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THE SPECTRUM PROBLEM II: TOTALLY TRANSCENDENTAL AND INFINITE DEPTH†

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ABSTRACT

We examine the main gap for the class of models of totally transcendental first-order theories, and compute the number of $\aleph_\varepsilon$-saturated models of power $\aleph_\alpha$ of a superstable T without the dop which is shallow but of depth $\geqq \omega$.

§1. Totally transcendental T

Hypothesis. T is totally transcendental.

We want to redo [2] for the class of models, instead of the $\boldsymbol{F}^a_{\aleph_0}$-saturated models, hence replacing $\boldsymbol{F}^a_{\aleph_0}$ by $\boldsymbol{F}^l_{\aleph_0}$ everywhere. The price is that we assume T is totally transcendental. We shall omit $\boldsymbol{F}^l_{\aleph_0}$ in expressions like "$\boldsymbol{F}^l_{\aleph_0}$-atomic".

1.1. Lemma. *Suppose $N \subseteq M \subseteq M'$, $M \neq M'$. Then for some $a \in M' - M$,* tp(a, M) *does not fork over N* or tp(a, M') *is orthogonal to N. In fact the type is strongly regular, and if it does not fork over N,* tp(a, N) *too is strongly regular.*

Proof. Among the formulas $\phi(x, \bar{a})$ such that $\bar{a} \in N$, $\phi(M, \bar{a}) \neq \phi(M', \bar{a})$ choose one with minimal $\alpha = R[\phi(x, \bar{a}), L, \aleph_0]$; α is $< \infty$ because T is totally transcendental, $\phi(x, \bar{a})$ exists as $x = x$ satisfies the requirement, and $\alpha \geqq 0$ as $\phi(M, \bar{a}) \neq \phi(M', \bar{a})$ implies $\exists x \phi(x, \bar{a})$, and w.l.o.g. $\mathrm{Mlt}[\phi(x, \bar{a}), L, \aleph_0] = 1$.

Among the formulas $\psi(x, \bar{b})$ such that $\bar{a} \subseteq \bar{b}$, $\bar{b} \in M$, $\psi(M, \bar{a}) \subseteq \phi(M, \bar{a})$, $\psi(M, \bar{b}) \neq \psi(M', \bar{b})$ choose one with minimal $\beta = R[\psi(x, \bar{b}), L, \aleph_0]$.

As before $0 \leqq \beta < \infty$, $\mathrm{Mlt}[\psi(x, \bar{b}), L, \aleph_0] = 1$, $\psi(x, \bar{b})$ exists since $\phi(x, \bar{a})$ satisfies the requirements. Choose $c \in M' - M$ such that $\vDash \psi[c, \bar{b}]$ so by V 3.19 (and 3.18, Ex. 3.10) tp(c, M) is strongly regular and choose an indiscernible set

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$\{\bar b_n{}^\wedge\langle c_\eta\rangle : n<\omega\}$ based on N, $\bar b_0{}^\wedge\langle c_0\rangle = \bar b^\wedge\langle c\rangle$. Note that $\mathrm{tp}(c,M)$ is the stationarization of $\mathrm{tp}(c,\bar b)$ over M, hence if $\mathrm{tp}(c,\bar b)$ is orthogonal to N we get the desired conclusion. Also if $\alpha=\beta$ we finish, so we can assume $\beta<\alpha$. So assume $\mathrm{tp}(c,\bar b)$ is not orthogonal to N, then by V 3.4 $\mathrm{tp}(c_n,\bar b_n)$ $(n<\omega)$ are pairwise not orthogonal. The types $\mathrm{tp}(c_n,\bar b_n)$ cannot be pairwise parallel, or then $\mathrm{tp}(c,M)$ does not fork over N, and we would finish the proof. So we assume $\mathrm{tp}(c_n,\bar b_n)$ $(n<\omega)$ are pairwise not parallel. Hence, by V 2.7 we can find n, $c_{i,l}$ $(l<3, i\leqq n)$ such that

(i) $c_{i,l}$ realizes the stationarization of $\mathrm{tp}(c_l,\bar b_l)$ over $N\cup\bigcup_{m<3}\bar b_m$ so $\vDash\psi[c_{i,l},\bar b_m]$ iff $l=m$,

(ii) $\{c_{i,l}: i\leqq n, l<3 \text{ but } (i,l)\neq(n,1),(n,2)\}$ is independent over $N\cup\bigcup_{m<3}\bar b_m$,

