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On Certain Indestructibility of Strong Cardinals and a Question of Hajnal

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Abstract. A model in which strongness of κ is indestructible under κ^+ – weakly closed forcing notions satisfying the Prikry condition is constructed. This is applied to solve a question of Hajnal on the number of elements of $\{\lambda^\delta | 2^\delta < \lambda\}$.

Strong cardinals were introduced by Mitchell [Mi]. H. Woodin was the initiator of using cardinals of this type in forcing constructions. As far as we know, all such applications were to reduce the strength of propositions known to be consistent assuming the existence of cardinals which are supercompact, huge, etc. Starting with n strong cardinals we shall construct a model satisfying the following: for some cardinal λ, $\lambda < \lambda^\omega < \lambda^{\omega_1} < \dots < \lambda^{\omega_n}$ and $2^{\omega_i} = \omega_{i+1}$ for every $i \in \omega$.

This answers a question of Hajnal. Hajnal [E-H-M-R] and later, and independently the second author [S1], observed that the set $\{\lambda^\delta | 2^\delta < \lambda\}$ is always finite. It is unclear how to apply supercompactness without appealing to strongness for this kind of results. The difficulty is that by Solovay [So], the Singular Cardinal Hypothesis holds above a strongly compact cardinal.

The construction of the model is based on the following indestructibility principle:

There is a model in which the strongness of κ is indestructible under κ^+-weakly closed forcings satisfying the Prikry condition.

This is similar to the Laver indestructibility of supercompactness by κ-directed closed forcing notions [L]. The forcing notions we allow are to some extent more closed, but on the other hand the use of forcing notions like those of Prikry, Magidor, and Radin is allowed.

1. The Basic Notions

Let us sketch the basic definition and facts about strong cardinals that we are going to use. We refer to the papers of Baldwin [B], Mitchell [Mi], and the book of Dodd [D] for a detailed presentation. The approach below follows Baldwin [B].

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A cardinal κ is β-strong iff there exists an elementary embedding $i: V \to M$ with critical point κ so that $M \supseteq V_\beta$. A cardinal κ is strong iff it is β-strong for every ordinal β.

A function $f: {}^\lambda\kappa \to V$ has support in x (where $x \subseteq \lambda$) if there is $f': {}^x\kappa \to V$ such that $f(a) = f'(a | x)$ for every $a \in {}^\lambda\kappa$. If $X \subseteq {}^\lambda\kappa$ then X has support in x if there exists $X' \subseteq {}^x\kappa$ such that for every $a \in {}^\lambda\kappa$, $a \in X$ iff $a | x \in X'$. For f having a finite support define $\operatorname{supp}(f)$ to be $\cap\{x \subseteq \lambda \mid f$ has a support in $x\}$. Similarly for $\operatorname{supp}(X)$.

Let $P_f({}^\lambda\kappa) = \{X \in P({}^\lambda\kappa) \mid X$ has a finite support$\}$. If U is an ultrafilter on (the Boolean algebra) $P_f({}^\lambda\kappa)$, then $\operatorname{Ult}(V, U)$ denotes the ultrapower of V by U using only functions of finite support. If it is well founded, then let us identify it with its transitive collapse. The following definition and two propositions are due to Baldwin [B].

Definition 1.1. An ultrafilter $U \subseteq P_f({}^\lambda\mu)$ is (κ, λ)-normal iff $\kappa < \lambda$ and

(a) $\operatorname{Ult}(V, U)$ is well founded and i is the canonical embedding of V in it.

(b) κ is the critical point of i.

(c) For all $\alpha < \lambda$ $[h_\alpha]_U = \alpha$, where $h_\alpha: {}^\lambda\kappa \to \kappa$ is defined by $h_\alpha(a) = a_\alpha$.

(d) If $\nu < \mu$ then $i(\nu) < \lambda$.

