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SATURATION OF ULTRAPOWERS AND KEISLER'S ORDER

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We try here to find the connection between how saturated is, or can be an ultrapower, and some properties of the theory of the model and of the ultrafilter. We deal also with similar problems for ultralimits, ultraproducts, limitultrapowers; and the existence of categorical pseudo-elementary calsses contained in given elementary classes. In another formulation, this is equivalent to the investigation of Keisler's order $\triangleleft$, and a generalization $\triangleleft^*$ defined here (see Def. 1.3 in § 1). Another generalization which was suggested – replacing ultrapowers by reduced limit powers, is not checked here. Almost all the results here (and more) appear in Shelah [13] §0, F, G (together with historical remakrs) and they appeared previously in the notices [15], [16]. We solved here, partially, question 25 (of Keisler), from Chang and Keisler [4]; and, equivalently, some questions and conjectures from Keisler [6]. The different sections here are quite unconnected, but §4 depends heavily on [13].

In Section § 1 we define notation. In Section § 2, we investigate $\triangleleft$ for uncountable theories. We find a way to deduce from theorems about $\triangleleft$ on countable theories theorems about $\triangleleft$ for uncountable theories. We proved that there is a non $\triangleleft$-minimal nor $\triangleleft$-maximal theory (2.13A), and that if G.C.H. fails (i.e. there is at least one λ, $2^\lambda > \lambda^+$), *then* there are two $\triangleleft$-incomparable theories (Th. 2.13B). (Those results answer questions of Keisler).

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In Section §3, we mainly prove that certain ultrapowers are not saturated.

Section §4 contains the main results. We affirm a conjecture of Keisler; characterizing countable ◁-minimal theories. We prove that if G.C.H. fails, there is a countable non ◁-minimal non ◁-maximal theory (Th. 4.10, 4.11). We find for models of countable stable theories, almost exactly how saturated are their ultrapowers (Th. 4.1). We also characterize the countable theories T, such that for some $T_1 \supset T$, the class of reducts of models of T_1 to the language of T is categorical in some $\lambda > |T_1|$.

In Section §5, we find, quite accurately, how saturated are ultralimits.

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§ 1. Notations

We shall mostly use the notations in Shelah [13] § 1. T will be a complete first-order theory with equality and with no finite models. The first-order language generated from L by adding the predicates $R_1, \ldots$ and the symbol functions $G_1, \ldots$ is denoted by $L \cup \{R_1, \ldots, G_1, \ldots\}$. Ultrafilters will be denoted by D, and we assume they are non-principal uniform and $\aleph_1$-incomplete, and D will be over I, if not mentioned otherwise. We shall use freely Łoś' theorem (see e.g. [1] or [4] on ultrapowers and ultraproducts). Elements of I will be denoted by i, s, t. In an abuse of notation if, for example, M_i is an L-model, $L_1 = L \cup \{P\}$, P_i a relation over $|M_i|$ for every $i \in I$, *then* (M_i, P_i) is an L_1-model and if $N = \prod_{i \in I} M_i/D$ then $(N, P^N) = \prod_{i \in I} (M_i, P_i)/D$. We shall denote elements of $\prod_{i \in I} M_i/D$ also as indexed sets $\langle a_i : i \in I\rangle$ and not always as equivalence classes of such indexed sets. Also if $a \in N = \prod_{i \in I} M_i/D$, then $a = \langle a[i] : i \in I\rangle$ and for $\bar{a} = \langle a_0, \ldots a_n \rangle$, $\bar{a}[i] = \langle a_0[i], \ldots, a_n[i]\rangle$. For $a \in M^I/D$, $\mathrm{eq}(a) = \{\langle s, t\rangle : a[s] = a[t]\}$, and for a filter G over $I \times I$, M^I_D/G is a submodel of M^I/D, whose set of elements is $\{a \in M^I/D : \mathrm{eq}(a) \in G\}$. This is defined and investigated in Keisler [9].

An ultrafilter D is (μ, λ)-regular if there is a family of λ subsets of I, which belong to D, and the intersection of every μ sets from the family is empty. D is regular if it is $(\aleph_0, |I|)$-regular.

For a model M the set $p = \{\varphi_k(\bar{x}, \bar{a}^k) : k < k_0\}$ $(\bar{a}^k \in |M|)$ is consistent over M, if for every finite $w \subset k_0$, $M \models (\exists \bar{x}) \bigwedge_{k \in w} \varphi_k(\bar{x}, \bar{a}^k)$. Such a consistent set is called a type over M. If all the $\bar{a}^k$ are from A, $A \subset |M|$, then p is a type over A. A sequence $\bar{c}$ realizes p if $\varphi(\bar{x}, \bar{a}) \in p$ implies $M \models \varphi[\bar{c}, \bar{a}]$. M realizes p if some $\bar{c} \in |M|$ realizes p, and if M does not realize p, it omits p.

M is λ-compact if it realizes every consistent type (over it) of cardinality $< \lambda$; M is λ-saturated if it realizes every (consistent) type over any subset $A \subset |M|$ $|A| < \lambda$. By Keisler [8] D is λ-good iff for every M, M^I/D is λ-compact; and every ($\aleph_1$-incomplete) D is $\aleph_1$-good, but not $|I|^{++}$-good. D is called good if it is $|I|^+$-good. M is λ-universal, if every set of λ formulas which is finitely satisfied in M is satisfied in M. M is $(< \lambda)$-universal if for every $\mu < \lambda$ M is μ-universal.

By [5] (or see e.g. [1], [4] or [6]) for every D_1, D_2 over I_1, I_2 we can define the ultrafilter $D_1 \times D_2$ over $I_1 \times I_2$ such that for every M,

$M^{I_1 \times I_2}/D_1 \times D_2$ is isomorphic to $(M^{I_1}/D_1)^{I_2}/D_2$. If D_1, D_2 are regular, then $D_1 \times D_2$ is regular, and for every λ, $D_1 \times D_2$ is λ-good iff D_2 is λ-good (see Keisler [10]).

After Keisler [6] we define:

1.1. Definition. $T_1 \triangleleft_\lambda T_2$ provided that: for every models M_1, M_2 of T_1, T_2, and $(\aleph_0, \lambda)$-regular ultrafilter D over λ, *if* M_2^λ/D is λ^+-compact, *then* M_1^λ/D is λ^+-compact.

1.2. Definition. $T_1 \triangleleft T_2$ if for every λ, $T_1 \triangleleft_\lambda T_2$.

A generalization of $\triangleleft$ is

1.3. Definition. $T_1 \triangleleft^* T_2$ if for every I, D, G, λ and $(\lambda^+ + |I|^+)$-saturated models M_1, M_2 of T_1, T_2, *if* $M^I_{2D}|G$ is λ^+-compact *then* $M^I_{1D}|G$ is λ^+-compact.

Keisler [6] shows: $T \triangleleft T$ (2.1a). T is $\triangleleft_\lambda$-minimal iff for every regular D over λ, and model M of T, M^λ/D is λ^+-compact (§4) and the theory of equality is $\triangleleft_\lambda$-minimal; and T is $\triangleleft_\lambda$-maximal iff for every non-good, $(\aleph_0, \lambda)$-regular D over λ, and model M of T, M^λ/D is not λ^+-compact, and e.g. the theory of numbers is $\triangleleft_\lambda$-maximal (Th. 3.1). He also shows that for $\lambda > \aleph_0$, no theory is both $\triangleleft_\lambda$-minimal and $\triangleleft_\lambda$-maximal.

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§ 2. Keisler's order for uncountable theories

Remark on notations.

We shall assume that different theories have languages without any common predicate or function symbol. So writing a formula, it is clear to what unique language it belongs. Let Φ denote an (indexed) set of formulas $\varphi(\bar{x})$; with repetitions possibly. Φ is of $L = L(T)$ if it is a set of formulas which belongs to L. We write $\Phi \subset L$.

2.1. Definition. $G: \langle \Phi_1, m_1 \rangle \leq \langle \Phi_2, m_2 \rangle$ holds, where $\Phi_1 \subset L(T_1)$, $\Phi_2 \subset L(T_2)$, provided that $\Phi_1 = \{\varphi_k(\bar{x}, \bar{z}^k) : k < k_0\}$ $G[\varphi_k(\bar{x}, \bar{z}^k)] = \Psi_k(\bar{y}, \bar{z}_k) \in \Phi_2$, $l(\bar{x}) = m_1$, $l(\bar{y}) = m_2$, and for every model M_1 of T_1, $\bar{a}^k \in |M_1|$, T_2 has a model M_2, and $\bar{b}^k \in |M_2|$ such that:

for every $w \subset k_0$ $(=\{l : l < k_0\})$
$\{\varphi_k(\bar{x}, \bar{a}^k) : k \in w\}$ is consistent over M_1
iff $\{\Psi_k(\bar{y}, \bar{b}^k) : k \in w\}$ is consistent over M_2 .

2.2. Definition. $\langle \Phi_1, m_1 \rangle \leq \langle \Phi_2, m_2 \rangle$ holds if there is G such that $G: \langle \Phi_1, m_1 \rangle \leq \langle \Phi_2, m_2 \rangle$ holds.

Remarks. A) Clearly by the compactness theorem $G: \langle \Phi_1, m_1 \rangle \leq \langle \Phi_2, m_2 \rangle$ holds iff for every finite $\Phi \subset \Phi_1$, $G|\Phi : \langle \Phi, m_1 \rangle \leq \langle \Phi_2, m_2 \rangle$ holds. B) In Definition 2.1 we can take M_1, M_2 as fixed λ-universal models.

Lemma 2.1. A) *If* $\langle \Phi_1, m_1 \rangle \leq \langle \Phi_2, m_2 \rangle$, $\Phi^1 \subset \Phi_1$, $\Phi_2 \subset \Phi^2$ *then* $\langle \Phi^1, m_1 \rangle \leq \langle \Phi^2, m_2 \rangle$.

B) *If* Φ^1 (Φ^2) *is the closure of* Φ_1 (Φ_2) *under conjunction and disjunction: then* $\langle \Phi_1, m_1 \rangle \leq \langle \Phi_2, m_2 \rangle$ *implies* $\langle \Phi^1, m_1 \rangle \leq \langle \Phi^2, m_2 \rangle$.

c) *If* $\langle \Phi_1, m_1 \rangle \leq \langle \Phi_2, m_2 \rangle$, *and* $\langle \Phi_2, m_2 \rangle \leq \langle \Phi_3, m_3 \rangle$ *then* $\langle \Phi_1, m_1 \rangle \leq \langle \Phi_3, m_3 \rangle$.

Proof. Immediate.

Theorem 2.2. A) *If for every* $\Phi_1 \subset L(T_1)$, $|\Phi_1| \leq \lambda$ *there is* $\Phi_2 \subset L(T_2)$ *and* $m_2 < \omega$ *such that* $\langle \Phi_1, 1 \rangle \leq \langle \Phi_2, m_2 \rangle$ *then* $T_1 \triangleleft_\lambda T_2$.

B) *From the hypothesis of* A) *we can conclude: if* M_1 *is a* κ*-compact model of* T_1, M_2 *a* $(<\kappa)$*-universal model of* T_2, D *a* (κ, λ)*-regular ultrafilter over* μ, *and* M_2^μ/D *is* λ^+*-compact, then* M_1^μ/D *is* λ^+*-compact.*

C) *In* B) *if* M_1 *is* λ^+*-compact,* M_2 λ*-universal, then the regularity of* D *is superfluous.*

D) *In the hypothesis of* A) *(and also* B), C)) *we can replace "for every* $\Phi_1 \subset L(T_1)$*," by "for every* $\Phi_1 \in K$ *" where* K *is a class of sets of formulas of* $L(T)$ *such that:*

if N_1 *is a non-*λ^+*-compact model of* T_1*, then there is a type* $p = \{\varphi_k(x, \bar{a}^k) : k < k_0 \leq \lambda\}$ *over* N_1 *which* N_1 *omit and* $\{\varphi_k(x, \bar{y}^k) : k < k_0\} \subset \Phi \in K$ *for some* Φ.

Remark. This and Theorem 2.5 generalize Keisler [6], Th. 2.1, p. 29. The generalization [6], Th. 2.3, p. 33, is seemingly incorrect. (On the one hand assume too little – an assumption like 2.2, and conclusion like 2.5; and on the other hand the pattern includes superfluous information). Nevertheless, the generalization goes easily.

Proof. We shall prove only the conclusion of C) by the hypothesis of D). The other cases follow or have similar proofs (or, alternatively, using Keisler [6], p. 29, Th. 2.1). So suppose M_1 is a λ^+-compact model of T_1, M_2 a λ-universal model of T_2, D an ultrafilter over μ, M_2^μ/D is λ^+-compact; and we should prove M_1^μ/D is λ^+-compact. Suppose this is not so, and we shall get a contradiction.

As $N_1 = M_1^\mu/D$ is not λ^+-compact, it omits a type (over N_1) $p = \{\varphi_k(x, \bar{a}^k) : k < k_0 \leq \lambda\}$. By the definition of K, we can assume $\Phi = \{\varphi_k(x, \bar{y}^k); k < k_0\} \subset \Phi_1 \in K$. By assumption there are $\Phi_2 \subset L(T_2)$, $G, m_2 < \omega$, such that $G : \langle \Phi_1, 1 \rangle \leq \langle \Phi_2, m_2 \rangle$. By Lemma 2.1A we can assume $\Phi = \Phi_1$. Let $G[\varphi_k(x, \bar{y}^k)] = \Psi_k(\bar{x}, \bar{z}^k)$ $(l(\bar{x}) = m_2)$.

By Definition 2.1, remembering M_2 is λ-universal, for every $i < \mu$ there are $\bar{b}^k[i] \in |M_2|$, $k < k_0$ such that:

for every $w \subset k_0$
$\{\varphi_k(x, \bar{a}^k[i]) : k \in w\}$ is consistent over M_1
iff $\{\Psi_k(\bar{x}, \bar{b}^k[i]) : k \in w\}$ is consistent over M_2.

As $\bar{b}^k[i]$ is defined for every $i < \mu$, $\bar{b}^k \in M_2^\mu/D$ is also defined.

