X_1 \subseteq X_2$, and $A_2 \cap X_1 \subseteq A_1$.

We claim that P_2 has the $\aleph_2$ $(= \aleph_4^{V_1})$ chain condition in V_2. Suppose not and let (A_α, X_α) $(\alpha < \aleph_4^{V_1})$ be an antichain in P_2. Let $X = \bigcup_\alpha X_\alpha$. Choose $X^* \in V_1$ with $X \subseteq X^*$ and $|X^*| = \aleph_4^{V_1}$. Choose in V_1 a map $\pi: X^* \to \aleph_4^{V_1}$ that is one-to-one and onto. For each α, let $B_\alpha = \pi[A_\alpha] = \{\pi(\beta) \mid \beta \in A_\alpha\}$ and let $Y_\alpha = \pi[X_\alpha]$. It is easily seen that each (B_α, Y_α) is a condition in P_2 and that (B_α, Y_α) $(\alpha < \aleph_2)$ is an antichain in P_2. Each Y_α is a subset of $\aleph_2$ and has ordertype $< \aleph_3^{V_1} < \aleph_2$. Hence we can find $\gamma_0 < \aleph_2$ and a subset $S \subseteq \aleph_2$ of size $\aleph_2$ such that for all $\alpha_1, \alpha_2 \in S$, $Y_{\alpha_1} \cap Y_{\alpha_2} \subseteq \gamma_0$. The cardinality of γ_0 is $\aleph_1$ and CH holds; therefore there are $\alpha_1, \alpha_2 \in S$ with $\alpha_1 \ne \alpha_2$ and $B_{\alpha_1} \cap \gamma_0 = B_{\alpha_2} \cap \gamma_0$. But then $(B_{\alpha_1}, Y_{\alpha_1})$ and $(B_{\alpha_2}, Y_{\alpha_2})$ are compatible, a contradiction.

We show that forcing with P_2 over V_2 adds no new reals. Before proceeding we fix some notation in V_1, since this is where we shall eventually be working. For $T \in P_1$ and $s \in T$ let $(T)_s = \{t \in T \mid t \text{ extends } s\}$. Similarly if $S \subseteq T$ let $(T)_S = \{t \in T \mid t$ extends some element of $S\}$. Suppose $T \in P_1$ and $f: T \to \lambda$ is a function from T into λ. T and f determine in a canonical fashion a term of a countable subset of λ. We denote this term by G_f. Conversely suppose τ is a term in $V_1^{P_1}$ for a countable subset of λ. Then for each $T' \in P_1$ there is $T \ge T'$, $f: T \to \lambda$, such that $T \Vdash G_f = \tau$. From this it follows that if $\tau \in V_1^{P_1}$ is a term for a condition $(A, X) \in P_2$ then for each $T' \in P_1$ one can find $T \ge T'$, $f: T \to \lambda$ and $Y \subseteq \lambda$, for which $|Y| = \aleph_2$ (in V_1) and $T \Vdash$ "$(G_f, Y) \in P_2$ and $(G_f, Y) \ge (A, X)$".

Now suppose b is a term in $V_2^{P_2}$ for a real. We regard b as a term for a subset of ω. Fix $(A, X) \in P_2$. We seek $(A', X') \ge (A, X)$ such that (A', X') determines b. To find (A', X') we work in V_1, so choose terms for b, (A, X) in V^{P_1} and fix $T \in P_1$.

Construct an infinite sequence $\langle T_0, f_0, X_0, S_0 \rangle, \ldots, \langle T_N, f_N, X_N, S_N \rangle, \ldots$, where $T_N \in P_1$, $X_N \subseteq \lambda$ of size $\aleph_2$, $S_N \subseteq T_N$ is an antichain and $f_N: (T_N)_{S_N} \to X_N \subseteq \lambda$, such that:

(1) $T \le T_0 \le \cdots \le T_N \le \cdots$ is an increasing chain in P_1.

(2) For each N, M, $S_N \subseteq T_M$ and is of size $\aleph_2$, each element of S_{N+1} extends some element of S_N and for each $s \in S_N$, $\{t \in S_{N+1} \mid t \text{ extends } s\}$ is of size $\aleph_2$.

(3) $T_0 \Vdash (G_{f_0}, X_0) \ge (A, X)$.

(4) $T_N \Vdash$ "$\langle G_{f_0} \cup \cdots \cup G_{f_N}, X_N \rangle$ decides $0 \in b, \ldots, N \in b$".

(5) For $s, t \in S_N$, $s \ne t$, range $(f_N \restriction (T_N)_s) \cap$ range $(f_N \restriction (T_N)_t) = \varnothing$.

(1) and (2) guarantee that $\bigcup_N S_N$ is a condition in P_1 that lies above each T_N.