We shall deal only with ultrafilters satisfying $\mu = \kappa$. If $X' = \{a \restriction \operatorname{supp}(X) \mid a \in X\}$, then let cX be the subset of ${}^n\kappa$ such that $(\gamma_0, \ldots, \gamma_{n-1}) \in cX$ iff $((\alpha_0, \gamma_0), \ldots, (\alpha_{n-1}, \gamma_{n-1})) \in X'$, where $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1}$ and $\operatorname{supp}(X) = \{\alpha_0, \ldots, \alpha_{n-1}\}$.

Proposition 1.2. *If U is (κ, λ)-normal then*

$$X \in U \quad \textit{iff} \quad \operatorname{supp}(X) \in i_U(cX)$$

(where $\operatorname{supp}(X)$ *is taken to be an increasing sequence).*

Proposition 1.3. *Let $j: V \to M$ (transitive) with critical point κ, and let $\lambda > \kappa$. Then there is a (κ, λ)-normal ultrafilter U and a elementary embedding k:* $\operatorname{Ult}(V, U) \to M$ *such that the following diagram commutes and $k | \lambda$ is the identity map.*

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[dr, "i_U"'] & & M. \\ & \operatorname{Ult}(V, U) \arrow[ur, "k"'] & \end{tikzcd}$$

Furthermore, there is only one such (κ, λ)-normal U.

Let us call a (κ, λ)-normal ultrafilter U over $P_f({}^\lambda\kappa)$ β-strong, if $V_\beta \subseteq \operatorname{Ult}(V, U)$.

If κ is β-strong, then by Proposition 1.3, there exists a (κ, β)-normal ultrafilter U over $P_f({}^\beta\kappa)$ which is β-strong. Note, that if cf., $\beta > \kappa$, then $\operatorname{Ult}(V, U)$ is closed under κ-sequences of its elements.

Let us describe now the way of iterating Prikry type forcing notions, which was introduced in [G1] and generalized in [S2]. Let $\langle P, \leqq, \leqq_* \rangle$ be a set with two partial orders so that $\leqq_* \subseteq \leqq$, i.e., $p \leqq_* q \to p \leqq q$ for every $p, q \in P$. $\langle P, \leqq \rangle$ is used as a forcing notion. If for every $p \in P$ and every statement σ of the forcing language there exists $q \in P$ $p \leqq_* q$, q decides σ ($q \parallel \sigma$), then let us say that P (or more precisely $\langle P, \leqq, \leqq_* \rangle$) satisfies the Prikry condition. Let α be a cardinal. Call P α-weakly closed if $\langle P, \leqq_* \rangle$ is α-closed. By Prikry [P] an α-weakly closed forcing satisfying the Prikry condition does not add new bounded subsets to α.

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Define now the iteration of forcing notions of such kind.

Let A be a set of inaccessible cardinals. Denote by A^l the closure of the set $A \cup \{\alpha+1 \mid \alpha \in A\}$. For every $\alpha \in A^l$ define $\mathscr{P}_\alpha$ by induction. Suppose that for every $\beta \in A^l \cup \alpha$, $\langle \mathscr{P}_\beta, \leqq, \leqq_*\rangle$ is defined. Let $\mathscr{P}_\alpha$ be the set of all elements p of the form $\{\underset{\sim}{p}_\alpha \mid \gamma \in g\}$ where

(1) g is a subset of $\alpha \cap A$;

(2) g has an Easton support, i.e., for every inaccessible $\beta \leqq \alpha$, $\beta > |\operatorname{dom} g \cap \beta|$;

(3) for every $\gamma \in \operatorname{dom} g$

$$p \mid \gamma = \{\underset{\sim}{p}_\beta \mid \beta \in \gamma \cap g\} \in \mathscr{P}_\gamma$$

and $p \mid \gamma \Vdash_{\mathscr{P}_\gamma}$ "$\underset{\sim}{p}_\gamma$ is a condition in a $\check{\gamma}$-weakly closed forcing notion $\underset{\sim}{Q}_\gamma$ satisfying the Prikry condition and if $\check{\gamma}'$ is the least element of $\check{A}$ above $\check{\gamma}$, then $|\underset{\sim}{Q}_\gamma| \leqq \check{\gamma}'$".