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Let $q = \{\Psi_k(\bar{x}, \bar{b}^k) : k < k_0\}$, and we shall show q is consistent over M_2. For let $w \subset k_0$, $|w| < \aleph_0$. We should prove $\models (\exists x) \bigwedge_{k \in w} \Psi_k(\bar{x}, \bar{b}^k)$. This follows from Łoś theorem, definition of $b^k[i]$ and consistency of p.

So $\{\Psi_k(\bar{x}, \bar{b}^k) : k \in w\}$ is consistent over M_2^μ/D. As this is true for every finite $w \subset k_0$, q is consistent over M_2^μ/D.

Now as M_2^μ/D is λ^*-compact, there is a sequence $\bar{c}$ from it that realizes q. We shall prove that p is realized in M_1^μ/D, and get the contradiction. Let for $i < \mu$

$$w[i] = \{k < k_0 : M_2 \models \Psi_k[\bar{c}[i], \bar{b}^k[i]]\} .$$

Clearly $q[i] = \{\Psi_k(\bar{x}, \bar{b}^k[i]) : k \in w[i]\}$ is consistent. So, as before, by the definition of the $\bar{b}^k[i]$, also $p[i] = \{\varphi_k(x, \bar{a}^k[i]) : k \in w[i]\}$ is consistent over M_1. As M_1 is λ^*-compact there is $c[i]$ that realizes $p[i]$. So $c \in M_1^\mu/D$ is defined. Now for every $k < k_0 : M_2^\mu/D \models \Psi_k[\bar{c}, \bar{b}^k]$ (By the definition of $\bar{c}$). Hence:

$$\{i < \mu : M_2 \models \Psi_k[\bar{c}[i], \bar{b}^k[i]]\} \in D \qquad \text{or}$$

$$\{i < \mu : k \in w[i]\} \in D \qquad \text{so by the definition of } c[i]$$

$$\{i < \mu : M_1 \models \varphi_k[c[i], \bar{a}^k[i]]\} \in D \qquad \text{hence}$$

$$M_1^\mu/D \models \varphi_k[c, \bar{a}^k] .$$

So c realizes p, contradiction.

2.3. Definition. Let $\Phi_1 \subset L(T_1)$, $\Phi_2 \subset L(T_2)$, $\Phi_1 = \{\varphi_k(\bar{x}, \bar{y}^k) : k < k_0\}$, $l(\bar{x}) = m_1$; G a function $G[\varphi_k(\bar{x}, \bar{z}^k)] = \Psi_k(\bar{y}, \bar{z}_k) \in \Phi_2$, $l(\bar{y}) = m_2$.

Then $G : \langle \Phi_1, m_1 \rangle \leq^* \langle \Phi_2, m_2 \rangle$ if for every model M_1 of T_1, and $\bar{a}_n^k \in M_1$, there are a model M_2 of T_2, and $\bar{b}_n^k \in M_2$ $(k < k_0, n < \omega)$ such that:

for every $w \subset k_0 \times \omega$
$\{\varphi_k(\bar{x}, \bar{a}_l^k) : \langle k, l \rangle \in w\}$ is consistent over M_1
iff $\{\Psi_k(\bar{y}, \bar{b}_n^k) : \langle k, n \rangle \in w\}$ is consistent over M_2.

2.4. Definition. Let $\langle \Phi_1, m_1 \rangle \leq^* \langle \Phi_2, m_2 \rangle$ holds if for some G, $G : \langle \Phi_1, m_1 \rangle \leq^* \langle \Phi_2, m_2 \rangle$ holds.

Lemma 2.3. A) *In Definition 2.3, we can replace* ω *by any* $\alpha > \omega$.

B) $G:\langle \Phi_1, m_1\rangle \leq^* \langle \Phi_2, m_2\rangle$ *implies* $G:\langle \Phi_1, m_1\rangle \leq \langle \Phi_2, m_2\rangle$

C) $\langle \Phi_1, m_1\rangle \leq^* \langle \Phi_2, m_2\rangle$ *implies* $\langle \Phi_1, m_1\rangle \leq \langle \Phi_2, m_2\rangle$

D) *If* Φ_1, Φ^1 *contain the same formulas (with a different number of repetitions) then*

$$\langle \Phi_1, m_1\rangle \leq^* \langle \Phi_2, m_2\rangle \Leftrightarrow \langle \Phi^1, m_1\rangle \leq^* \langle \Phi_2, m_2\rangle$$

E) *If* $\Phi^1 \subset \Phi_1$, $\Phi_2 \subset \Phi^2$ *then* $\langle \Phi_1, m_1\rangle \leq^* \langle \Phi_2, m_2\rangle$ *implies* $\langle \Phi^1, m_1\rangle \leq^* \langle \Phi^2, m_2\rangle$.

F) $\langle \Phi_1, m_1\rangle \leq^* \langle \Phi_1, m_1\rangle$ *(by the identity map).*

Proof. Immediate.

Lemma 2.4. *The following statements about* T_1, T_2 *are equivalent.*

A) *For every* $\Phi_1 \subset L(T_1)$ *there are* $\Phi_2 \subset L(T_2)$ *and* $m_2 < \omega$ *such that* $\langle \Phi_1, 1\rangle \leq \langle \Phi_2, m_2\rangle$.

B) *For every* $\Phi_1 \subset L(T_1)$, $|\Phi_1| \leq |T_1| + |T_2|^+$ *there are* $\Phi_2 \subset L(T_2)$ *and* m_2 *such that* $\langle \Phi_1, 1\rangle \leq \langle \Phi_2, m_2\rangle$.

C) *For every* $\Phi_1 \subset L(T_1)$ *there are* $\Phi_2 \subset L(T_2)$ *and* m_2 *such that* $\langle \Phi_1, 1\rangle \leq^* \langle \Phi_2, m_2\rangle$.

D) *For every* $\Phi_1 \subset L(T_1)$, $|\Phi_1| \leq |T_1|$ *there are* $\Phi_2 \subset L(T_2)$ *and* m_2 *such that* $\langle \Phi_1, 1\rangle \leq^* \langle \Phi_2, m_2\rangle$.

E) *Let* Φ_0 *be the set of formulas* $\varphi(x, \bar{y}) \in L(T_1)$ *(clearly* $|\Phi_0| = |T_1|$*). There are* $\Phi_2 \subset L(T_2)$, m_2 *such that* $\langle \Phi_0, 1\rangle \leq^* \langle \Phi_2, m_2\rangle$.

Proof. Clearly A → B, C → D → E. So we should prove B → C, E → A only.

Suppose E) holds, and we shall prove A). Let $\Phi_1 \subset L(T_1)$; clearly Φ_1 has a subset Φ such that every formula which appears in Φ_1 appears in Φ exactly once. Hence $\Phi \subset \Phi_0$ [of E)], so by Lemma 2.3E $\langle \Phi, 1\rangle \leq^* \langle \Phi_2, m_2\rangle$ [Φ_2 – of E)]. By Lemma 2.3D also $\langle \Phi_1, 1\rangle \leq^* \langle \Phi_2, m_2\rangle$, and so by 2.3C $\langle \Phi_1, 1\rangle \leq \langle \Phi_2, m_2\rangle$. So A) holds.

Now suppose that B) holds, and we shall prove C). Let $\lambda = |T_1| + |T_2|^+$, and let $\Phi_1 \subset L(T_1)$. We should prove that there are $\Phi_2 \subset L(T_2)$, m_2 such that $\langle \Phi_1, 1\rangle \leq^* \langle \Phi_2, m_2\rangle$. By Lemma 2.3D we can assume without loss of generality that no formula appears in Φ_1 twice, hence $|\Phi_1| \leq$

$|T_1| \leq \lambda$. Let $\Phi_1 = \{\varphi_k(x, \bar{y}^k) : k < k_0\}$. Let $\Phi^1 \subset L(T_1)$ be such that every formula of $L(T_1)$ appears in it exactly $|T_2|^+$ times. By B) there are $\Phi_2 \subset L(T_2)$, m_2 and G such that $G : \langle \Phi^1, 1\rangle \leq \langle \Phi_2, m_2\rangle$. Now each formula $\varphi_k(x, \bar{y}^k) \in L(T)$ appears in Φ^1 $|T_2|^+$ times, but there are $|T_2|$ formulas in $L(T_2)$. So for some $\Psi_k(\bar{x}, \bar{z}^k) \in L(T_2)$, for $|T_2|^+$ appearances of $\varphi_k(x, \bar{y}^k)$ in Φ^1, $G[\varphi_k(x, \bar{y}^k)] = \Psi_k(\bar{x}, \bar{z}^k)$. So define $G_1 : \Phi_1 \to \Phi_2$ by $G_1[\varphi_k(x, \bar{y}^k)] = \Psi_k(\bar{x}, \bar{z}^k)$. It is easy to check that $G_1 : \langle \Phi_1, 1\rangle \leq^* \langle \Phi_2, m_2\rangle$.

Theorem 2.5. A) *If Φ_0 is the set of all formulas in $L(T_1)$, and for some $\Phi_2 \subset L(T_2)$, $m_2 < \omega$, $\langle \Phi_0, 1\rangle \leq^* \langle \Phi_2, m_2\rangle$ then $T_1 \lhd^* T_2$.*

B) *In fact it suffices to demand that there are Φ_i $i < i_0$ such that: if M_1 is a non-λ^+-compact model of T_1, then there is a type p over M_1, $p = \{\varphi_k(x, \bar{a}^k) : k < k_0 < \lambda^+\}$, such that for some $i < i_0$ every $\varphi_k(x, \bar{y}^k) \in \Phi_i$; and there are $\Phi_{2,i} \subset L(T_2)$ $m_{2,i} < \omega$ such that $\langle \Phi_i, 1\rangle \leq^* \langle \Phi_{2,i}, m_{2,i}\rangle$.*

Proof. It is very similar to that of Theorem 2.2, so we omit it. The only differences between the proofs are that here we cannot treat each $i < \mu$ separately, but all together; and that we use $\leq^*$ instead of $\leq$ and Lemmas 2.3, 2.4 are also used.

Theorem 2.6. A) *If T has the strict order p. (see Shelah [13], Def. 4.2) then $T_{\text{ord}} \lhd^* T$, hence $T_{\text{ord}} \lhd T$. Also the other conclusions of 2.2 hold for $T_1 = T_{\text{ord}}$, $T_2 = T$.*

B) *If T has the independence p (Shelah [13], Def. 4.1) then $T_{\text{ind}} \lhd^* T$ hence $T_{\text{ind}} \lhd T$. Also the other conclusion of 2.2 holds for $T_1 = T_{\text{ind}}$, $T_2 = T$.*

C) *If T is unstable (Shelah [13], Def. 2.1D) then $T_{\text{ind}} \lhd^* T$ or $T_{\text{ord}} \lhd^* T$ (or both hold).*

Remark. T_{ord} is the theory of the rational order. T_{ind} is defined in [13] Th. 4.7.

Proof. A) and B) imply C) by [13], Th. 4.1. Now it is easy to check that for T_{ord}, $i_0 = 1$, $\Phi_0 = \{x < y, \neg x < y\}$ satisfies the requirement of 2.5B; and for T_{ind}, $\Phi_0 = \{P(x), z_1 E x, \neg z_2 E x\}$, $\Phi_1 = \{\neg P(x), x E z_1, \neg x E z_2\}$, $i_0 = 2$ satisfy those requirements. Hence the conclusion follows by 2.5B.

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2.5. Definition. A complete theory T is simple if it satisfies the following.

A) In L(T) there are one two-place predicate xEy, and one-place predicates. For every model M of T, E^M is an equivalence relation over $|M|$. (Also the equality sign $\in$ L(T)). For a model M of T, $a \in M$ let

$$[a]_M = \{b \in M : M \models b\,E\,a, \text{ for every predicate } P(x) \text{ of } L(T),$$
$$M \models P(a) \equiv P(b)\}.$$

B) There is a model M of T such that for every $a \in M$, $[a]_M$ is infinite.

C) There is a model M of T such that for every $a \in M$, there are infinitely many $b \in M$ from different E-equivalence classes which realize the same type.

Lemma 2.7. *Let T be a simple theory.*

A) *If M is a model of T, $a \in M$, then any permutation of $[a]_M$ is an automorphism of M.*

B) *Every formula (of L(T)) is equivalent to a boolean combination of formulas of the following forms*

$$1)\ x = y, \qquad 2)\ x\,E\,y, \qquad 3)\ P(x),$$

$$4)\ (\exists y)[xEy \wedge \bigwedge_{j<n} P_j(y) \wedge \bigwedge_{j<m} \neg P^j(y)]\ .$$

C) *T is stable in every $\lambda \geq 2^{|T|}$ (stable – see [13], Def. 2.1D). So T is superstable.*

Proof. Immediate.

Lemma 2.8. *Suppose M is a non λ^+-compact model of a simple theory T. Then M omit a type p (over M) which is of one of the following forms.*

$$1)\ p = \{xEa\} \cup \{P_k(x)^{\eta_1(l)} : l < l_0 \leq \min(\lambda, |T|)\} \cup \{x \neq c_k : k < k_0 \leq \lambda\}$$

$$2)\ p = \{P_k(x)^{\eta_1(l)} : l < l_0 \leq \min(\lambda, |T|)\} \cup p_0 \cup \{\neg xEc_k : k < k_0 \leq \lambda\}$$

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where p_0 consist of formulas of the fourth form from Lemma 2.7B, and negations of such formulas. (η is a sequence of ones and zeroes, $\varphi^0 = \varphi$, $\varphi^1 = \neg\varphi$)

Proof. As M is not λ^+-compact, M omits a 1-type q, $|q| \le \lambda$. Without lose of generality suppose $|L(T)| \le |q| + \aleph_0 = |q|$, because otherwise we can replace M by an appropriate reduct. So there is $A \subset |M|$, $|A| \le |q| \le \lambda$ such that q is a type over A, so there is a type $q_1 \in S(A)$, $q \subset q_1$, and clearly q_1 is also omitted.