(iii) $c_{n,0}=c$,

(iv) $\vDash\theta[c_{n,0},c_{n,1},\cdots,c_{i,l},\cdots,\bar b_l,\cdots,\bar d]_{i<n,\,l=0,1}$
$\vDash\theta[c_{n,0},c_{n,2},\cdots,c_{i,l},\cdots,b_l,\cdots,\bar d]_{i<n,\,l=0,2}$

where $\bar d\in N$, and $\theta(x,c_{n,m},\cdots,c_{i,l},\cdots,\bar b_l,\cdots,\bar d)$ (for $m=1,2$) forks over M; and so w.l.o.g. for every $c'_{n,m},\cdots$

$$R[\theta(x,c'_{n,m},\cdots),\theta,2]<n^*=R[\mathrm{tp}(c,M),\theta,2]=R[\mathrm{tp}(c,\bar b),\theta,2].$$

Now remember that every type which does not fork over a model is finitely satisfiable in it (III 0.1). So we can define first b'_1, $\bar b'_2\in N$, then $c'_{i,l}\in M$ $(i<n, l<3)$ (letting $\bar b'_0=\bar b=\bar b_0\in M$) and at last define $c'_{n,1}$, $c'_{n,2}\in M'$, each time preserving all relevant information (define the exact demands looking at what follows for what is needed, and go in the reverse order of the definition).

Then by (iv), $\mathrm{tp}(c,M\cup\{c'_{n,l}\})$ forks over M (for $l=1,2$), hence $c'_{n,l}\in M'-M$, and of course $\vDash\psi[c'_{n,l},\bar b'_l]\wedge\neg\psi[c'_{n,l},\bar b'_{3-l}]$ for $l=1,2$.

Now the formula $\psi(x,\bar b'_l)\wedge\neg\psi(x,b'_{3-l})$ satisfies the requirements on $\phi(x,\bar a)$ and $\psi(x,\bar b'_l)\vdash\phi(x,\bar a)$, hence by α's minimality, $R[\psi(x,\bar b'_l)\wedge\neg\psi(x,b'_{3-l}),L,\aleph_0]$ is α. However, we have two such formulas $(l=1, l=2)$, both extend $\psi(x,\bar b)$ and are contradictory, But this contradicts $\mathrm{Mlt}[\phi(x,\bar a),L,\aleph_0]=1$.

1.2. Claim. (1) *Suppose $N\subseteq A$, $p\in S^m(N)$ is orthogonal to* $\mathrm{tp}_*(A,N)$ *and M is prime over A; then p is orthogonal to* $\mathrm{tp}_*(M,N)$.

(2) *Suppose $N\subseteq M$, $\mathrm{tp}(\bar a,M)$ is regular not orthogonal to N, and M' is prime over $M\cup\bar a$. Then there is $b\in M$, $\mathrm{tp}(b,M)$ does not fork over N.*

Remark. Note in 1.2(1) that this is stronger than weak orthogonality. A similar claim holds for $\boldsymbol{F}^l_\kappa$ (i.e., N $\boldsymbol{F}^l_\kappa$-saturated, M $\boldsymbol{F}^l_\kappa$-primary).

Proof. (1) It suffices to prove that if $\mathrm{tp}(\bar a,A)$ is isolated, then $\mathrm{tp}_*(A\cup\bar a,N)$, p are orthogonal. Let M be $\boldsymbol{F}^a_{\aleph_0}$-saturated, $N\subseteq M$, $\mathrm{tp}_*(M,A)$ does not fork over

N, hence $(A, A \cup M)$ satisfies the Tarski–Vaught condition. Let p' be the stationarization of p over M. As p, $\mathrm{tp}_*(A,N)$ are orthogonal, clearly p', $\mathrm{tp}_*(A,M)$ are weakly orthogonal. Easily $\mathrm{tp}(\bar{a},A) \vdash \mathrm{tp}(\bar{a}, M \cup A)$, hence by V 3.2 $\mathrm{tp}_*(A \cup \bar{a}, M)$, p' are weakly orthogonal, hence orthogonal, but p, p' are parallel and so are $\mathrm{tp}_*(A \cup \bar{a}, M)$, $\mathrm{tp}_*(A \cup \bar{a}, N)$ so we finish.