Let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma \in g\rangle$, $q = \langle \underset{\sim}{q}_\gamma \mid \gamma \in f\rangle$ be elements of $\mathscr{P}_\alpha$. Then $p \geqq q$ (p is stronger than q) if the following holds:

(1) $g \supseteqq f$

(2) for every $\gamma \in f$

$$p \mid \gamma \Vdash_{\mathscr{P}_\gamma} \text{"}\underset{\sim}{p}_\gamma \leqq \underset{\sim}{q}_\gamma \text{ in the forcing } \underset{\sim}{Q}_\gamma\text{"}$$

(3) There exists a finite subset b of f so that for every $\gamma \in f - b$

$$p \mid \gamma \Vdash_{\mathscr{P}_\gamma} \text{"}\underset{\sim}{p}_\gamma \leqq \underset{\sim}{q}_\gamma \text{ in the ordering } \leqq^* \text{ of } \underset{\sim}{Q}_\gamma\text{"}.$$

Let $p_* \geqq q$ if $p \geqq q$ and the set b in (3) is empty.

It follows from [G1], that $\langle \mathscr{P}_\alpha, \leqq, \leqq_*\rangle$ satisfies the Prikry condition.

2. Making the Strongness of κ Indestructible

Let κ be a strong cardinal. Assume *GCH*.

Our aim will be to define a forcing which makes a strongness of κ indestructible under κ^+-weakly closed forcings satisfying the Prikry condition. The construction will be similar to those of Laver [L]. The point will be to show that κ remains strong in forcing extensions.

The following lemma was proved by Laver [L] for a supercompact κ and a supercompact ultrafilter U on $[\lambda]^{<\kappa}$, but actually his proof works also for a strong κ.

Lemma 2.1. *Let κ be a strong cardinal. Then there is $f: \kappa \to V_\kappa$ such that for every x and every $\lambda \geqq |TC(x)|$ (where $TC(x)$ is the transitive closure of x), there is a (κ, λ)-normal λ-strong ultrafilter U on $P_f({}^\lambda\kappa)$ such that $(i_U f)(\kappa) = x$.*

Let us fix some $f: \kappa \to V_\kappa$ as in the lemma. Let A be the set of all inaccessible cardinals $\vartheta < \kappa$ so that V_ϑ is closed under f. Define A^l as in Sect. 1. We define by induction an iteration $\langle \mathscr{P}_\alpha, Q_\alpha \mid \alpha \in A^l\rangle$ and also ordinals $\lambda_\alpha, \alpha \in A^l$. Suppose that $\alpha \in A^l$ $\mathscr{P}_\beta$ are defined for every $\beta \in A^l \cap \alpha$. If α is a limit point of A^l, then let $\mathscr{P}_\alpha$ be the limit of the $\mathscr{P}_\beta$'s as it is defined in Sect. 1. Set $\lambda_\alpha = \bigcup_{\beta < \alpha} \lambda_\beta$. Suppose that α is a successor of β in A^l. If $\alpha \neq \beta+1$, then let $Q_\beta = \{\phi\}$, $\mathscr{P}_\alpha = \mathscr{P}_\beta * Q_\beta$ and $\lambda_\alpha = \max(\alpha, \lambda_\beta)$. Suppose that $\alpha = \beta+1$. Set $Q_\alpha = \{\phi\}$ unless

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(1) $f(\beta)=\langle \underset{\sim}{Q}, \lambda\rangle$, where λ is an ordinal and $\underset{\sim}{Q}$ a $\mathscr{P}_\beta$-name of a forcing notion so that $\Vdash_{\mathscr{P}_\beta}$ "$\underset{\sim}{Q}$ is β-weakly closed forcing satisfying Prikry condition and $\lambda>|\underset{\sim}{Q}|$", and

(2) $\lambda<\beta$, $\lambda_\alpha<\beta$.