It is clear that *if* $q_2 \subset q_1$ and:

> for every $\varphi \in q_1$ there are $\Psi_1, \ldots, \Psi_n \in q_2$ such that $M \models (\forall x)\left[\bigwedge_{m=1}^{n} \Psi_m \to \varphi\right]$ (x – the only free variable in the formulas of q_1),

then M omits also q_2.

So if q_2 is a subtype of q_1 consisting of the formulas of the forms mentioned in 2.7B and their negations, then clearly M omits q_2.

Now our proof split to two cases, according to whether some $x\,E\,a$ belong to q_2 or not.

Case I. $x Ea \in q_2$. Clearly no formula $x = c$ belongs to q_2 (otherwise c will realize q_2). So for every $c \in A$, $(x \ne c) \in q_1 \in S(A)$ hence $(x \ne c) \in q_2$. Clearly if $\varphi = x Ea_1 \in q_2$ then as q_2 is consistent over M, $M \models (\forall x)[x Ea \to \varphi]$. Similarly if $\varphi = \neg x Ea_1 \in q_2$, $M \models (\forall x)(x Ea \to \varphi)$.

Similar implications hold if $\varphi \in q_2$ is of the form $(\exists y)[x Ey \wedge \bigwedge_l P_l(y) \wedge \bigwedge_l \neg P^l(y)]$ or its negation. So if p is the subtype of q_2 consisting of the formulas $x Ea$, $P_k(x)$ [if $P_k(x) \in q_2$] $\neg P_k(x)$ [if $\neg P_k(x) \in q_2$] and $x \ne c$ for $c \in A$ then M omits p, and p is of the form 1); and $|p| \le |q_1| \le \lambda$.

Case II. For no a $x\,E\,a \in p$. Clearly for every $c \in A$, $x \ne c$, $\neg x Ec \in q_2$ and $M \models (\forall x)(\neg x Ec \to x \ne c)$. Hence it is clear that $p = q_2 - \{x \ne c : c \in A\}$ is omitted in M and it is of the form 2).

Lemma 2.9. *If M is a λ-compact model of a simple theory T, and $N = M^I_D | G$ is $|T|^+$-compact then N is λ-compact. (In fact it is $\lambda^I_D | G$-compact.)*

Proof. If $\lambda \leq |T|^+$, then there is nothing to be proved. So suppose $\lambda > |T|$. Assume N is not λ-compact and we shall get a contradiction. By the previous lemma we can assume N omits a type p which is of one of the forms mentioned there. So we have two cases.

Case I. M omits p (which is consistent over M) where

$$p = \{xEa\} \cup \{P_l(x)^{\eta(l)} : l < |T|\} \cup \{x \neq c_k : k < k_0 < \lambda\}$$

(there are $|T|$ one place predicates in $|T|$); clearly it suffices to prove that at least λ-elements of N realize p_1, where

$$p_1 = \{xEa\} \cup \{P_l(x)^{\eta(l)} : l < |T|\}.$$

As $|p_1| \leq |T|$ and N is $|T|^+$-compact, some $b \in N$ realize p_1. As M is λ-compact, for every $i \in I$, $[b[i]]_M$ is a set of cardinality λ. So we can define for every $k < \lambda$, $i \in I$, an element $b_k[i] \in M$ such that: $k \neq l \Rightarrow b_k[i] \neq b_l[i]$; $b[i] = b[j] \Rightarrow b_k[i] = b_k[j]$. Hence for every k, $b_k \in |M|^I$ is defined, and $\mathrm{eq}(b_k) = \mathrm{eq}(b) \in G$ hence $b_k \in N$. It is also clear that each b_k belongs to $[b]_N$, and $k \neq l \Rightarrow b_k \neq b_l$. As every element in $[b]_N$ realizes p_1, p_1 is realized $\geq \lambda$ times in N. Hence p is realized in N, contradiction.

Case II. M omits p which is of form 2) from Lemma 2.8. The proof is similar to that of Case I, except that here we should find λ non-E-equivalent elements of N realizing a type over the empty set. Here we use part C) of Definition 2.5 instead of Part B).

The proof that N is $\lambda^I_D | G$-compact is similar, so we omit it.

Corollary 2.10. A) *A simple countable theory is $\triangleleft^*$-minimal, and hence $\triangleleft$-minimal.*

B) *if M is a model of a simple theory T, D a $|T|^+$-good ultrafilter on μ, then M^μ/D is $\aleph_0^\mu/D$-compact. Hence if D is $(\aleph_0, \mu)$-regular, M^μ/D is 2^μ-compact.*

Proof. Immediate.

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Theorem 2.11. *For every theory T_1 and cardinal λ there is a simple theory T_2 such that $T_1 \lhd_\lambda T_2 \lhd_\lambda T_1$. If $|T_1| \leq \lambda$ then also $|T_2| \leq \lambda$. Moreover if D is a $(\aleph_0, \lambda)$-regular ultrafilter over μ, M_1 a model of T_1, M_2 a model of T_2 then M_1^μ/D is λ^+-compact iff M_2^μ/D is λ^+-compact.*

Proof. We shall deal only with the case $|T_1| \leq \lambda$. The other case follows from Theorem 2.12.

Let Φ_1 be the set of formulas of T_1 each repeated λ times. Clearly $|\Phi_1| = \lambda$. It is also clear that if for some $\Phi_2 \subset L(T_2)$, $\langle \Phi_1, 1\rangle \leq \langle \Phi_2, 1\rangle$ then $T_1 \lhd_\lambda T_2$. (Because if $\Phi^1 \subset L(T_1)$, $|\Phi^1| \leq \lambda$ then $\Phi^1 \subset \Phi_1$, and our conclusion follows by 2.1, 2.2).

Let $\Phi_1 = \{\varphi_k(x, \bar{y}^k) : k < \lambda\}$.

We shall now define a model M_2, and T_2 will be its theory. We list the properties of M_2 we need, and it is trivial that M_2 exists:

1) The relations of M_2 are an equivalence relation $E = E^{M_2}$, and for each $k < \lambda$ a monadic relation $P_k = P_k^{M_2}$.

2) For every $a \in M_2$, $[a]_{M_2}$ is infinite.

$$[[a]_{M_2} = \{b : b \in M_2, aEb, \text{ and } P_k(a) \equiv P_k(b) \text{ for every } k < \lambda\}]$$

3) For every model M_1 of T_1 and $\bar{a}_k \in M_1$, $k < \lambda$ there are infinitely many $a \in M_2$ such that they are not E-equivalent and

(*) for every $w \subset \lambda$, $\eta \in {}^\lambda 2$

$\{\varphi_k(x, \bar{a}_k)^{\eta(k)} : k \in w\}$ is consistent over M_1, *iff*

$\{xEa \wedge P_k(x)^{\eta(k)} : k \in w\}$ is consistent over M_2.

4) For every $a \in M_2$ there are a model M_1 of T_1 and $\bar{a}_k \in M_1$ $k < \lambda$ such that (*) holds.

Remark. We can replace "for every M_1" by a fixed λ-universal model M_1 of T_1.

Now let T_2 be the theory of M_2. Clearly T_2 is simple, $|T_2| = \lambda$. Let $\Phi_2 = \{xEy \wedge P_k(x) : k < \lambda\}$.

By 3) in the Definition of M_2, $\langle\Phi_1, 1\rangle \le \langle\Phi_2, 1\rangle$. Hence by 2.2A $T_1 \lhd_\lambda T_2$. By 2.2B, if D is $(\aleph_0, \lambda)$-regular, over I; M_2^I/D is λ^+-compact implies M_1^I/D is λ^+-compact. We should prove that M_2^I/D is not λ^+-compact implies M_1^I/D is not λ^+-compact. By Lemma 2.8 there are two cases.

Case I. $N_2 = M_2^I/D$ omits a type p (which is consistent over N_2)

$$p = \{xEa\} \cup \{P_k(x)^{\eta(k)} : k \in w \subset \lambda\} \cup \{x \ne c_k : k < k_0 \le \lambda\}$$

By extending the type we can assume $w = \lambda$. Let $p_1 = \{xEa\} \cup \{P_k(x)^{\eta(k)}$ $k < \lambda\}$.

As in the proof of Lemma 2.9 it follows that N_2 omits p_1. By condition 4) in the definition of M_2,

$$\langle\langle xEa \wedge P_k(x)^{\eta(k)} : k < \lambda\rangle, 1\rangle \le \langle\langle\varphi_k(x, \bar{y}^k)^{\eta(k)} : k < \lambda\rangle, 1\rangle$$

(We extend $L(T_\lambda)$ to include a, temporarily, and also extend Γ_λ accordingly.) So by Theorem 2.2, in fact, M_1^I/D is also not λ^+-compact.

Case II. M_2^I/D omits p [p_0 as in 2.8, 2)].

$$p = \{P_k(x)^{\eta(k)} : k \in w \subset \lambda\} \cup \{\neg xEc_k : k < k_1 \le \lambda\} \cup p_0$$

Let $p_1 = \{P_k(x)^{\eta(k)} : k \in w \subset \lambda\} \cup p_0$.

By the proof of 2.9, M_2^I/D omits p_1. But by Keisler [6], Th. 1.5, M_2^I/D is λ-universal, contradiction.

Theorem 2.12. *For every set $\{T_k : k < k_0\}$ of theories there is at least upper bound for each of the orderings $\lhd^*$, $\lhd$, $\lhd_\lambda$. Its cardinality is $\le \Sigma_k |T_k|$.*

Proof. Let Q_k, $k < k_0$ be k_0 new one-place predicates. Let

$$T = \{\neg(\exists x)[Q_k(x) \wedge Q_l(x)] : k, l < k_0, k \ne l\} \cup \{\Psi^{Q_k} : \Psi \in T_k, k < k_0\} \cup \{(\forall x_1 \dots x_n)[R(x_1, \dots, x_n) \to \bigwedge_{i=1}^{n} Q_k(x_i)] : R \text{ of } L(T_k)\}$$

[Ψ^Q is Ψ relativized to Q – $(\exists x)\varphi$ is replaced by $(\exists x)(Q(x) \wedge \varphi)$].

It is clear that T satisfies our demands.

Using the last two theorems we can prove many properties of the order $\lhd$ between theories, if we know something about the order among countable theories.

Theorem 2.13. A) *For every* λ *there is a simple theory* T_λ, $|T_\lambda| = \lambda$ *such that* T_λ *is* $\lhd_\lambda$*-maximal. Hence if* $\lambda < \mu$, $T_\lambda \lhd T_\mu$ *but not* $T_\mu \lhd T_\lambda$. *So there is an (uncountable) theory which is not* $\lhd$*-minimal nor* $\lhd$*-maximal.*

B) *If there is a countable theory* T *which is not* $\lhd$*-minimal nor* $\lhd$*-maximal* (see Th. 4.11) *then there are* $\lhd$*-incomparable theories.*

Proof. A) Let T^1 be the (full) theory of numbers. By Keisler [6] T^1 is $\lhd$-maximal, and if M^1 is a model of T^1, D an $(\aleph_0, \lambda)$-regular ultrafilter on λ, then $(M^1)^\lambda/D$ is λ^+-saturated iff D is λ^+-good. By 2.11, for every λ there is a simple theory T_λ, $|T_\lambda| = \lambda$, such that $T^1 \lhd_\lambda T_\lambda \lhd_\lambda T^1$. By the construction (and also by Th. 2.11 itself) it is clear that for $\lambda < \mu$, $T_\lambda \lhd T_\mu$. Not $T_\mu \lhd T_\lambda$, follow from the existence of λ^+-good but not λ^{++}-good $(\aleph_0, \mu)$-regular ultrafilters on μ.

This is by 2.10B and the definitions. The existence of such D follows from Kunen [12], and Keisler [10].

B) By 4.1B we can choose such T, such that if M is any model of T, D a $(\aleph_0, \lambda)$-regular ultrafilter over λ, then M^λ/D is not λ^+-compact iff for some n_i, $\aleph_0 \le \Pi n_i/D \le \lambda$. Hence by 2.10B ($M_1$ from 2.11) M_1^I/D is μ^+-compact, but M^I/D is not μ^+-compact. So not $T \lhd T_\lambda$.

On the other hand as T is not maximal, there is an ultrafilter D over a set I, such that D is not good, but $\aleph_0 \le \Pi n_i/D \Rightarrow |I| < \Pi n_i/D$. Define $\lambda = |I|$. So M_1^I/D is $|I|^+$-compact, but as D is not good, $\lambda = |I|$, M^I/D is not $|I|^+$-compact. So not $T_\lambda \lhd T_\mu$.

Conjecture. Every theory is the least upper bound of a set of $\le 2^{\aleph_0}$ countable theories and a simple theory of cardinality $|T|$.

§ 3. Unsaturated Ultrapowers

Theorem 3.1. *Let T be with the* f.c.p., $\mu = \prod m_i/D$, *D an ultrafilter over I. Then M^I/D is not μ^+-compact, hence is not $(2^{|I|})^+$-compact.*

Remark. The f.c.p. was first defined in Keisler [6], p. 38. This is essentially Theorem 4.1, p. 39 Keisler [6], and we repeat it for completeness only.

Proof. Let $\lambda = \min\{\prod n_i/D : \prod n_i/D \geq \aleph_0\}$ and $\lambda = \prod n_i/D$. By the definition of f.c.p., there is a formula $\varphi(x, \bar{y})$ of $L(T)$, such that for arbitrarily large natural numbers n, the following holds:

(*) there are $\bar{a}_n^0, \ldots, \bar{a}_n^{n-1}$ such that

$$M \models \neg(\exists x) \bigwedge_{i=0}^{n-1} \varphi(x, \bar{a}_n^i)$$

and for $j < n$ $$M \models (\exists x) \bigwedge_{\substack{i=0 \\ i \neq j}}^{n-1} \varphi(x, \bar{a}_n^i)$$

Let for every $i \in I$, $f(i)$ be the maximal number $\leq n_i$ for which (*) holds. Hence $f(i) \leq n_i$, hence $\prod f(i)/D \leq \prod n_i/D = \lambda$. On the other hand for every n^0 there is $n^1 \geq n^0$ for which (*) holds. So $n_i \geq n^1$ implies $f(i) \geq n^1 \geq n^0$. So

$$\{i : n_i \geq n^1\} \subset \{i : f(i) \geq n^1\} \subset \{i : f(i) \geq n^0\}$$

As $\prod n_i/D \geq \aleph_0$, $\{i : n_i \geq n^1\} \in D$, hence $\{i : f(i) \geq n^0\} \in D$, hence $\prod f(i)/D \geq n^0$. As n^0 is arbitrary, $\prod f(i)/D \geq \aleph_0$, so by the definition of λ, $\prod f(i)/D = \lambda$.