Suppose $\langle T_N, f_N, X_N, S_N\rangle$ is given: we find $\langle T_{N+1}, f_{N+1}, X_{N+1}, S_{N+1}\rangle$.

Choose $S_{N+1} \subseteq T_N$ an antichain in T_N satisfying (2). Enumerate S_{N+1} in length $\aleph_2$, $s_1 \cdots s_\alpha \cdots, \alpha < \aleph_2$. By induction on α construct a sequence $\langle f^\alpha, T^\alpha, X^\alpha\rangle$ such that $T^\alpha \subseteq (T_N)_{s_\alpha}$ is a condition in P_1, $X_N \subseteq X^\alpha \subseteq \lambda$, X^α of size $\aleph_2$, f^α: $T^\alpha \to X^\alpha \subseteq \lambda$, $T^\alpha \Vdash$"$\langle G_{f_0} \cup \cdots \cup G_{f_N} \cup G_{f^\alpha}, X^\alpha\rangle$ decides $0 \in b, \ldots, N+1 \in b$", and finally such that range $(f^\alpha) \cap X_N \cap (\bigcup_{\beta<\alpha} X^\beta) = \varnothing$.

Set $T_{N+1} = \bigcup_\alpha T^\alpha \cup S_0 \cup \cdots \cup S_N$, $f_{N+1} = \bigcup_\alpha f^\alpha$ and $X_{N+1} = \bigcup_\alpha X^\alpha$. It is clear that $\langle T_{N+1}, f_{N+1}, X_{N+1}, S_{N+1}\rangle$ is as required.

Let $T^\infty = \bigcup_N S_N$. Hence $T^\infty \in P_1$ and $T^\infty \geq T_N$ for each N. We claim that

$$T^\infty \Vdash \text{“}\langle \textstyle\bigcup_N G_{f_N}, \bigcup_N X_N\rangle \in P_2\text{”}.$$

For this it suffices to see that for each $Y \subseteq \lambda$ with $|Y| = \aleph_1$, $T^\infty \Vdash$"$(\bigcup_N G_{f_N}) \cap Y$ is finite". It is routine to verify that condition (5) guarantees this. Finally, from all of this it follows that $T^\infty \Vdash$"$\langle \bigcup_N G_{f_N}, \bigcup_N X_N\rangle$ decides b" and therefore forcing with P_2 over V_2 adds no reals.

Suppose G_2 is a generic subset of P_2 over V_2. Let $V_3 = V_2[G_2]$. V_3 and V_2 have the same cardinals and reals.

Let $(\alpha_N : N < \omega)$ be the cofinal "Namba" sequence through $\aleph_2^{V_1}$ defined by G_1. Let $P_3 \in V_3$ be a c.c.c. forcing notion for coding the sequence $(\alpha_N : N < \omega)$ by a real. Since CH holds in V_3 one can use the technique of "almost disjoint" forcing to define P_3 (see [JS] for further details).

Let a^* be a real generic over V_3 for P_3, and let $V_4 = V_3[a^*]$. Let $(r_\alpha : \alpha < \lambda)$ be a generic sequence of Cohen reals over V_4. Each r_α we view as an infinite sequence of zeros and ones, i.e. as a function from ω onto $\{0,1\}$.

Let $f_0 = \bigcup_{f \in G_0} f$, $A^* = \bigcup\{A \mid (A, X) \in G_2$ for some $X\}$. Thus f_0 is a function, $f_0 : \lambda \to \{0,1\}$, and A^* is a subset of λ with size λ. A^* has the additional property that for each $Y \subseteq \lambda$, $Y \in V_1$, $|Y|^{V_1} = \aleph_1^{V_1}$, $A^* \cap Y$ is finite. The key point is that for each (open) dense set $D \subseteq P_0$, $D \in V_0$, there is a dense set $S \subseteq D$, $S \in V_0$, such that for every $f \in S$ and any $g \in P_0$, if $\{\beta \mid f(\beta) \neq g(\beta)\}$ is finite and $\operatorname{dom} f = \operatorname{dom} g$, then $g \in D$. Using this fact we define a function $f_0^* : \lambda \to \{0,1\}$ such that f_0^* determines a V_0-generic subset of P_0. Define f_0^* by:

$$f_0^*(\beta) = \begin{cases} r_\alpha(N) & \text{if } \beta = \omega_2^{V_0} \cdot \alpha + \alpha_N \text{ for some } \alpha \in A^*, \\ f_0(\beta) & \text{otherwise.} \end{cases}$$

Let $Z = \{\beta \mid f_0(\beta) \neq f_0^*(\beta)\}$. For each $Y \in V_0$ with $|Y|^{V_0} = \aleph_1^{V_0}$, $Y \cap Z$ is finite. Hence f_0^* does determine a V_0-generic subset of P_0, as desired.