If (1) and (2) hold, then let $\underset{\sim}{Q}_\beta=\underset{\sim}{Q}$ $\mathscr{P}_\alpha=\mathscr{P}_\beta * \underset{\sim}{Q}_\beta$ and $\lambda_\alpha=\lambda$.

Set $\mathscr{P}_\kappa$ to be the limit of $\mathscr{P}_\alpha$'s of Sect. 1.

Suppose that $\phi\Vdash_{\mathscr{P}_\kappa}$ "$\langle \underset{\sim}{Q}, \leqq^Q, \leqq^Q_*\rangle$ is a κ^+-weakly closed forcing notion satisfying the Prikry condition."

We want to show that κ remains strong in $V^{\mathscr{P}_\kappa * Q}$. It is enough to show that for arbitrarily large λ κ remains λ-strong. So let λ be a regular cardinal above $\max(\kappa, TC(\langle \underset{\sim}{Q}, \leqq^Q, \leqq^Q_*\rangle))$. Using Lemma 2.1, find a (κ,λ)-normal, λ-strong ultrafilter U over $P_f({}^\lambda\kappa)$, *so that* $i_U(f)(\kappa)=\langle\langle \underset{\sim}{Q}, \leqq^Q, \leqq^Q_*\rangle, \lambda\rangle$. Denote i_U by i and $\mathrm{Ult}(V,U)$ by M. Also let us drop the upper index $\underset{\sim}{Q}$ from $\leqq^Q$ and $\leqq^Q_*$.

We call a subset D of a forcing notion $\langle P, \leqq^P, \leqq^P_*\rangle$ $*$-dense if it is dense is the ordering $\leqq^P_*$. Notice, that if P satisfies the Prikry condition, then the set $\{p\in P \mid p \Vert \sigma\}$ is $*$-dense for every statement σ of the forcing language.

Lemma 2.2. *There is a sequence* $\langle \underset{\sim}{D}_\alpha | \alpha<\kappa^+\rangle$ *so that for every* $\alpha<\kappa^+$

(1) $\underset{\sim}{D}_\alpha\in M$,

(2) *in* M, $\phi\Vdash_{\mathscr{P}_{\kappa+1}}$ "$\underset{\sim}{D}_\alpha$ *is a $*$-dense subset of* $i(\mathscr{P}_\kappa)/\mathscr{P}_{\kappa+1}$"

(3) *for every* $\underset{\sim}{D}\in M$ *so that in* M $\phi\Vdash_{\mathscr{P}_{\kappa+1}}$ "$\underset{\sim}{D}_\alpha$ *is a $*$-dense subset of* $i(\mathscr{P}_\kappa)/\mathscr{P}_{\kappa+1}$" *there exists* $\alpha<\kappa^+$ *s.t. in* M $\phi\Vdash_{\mathscr{P}_{\kappa+1}} \underset{\sim}{D}\supseteq\underset{\sim}{D}_\alpha$".

Proof. Consider the following set $X=\{\underset{\sim}{D}\in M \mid$ in M $\phi\Vdash_{\mathscr{P}_{\kappa+1}}$ "$\underset{\sim}{D}$ is a dense $*$-dense subset of $i(\mathscr{P}_\kappa)/\mathscr{P}_{\kappa+1}$" and there exists a function $g:\kappa\to\mathscr{P}(V_\kappa)$ so that $\underset{\sim}{D}=i(g)(\kappa)\}$.