(2) If the conclusion fails, then by 1.1 for some $b \in M'$, $q = \mathrm{tp}(b, M)$ is orthogonal to N. Then q is orthogonal to $\mathrm{tp}(\bar{a}, M)$ hence by 1.2(1), $\mathrm{tp}_*(M', M), q$ are orthogonal; contradiction.

1.3. Claim. *Suppose $N \subseteq N_0, N_1$, and $N_0, N_1 \subseteq M$ and $\{N_0, N_1\}$ is independent over N. Then at least one of the following occurs:*

(a) *M is prime and minimal over $N_0 \cup N_1$;*

(b) *there is $\bar{a} \in M$, $\bar{a} \notin N$, $\mathrm{tp}(\bar{a}, N_0 \cup N_1)$ does not fork over N;*

(c) *there is $l \in \{0,1\}$ and $\bar{a} \in M$, $\bar{a} \notin N$, $\mathrm{tp}(\bar{a}, N_l)$ is orthogonal to N and $\mathrm{tp}(\bar{a}, N_0 \cup N_1)$ does not fork over N_l;*

(d) *there is $\bar{a} \in M$, $N_0 \cup N_1 \subseteq M' \subseteq M$, $\bar{a} \notin M'$, M' prime over $N_0 \cup N_1$, and $\mathrm{tp}(\bar{a}, M')$ is orthogonal to N_0 and to N_1.*

Proof. Choose $M' \subseteq M$ prime over $N_0 \cup N_1$. If $M' = M$ is also minimal then (a) holds. If $M' = M$ but it is not minimal, there is M'', $N_0 \cup N_1 \subseteq M'' \subsetneqq M'$, so w.l.o.g. $M' \neq M$. Apply 1.1 to N, M', M so there is $\bar{a} \in M$, $\bar{a} \notin M'$, $\mathrm{tp}(\bar{a}, M')$ does not fork over N or $\mathrm{tp}(\bar{a}, M')$ is orthogonal to N. In the first case (b) holds. In the second case w.l.o.g. M is prime over $M' \cup \bar{a}$, so by 1.2(1) for every $\bar{a}' \in M$, $\mathrm{tp}(\bar{a}', M')$ is orthogonal to N. Apply 1.1 to N_0, M', M, so there is $\bar{a}_0 \in M$, $\bar{a}_0 \notin M'$ such that $\mathrm{tp}(\bar{a}_0, M')$ does not fork over N_0 or is orthogonal to N_0. In the first case (c) holds, in the second case we can w.l.o.g. assume that for every $\bar{a}' \in M$ $\mathrm{tp}(\bar{a}', M')$ is orthogonal to N_0 (by 1.2(1), as before). Now apply 1.1 to N_1, M', M and we either get that (c) holds or that w.l.o.g. for every $\bar{a}' \in M$, $\mathrm{tp}(\bar{a}', M')$ is orthogonal to N_1. In the last case any $\bar{a} \in M - M'$ satisfies (d), so we finish.

1.4. Claim. *If T does not have the* dop, *then in* 1.3, *case* (d) *is impossible.*

Proof. Choose $\boldsymbol{F}^a_{\aleph_0}$-saturated N^*, N^*_0, N^*_1 such that $N \subseteq N^*$, $\mathrm{tp}(N^*, M)$ does not fork over N, and $N^* \cup N_l \subseteq N^*_l$, $\mathrm{tp}(N^*_l, M \cup N_{1-l})$ does not fork over $N_l \cup N^*_0$.

By the uniqueness, the prime model M' is $\boldsymbol{F}^l_{\aleph_0}$-constructible over $N_0 \cup N_1$.

Clearly $(N_0 \cup N_1, N^*_0 \cup N^*_1)$ satisfies the Tarski–Vaught condition. Now if $\bar{b} \in M'$, $\mathrm{tp}(\bar{b}, N_0 \cup N_1)$ is isolated hence $\mathrm{tp}(\bar{b}, N_0 \cup N_1) \vdash \mathrm{tp}(\bar{b}, N^*_0 \cup N^*_1)$, and so $\mathrm{tp}(\bar{b}, N^*_0 \cup N^*_1)$ is isolated. We can easily conclude that M' is $\boldsymbol{F}^l_{\aleph_0}$-constructible over $N_0^* \cup N_1^*$. Hence there is M^*, $F^a_{\aleph_0}$-prime over $N_0^* \cup N_1^*$, $M' \subseteq M^*$. W.l.o.g. $\mathrm{tp}(\bar{a}, M^*)$ does not fork over M', hence if we prove that $\mathrm{tp}(\bar{a}, M')$ is orthogonal to N_0^* and to N_1^* we get a contradiction by [2] §2. Let $l \in \{0,1\}$; clearly $\mathrm{tp}_*(N_l^*, M')$ does not fork over N_l, and so by [2] 1.1 $\mathrm{tp}(\bar{a}, M')$ is orthogonal to N_l^*, so we finish.