Let $P^i = \{\bar{a}_{f(i)}^0, \ldots, \bar{a}_{f(i)}^{f(i)-1}\}$. It is easy to see that the models (M, P^i) satisfy the following sentences

(i) $\neg(\exists x)(\forall \bar{y})[P(\bar{y}) \rightarrow \varphi(x; \bar{y})]$

(ii) $(\forall \bar{y})[P(\bar{y}) \rightarrow (\exists x)(\forall \bar{z})(P(\bar{z}) \wedge \bar{y} \neq \bar{z} \rightarrow \varphi(x; \bar{z}))]$.

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Let $(N, P^N) = \Pi_i (M, P^i)/D$. Clearly $|P^N| = \Pi |P^i|/D = \Pi f(i)/D = \lambda$. As the sentences (i), (ii) are satisfied by every (M, P^i), they are satisfied by (N, P^N). So $p = \{\varphi(x, \bar{a}) : \bar{a} \in P^N\}$ is a type over N, (by (ii)) but is omitted (by (i)), and $|p| = |P^N| = \lambda \leq \mu$. So N is not μ^+-compact.

Theorem 3.2. *Let M be a model of T, T has the f.c.p. . Let $\varphi(x; \bar{y}) \in L(T)$, and P, the set of $n < \omega$ for which (*) (from 3.1) is satisfied, is infinite. Let $(N_1, <, P^N) = (\omega, <, P)^I_D | G$, $a \in P^N$, $\mu = |\{b \in N : b < a\}|$. Then over $M^I_D | G$ there is a type p, $|p| = \mu$, which is omitted, but $q \subset p$, $q \neq p \Rightarrow q$ is realized. Moreover, p consists of formulas of the form $\varphi(x, \bar{a})$ only.*

Proof. Clear from 3.1.

Theorem 3.3. *Let M be a model of an unstable theory T, $\Pi_{i \in I} m_i/D < 2^\lambda$. Then M^I/D is not λ^+-compact.*

Proof. Let $\mu = \min\{\Pi n^i/D : \Pi n^i/D \geq \aleph_0\}$, $\mu = \Pi n^i/D < 2^\lambda$, $n_i = [\log_2 n^i - 1]$ ($[x]$ - the integral part of x). Clearly $\aleph_0 \leq \Pi n_i/D \leq \Pi n^i/D = \mu$, hence by the definition of μ, $\Pi n_i/D = \mu$. By [13], Th. 4.1A there is a formula $\varphi = \varphi(x; \bar{y}) \in L(T)$ which has the strict order p, or the independence p. For simplicity let $\varphi = \varphi(x; y)$.

By the definitions for every $i \in I$ there are elements $a_i^0, \ldots, a_i^{n_i - 1}$ of M such that:

(i) if φ has the independence p, then for every $w \subset n_i$,

$$\{\varphi(x, a_i^k)^{\text{if } (k \in w)} : k < n_i\} \text{ is consistent over } M$$

(ii) if φ has not the independence p, (hence has the strict order p) for $k, l < n_i$

$$M \vDash (\exists x)[\neg\varphi(x, a_i^k) \wedge \varphi(x, a_i^l)] \text{ iff } k < l$$

Let $P_i = \{a_i^k : k < n_i\}$, and $S_i \subset |M|$ be such that:

(1) for every $a \in M$ there is $b \in S_i$ such that:

$$\text{for every } c \in P_i,\ M \vDash \varphi[a, c] \equiv \varphi[b, c]$$

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(2) there are no $a, b \in S_i$, $a \neq b$, such that:

$$c \in P_i \Rightarrow M \models \varphi[a, c] \equiv \varphi[b, c].$$

Clearly $n_i \leq |S_i| \leq 2^{n_i} \leq n^i$ as $|P_i| = n_i$. Let $N = M^I/D$, $(N, P^N, S^N) = \prod_{i \in I}(M, P_i, S_i)/D$. Clearly $|P^N| = \prod |P_i|/D = \prod n_i/D = \mu$, and $|S^N| = \prod |S_i|/D \leq \prod n^i/D = \mu$, $|S^N| \geq \prod n_i/D = \mu$, so $|S^N| = \mu$.

Now we split the proof to two cases.

Case I. $\varphi(x; y)$ has the independence p. Let $\lambda_1 = \min(\lambda, \mu)$, and choose $A \subset P^N$, $|A| = \lambda_1$. By the definition of the P_i's, clearly for every $B \subset A$, $p_B = \{\varphi(x, a)^{\text{if } (a \in B)} : a \in A\}$ is consistent over N. Now by the definition of the S_i, if p_B is realized in N, it is realized by some element of S^N. Hence the number of types p_B, which are realized in N is $\leq |S| = \mu$ (because $B_1 \neq B_2$ implies no elements realized both p_{B_1} and p_{B_2}). On the other hand the number of such types is $|\{B : B \subset A\}| = 2^{|A|} = 2^{\lambda_1}$. Clearly $2^\mu > \mu$, and by hypothesis and definition of μ, $2^\lambda > \mu$; hence $2^{\lambda_1} > \mu$. So for some $B \subset A$, N omit p_B, and as $|p_B| = \lambda_1 \leq \lambda$, N is not λ^+-compact.

Case II. $\varphi(x, y)$ has not the independence p, hence has the strict order p.

Let us assume N is λ^+-compact.

Clearly the formula $y < z = (\exists x)[\neg\varphi(x, y) \wedge \varphi(x, z)]$ define an order on P^N. It is easily seen that for every $a \in N$, either $c \in P^N \Rightarrow N \models \varphi(a, c)$ *or* there is $b \in P^N$ such that $c \in P^N \Rightarrow N \models \varphi(a, c) \equiv b < c$ [as the corresponding sentence holds in every (M, P_i)]. Hence *if* there is a set of formulas $\{P(x)\} \cup \{x < c : c \in C_1 \subset P^N\} \cup \{c < x : c \in C_2 \subset P^N\}$ which is finitely satisfied in (N, P^N, S^N) but not realized in it, *then* N will not be λ^+-compact, contradiction. So there is no such set of formulas.

Now we define by induction on $l(\eta)$, $\eta \in {}^{\lambda >}2$ elements $a_\eta b_\eta \in P^N$ such that:

(1) for every n, $(N, P) \models (\exists y_1 \dots y_n)[\bigwedge_{i=1}^{n} P(y_i) \wedge a_\eta < y_1 \wedge y_1 < y_2 \wedge \dots$

$\dots \wedge y_n < b_\eta [$

(2) if $k < l(\eta)$ then $a_{\eta|k} < a_\eta < b_\eta < b_{\eta|k}$

(3) $a_{\eta\langle 0\rangle} < b_{\eta\langle 0\rangle} < a_{\eta\langle 1\rangle} < b_{\eta\langle 1\rangle}$

Clearly the definition is possible hence

$$2^\lambda > \mu = |P^N| \geq |\{a_\eta : \eta \in {}^\lambda 2\}| = |\{\eta : \eta \in {}^\lambda 2\}| = 2^\lambda$$

contradiction. So M^I/D is not λ^+-compact, also in the second case.

Theorem 3.4. *Suppose T has the strict order p, M is a λ^+-universal model of T, D an $(\aleph_0, \lambda)$-regular ultrafilter on λ. Then M^λ/D is not λ^{++}-compact.*

Remarks. 1) this theorem was proved independently by Keisler and the author.

2) The demand of λ^+-universality of M is necessary, because by an unpublished result of Solovay, it is consistent with ZFC + $2^{\aleph_0} > \aleph_1$, that there is an ultrafilter D on ω such that got any countable model M of a countable language, M^ω/D is saturated.

Also, for a weaker result that follows from **ZFC**, see [17].

Proof. Let $\mu = \beth_{(\lambda^+)}$. Note that $\mu^\lambda = \mu$, $\mu^{\lambda^+} > \mu$, and w.l.o.g. $\mu > |T|$. If M^λ/D is not λ^+-compact, the theorem holds. So assume it is λ^+-compact. So by Theorem 2.6, if N is a model of T_{ord} (the theory of dense order) then N^I/D is λ^+-compact. Let $J = {}^{\omega >}(\mu^* + \mu)$, ($\mu^*$ is μ with inverse order). Let $<$ order J by the lexicographic order. Note that $\langle J, < \rangle$ satisfies

(i) J is dense without last and first element

(ii) $s < t$, $s, t \in J$ implies there are $s_i, t_i, i < \mu$ such that

$$i < j < \mu \Rightarrow s < s_i < t_i < s_j < t_j < t .$$

W.l.o.g. assume M is μ^+-saturated. Now as T has the strict order p, and M is universal, there is $\varphi(\bar{x}, \bar{y}) \in L(T)$ and $\bar{a}_s \in |M|$ for $s \in J$ such that:

(iii) $M \models (\exists \bar{x})[\neg\varphi(\bar{x}, \bar{a}_s) \wedge \varphi(\bar{x}, \bar{a}_t)]$ iff $s < t$.

Let $P^M = \{\bar{a}_s : s \in J\}$, $<^M = \{\langle \bar{a}_s, \bar{a}_t \rangle : s < t\}$, and $(N, P^N, <^N) = (M, P^M, <^M)^\lambda/D$. Note that $<^M$ order P^M is in a dense order without first and last element, hence $(P^N, <^N)$ is λ^+-saturated. Notice that also (ii) is satisfied by $(P^N, <^N)$. So we can define $\bar{a}_\eta, \bar{b}_\eta \in P^N$ for $\eta \in {}^{\lambda^+ >}\mu$ such that:

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(A) If $k < l(\eta)$, $\tau = \eta | k$ then $\bar{a}_\tau < \bar{a}_\eta < \bar{b}_\eta < \bar{b}_\tau$

(B) If $i < k < \mu$, then $\bar{a}_{\eta^\frown\langle i\rangle} < \bar{b}_{\eta^\frown\langle i\rangle} < \bar{a}_{\eta^\frown\langle k\rangle} < \bar{b}_{\eta^\frown\langle k\rangle}$.

Now for every $\eta \in {}^{\lambda^+}\mu$, the type

$$p_\eta = \{\neg\varphi(\bar{x}, \bar{a}_{\eta|l}) : l < \lambda^+\} \cup \{\varphi(\bar{x}, \bar{b}_{\eta|l}) : l < \lambda^+\}$$

is consistent over N, and $|p_\eta| = \lambda^+$. If any p_η is omitted – the conclusion of the theorem holds. So p_η is realized by c_η, and clearly $\eta, \tau \in {}^{\lambda^+}\mu$, $\eta \neq \tau$ implies $c_\eta \neq c_\tau$. As in the proof of Theorem 3.3 (the use of S) we see that in N at most μ types $\subset \{\varphi(\bar{x}, \bar{a})^i : i \in 2, \bar{a} \in P^N\}$ are realized. Contradiction.

Lemma 3.5. *If T is $\triangleleft_\lambda$-minimal, $\mu \leq \lambda$, then T is $\triangleleft_\mu$-minimal.*

Proof. By Keisler [6] T is not $\triangleleft_\kappa$ minimal iff there is an $(\aleph_0, \kappa)$-regular ultrafilter D on κ, and a model M of T such that M^κ/D is not κ^+-compact. Assume T is not $\triangleleft_\mu$-minimal. So there is a $(\aleph_0, \mu)$-regular ultrafilter on μ, and a model M of T such that M^μ/D is not μ^+-compact. Let D_1 be a $(\aleph_0, \lambda)$-regular ultrafilter on λ, $D_2 = D_1 \times D$, $I = \lambda \times \mu$, so D_2 is an ultrafilter on I, $|I| = \lambda$ D_2 is $(\aleph_0, \lambda)$-regular and $M^I/D_2 = (M^\lambda/D_1)^\mu/D$ is not μ^+-compact.

Hence not λ^+-compact. So T is not $\triangleleft_\lambda$-minimal. Contradiction.

§ 4. Saturation of ultrapowers and categoricity of pseudo-elementary classes

Theorem 4.1. *Let T be countable theory, M_i a model of T for every $i \in I$, and D an ultrafilter over I. Let $N = \Pi_{i \in I} M_i/D$. Then*
A) *If T has not the* f.c.p., $\lambda = \aleph_0^I/D$, *then N is λ-saturated*
B) *If T is stable and has the* f.c.p. *then N is* max *λ-saturated where*

$$\lambda = \min \{\Pi n_i/D : \Pi n_i/D \geq \aleph_0\}.$$

C) *If T has not the* f.c.p., *each M_i is μ saturated, and*

$$\lambda = \mu^I/D \text{ then } N \text{ is } \lambda\text{-saturated.}$$

D) *For every finite $\Delta \subset L(T)$ let*

$$\lambda_i(\Delta) = \min \{|p| : p \text{ is } \Delta\text{-1-type over } M_i \text{ which is omitted by } M_i\}$$

$$\lambda^* = \min \{\Pi \lambda_i(\Delta)/D : \Delta \subset L(T), |\Delta| < \aleph_0\}.$$

Let λ be the first cardinal, $\lambda = \Pi \lambda^i/D$ for some λ^i, and for every finite $\Delta \subset L(T)$, $\{i : \lambda^i \leq \lambda_i(\Delta)\} \in D$.

Then if T has not the f.c.p., *N is λ-saturated, but not $(\lambda^*)^+$-saturated.*

Remarks. 1) Clearly the results, except D, are the best possible. For example in A), if we choose the M_i as countable models, then $\|N\| = \aleph_0^I/D = \lambda$, hence N is not λ^+-saturated.