Let $\langle \underset{\sim}{D}_\alpha|\alpha<\kappa^+\rangle$ be an enumeration of X. Let $\underset{\sim}{D}$ be as in (3). Since $D\in M$ and $M=\mathrm{Ult}(V,U)$, for some $g:\kappa^n\to\mathscr{P}(V_\kappa)$ and some $\kappa_1,\ldots,\kappa_{n-1}$, $\kappa<\kappa_1<\ldots<\kappa_{n-1}<\lambda$ $\underset{\sim}{D}=i(g)(\kappa,\kappa_1,\ldots,\kappa_{n-1})$. The forcing $i(\mathscr{P}_\kappa)/\mathscr{P}_{\kappa+1}$ is, in M, λ^+-weakly closed. Define $\underset{\sim}{D}'=\cap\{i(g)(\kappa,\alpha_1,\ldots,\alpha_{n-1}) \mid \kappa<\alpha_1<\ldots<\alpha_{n-1}<\lambda,\quad i(g)(\kappa,\alpha_1,\ldots,\alpha_{n-1})$ is a $\mathscr{P}_{\kappa+1}$-name of a $*$-dense number of $i(\mathscr{P}_\kappa)/\mathscr{P}_{\kappa+1}\}$. Then $\underset{\sim}{D}'\in M$, it is $*$-dense in $i(\mathscr{P}_\kappa)/\mathscr{P}_{\kappa+1}$ and, since $\lambda=i\ (<\lambda_\alpha|\alpha\langle\kappa>)(\kappa)$, $D'\in X$. $\square$

Let $G*H$ be a generic subset of $\mathscr{P}_\kappa * \underset{\sim}{Q}$ over V. Since $V_\lambda\subseteq M$, $<\underset{\sim}{Q}, \leqq, \leqq_*>\in M$ and $\mathscr{P}_{\kappa+1}=\mathscr{P}_\kappa*\underset{\sim}{Q}$. Also $V_\lambda^{V[G*H]}\subseteq M[G*H]$. (Clearly $G*H$ is also a generic subset of $\mathscr{P}_{\kappa+1}$ over M). Define in $V[G*H]$ a sequence $\langle r_\alpha|\alpha<\kappa^+\rangle$ so that $r_\alpha\in \underset{\sim}{D}_\alpha[G*H]$ and for every $\beta\geqq\alpha$ $r_\alpha\leqq r_\beta$. Clearly $\langle r_\alpha|\alpha<\beta\rangle\in M[G*H]$ for every $\beta<\kappa^+$. We use the fact that $\langle \underset{\sim}{D}_\alpha|\alpha<\beta\rangle\in M$ for $\beta<\kappa^+$ which holds since $V\cap{}^\kappa M\subseteq M$.

We would like now to extend U to an ultrafilter U^* over $\mathscr{P}_f({}^\lambda\kappa)$ in $V[G,H]$. Notice, that $\mathscr{P}_f({}^\lambda\kappa)$ in sense of $V[G]$ is the same as $\mathscr{P}_f({}^\lambda\kappa)$ of $V[G,H]$, since $\underset{\sim}{Q}$ does not add new subsets to κ. Let U^* consists of all $X\in\mathscr{P}_f({}^\lambda\kappa)$ so that for some $\mathscr{P}_\kappa$-name $\underset{\sim}{X}$ of X, $\alpha<\kappa^+$ and $p\in G*H$, in M

$$\langle p, r_\alpha\rangle\Vdash_{i(\mathscr{P}_\kappa)} (\mathrm{supp}(X))\check{\in} i(cX)\,.$$

Clearly, U^* is an ultrafilter over $\mathscr{P}_f({}^\lambda\kappa)$ extending U.