1.5. Claim. (1) *Suppose $N \subseteq M$, $\bar{a} \not\subseteq M$, M' is prime over $M \cup \bar{a}$. Then there is $b \in M' - M$, $\mathrm{tp}(b, M)$ does not fork over N, $\mathrm{tp}(b, N)$ strongly regular and not orthogonal to $\mathrm{tp}(\bar{a}, M)$, provided that*

(a) $\mathrm{tp}(\bar{a}, M)$ *is regular not orthogonal to N, or at least*

(b) $\mathrm{tp}(\bar{a}, M)$ *is orthogonal to every $p \in S^m(M')$ which is orthogonal to N.*

(2) *Every type which is not orthogonal to N is not orthogonal to some strongly regular $p \in S^m(N)$.*

Proof. (1) Easily (a) implies (b), so assume (b) holds. By 1.1 there is $b \in M' - M$ as required except that maybe $\mathrm{tp}(b, M)$ is orthogonal to N. But then by (b) $\mathrm{tp}(b, M)$, $\mathrm{tp}(\bar{a}, M)$ are orthogonal, hence by 1.2, $\mathrm{tp}(b, M)$, $\mathrm{tp}_*(M', M)$ are orthogonal, hence weakly orthogonal; contradiction.

(2) Easy.

1.6. Theorem. *The lemmas* [2] 3.1, 3.2 *hold for $F^l_{\aleph_0}$-primeness.*

Proof. Straightforward: when in §3 we use the failure of the dop, we here use 1.3, 1.4, and where in 1.3 we used V 1.12 here we use 1.5.

As there are at least as many models as there are $F^a_{\aleph_0}$-saturated models, obviously (by [2] 2.5, [2] 5.1)

1.7. Theorem. *If T has the dop or is deep, then $I(\lambda, T) = 2^\lambda$ for $\lambda \geqq \lambda(T) + \aleph_1$.*

Now we shall deal with [2] §4.

1.8. Definition. $K^x_\lambda = \{(N, N', \bar{a}) : N \subseteq N'$ are F^x_λ-saturated models, $\bar{a} \in N$, $\bar{a} \not\subseteq N'$, N' is F^x_λ-atomic over $N \cup \bar{a}\}$, $K^{r,x}_\lambda = \{(N, N', \bar{a}) \in K^x_\lambda : \mathrm{tp}(\bar{a}, N)$ is regular$\}$.

1.9. Lemma. (1) *If $(N, N', \bar{a}) \in K$, $K \in \{K^l_{\aleph_0}, K^l_\lambda, K^{r,l}_\lambda\}$ then* $\mathrm{Dp}((N, N', \bar{a}), K) \leqq \mathrm{Dp}(\mathrm{tp}(\bar{a}, N), K^a_{\aleph_0})$.

(2) *If in* (1), *N is $F^a_{\aleph_0}$-saturated, then* $\mathrm{Dp}((N, N', \bar{a}), K) = \mathrm{Dp}(\mathrm{tp}(\bar{a}, N), K^{r,a}_{\aleph_0})$.

Remark. Look at [2] Definition 4.1, 4.3, Lemma 4.4.

Proof. (1) For simplicity we concentrate on $K = K^l_{\aleph_0}$, we prove by induction on α that if $(N_x, N'_x, \bar{a}) \in K^x_{\aleph_0}$, $N_t \subseteq N_a$, $N'_t \subseteq N'_a$, $\{N_a, N'_t\}$ is independent over N_t, then

$$\mathrm{Dp}((N_\iota, N'_\iota, \bar a), K^\iota_{\aleph_0}) \geqq \alpha \quad \text{implies} \quad \mathrm{Dp}((N_a, N'_a, \bar a), K^a_{\aleph_0}) \geqq \alpha.$$

For α zero, limit or successor of limit there is no problem. So let $\alpha = \beta + 1$, β non-limit.