Let $W = V_0[f_0^*] = L[f_0^*]$, $V = V_0[f_0^*, a^*] = L[f_0^*, a^*]$. Within V, $(\alpha_N : N < \omega)$ can be computed from a^*, and from this r_α can be computed for each $\alpha \in A^*$. Hence $V \models 2^{\aleph_0} = \lambda$.

This completes the proof of Theorem 1. □

If we work within a theory stronger than ZFC then we can find a pair (W, V) satisfying (1) and (2). Specifically we prove the following:

THEOREM 2. *Assume ZFC and that there is an inaccessible 2-Mahlo cardinal in L. Then there is a pair (W, V) of generic extensions of L satisfying (1) and (2).*

PROOF. The proof of this is very similar to the proof of the next theorem (Theorem 3). Let κ be an inaccessible 2-Mahlo cardinal in L. Using an Easton style product, force over L to add a generic subset of λ^{++} for every inaccessible cardinal λ

that is less than κ (use the appropriate notion of forcing as defined in the proof of Theorem 3). Let L_1 denote the generic extension. Thus GCH holds in L_1 and κ is an inaccessible 2-Mahlo cardinal in L_1. Using Theorem 7.3 of [Sh, Chapter XI] one can force over L_1 to make $\kappa:\aleph_2$, in such a way that in the generic extension $L_1[G]$ there is a closed unbounded set $C \subseteq \aleph_2$ of inaccessible cardinals of L_1. Using this the proof is similar to the proof of Theorem 3. □

We now leave the confines of L in order to find even nicer pairs (W, V). Our goal is a pair satisfying (1), (2) and (3), and we will use measurable cardinals. First we use $0^\#$ to improve Theorem 2. Theorem 2 has the defect that for the pair (W, V) produced we can only make $2^{\aleph_0} = \aleph_2$ in V.

THEOREM 3. *Assume ZFC and that $0^\#$ exists. Then there is a pair (W, V) satisfying* (1) *and* (2), *$W \subseteq L[0^\#]$, V is a (c.c.c.) generic extension of $L[0^\#]$ and further V may be chosen with $2^{\aleph_0}$ arbitrarily large.*

PROOF. Fix λ a regular cardinal in $L[0^\#]$ with $\lambda \geq \aleph_2^{L[0^\#]}$.

Suppose δ is a cardinal in L, $\delta < \lambda$. Define a partial order P_δ in L by

$$P_\delta = \{f \in L \mid f:(\delta,\alpha) \to \{0,1\} \text{ for some } \delta < \alpha < \delta^{++} \text{ where } (\delta,\alpha) = \{\beta \mid \delta < \beta < \alpha\}\}.$$

Define a partial order P in L by $P = \{F \in L \mid F$ is a function with domain a set of limit cardinals in L, $|\text{Dom } F \cap \kappa|^L < \kappa$ for every inaccessible cardinal κ in L, $\text{Dom } F \subseteq \lambda$ and $F(\delta) \in P_\delta\}$.

Both P_δ and P are ordered naturally, i.e. P_δ is ordered by inclusion, P as a product with restricted support.

Let $\kappa_0 = \aleph_1^{L[0^\#]}$. κ_0 is an inaccessible cardinal in L. Let Q denote the partial order of forcing conditions for collapsing all cardinals $<\kappa_0$ to ω. Take for Q the usual Levy conditions so that forcing with G makes κ_0, $\aleph_1$.

By Beller, Jensen and Welch [BJW] there is a set $G_P \times G_G \in L[0^\#]$, $G_P \subseteq P$, $G_Q \subseteq Q$ such that $G_P \times G_Q$ is a generic subset of $P \times Q$ over L.

Choose $C \in L[0^\#]$, $C \subseteq \lambda$, a closed unbounded set of inaccessible cardinals of L. It is easy to verify that for each $F \in P$, dom $F \cap C$ is finite.

Suppose $(r_\alpha:\alpha < \lambda)$ is a generic sequence of Cohen reals over $L[0^\#]$.

Let $F_0 = \bigcup_{F\in G_P} F$. Thus F_0 is a function $F_0:\text{dom } F_0 \to \{0,1\}$, where $\text{dom } F_0 = \bigcup_{\delta<\lambda}(\delta,(\delta^{++})^L)$ (δ a limit cardinal in L). For each $\delta \in C$ let $(\alpha_N^\delta:N < \omega) \in L[0^\#]$ be a cofinal sequence through $(\delta^{++})^L$. Define F_0^* by:

$$F_0^*(\beta) = \begin{cases} r_\delta(N) & \text{if } \beta = \alpha_N^\delta,\ \delta \in C, \\ F_0(\beta) & \text{otherwise.} \end{cases}$$

Much as in the proof of Theorem 1 it is easily seen that F_0^* determines a generic subset of P over L. Call it G_P^*. Similarly it follows that $G_P^* \times G_Q$ is a generic subset of $P \times Q$ over L.