Lemma 2.3. $\mathrm{Ult}(V[G*H], U^*)$ *is well-founded.*

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Proof. For every countable or finite $a \subseteqq \lambda$ in $V[G*H]$ consider U_a over $[\kappa]^{otp(a)}$ defined as follows:

$X \in U_a$ iff for some $p \in G_\kappa * H$, $\alpha < \kappa^+$, a $\mathscr{P}_\kappa$-name $\underset{\sim}{X}$ of X and a $\mathscr{P}_{\kappa+1}$-name $\underset{\sim}{a}$ of a, in M

$$\langle p, r_\alpha \rangle \Vdash_{\overline{i(\mathscr{P}_\kappa)}} \underset{\sim}{a} \in i(\underset{\sim}{X}).$$

The standard arguments show that U_a is a κ-complete ultrafilter in $V[G_\kappa * H]$. Denote $\mathrm{Ult}(V[G*H], U_a)$ by M_a and the canonical embedding by i_a. If $a \subseteqq b \subseteqq \lambda$, $|b| \leqq \aleph_0$, then there is an embedding $k_{ab}: M_a \to M_b$ making the diagram

$$\begin{tikzcd} & M_a \arrow[d] \\ V[G,H] \arrow[ur, "i_a"] \arrow[r, "i_b"'] & M_b \end{tikzcd}$$

commutative. Just set $k_{ab}([h]_{U_a}) = [h']_{U_b}$ where $h'(x) = h(x|a)$.

It is easy to see that $\langle M_a, \pi_{ab} | a \subseteqq b \subseteqq \lambda, |b| \leqq \aleph_0 \rangle$ forms a direct system with a well-founded limit and this limit is isomorphic to $\mathrm{Ult}(V[G*H], U^*)$. $\square$

Denote the transitive collapse of $\mathrm{Ult}(V[G*H], U^*)$ by M^* and i_{U^*} by i^*. In order to finish the proof we need to show that $V_\lambda^{V[G*H]} \subseteqq M^*$. Let us prove first the following.

Lemma 2.4. *V_λ in sense of V is contained in M^*.*

Proof. Let $g: [\kappa]^n \to V_\kappa$ be a function of $V[G,H]$ so that for almost all $\alpha_0 < \ldots < \alpha_{n-1} < \kappa$ $g(\alpha_0, \ldots, \alpha_{n-1}) \in V_{\lambda_{\alpha_0}}$. Let $a \in [\lambda]^n$.

Claim. For some $h \in V$ $i^*(g)(a) = i^*(h)(a)$.

Proof. Let $\underset{\sim}{g}$ be a $\mathscr{P}_\kappa$-name of g so that in M

$$\phi \Vdash \text{“} i(\underset{\sim}{g})(\check{a}) \in \check{V}_\lambda \text{”}.$$

Since the forcing above λ is λ^+-weakly closed, the set $\underset{\sim}{D} = \{p \in j(\mathscr{P}_\kappa)/\mathscr{P}_{\kappa+1} | \text{ for some } t \in V_\lambda\ p \Vdash i(\underset{\sim}{g})(\check{a}) = \check{t}\}$ is $*$-dense. So by Lemma 2.2, for some $\alpha < \kappa^+$ $\underset{\sim}{D}_\alpha \subseteqq \underset{\sim}{D}$. Then there are $p \in G*H$ and $t \in V_\lambda$ so that $\langle p, r_\alpha \rangle \Vdash_{\overline{\mathscr{P}_{j(\kappa)}}} i(\underset{\sim}{g})(\check{a}) = \check{t}$. Since $V_\lambda \subseteqq M$, $t \in M$ and there is $h: [\kappa]^n \to V_\kappa$ in V so that $i(h)(a) = t$. But then $i^*(g)(a) = i^*(h)(a)$. $\square$ of the claim.

Hence every element of $V_{i^*(\langle \lambda_\alpha | \alpha \langle \kappa \rangle)(\kappa)}$ in M^* is represented by some $h: [\kappa]^n \to V_\kappa$, $h(\alpha_0, \ldots, \alpha_{n-1}) \in V_{\lambda_{\alpha_0}}$ which belongs to V. Since $U^* \supseteqq U$, it implies that for every such h $i(h)(a) = i^*(h)(a)$. Also every element of V_λ is of the form $i(h)(a)$ for some h, a as above. So $V_\lambda \subseteqq M^*$.

The lemma implies that U^* is (κ, λ)-normal ultrafilter.