So there is $\bar b \not\in N'_\iota$, $\mathrm{tp}(\bar b, N'_\iota)$ orthogonal to N_ι, and N''_ι prime over $N'_\iota \cup \bar b$, $\mathrm{Dp}((N'_\iota, N''_\iota, \bar b), K^\iota_{\aleph_0}) \geqq \beta$.

W.l.o.g. $\mathrm{tp}(\bar b, N'_a)$ does not fork over N'_ι. Then N''_ι is $\boldsymbol{F}^\iota_{\aleph_0}$-constructible over $N'_a \cup \bar b$, hence there is N''_a $\boldsymbol{F}^a_{\aleph_0}$-prime over $N'_a \cup \bar a$, $N''_\iota \subseteq N''_a$.

Now use the induction hypothesis.

(2) For each tree I (of sequences of ordinals) satisfying $\langle i \rangle \in I$ iff $i = 0$, $\mathrm{Dp}(\langle\ \rangle, I) = \mathrm{Dp}(\mathrm{tp}(\bar a, N), K^a_{\aleph_0})$, we can find a $\boldsymbol{F}^a_{\aleph_0}$-representation $\langle N^I_\eta, \bar a^I_\eta : \eta \in I \rangle$, such that $N^I_{\langle\ \rangle} = N$, $\bar a^I_{\langle\ \rangle} = \bar a$, and let M_I be $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_{\eta \in I} N^I_\eta$. By the proof of [2] 3.1, there is no $\bar b \in M_I - N$, $\{\bar b, \bar a\}$ independent over N. Hence by 1.6, M_I has an $\boldsymbol{F}^\iota_{\aleph_0}$-representation $\langle M^I_\eta, \bar a^I_\eta : \eta \in J_I \rangle$, $M^I_{\langle\ \rangle} = N$, $\bar a^I_{\langle 0 \rangle} = \bar a$. $\langle i \rangle \in J_I \Leftrightarrow i = 0$. Now counting the number of M_I for $|I| = \aleph_\alpha$, α large enough, $\alpha < \aleph_\alpha$, $|\alpha| = |\alpha|^{2^{|T|}}$, we get the missing inequality.

1.10. THEOREM. *If T is shallow without the dop then $I(\aleph_\alpha, T) \leqq \beth_\gamma(|\alpha|^{|T|})$ where $\gamma = \mathrm{Dp}(T, K^\iota_{\aleph_0})$.*

PROOF. By 1.6, just like [2] 4.7.

§2. Infinite depth

Hypothesis. T is superstable shallow and without the dop.

Here we get lower bounds for $I(\aleph_\alpha, T)$, $I^a_{\aleph_\beta}(\aleph_\alpha, T)$ for the case mentioned in the title.

At first glance it may look surprising that as long as $\beta < \alpha$ its value has no influence. The point is that, if $l.y\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\boldsymbol{F}^a_{\aleph_0}$-representation of an $\boldsymbol{F}^a_{\aleph_\beta}$-saturated model, we know that each $\eta \in I^-$ has $\geqq \aleph_\beta$ immediate successors, but there is no restriction on how many immediate successors of $\eta \in (I^-)^-$ have $\aleph_{\beta+1}$ immediate successors.

Note that for countable T, the situation is considerably simplified. More generally if $\aleph_\alpha$ is big enough (if $|T| < \beth_\omega$-always) we get the exact number.

2.1. THEOREM. *Suppose $\aleph_\alpha > \lambda(T) + \aleph_1$, $\alpha \geqq \omega$, $\mathrm{Dp}(T) \geqq \omega$, $\beta < \alpha$ then $I^a_{\aleph_\beta}(\aleph_\alpha, T) \geqq \beth_{\mathrm{Dp}(T)}(|\alpha| + \aleph_0$.*

2.1A. REMARK. (1) We should have written $\min\{\beth_{\mathrm{Dp}(T)}(\alpha), 2^{\aleph_\alpha}\}$, but we shall ignore this for notational simplicity.

(2) If $|T| < \beth_n$ for some n, or even $|T| < \beth_\alpha$, $\alpha + \mathrm{Dp}(T) = \mathrm{Dp}(T)$, the equality holds.

(3) The theorem, of course, holds for $I'_{\aleph_\beta}$ when T is totally transcendental, and similarly for 2.1 A(2).