Let $W = L[G_P^* \times G_Q]$ and $V = L[G_P^* \times G_Q, 0^\#]$.

It is clear that $r_\delta \in L[G_P^* \times G_Q, 0^\#]$ for each $\delta \in C$; hence $V \models 2^{\aleph_0} = \lambda$. V is also clearly a c.c.c. extension of $L[0^\#]$, $V = L[0^\#,(r_\alpha:\alpha < \lambda)]$. □

We now use measurable cardinals to find a pair (W, V) satisfying (1), (2) and (3).

THEOREM 4. *Assume ZFC + GCH and that there are λ many measurable cardinals ($\lambda > \aleph_1$). Then there is a pair (W, V) of cardinal-preserving generic extensions of the universe satisfying* (1), (2) *and* (3) *with $2^{\aleph_0} = \lambda$ in V.*

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PROOF. Let $V_0 \models$ "ZFC + GCH + there are λ many measurable cardinals, $\lambda > \aleph_1$". Working in V_0, fix a set S of measurable cardinals, S of size λ and such that no element of S is a limit point of S. Define a notion of forcing P_0 in V_0 as follows. For each $\kappa \in S$ fix a normal measure μ_κ on κ. Let Q_κ denote the corresponding partial order of Prikry conditions. Put $P_0 = \{F \in V_0 \mid F: S \to V_0$ such that $F(\kappa) \in Q_\kappa$ for each $\kappa \in S$ and $F(\kappa)$ is a condition in the form of a subset of κ for all but finitely many $\kappa\}$. Thus P_0 is the usual "Prikry" style product of the Q_κ.

Suppose $G_0 \subseteq P_0$ is generic over V_0. Let $V_1 = V_0[G_0]$. Thus V_0 and V_1 have the same cardinals and reals. $V_1 \models$ GCH, so by Beller, Jensen and Welch [BJW] it is possible to force to find a real, a^*, class generic over V_1 such that $V_1 \subseteq L[a^*]$ ($V_1 \subseteq V_0[a^*]$ will suffice) and such that $V_0[a^*]$ ($= L[a^*]$) is a cardinal-preserving extension of V_1.

Let $(r_\alpha: \alpha < \lambda)$ be a generic sequence of Cohen reals over $L[a^*]$. As usual each r_α we view as a function from ω onto $\{0, 1\}$.

G_0 may be interpreted as a λ-sequence $(s_\alpha: \alpha < \lambda)$, where each s_α is a Prikry sequence through κ_α, the αth element of s. For each $\alpha < \lambda$ define a subsequence s_α^* of s_α by $s_\alpha^* = \{\beta \mid \beta = s_\alpha(N)$ and $r_\alpha(N) = 1\}$. Thus s_α^* is the subsequence of s_α corresponding to r_α. It is routine to show that $(s_\alpha^*: \alpha < \lambda)$ defines a generic subset G_0^* of P_0 over V_0.

Let $V = V_0[G_0^*]$ and $W = V_0[G_0^*, a^*]$. Thus V and W are cardinal preserving extensions of V_0, $W \models$ GCH and $V \models 2^{\aleph_0} = \lambda$. □

There are endless possible variations of Theorem 4. We state one and leave the others to the reader's imagination.

Using the methods in the proof of Theorem 4 one can find a pair (W, V) such that W, V have the same cardinals, $V = W[r]$ for some real r, $W \models$ "ZFC + GCH" and $V \models$ "CH + $2^{\aleph_1}$ is large".

The essence of all of this is that if one allows inner models of large cardinals then there seems to be very little that can be deduced about $V[r]$, for r a real, even given that V and $V[r]$ have the same cardinals and that $V \models$ GCH.

In closing we suggest another problem. Suppose $V[r]$ is a cardinal-preserving extension of V obtained by adding a single real and that GCH holds in V. Can GCH fail everywhere in $V[r]$?

REFERENCES

[BJW] A. BELLER, R. B. JENSEN and P. WELCH, ***Coding the universe***, London Mathematical Society Lecture Note Series, no. 47, Cambridge University Press, Cambridge, 1982.

[JS] R. B. JENSEN and R. SOLOVAY, *Some applications of almost disjoint sets*, ***Mathematical logic and the foundations of set theory (proceedings of an international colloquium, Jerusalem, 1968***; Y. Bar-Hillel, editor), North-Holland, Amsterdam, 1970, pp. 84–104.

[Sh] S. SHELAH, ***Proper forcing***, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin, 1982.

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