Lemma 2.5. *U^* is λ-strong.*

Proof. Since $V_\lambda^{V[G,H]} = V_\lambda[G,H]$ it is enough to show that $G*H \in M^*$. By Lemma 2.4, $i^*(f)(\kappa) = i(f)(\kappa) = \langle \underset{\sim}{Q}, \lambda \rangle$. So, $\mathscr{P}_{\kappa+1} = \mathscr{P}_\kappa * \underset{\sim}{Q}$ also in M^*. Consider the function $g: \kappa \to V_\kappa[G]$ so that $g(\alpha) = G_\alpha * H$ where $G_\alpha = \tilde{G} \cap \mathscr{P}_\alpha$ and $H_\alpha = G \cap Q_\alpha$. Let us show that $i^*(g)(\kappa) = G*H$. It is enough to prove that $G*H \subseteqq i^*(g)(\kappa)$. Let $p \in G_\kappa * H$. Find $h': \kappa^n \to \mathscr{P}_\kappa$, $h' \in V$ and $\kappa_0 < \kappa_1 < \cdots < \kappa_{n-1} < \lambda$ so that

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$i(h')(\kappa_0,\dots,\kappa_{n-1})=p$. Let $h:{}^\lambda\kappa\to\mathscr{P}_\kappa$ be defined by $h(a)=h'(a|\langle\kappa_0,\dots,\kappa_{n-1}\rangle)$ for every $a\in{}^\lambda\kappa$. Consider a set

$$X=\{a\in{}^\lambda\kappa\,|\,h(a)\in G_{a(\kappa)}*H_{a(\kappa)}\}.$$

Clearly, $X\in\mathscr{P}_f({}^\lambda\kappa)$ and $\operatorname{supp}X=\{\kappa_0,\dots,\kappa_{n-1}\}$. Now, in M

$$p\Vdash_{\overline{i(\mathscr{P}_\kappa)}}\{\kappa_0,\dots,\kappa_{n-1}\}\in i(c\underset{\sim}{X})$$

since $p\Vdash_{\overline{i(\mathscr{P}_\kappa)}}\check{p}\in \underset{\sim}{G}_\kappa*\underset{\sim}{H}_\kappa$, and $i(h')(\kappa_0,\dots,\kappa_{n-1})=p$. Hence $X\in U^*$ and $\{\kappa_0,\dots,\kappa_{n-1}\}\in i^*(cX)$. So $i^*(h')(\kappa_0,\dots,\kappa_{n-1})\in i^*(g)(\kappa)$. But by Lemma 2.4 $i^*(h')(\kappa_0,\dots,\kappa_{n-1})=i(h')(\kappa_0,\dots,\kappa_{n-1})=p$. $\square$

We have thus completed the proof of the strongness of κ in $V[G*H]$.

Let us mention in conclusion two possible generalizations of the present construction to higher cardinals.

The following cardinals were introduced by Baldwin [B]: κ is 1-hyperstrong iff κ is strong; κ is $(\beta+1)$-hyperstrong iff for every $x\in V$ there exists $j:V\to M$, $\operatorname{crit}(j)=\kappa$ $x\in M$ and $M\models\kappa$ is β-hyperstrong; if γ is a limit ordinal, then κ is γ-hyperstrong; iff κ is β-hyperstrong for every $\beta<\gamma$; κ is hyperstrong iff κ is β-hyperstrong for every β. Under the same lines it is possible to make the hyperstrongness of κ indestructible under κ^+-weakly closed forcing satisfying the Prikry condition.

Let $A\subseteqq{}^\kappa\kappa$. A cardinal κ is called A-Shelah if for every $f\in A$ there exists $j:V\to M$, $\operatorname{crit}(j)=\kappa$ and $M\supseteqq V_{j(f)(x)}$. For $A={}^\kappa\kappa$ such cardinals were defined in [S-W]. It is possible to make the property of being A-Shelah indestructible under κ^+-weakly closed forcing satisfying the Prikry condition. Also starting with a Shelah cardinal κ it is possible to construct a generic extension $V[G]$ where being a $({}^\kappa\kappa\cap V)$-Shelah is indestructible under κ^+-weakly closed forcing satisfying the Prikry condition.