2) Instead demanding T is countable, we can demand D is $|T|^+$-good. By Theorem 2.3 this is necessary.

Proof. Notice: as T is countable, for every model M of T and cardinality $\kappa > \aleph_0$, M is κ-compact iff M is κ-saturated.

Now in case B), N is not λ^+-saturated by Theorem 3.1. Similarly we can prove in Case D) N is not $(\lambda^*)^+$-saturated. So it remains to prove that in all cases N is λ-saturated.

Clearly N is $\aleph_1$-saturated. By [13] Th. 5.16, as T is countable and

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stable, it suffices to prove:

if $\{c_i : i < \omega\} \subset |N|$ is an indiscernible set ([13], Def. 5.1, 5.2), then it can be extended in N to an indiscernible set of cardinality λ.

For every $i \in I$ let us choose a family S_i of subsets of $|M_i|$ such that:

1) $|S_i| = \|M_i\|$

2) every finite subset of $|M_i|$ belongs to S_i

3) for every finite $\Delta \subset L(T)$, $n < \omega$, *if* $w \in S_i$ is Δ-n-indiscernible set, $0 \leq \mu \leq \|M_i\|$ and there is a Δ-n-indiscernible set w', $w \subset w' \subset |M_i|$, $|w'| = \mu$, *then* there is $w'' \in S_i$, $|w''| = \mu$, $w \subset w'' \subset |M_i|$ and w'' is Δ-n-indiscernible set.

Let $|M_i| = \{a^i_j : j < \|M_i\|\}$, $S_i = \{w^i_j : j < \|M_i\|\}$. Let us define the relation $\in^i$ on $|M_i| : \in^i = \{\langle a^i_j, a^i_k\rangle : a^i_j \in w^i_k\}$. We shall write $x \in y$ instead of $\in(x, y)$. In the language $L = L(T) \cup \{\in\}$, clearly there is a formula $\varphi_{\Delta,n}(x)$ meaning $\{y : y \in x\}$ is a Δ-n-indiscernible set, for every finite Δ, n.

Now for every $i \in I$ we define P^i_Δ according to the part of the theorem we want to prove; in

A) $P^i = \{a^i_k : |w^i_k| \geq \aleph_0\}$, in

B) $P^i = \{a^i_k : k < \|M_i\|\} = |M_i|$, in

C) $P^i = \{a^i_k : |w^i_k| \geq \mu\}$, in

D) $P^i = \{a^i_k : |w^i_k| \geq \lambda^i\}$

where λ^i are defined such that $\Pi\lambda^i/D \geq \lambda$, and for every finite Δ, $\{i : \lambda^i \leq \lambda_i(\Delta)\} \in D$.

Now the following hold

(*) For every finite $\Delta \subset L(T)$, $n < \omega$ there is $m = m(\Delta, n) < \omega$ such that the set of i's for which the following holds belongs to D:

(**) For every Δ-n-indiscernible set w^i_k, $|w^i_k| \geq m$, there is a Δ-n-indiscernible set w^i_j, $w^i_j \subset w^i_j \in P^i$.

Let us prove it. In part B) it is trivial. In the other parts T has not the f.c.p., so in part A) it follows from [13] Th. 5.5C, in part C) from 5.5B, and in part D) from the proof of Th. 5.5A in [13]. Notice that except in D) (**) holds for every i.

Now clearly (**) is equivalent to a first-order sentence in $L' = L \cup \{\in\} \cup \{P\}$. Let $N' = (N, \in^N, P^N) = \Pi(M_i, \in^i, P^i)/D$. Clearly N' is $\aleph_1$-saturated.

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By (*) clearly the sentences corresponding to (**) are satisfied by N'. Remember we say it suffices to prove that $\{c_i : i < \omega\}$ can be extended in N to an indiscernible set of cardinality λ. As $\{c_i : i < \omega\}$ is an indiscernible set, for every Δ, n it is a Δ-n-indiscernible set. Hence every finite subset of

$$p = \{c_i \in x : i < \omega\} \cup \{\varphi_{\Delta,n}(x) : \Delta \subset L(T),$$
$$|\Delta| < \aleph_0, n < \omega\} \cup \{P(x)\}$$

is satisfies in N', hence p is satisfied in N', say by b. As for every Δ, n, $N' \vDash \varphi_{\Delta,n}(b)$, clearly $w = \{a \in |N| : N' \vDash a \in b\}$ is an indiscernible set, and of course $\{c_i : i < \omega\} \subset w$. As $N' \vDash P[b]$, and $|w| \geq |\{c_i : i < \omega\}| = \aleph_0$, clearly $|w| \geq \lambda$ (the check for each part is easy). So we prove the theorem.

It will be more satisfactory if in 4.1D, $\kappa = \lambda^*$. (This holds if $M_i = M$). For this it suffices to prove

Conjecture A. Let $\langle J, < \rangle = \langle \mu, < \rangle^I / D$. ($<$ - the natural order on ordinals.) For $a \in J$, let $|a| = |\{b \in J : b < a\}|$. Suppose $a_n \in J$ for $n < \omega$, $|a_n| = |a_0|$. *Then* there is $a \in J$, $a \leq a_n$ and $|a| = |a_0|$.

Theorem 4.2. *Let M be a λ-compact model of T, $|T| \leq |I|$, $N = M^I/D$. If N is $(2^{|I|})^+$-compact, then N is λ^I/D-saturated.*

Remarks. 1) This affirms conjecture 4D of Keisler [6], p. 41, which says that N is λ-saturated.

2) For countable T, this theorem follows from Theorems 3.1, 4.1C.

3) Here the proof works also for $\aleph_1$-complete ultrafilter D.

Proof. As N is $(2^{|I|})^+$-compact, by 3.1, T has not the f.c.p. Hence T is stable ([13], Th. 3.8A). As N is $(2^{|I|})^+$-compact, $|I| \leq |T|$, clearly every infinite indiscernible set can be extended to one with cardinality $\geq (2^{|I|})^+$. By [13], 5.16 and 5.11 (remembering that by [13] Th. 4.1A T has not the independence p). It suffices to prove that:

If W_1 is an indiscernible set in N, $|W_1| \geq (2^{|I|})^+$, then there is an indiscernible set W_2, $|W_1 \cap W_2| \geq \aleph_0$, $|W_2| \geq \lambda^I/D$.

Let $\{a_k : k < (2^{|I|})^+\} \subset W_1$. Now the following statement will be proved later.

(*) there is an infinite $w \subset (2^{|I|})^+$ such that for every $i \in I$, $\{a_k[i] : k \in w\}$ is an indiscernible set in M.

We can assume $\lambda > |T|$, as otherwise the conclusion of the theorem is trivial. For every $i \in I$ let P^i be a maximal indiscernible set $\{a_k[i] : k \in w\} \subset P^i \subset |M|$. As M is λ-compact, $\lambda > |T|$, clearly $|P^i| \geq \lambda$. Let $(N, P^N) = \Pi(M, P^i)/D$. Clearly $|P| = \Pi|P^i|/D \geq \lambda^I/D$. Now for every finite $\Delta \subset L(T)$, $n < \omega$, the statement "P is a Δ-n-indiscernible set" is elementary, hence P is an indiscernible set. So $P \subset |M|$, $\{a_k : k \in w\} \subset P$, hence $|P \cap W_1| \geq |\{a_k : k \in w\}| \geq \aleph_0$. So P satisfies the conditions for W_2. Hence we should prove only (*).

As T is stable, by [13], Th. 2.13, $|B| \leq 2^{|I|}$ implies $|S(B)| \leq (2^{|I|})^{|T|} = 2^{|I|}$. It is also clear that for $B_i \subset |M|$, $|B_i| \leq 2^{|I|}$ for every $i \in I$: $|\Pi_{i\in I} S(B_i)| = \Pi_{i\in I} |S(B_i)| \leq (2^{|I|})^{|I|} = 2^{|I|}$.

Define for $k \leq |I|^+$, sets $w_k \subset (2^{|I|})^+$ by induction:

1) $w_0 = \{\ \}$, $w_\delta = \bigcup_{l<\delta} w_l$ for a limit ordinal δ.

2) Let w_α be defined. Then for every $l < (2^{|I|})^+$ there is a unique $k \in w_{\alpha+1}$ such that: for every $i \in I$, $a_k[i]$, $a_l[i]$ realizes the same type in M over $\{a_j[i] : j \in w_\alpha\}$.

Clearly for every k, $|w_k| \leq 2^{|I|}$. Choose $\alpha_0 < (2^{|I|})^+$, $\alpha_0 \notin w_{|I|^+}$. For every $\alpha < |I|^+$, let k_α be the ordinal such that for every $i \in I$, $a_{\alpha_0}[i]$, $a_{k_\alpha}[i]$ realizes the same type over $\{a_j[i] : j \in w_\alpha\}$ and $k_\alpha \in w_{\alpha+1}$. Clearly for every i, $\alpha \leq \beta < \gamma < |I|^+$, $a_{k_\beta}[i]$, $a_{k_\gamma}[i]$ realizes the same type in M over $\{a_{k_l} : l < \alpha\}$.

By [13], Th. 5.17, for every i, there is $l(i) < |I|^+$ such that $\{a_{k_\alpha}[i] : l(i) \leq \alpha < |I|^+\}$ is an indiscernible set. Let $l_0 = \sup_{i\in I} l(i)$, $w = \{k_\alpha : l_0 \leq \alpha < |I|^+\}$. Clearly this is the w required in (*).

Remark. We can in fact find such w of cardinality $(2^{|I|})^+$.

Theorem 4.3. *If T is countable, superstable, and has not the* f.c.p., *then there is T_1, $T \subset T_1$, $|T_1| = 2^{\aleph_0}$ such that* PC(T_1, T) *is categorical in every cardinality $\geq 2^{\aleph_0}$. Moreover every model in* PC(T_1, T) *of cardinality $> \aleph_0$ is saturated.*

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Remark. $PC(T_1, T)$ is the class of reducts to $L(T)$ of models of T_1. Note that by Theorem 4.8, and by [13] Section 0, G.7, G.10: the theorem is the best possible.

Proof. Let M be a countable model of T. We expand M to M_1 by adding names for all the possible relations and functions over $|M|$ (i.e. M_1 is a complete model). Let L_1 be the language of M_1, and T_1 the theory of M_1 (i.e. the set of sentences from L_1 that M_1 satisfied). Clearly T_1 contains its Skolem functions.

Let N_1 be any uncountable model of T_1, and let N be the reduct of N_1 to $L(T)$. It suffices to prove that N is saturated (as by Morley and Vaught [18], every two saturated models of the same complete theory, which are of the same cardinality *are isomorphic*). So let p be any 1-type over N, $|p| < \|N\|$, and it suffices to prove that p is realized in N.

Let p_1 be any extension of p to a complete type over $|N|$, and let $\varphi(x, \bar{a}) \in p_1$ be such that $\mathrm{Deg}\{\varphi(x, \bar{a})\} = \mathrm{Deg}\, p_1$. (see [13], Def. 6.3, Lemma 6.2A, 6.2B). Let $|M| = \{a_i : i < \omega\}$, and let c_i, $i < \omega$ be individual constants in L_1 such that $c_i^{M_1} = a_i$. Clearly there is $a^0 \in |N_1|$, $a^0 \neq c_i^{N_1}$ for $i < \omega$. Define $A = \{F^{N_1}[\bar{a}, a^0] : F \text{ a function symbol in } L_1\}$. Clearly the submodel N_1^* of N_1, $|N_1^*| = A$, is an elementary submodel if N_1 (by the definition of T_1 and Tarski–Vaught Test). Let N^* be the reduct of N_1^* to $L(T)$. Clearly N^* is an elementary submodel of N. We shall show now

(*) N_1^* is $\aleph_1$-compact, hence N^* is $\aleph_1$-saturated.

So let q be a countable type over N_1^*, and we should prove it is realized in N_1^*. Let $q = \{\varphi_i(x, a_0^i, \ldots, a_{n_i}^i) : i < \omega\}$.

As every $a_j^i \in A$, for some $F_{ij} \in L_1$, $a_j^i = F_{i,j}^{M}[\bar{a}, a^0]$. So by substituting we get $q = \{\Psi_i(x, \bar{a}, a^0) : i < \omega\}$. Remembering $|M| = \{a_i : i < \omega\}$, $c_i^{M_1} = a_i$, M_1 is complete; it is clear that there is a function symbol G in L_1 such that for every $a_n, \bar{b}, b^0$ from $|M|$, $G^{M_1}(a_n, \bar{b}, b^0)$ realizes $\{\Psi_i(x, \bar{b}, b^0) : i < m\}$ for the maximal possible $m \leq n$. Clearly for every n

$$M_1 \models (\forall \bar{z})(\forall y)\Big[\bigwedge_{i=0}^{n-1} y \neq c_i \wedge (\exists x) \bigwedge_{i=0}^{n-1} \Psi_i(x, \bar{z}) \rightarrow$$

$$\rightarrow \bigwedge_{i=0}^{n-1} \Psi_i(G(y, \bar{z}), \bar{z})\Big]$$

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As $a^0 \neq c_i^{N^*}$ for $i < \omega$, clearly $G^{N^*}(a^0, \bar{a}, a^0)$ realizes q. So we prove (*).