Proof. We shall define for every $W = (N, N', \bar{a}) \in K'_{\aleph_0}$ such that N is $\boldsymbol{F}^a_{\aleph_0}$-prime over ϕ a set $H(W)$ and a partition of it $\langle H_\zeta(W) : \zeta < \zeta_W \rangle$, and an $\boldsymbol{F}^a_{\aleph_0}$-representation $\langle N^{W,Y}_\eta, a^{W,Y}_\eta : \eta \in I^{W,Y} \rangle$ for any $Y \in H(W)$. The definition is by induction on the depth $\zeta = \mathrm{Dp}(N, N', \bar{a})$. For notational simplicity assume $\mathrm{Dp}(T) < \beth_\omega$ and $\mathrm{Dp}(T) \leqq \aleph_\alpha + 1$.

$\zeta = 0$. Let $H(W) = \{\aleph_\beta, \aleph_\alpha\}$, $I^{W,\aleph_\alpha} = \{\langle\ \rangle, \langle i \rangle : i < \aleph_\alpha\}$, $I^{W,\aleph_\beta} = \{\langle\ \rangle, \langle i \rangle : i < \aleph_\beta\}$, $N^{W,Y}_{\langle\ \rangle} = N$.

$\{\bar{a}^{W,Y}_\eta : \eta \in I^{W,Y}\}$ is an independent set over N of sequences realizing $\mathrm{tp}(\bar{a}, N)$, and $N^{W,Y}_\eta$ is $\boldsymbol{F}^a_{\aleph_0}$-prime over $N \cup \bar{a}^{W,Y}_\eta$, for $\eta \in I^{W,Y} - \{\langle\ \rangle\}$. Let $\zeta_W = 1$.

$\zeta = 1$. Let $V = (N', N'', \bar{a}') \in K'$ be such that $N' <_N N''$, $\mathrm{Dp}(N', N'', \bar{a}') = 0$. Let $H(W) = \{\langle \chi, I^{V,\aleph_\alpha} \rangle : 0 \leqq \chi \leqq \aleph_\alpha\}$ (so χ may be finite) and if $Y = \langle \chi, I^{V,\aleph_\alpha} \rangle$ then

$$I^{W,Y} = \{\langle\ \rangle\} \cup \{\langle \gamma \rangle ^\wedge \eta : \eta \in I^{V,\aleph_\alpha} \text{ and } \gamma < \chi \text{ or } \eta \in I^{V,\aleph_\beta} \text{ and } \chi \leqq \gamma < \aleph_\alpha\}$$

and define the representation accordingly, and let $\zeta_W = 1$.

$\zeta = \xi + 1$, *ξ successor.* Let $V = (N', N'', \bar{a}'') \in K'$, $N' <_N N''$, $\mathrm{Dp}(N', N'', \bar{a}'') = \xi$. We let $H(W) \subseteq \{Y : Y \subseteq H_0(V), |Y| \leqq \aleph_\alpha\}$ be such that:

(a) $|H(W)| = \mathrm{Min}\{|\mathscr{P}(H_0(V))|, 2^{\aleph_\alpha}\}$,

(b) all $Z \in H(W)$ have the same power $\leqq \aleph_\alpha$,

(c) for every $Z \in H(W)$, any two members have an infinite symmetric difference.

Let for $Z \in H(W)$, $Z = \{Y_i : i < i_0\}$,

$$I^{Z,W} = \{\langle\ \rangle\} \cup \{\langle \omega_\alpha i + j \rangle ^\wedge \eta : i < i_0, j < \aleph_\alpha \text{ and } \eta \in I^{V,Y_i}\}$$

and the representation is defined accordingly.

What about the partition? As we shall see $H(W)$ is infinite, so let $\zeta_W = |H(W)| + 1$, $H_\xi(W)$ have power $|H(w)|$ for every $\xi < \zeta_W$.

$\zeta = \delta + 1$, *δ limit.* There is a set $S \subseteq \{i + 1 : i < \delta\}$, unbounded, and for every $\alpha \in S$, $V(\alpha) = (N', N''_\gamma, \bar{a}_\gamma) \in K'_{\aleph_0}$, $N' <_N N''_\gamma$, $\mathrm{Dp}(N', N''_\gamma, \bar{a}_\gamma) = \gamma$. As $\mathrm{Dp}(T)$ was assumed to be smaller than $\aleph_\alpha + 1$, and by the computation below, $\mathrm{Dp}(W) \in S \Rightarrow \mathrm{Dp}(T) < |H(W)|$). Let