As N^* is $\aleph_1$-saturated; by [13], 6.8A, 6.8D, we can find $B \subset N^*$, $|B| = \aleph_0$ such that $p_1 | B$ is fixed ([13], Def. 6.5), and we can define $b_i \in N^*$ for $i < \omega$, such that b_i realizes $p_1 | (B \cup \{b_j : j < i\})$. By the definition of a fixed type we can define b_i, $\omega \leq i < \omega + \omega = \omega 2$ such that b_i realizes over $|N| \cup \{b_j : j < i\}$ a type p_i, $p_1 \subset p_i$, $\mathrm{Deg}\, p_1 = \mathrm{Deg}\, p_i$. By [13], Th. 6.12A, $\{b_i : i < \omega 2\}$ is an indiscernible set over B. By [13] Th. 4.1 T has not the independence p. So $\theta(x, c) \in p_1$ implies $\vDash \theta[b_i, c]$ for $\omega < i < \omega 2$. So $\{i < \omega 2 : \vDash \theta[b_i, \bar{c}]\}$ is infinite, so by [13], Th. 5.9, $\{i < \omega 2 : \vDash \neg\theta[b_i, \bar{c}]\}$ is finite, so $\{i < \omega : \vDash \neg\theta[b_i, \bar{c}]\}$ is finite. So if W is an indiscernible set in N, $b_i \in W$ for $i < \omega$, then $\theta(x, \bar{c}) \in p_1$ implies $\{b \in W : N \vDash \neg\theta(b, \bar{c})\}$ is finite. So clearly it suffices to prove that $\{b_i : i < \omega\}$ can be extended in N (not N_1) to an indiscernible set of cardinality $\|N\|$. (Because then all but $\leq |p| + \aleph_0$ elements of the set will realize p.)

Let S be a family of subsets of $|M|$ such that

1) $|S| = \aleph_0$
2) every finite subset of $|M|$ belongs to S.
3) If W is a finite Δ-n-indiscernible subset of M, (Δ a finite subset of L_1), and W can be extended to an infinite Δ-n-indiscernible set in M, *then* there is such extension which belongs to S.

Let $S = \{W_i : i < \omega\}$, and noting $|M| = \{a_i : i < \omega\}$ let $\in^{M_1} = \{\langle a_i a_j \rangle : a_i \in W_j\}$. $P^{M_1} = \{a_j : |W_j| = \aleph_0\}$, where $\in$, P belongs to L_1 and let $F \in L_1$ be such that for every $a_j \in P^{M_1}$, $F^{M_1}(x, a_j)$ is a function from W_j onto $|M|$; and we write $x \in y$ instead of $\in(x, y)$. Clearly for every finite $\Delta \subset L(T)$, $n < \omega$, there is a formula $\varphi_{\Delta,n}(x)$ in L_1 saying that $\{y : y \in x\}$ is a Δ-n-indiscernible set. Let

$$q = \{\varphi_{\Delta,n}(x) : \Delta \subset L(T), n < \omega, |\Delta| < \aleph_0\} \cup$$

$$\cup \{b_i \in x : i < \omega\} \cup \{P(x)\}.$$

It suffices to prove that q is consistent over N_1^*. Because as N_1^* is $\aleph_1$-compact, q is realized, by some element $b \in N_1^*$. Hence $W = \{c \in N_1 : N_1 \vDash c \in b\}$ is an indiscernible set (as $N_1^* \vDash \varphi_{\Delta,n}(b)$, N_1^* is an elementary submodel of N_1). Clearly $b_i \in W$ for $i < \omega$. Also $|W| = \|N\|$ as $N_1 \vDash P[b]$ [using $F^{N_1}(x, b)$].

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Now in order to prove that q is consistent over N_1^* it suffices to prove that every finite subset of it is consistent. By [13], Lemma 5.1C instead of a finite number of $\varphi_{\Delta,n}(x)$ we can take one. So it suffices to prove the consistency of

$$q' = \{P(x), \varphi_{\Delta,n}(x)\} \cup \{b_i \in x : i < m < \omega\}.$$

By [13] Lemma 5.5C for every finite Δ, n there is $r = r(\Delta, n) < \omega$ such that: if $m \geq r$, $\{b_0, \dots, b_m\}$ is a Δ-n-indiscernible set in M, then there is an infinite Δ-n-indiscernible set in M which extends $\{b_0, \dots, b_m\}$. So for $r \geq r(\Delta, n)$

$$M_1 \models (\forall x)(\forall y_0 \dots y_r)\left[\left(\bigwedge_{i<j} y_i \neq y_j \wedge \varphi_{\Delta,n}(x) \wedge \bigwedge_{i\leq r} y_i \in x\right) \rightarrow\right.$$

$$\left.\rightarrow (\exists y)\left(\varphi_{\Delta,n}(y) \wedge P(y) \wedge \bigwedge_{i\leq r} y_i \in y\right)\right]$$

This clearly implies the consistency of q', as $\{b_i : i < \omega\}$ is an indiscernible set (in $I(T)$) and for every $c_1 \dots c_n \in N_1$ there is $c \in N_1$ such that $N_1 \models (\forall x)(x \in c \equiv \bigvee_{i=1}^{n} x = c_i)$.

The following theorems have similar proofs, so we omit them.

Theorem 4.4. A) *If T is countable, without the* f.c.p., *and stable in $\aleph_0$ (i.e. totally transcendental) then there is T_1, $T \subset T_1$, $|T_1| = \aleph_0$, such that* PC(T_1, T) *is categorical in every $\lambda \geq \aleph_0$, and every model of it is saturated.*

B) *If T has the* f.c.p., *is countable and stable in $\aleph_0$, $\lambda \geq 2^{\aleph_0}$ then there is T_1, $T \subset T_1$, $|T_1| = \lambda$ such that* PC(T_1, T) *is categorical in λ and every model of it of cardinality λ is saturated.*

Theorem 4.5. *If T is countable and superstable, then there is T_1, $T \subset T_1$, $|T_1| = 2^{\aleph_0}$ such that* PC(T_1, T) *is categorical in $2^{\aleph_0}$, and every model of it of cardinality $2^{\aleph_0}$ is saturated.*

Remark. We use the following fact: if M_1 is a complete model, which expands $\langle \omega, < \rangle$, N_1 is an uncountable model of the theory of M_1, $a \in |N_1|$, $|\{b \in N_1 : b < a\}| \geq \aleph_0$ *then* $|\{b \in N_1 : b < a\}| \geq 2^{\aleph_0}$.

Theorem 4.6. *Let M be a model of a countable and superstable theory T, $N = M^I_D | G$, $\|N\| > \aleph_0$, $N \neq M$. Then*
A) *N is $\beth_1$-saturated.*
B) *If T has not the* f.c.p., *M is λ-compact then N is $\lambda^I_D | G$-compact.*
C) *If $\langle J, < \rangle = \langle \omega, < \rangle^I_D | G$, and for no $s \in J$, $\aleph_0 \leq |\{b \in J : \langle J, < \rangle \models b < a\}| < \lambda$, then N is λ-saturated.*

Theorem 4.7. A) *Let M be a countable model of a stable theory T which has the* f.c.p., *and $\Delta \subset L(T)$ be finite. Let p be a Δ-1-type over $N = M^I_D | G$ which is omitted by N; but every $q \subset p$, $|q| < |p|$ is realized by N; and $|p|$ is regular. Then there is*

$$s \in \langle \omega + 1, < \rangle^I_D | G \text{ such that } |p| = |\{t : \langle \omega + 1, < \rangle^I_D | G \models t < s\}|$$

Remark. 1) This theorem is a converse to Theorem 3.2.
2) For uncountable M, we should replace ω+1 by λ+1, $\lambda = \|M\|$.

Proof. By [13] Th. 5.9A there are finite Δ_1, n_1 such that:
(*) If $\varphi(x, \bar{y}) \in \Delta$, $\{a_i : i < \alpha\}$ is a Δ_1-n_1-indiscernible set in N *then* for every $\bar{b}$ from N either $|\{i < \alpha : N \models \varphi[a_i, \bar{b}]\}| < n_1$ or $|\{i < \alpha : N \models \neg \varphi[a_i, \bar{b}]\}| < n_1$.

By [13], Th. 5.10 there are finite Δ_2, n_2 such that
(**) (i) every Δ_2-n_2-indiscernible set is a Δ_1-n_1-indiscernible set, $n_2 \geq n_1$.
(ii) if W_i is a Δ_i-n_i-indiscernible set in N, $i = 1, 2$ and $|W_1 \cap W_2| \geq n_2$ $\dim(W_2, \Delta_2, n_2, N) \geq \aleph_0$ *then* $\dim(W_1, \Delta_1, n_1, N) \geq \dim(W_2, \Delta_2, n_2, N)$ ([13], Def. 5.4 define dim).

Similarly we can define finite Δ_3, n_3 which will relate to Δ_2, n_2 just as Δ_2, n_2 relate to Δ_1, n_1.

Now let $p = \{\varphi_i(x, \bar{a}^i) : i < |p|\}$. (So for every i, $\varphi_i(x, \bar{y})$ belongs to Δ, or is the negation of a formula from Δ.) For every $j < |p|$ let $p_j = \{\varphi_i(x, \bar{a}^i) : i < j\}$. By our assumption each p_j is realized by some $b_j \in N$. As $|p|$ is regular, by [13], Th. 5.8 there is $w \subset |p|$, $|w| = |p|$ such that $W_1 = \{b_j : j \in w\}$ is Δ_3-n_3-indiscernible set (hence also Δ_2-n_2- and Δ_1-n_1-indiscernible set). Clearly $\dim(W_1, \Delta_1, n_1, N) \geq |p|$. *Let us prove that the equality holds.* Otherwise there is W^1, $W_1 \subset W^1$, $|W^1| > |p|$ and

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W^1 is also Δ_1-n_1-indiscernible. Now $\varphi_i(x, \bar{a}^i) \in p$ implies $i < j < |p| \Rightarrow N \models \varphi_i[b_j, \bar{a}^i]$, hence $|\{b \in W_1 : \models \varphi_i[b, \bar{a}^i]\}| \geq \aleph_0$ hence $|\{b \in W^1 : \models \varphi_i[b, \bar{a}^i]\}| \geq \aleph_0$, hence by (*) $|\{b \in W^1 : N \models \neg\varphi_i[b, \bar{a}^i]\}| < n_1$. So the number of $b \in W^1$ which do not realize p is $\leq n_i\ |p| < |W^1|$, so p is realized in N. Contradiction. So $\dim(W_1, \Delta_1, n_1, N) = |p|$.

Let us choose in M any countable set $P^M = \{a_i : i \leq \omega\}$, and define an order relation $<^M = \{\langle a_i, a_j\rangle : i < j\}$ (we write $x < y$ instead $<(x, y)$). We also define a relation Q^M such that: if $\{c_1, \ldots, c_{n3}\}$ is a Δ_2-n_2-indiscernible set in M, then $\{c \in M : \langle c, c_1, \ldots, c_{n3}\rangle \in Q^M\}$ is a maximal Δ_2-n_2-indiscernible set in M, and it includes $c_1, \ldots, c_{n3}$. Let us define also a function F^M such that: for every $c_1, \ldots, c_{n3} \in M$, let $W = \{c \in M : \langle c, c_1, \ldots, c_{n3}\rangle \in Q^M\}$; now $|W| = r < \omega$ implies $F^M(c_1, \ldots, c_{n3}) = a_{r+1}$ and $|W| \geq \aleph_0$ implies $F^M(c_1, \ldots, c_{n3}) = a_\omega$. We also define H^M such that if $F^M(c_1, \ldots, c_{n3}) = a_{r+1}$, $H^M(x, c_1, \ldots, c_{n3})$ will be a one-to-one function from $\{a_i : i < r\}$ onto $\{c \in M : \langle c, c_1, \ldots, c_{n3}\rangle \in Q^M\}$; and if $F^M(c_1, \ldots, c_{n3}) = a_\omega$, $H^M(x, c_1, \ldots, c_{n3})$ will be a one-to-one function from $\{a_i : i < \omega\}$ onto $\{c \in M : \langle c, c_1, \ldots, c_{n3}\rangle \in Q^M\}$. Let

$$N_1 = (N, P^N, <^N, F^N, Q^N, H^N) = (M, P^M, <^M, F^M, Q^M, H^M)^I_D | G$$

Let us choose n_3 different element of $W_1 (\subset |N|) - c_1, \ldots, c_{n3}$. Let $W_2 = \{c \in |N_1| : N_1 \models Q[c, c_1, \ldots, c_{n3}]\}$. Clearly W_2 is a maximal Δ_2-n_2-indiscernible set, hence $\dim(W_2, \Delta_2, n_2, N) = |W_2|$. Let $a = F^{N_1}[c_1, \ldots, c_{n3}]$, and $\lambda = |\{b \in P^{N_1} : N_1 \models b < a\}|$. Clearly, (using H) $|W_2| = \lambda$. It is also clear that $c_1, \ldots, c_{n3} \in W_2$, hence $|W_1 \cap W_2| \geq n_3$.

As W_1 is Δ_i-n_i-indiscernible set for $i = 1, 2, 3$.

(i) $|p| = |W_1| \leq \dim(W_1, n_3, \Delta_3, N) \leq \dim(W_1, \Delta_1, n_1, N) = |p|$

As $|W_1 \cap W_2| \geq n_3$, and W_1 is infinite, by the definition of $\Delta_3 n_3$.

(ii) $|W_2| = \dim(W_2, \Delta_2, n_2, N) \geq \dim(W_1, \Delta_3, n_3, N)$

Hence W_2 is infinite. As $|W_1 \cap W_2| \geq n_3 \geq n_2$, by (**).

(iii) $\dim(W_1, \Delta_1, n_1, N) \geq \dim(W_2, \Delta_2, n_2, N)$.

By (i), (ii), (iii), $|p| = \dim(W_1, \Delta_1, n_1, N) = |W_2| = \lambda$. So we prove the theorem: $|p| = \lambda$. *Remark*: We could choose $P^M = |M|$.

Conjecture 4B. The theorem holds also if $|p|$ is singular.

Theorem 4.8. *Suppose T is stable and has the* f.c.p. *Let $\aleph_\alpha \geq |T_1| + \aleph_\beta$, $\aleph_\beta = 2^{\aleph_0}$, and $T \subset T_1$. Then in* PC(T_1, T) *there are at least $2^{|\alpha - \beta|}$ non-isomorphic models of cardinality $\aleph_\alpha$.*

Proof. Follows immediately from Theorems 3.1, 4.8, (and 4.1A if $\aleph_\beta$ is singular) depending on the following.

For $s \in P$, where $P \subset J$, $<$ order J, define $|s| = |\{t : \langle J, <\rangle \models t < s\}|$ $\mathrm{SP}(\langle J, <, P\rangle) = \{|s| : s \in P,\ |s| \text{ is infinite and regular, or } |s| = 2^{\aleph_0}\}$.

Let K be a set of regular cardinals $\geq 2^{\aleph_0}$, and may be also $2^{\aleph_0}$; and assume there is a greatest cardinal in K, and let P be a set of natural numbers. *Then* there are I, D, G such that

$$K = \mathrm{SP}(\langle \omega, <, P\rangle^I_D | G), \quad \aleph_0{}^I_D | G = \max\{\lambda : \lambda \in K\}.$$

Theorem 4.9. *If T is not $\triangleleft_\lambda$-minimal, then it is not $\triangleleft_\mu$-minimal for every $\mu \geq \min(2^{|T|}, \lambda)$.*

Remark. If T is countable, stable and with the f.c.p., T is $\triangleleft_\lambda$-minimal iff $\lambda < 2^{\aleph_0}$.

Proof. If $\mu \geq \lambda$, the conclusion follows by Lemma 3.5. So we can assume $\lambda > \mu \geq 2^{|T|}$; and by the same lemma it suffices to prove the theorem for the case $\mu = 2^{|T|}$. So let $\lambda > \mu = 2^{|T|}$, T is $\triangleleft_\mu$-minimal but not $\triangleleft_\lambda$-minimal.

As T is not $\triangleleft_\lambda$-minimal, by Keisler [6] there is an $(\aleph_0, \lambda)$-regular ultrafilter D over λ, such that for every model N of T, N^λ/D is not λ^+-compact. Let M be a λ^+-saturated model of T; $\{I_k : k < \lambda\} \subset D$ a family of sets, the intersection of any infinite subfamily of it is empty.

Suppose first M^λ/D is not $|T|^+$-compact. Then there is $A \subset |M^\lambda/D|$, $|A| \leq |T|$, such that M^λ/D omit a type over A. Without loss of generality there is $\mathrm{eq} \subset \lambda \times \lambda$, such that for every $a \in A$, $\mathrm{eq}(a) \supset \mathrm{eq}$ and eq has $|T|$ equivalence classes. Let G be the filter over $\lambda \times \lambda$ generated by eq. Then also $M^\lambda_D | G$ is not $|T|^+$-compact, and clearly for some filter D_1 over $|T|$, $M^\lambda_D | G$ is isomorphic to $M^{|T|}/D_1$; so T is not $\triangleleft_{|T|}$-minimal hence not $\triangleleft_\mu$-minimal.

Assume now M^λ/D is $|T|^+$-saturated. By [13], 5.16, there is an indiscernible set $W = \{a_n : n < \omega\}$ in M^λ/D, $\dim(W, M) \leq \lambda$. Without loss of generality there is an equivalence relation eq $\subset \lambda \times \lambda$ with $\leq |T|$ equivalence classes such that $\mathrm{eq}(a_n) \supset \mathrm{eq}$ for $n < \omega$. Let G be the filter over $\lambda \times \lambda$ generated by eq. Clearly $M^\lambda_D | G$ is an elementary submodel of M^λ/D (Keisler [9]) and $W \subset M^\lambda_D | G$. It is also clear that for some ultrafilter D_1 over $|T|$, $M^\lambda_D | G$, $N = M^{|T|}/D_1$ are isomorphic. As M is λ^+-saturated, $\lambda > 2^{|T|}$, it suffices to prove $M^\lambda_D | G$ is not $(2^{|T|})^+$-saturated. If it was, by Lemma 4.2 it will be λ^+-saturated, hence $\lambda \geq \dim(W, M^\lambda/D) \geq \dim(W, M^\lambda_D | G) \geq \lambda^+$. Contradiction.

Now we shall try to deduce some results on $\triangleleft$.

Theorem 4.10. A) *Let T be countable. T is $\triangleleft$-minimal iff T has not the* f.c.p.

B) *For $\lambda \geq 2^{\aleph_0}$, T is $\triangleleft_\lambda$-minimal iff T has not the* f.c.p.

C) *If $\aleph_0 < \lambda < 2^{\aleph_0} < 2^\lambda$, T is $\triangleleft_\lambda$-minimal iff T is stable.*

D) *If $\aleph_0 < \lambda < 2^{\aleph_0}$, then if T is stable, it is $\triangleleft_\lambda$-minimal, and if it is $\triangleleft_\lambda$-minimal it has not the strict order p.*

Proof. A, B) Follow from 4.1A and from 3.1 with product of ultrafilters.

C) Follows from 4.1A, B and from 3.3 with product of ultrafilters.

D) Follows from 4.1A, B and from 4.4 with product of ultrafilters.

Theorem 4.11. *There is a non-$\triangleleft$-minimal or $\triangleleft$-maximal countable theory T, iff there is a non-good ultrafilter D, such that $\lambda = \Pi n_i/D \geq \aleph_0$ implies $\lambda > |I|$ (if G.C.H. fails, there is such D).*

Proof. If there is no such D, by 4.1 every T with the f.c.p. is $\triangleleft$-maximal; so by 4.10A every countable theory is either $\triangleleft$-minimal or $\triangleleft$-maximal. If there is such D, every stable countable T with the f.c.p. is not $\triangleleft$-minimal (by 4.1A) nor $\triangleleft$-maximal (by 4.1). By [13] Th. 3.9A or Keisler [6], p. 44, 45 there is such T.

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§ 5. Saturation of Ultralimits

For every M and D, there is an elementary embedding of M into M^I/D–$a \to f_a/D$ where $f_a(i) = a$ for every $i \in I$. Hence we can look at M^I/D as an elementary extension of M; and can repeat extending the models by taking ultrapowers and at limit stages take union. So we get an increasing elementary extension of models, which are ultralimits of M. For simplicity, all the ultrapowers will be with the same ultrafilter D. This notion was defined and investigated in Kochen [11], Keisler [9] §5.

Let us make the definition more precise.

5.1. Definition. $\mathrm{UL}(M, D, \alpha)$ will be defined by induction on α, such that for $\beta < \alpha$, $\mathrm{UL}(M, D, \beta)$ is an elementary submodel of $\mathrm{UL}(M, D, \alpha)$.

1) for $\alpha = 0$, $\mathrm{UL}(M, D, \alpha) = M$

2) for α a limit ordinal, $\mathrm{UL}(M, D, \alpha) = \bigcup_{\beta<\alpha} \mathrm{UL}(M, D, \beta)$

3) for $\alpha = \beta+1$, $\mathrm{UL}(M, D, \alpha)$ will be isomorphic to $\mathrm{UL}(M, D, \beta)^I/D$, and the isomorphism F_β takes each $f_a/D \in \mathrm{UL}(M, D, \beta)$ to $a \in \mathrm{UL}(M, D, \beta) \subset \mathrm{UL}(M, D, \alpha)$ (f_a is defined by $f_a(i) = a$).

Notation: At most of the time M and D are fixed, we let $M_\alpha = \mathrm{UL}(M, D, \alpha)$ and F_α the isomorphism mentioned in 3). We assume also M is a model of T.

Clearly we can assume that for every α, β, $\mathrm{UL}(M, D, \alpha+\beta) = \mathrm{UL}(M_\alpha, D, \beta)$.

We shall try here to find how compact the ultralimits are, by properties of the oridnal, the ultrafilter and the the theory of the model. As $M_{\alpha+1}$ is isomorphic to M_α^I/D, we shall restrict ourselves to M_δ for limit ordinals δ.

The following theorem is well known.

Theorem 5.1. *If the cofinality of δ, $\mathrm{cf}(\delta)$, is μ, and for every $\lambda < \mu$, D is $(\aleph_0, \lambda)$-regular, <u>then</u> M_δ is μ-compact.*

Proof. Let p be a type over M_δ of cardinality $<\mu$. Then clearly p is a type over M_β for some $\beta < \delta$. As D is $(\aleph_0, |p|)$-regular, p is realized in

$M_{\beta+1}$ (see, e.g. Keisler [6], Sec. 1), hence p is realized in M_δ. So every type over M_δ of cardinality $<\mu$ is realized in M_δ; hence M_δ is μ-compact.

Theorem 5.2. *If T is unstable, $\mu = \mathrm{cf}(\delta)$ then M_δ is not μ^+ compact.*

Proof. As mentioned in Section 1, M_1 should be $\aleph_1$-compact (remember we deal only with $\aleph_1$-incomplete ultrafilters). As T is unstable, by [13], Th. 2.13, (1), (3); there is a formula $\varphi(x, \bar{y})$ and sequences $\bar{a}^0$, $\bar{a}^0$, ..., $\bar{a}^n$, ... from M_1 (all of the length of $\bar{y}$) such that:

for every $m < \omega$, $\{\varphi(x, \bar{a}_n)^{\text{if } (n \geq m)} : n < \omega\}$

is consistent over M_1.

As $\mathrm{cf}(\delta) = \mu$, let $\delta = \bigcup_{k<\mu} \alpha_k$, where $k < l < \mu$ implies $1 < \alpha_k < \alpha_l$. We shall now define by induction on k sequence $\bar{a}^k$ such that

1) $\bar{a}^k \in M_{\alpha_k+1}$, $\bar{a}^k \notin M_{\alpha_k}$.

2) $\{\neg\varphi(x, \bar{a}_n) : n < \omega\} \cup \{\varphi(x, \bar{a}^k)\}$ is not realized by any element of M_{α_k}.

3) for every $m < \omega$, $p_k^m = \{\varphi(x, \bar{a}_n)^{\text{if } (n \geq m)} : n < \omega\} \cup \{\varphi(x, \bar{a}^l) : l \leq k\}$ is consistent (over $M_{\alpha_k} + 1$).

If we shall succeed in defining the $\bar{a}^k$'s then clearly by 3) $p = \{\neg\varphi(x, \bar{a}_n) : n < \omega\} \cup \{\varphi(x, \bar{a}^l) : l < \mu\}$ is consistent (over M_δ), because every finite subset of p is a subtype of p_k^m. On the other hand if p is realized in M_δ, then it is realized in M_β for some $\beta < \delta$, so there is $k < \mathrm{cf}(\delta)$. $\beta < \alpha_k < \delta$. Hence p is realized in M_{α_k}, contradiction to 2). Hence p is a consistent type over M_δ, which M_δ omits, and $|p| = \aleph_0 + \mu < \mu^+$. So M_δ is not μ^+-compact.

It remains only to define $\bar{a}^k$, assuming $\bar{a}^l$ for $l < k$ has been defined. As D is $\aleph_1$-incomplete there are $I_n \in D$, $I_{n+1} \subset I_n$, $I_0 = I$, $\bigcap_{n<\omega} I_n = \emptyset$. Let us define $\bar{a} \in M_{\alpha_k}^I/D$: if $i \in I_n - I_{n+1}$, then $\bar{a}[i] = \bar{a}_n$, so $\bar{a} = \langle \bar{a}[i] : i \in I\rangle/D$, and $\bar{a}^k = F_{\alpha_k}(\bar{a})$. Let us check conditions 1), 2), 3) are satisfied.

Clearly $\bar{a}^k \in M_{\alpha_k+1}$. Now for any $n < \omega$, $\{i \in I : \bar{a}[i] = \bar{a}_n\} = I_n - I_{n+1} \notin D$ hence $\bar{a}^k \notin M_{\alpha_k}$. So 1) is satisfied.

For proving 2) suppose $c \in M_{\alpha_k}$ realizes $q = \{\neg\varphi(x, \bar{a}_n) : n < \omega\} \cup \{\varphi(x, \bar{a}^k)\}$. Then

$$\{i : M_{\alpha_k} \models \varphi[F^{-1}(c)[i], \bar{a}[i]]\} \in D$$

that is

$$\{i : M_{\alpha_k} \models \varphi[c, \bar{a}[i]]\} \in D .$$

Hence for some i, $M_{\alpha_k} \models \varphi[c, \bar{a}[i]]$, and $\bar{a}[i] = \bar{a}_n$ for some n. But as M_{α_k+1} elementarily extend M_{α_k}. $M_{\alpha_k+1} \models \varphi[c, \bar{a}[i]]$. So c does not realize q, contradiction, hence 2) holds. Part 3) has a similar proof. So we finish the definition and the proof.

5.2. Definition. Let $\mu(D)$ be the first cardinal μ such that D is μ-descendingly complete, that is, μ is the first cardinality such that $I_k \in D$, $k < l \Rightarrow I_l \subset I_k$, implies $\bigcap_{k<\mu} I_k \neq \emptyset$ (equivalently $\bigcap_{k<\mu} I_k \notin D$).

Notice if D is $(\aleph_0, \kappa)$ regular, then $\kappa < \mu(D)$; also $\mu(D) \leq |I|^+$. Note also that $\mu(D)$ should be regular.

Theorem 5.3. *If $\mu \leq \mu(D)$, $\mu \leq \mathrm{cf}(\delta)$ then M_δ is μ-compact.*

Remark. I don't know whether this is known.

Proof. Let p be a type over M_δ, $|p| < \mu$, and we shall prove that p is realized in M_δ, and so prove the theorem.

As $|p| < \mu \leq \mathrm{cf}(\delta)$, p is a type over M_α for some $\alpha < \delta$. Let $|p| = \aleph_\beta$. We shall prove by induction on $\gamma \leq \beta$, that

(*) every subtype of p of cardinality $\leq \aleph_\gamma$ is realized in $M_{\alpha+\gamma+1}$.

As $\beta \leq \aleph_\beta = |p| < \mu \leq \mathrm{cf}(\delta)$, $\alpha + \beta + 1 < \delta$, hence by proving this we shall prove that p is realized in M_δ.

Suppose we have proved (*) for every $\gamma_1 < \gamma$. Hence every subtype of p of cardinality $< \aleph_\gamma$ is realized in $M_{\alpha+\gamma}$ (remember every model is $\aleph_0$-compact, hence every finite subtype of p is realized in M_α). Let q be any subtype of p of cardinality $\aleph_\gamma$, $q = \{\varphi_k(x, \bar{a}_k) : k < \aleph_\gamma\}$, and we shoudl prove q is realized in $M_{\alpha+\gamma+1}$. By the induction hypothesis for every $k < \aleph_\gamma$, there is $c_k \in M_{\alpha+\gamma}$ which realize $\{\varphi_l(x, \bar{a}_l) : l < k\}$. As

$\aleph_\gamma \leq |p| < \mu \leq \mu(D)$ there is a decreasing sequence I_k, $k < \aleph_\gamma$, $I_k \in D$ $\bigcap_{k<\aleph_\gamma} I_k = \emptyset$, and $I_0 = I$. Let us define $c \in (M_{\beta+\gamma+1})^I/D$:

if $i \in \bigcap_{l<k} I_l - I_k$ then $c[i] = c_k$ (clearly c is well defined). Now clearly $F_{\alpha+\gamma}(c) \in M_{\alpha+\gamma+1}$ realize q, as for every $k < \aleph_\gamma$

$$\{i : M_{\alpha+\gamma} \models \varphi[c[i], \bar{a}_k]\} \supset \{i : i \in \bigcap_{j<l} I_j - I_l, l > k\} = I_{k+1} \in D$$

So q is realized in $M_{\alpha+\gamma+1}$; so p is realized in M_δ.

5.3. Definition. A model N strongly omits a type p (over it) if no subtype of p of cardinality $|p|$ is realized in N.

Lemma 5.4. A) *If M strongly omits p $|p| = \mu(D)$, then also M_1 strongly omits p.*

B) *If M_α strongly omits p, $|p| = \mu(D)$, $\alpha < \beta$ then also M_β strongly omits p.*

C) *In A), B) instead of $|p| = \mu(D)$, it suffices to assume that there are no $I_k \in D$ for $k < |p|$, $k < l \Rightarrow I_l \subset I_k$, $\bigcap_{k<|p|} I_k = \emptyset$; and $|p|$ is regular.*

Proof. We shall prove A), as B), C) have similar proofs.

Suppose A) fails, so $c_1 \in M_1$ realize $q \subset p$, $|q| = |p|$. Let $c_1 = F_0(c)$, $q = \{\varphi_k(x, \bar{a}_k) : k < |q|\}$. So clearly for every $k < |q| = |p|$

$$\{i : i \in I, M \models \varphi_k[c[i], \bar{a}_k]\} \in D$$

It is also clear that for every $i \in I$

$$q(i) = \{\varphi_k(x, \bar{a}_k) : M \models \varphi_k[c[i], \bar{a}_k]\}$$

is a subtype of q, hence of p, which is realized in M; hence $|q(i)| < |p|$. As $|p| = \mu(D)$ is regular, for every $i \in I$ there is a bound $k(i) < |p|$ to $\{k : M \models \varphi_k[c[i], \bar{a}_k]\}$. Let, for $l < |p|$, $I_l = \{i : k(i) \geq l\}$. Clearly I_l, $l < |q|$ is a decreasing sequence, and by the definition of $k(i)$, $\bigcap_{l<|p|} I_l = \emptyset$. In addition each $I_l \in D$ as $I_l = \{i : k(i) \geq l\} \supset \{i : M \models \varphi_l[c[i], \bar{a}_l]\} \in D$.

So we get a contradiction to the definition of $\mu(D)$.

Theorem 5.5. *If T is unstable, $\delta \geq \mu(D)$, then M_δ is not $\mu(D)^+$-compact. Moreover there is a type over $M_{\mu(D)}$ of cardinality $\mu(D)$ which M_δ* strongly omits.

Proof. As it is similar to 5.2, 5.7 we omit it.

Conclusion 5.6. If T is unstable, $\mu = \min(\mathrm{cf}(\delta)\ \mu(D))$, then M_δ is maximally μ-compact.

Proof. Immediate by 5.2, 5.3, and 5.5.

5.4. Definition. T satisfies $(C*\lambda)$ if: there are an increasing sequence of sets A_k, $k \leq \lambda$; a type $p \in S(A_\lambda)$ ([13], sec. 1) such that for every $k < \lambda$ there is a formula $\varphi_k(x, \bar{y}_k)$ and a infinite-indiscernible set over A_k ([13], Def. 5.2), $\{\bar{a}_{k,n} : n < \omega\}$ such that $\bar{a}_{k,0}, \bar{a}_{k,1} \in A_{k+1}$, and $\neg\varphi_k(x, \bar{a}_{k,0}), \varphi_k(x, \bar{a}_{k,1}) \in p$.

5.5. Definition. $\kappa(T)$ is the first cardinality κ such that T does not satisfy $(C*\kappa)$.

Remark. $(C*\lambda)$ was defined and investigated in [14]. By [14], Th. 4.4 for stable T, and $\lambda \geq 2^{|T|}$: T is stable in λ iff $\lambda = \sum_{\kappa<\kappa(T)} \lambda^\kappa$.

Theorem 5.7. *If $\kappa(T) > \mu = \min[\mu(D), \mathrm{cf}(\delta)]$ T is stable, then M_δ is maximally μ-compact*

Proof. By Theorem 5.3, M_δ is μ-compact, so we should prove only that M_δ is not μ^+-compact. By hypothesis T satisfies $(C*\mu)$, so there are A_k, $k \leq \mu$, $p \in S(A_\lambda)$, $\varphi_k(x, \bar{y}_k)$, and $\bar{a}_{k,n}$, $k < \mu$, $n < \omega$; such that

$k < l \Rightarrow A_k \subset A_l$; $\{\bar{a}_{k,n} : n < \omega\}$ is an indiscernible set over A_k, $\bar{a}_{k,0}, \bar{a}_{k,1} \in A_{k+1}$ and $\neg\varphi_k(x, \bar{a}_{k,0}), \varphi_k(x, \bar{a}_{k,1}) \in p$.

Clearly it suffices to prove the theorem for the case $L = L(T)$ is the minimal language containing all the formulas $\varphi_k(x, \bar{y}_k)$; so $|L| \leq \mu$.

Choose $\alpha_k < \delta$ for $k < \mathrm{cf}(\delta)$ such that $\delta = \bigcup_{k<\mathrm{cf}(\delta)} \alpha_k$.

Let us define: a function H is elementary if for every $\varphi \in L$, $a_1, \ldots, a_n$:

$$\models \varphi[a_1, \ldots, a_n] \text{ iff } \models \varphi[H(a_1), \ldots, H(a_n)]$$

and let

$$H(\langle a_1, \ldots, a_n \rangle) = \langle H(a_1), \ldots, H(a_n) \rangle.$$

Now we define by induction an increasing sequence of elementary functions H_k and ordinals $\beta_k < \delta$ for $k \leq \mu$ such that:

1) the domain of H_k is $\bigcup_{l<k} \text{Range}(\bar{a}_{l,0}\hat{\ }\bar{a}_{l,1})$

2) the range of H_k is included in M_{β_k}, $\beta_k \geq \alpha_k$

3) if $k < l < \mu$, then $\beta_k < \beta_l < \delta$, and for every $c \in M_{\beta_k}$,

$$M_\delta \models \varphi_l[c, H_{l+1}(\bar{a}_{l,0})] \equiv \varphi_l[c, H_{l+1}(\bar{a}_{l,1})] .$$

For $k = 0$, H_k will be the void function, $\beta_0 = \alpha_0$.

For a limit ordinal k $H_k = \bigcup_{l<k} H_l$, $\beta_k = \max(\alpha_k, \bigcup_{l<k} \beta_l)$. [If $k < \mu$, $\beta_k < \mu$ because $\mu \leq \text{cf}(\delta)$].

Suppose H_k, β_k are defined, $k < \mu$, and we shall define H_{k+1}, β_{k+1}. We first show:

(*) there is $\beta < \delta$ such that we can extend H_k to an elementary function H^* from $\text{Dom}\, H_k \cup \bigcup \{\text{Range}\, \bar{a}_{k,n} : n < \omega\}$ into M_β.

If $\mu = \aleph_0$, this is true, as for every N, N^I/D is $\aleph_1$-compact, so $\beta = \beta_k + 1$ will suffice. So assume $\mu > \aleph_0$. We define now by induction on n an increasing sequence of functions H^n from $\text{Dom}\, H_k \cup \bigcup \{\text{Range}\, \bar{a}_{k,m} : m < n\}$ into M_δ. If we have defined H^n, and cannot define H_{n+1}, this means M_δ is not μ^+-compact [as it omits

$$\{\varphi(\bar{x}, F(\bar{c})) : \varphi \in L, \bar{c} \in \text{Dom}\, H^n, \models \varphi[\bar{a}_{k,n}, \bar{c}]\}]$$

and so the conclusion of the theorem holds. So we can assume H^n is defined for every n and let $H^* = \bigcup_{n<\omega} H^n$. Clearly H^* is an elementary function, with the appropriate domain into M_δ. As μ is regular (as $\mu(D)$, cf (δ) are regular) $\mu > \aleph_0$, H^* is into M_β for some $\beta < \delta$.

So we proved (*).

Define $\beta_{k+1} = \max(\beta, \alpha_k)$. Let $I_n \in D$, $I_n \supset I_{n+1}$, $I_0 = I$, $\bigcap_{n<\omega} I_n = \emptyset$ (they exist as D is $\aleph_1$-incomplete). Define $H_{k+1}(\bar{a}_{k,0} \hat{} \bar{a}_{k,1}) \in M_{\beta_{k+1}}$ as $F_{\beta_k}(\bar{c})$, where $\bar{c} \in M_{\beta_k}{}^I/D$ is defined as follows: if $i \in I_n - I_{n+1}$, $\bar{c}[i] = H^*(\bar{a}_{k,n} \hat{} \bar{a}_{k,n+1})$. It is easy to verify H_{k+1}, β_{k+1} satisfies the induction conditions.

Now

$$p = \{\varphi_k(x, H_{k+1}(\bar{a}_{k,0})) \equiv \neg \varphi_k(x, H_{k+1}(\bar{a}_{k+1})) : k < \mu\}$$

is a consistent type over M_{β_n}, and it is strongly omitted by M_{β_μ}. As $\beta_\mu \leq \delta$, by Lemma 5.4, also M_δ omits the type, so M is not μ^*-compact.

It is natural to conjecture that if $\kappa(T) \leq \mu$, $\mu = \min[\mu(D), \mathrm{cf}(\delta)]$, and, $\alpha, \beta < \delta \Rightarrow \alpha + \beta < \delta$, *then* M_δ is UL($\aleph_0$, D, δ)-saturated (UL($\aleph_0$, D, δ) - the cardinality of UL(M, D, δ) for every countable M) [this would generalize 4.1A]. But this is not true. T may be superstable [$\kappa(T) = \aleph_0$] or even simple [Def. 2) and M or M_1 will omit strongly a type of cardinality $\mu(D)$. However

Theorem 5.8. *Suppose* $\kappa(T) \leq \min[\mu(D), \mathrm{cf}(\delta)]$, *$D$ is* $(\aleph_0, |T|)$*-regular ultrafilter*; $\alpha, \beta < \delta \Rightarrow \alpha + \beta < \delta$. *Then M_δ is λ-saturated, where* $\lambda = \mathrm{UL}(\aleph_0, D, \delta)$.

Remark. 1) For every δ_1 there are δ_2, δ; $\delta_1 = \delta_2 + \delta$; $\alpha, \beta < \delta \Rightarrow \alpha + \beta < \delta$, and $\mathrm{UL}(M, D, \delta_1) = \mathrm{UL}(M_{\delta_2}, D, \delta_1)$. So the restriction on δ is natural.

2) Clearly $\lambda > |T|$, so it suffices to prove M_δ is λ-compact.

Proof. Let p be a type over M_δ, $|p| < \lambda$. We should prove p is realized in M_δ. Let q be any extension of p in $S(|M_\delta|)$.

Notice that if $|B| < \kappa(T) \leq \mathrm{cf}(\delta)$, $B \subset M_\delta$, then for some $\alpha < \delta$, $B \subset M_\alpha$. Hence by Shelah [19] there is $\alpha < \delta$ s.t. for every $\varphi = \varphi(x, \bar{y}) \in L$, $\mathrm{Rank}_\varphi(q|\varphi) = \mathrm{Rank}_\varphi[(q|M_\alpha)|\varphi]$ (see [13], Def. 2.4, 2.5, and Th. 2.13, $p|\varphi$ is the maximal φ-type contained in p, $p|A$ - the maximal type over A contained in p). So by [13], 2.5B; there is a set $B \subset M_\alpha$, $|B| \leq T$, such that for every φ, $\mathrm{Rank}_\varphi(q|\varphi) = \mathrm{Rank}_\varphi[(q|B)|\varphi]$. Now we can define a_n for $n < \omega$ such that:

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1) a_n realizes $q \restriction (B \cup \{a_m : m < n\})$

2) if $\delta > \omega$, $a_n \in M_{\alpha+n+1}$

As D is $(\aleph_0, |T|)$-regular, this is possible. As in the proof of 4.1, and in [13], 5.16, it follows that:

if $\varphi(x, \bar{b}) \in q$, then $\{n < \omega : \models \neg\varphi(a_n, \bar{b})\}$ is finite,
and $\{a_n : n < \omega\}$ is an indiscernible set over B.

Suppose for a moment $\delta > \omega$. Let $P = \{a_n : n < \omega\} \subset M_{\alpha+\omega}$ (as $\alpha < \delta$, $\omega < \delta$; $\alpha + \omega < \delta$). Let $(M_\delta, P^\delta) = \mathrm{UL}((M_{\alpha+\omega}, P), D, \delta)$ (remember $\delta = \alpha + \omega + \delta$). Clearly P^δ extends P and is an indiscernible set over ϕ. So $\varphi(x, \bar{b}) \in p$ implies $\varphi(x, \bar{b}) \in q$ implies $\{a : a \in P^\delta, \models \neg\varphi(a, \bar{b})\}$ is finite. So all except $|p| \cdot \aleph_0 < \lambda$ members of P^δ realize p. As $|P^\delta| = \mathrm{UL}(\aleph_0, D, \delta) = \lambda$, the theorem follows and we remain only with the case $\delta = \omega$; and we can define the a_n's simultaneously in $M_{\alpha+1}$ and the proof goes in the same way.

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