H. Woodin showed that adding Cohen subsets to κ may preserve the strongness of κ. It is possible to incorporate his construction in above. This will provide a model in which a strongness of κ is indestructible under κ^+-weakly closed forcing notions satisfying the Prikry condition and the forcing adding any number of Cohen subsets to κ.

Notice, that by Mitchell [Mi] it looks like collapsing κ^+ to κ destroys even measurability of κ unless there exists an inner model with cardinals stronger than strong.

3. An Application to a Question of Hajnal

Hajnal, see [E-H-M-R], and Shelah, see [S1, p. 164], showed independently that the $\{\lambda^\delta|2^\delta<\lambda\}$ is always finite. Hajnal asked if this set can contain more than two elements.

Theorem 3.1. *Let $2<n\in\omega$. Suppose that there are n strong cardinals. Then there exists a generic extension satisfying the following:*

(a) *GCH below $\aleph_\omega$*

(b) *for some λ $\lambda<\lambda^\omega<\lambda^{\omega_1}<\dots<\lambda^{\omega_n}$.*

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Proof. Let n be fixed. Suppose that $\kappa_0 < \kappa_1 < \cdots < \kappa_n$ are strong cardinals and GCH holds. Using Sect. 2, define a generic extension V_1 of V so that for every $i<n$ the strongness of κ_i is indestructible under κ_i^+-weakly closed forcing satisfying the Prikry condition, (e.g., go up by induction. The forcing above κ_{i-1} is κ_{i-1}^+-weakly complete). Note that as in Levy-Solovay [L-So] a forcing of power $<\kappa$ does not destroy strongness or indestructible strongness of κ. Let $\lambda = \kappa_n$. Using Woodin method (see [G2]). Blow up 2^λ to $\lambda^{+(n+1)}$ preserving the strongness of λ by κ_{n-1}^+-closed forcing. Now use the Magidor forcing [Ma] with conditions above κ_{n-1} to change the cofinality of λ to ω_n. Denote the combination of these two forcings by P_n. P_n is a κ_{n-1}^+-weakly closed forcing satisfying the Prikry condition. So all κ_i's $(i<n)$ remain strong in $V_1^{P_n}$.

Now, in the similar fashion, define P_{n-1} which is κ_{n-2}^+-weakly closed, satisfies the Prikry condition, blows $2^{\kappa_{n-1}}$ to λ^{+n} and changes the cofinality of κ_{n-1} to ω_{n-1}. Then $P_n * P_{n-1}$ is a κ_{n-2}^+-weakly closed forcing satisfying the Prikry condition. So all κ_i's with $i<n-1$ remain strong in $V_1^{P_n * P_{n-1}}$. Continue, define P_{n-2} and so on. Let $V_2 = V_1^{P_n * P_{n-1} * \cdots * P_0}$. Then, in V_2, $\lambda^\omega = \lambda^+$, $\lambda^{\omega_1} = \lambda^{++}, \ldots, \lambda^{\omega_n} = \lambda^{+(n+1)}$. So V_2 is the desired model. □

Working harder, it is possible to project the above to $\aleph_{\omega_n}$. Namely, the following holds.

Theorem 3.2. *There is a generic extension of a model with n strong cardinals so that*

(a) $\aleph_\omega$ *is strong limit cardinal*

(b) $\aleph_{\omega_n} < \aleph_{\omega_n}^{\omega} < \aleph_{\omega_n}^{\omega_1} < \cdots < \aleph_{\omega_n}^{\omega_n}$.

Proof. Extend V to V_1 as above. Define P_n to be the Woodin forcing for making $2^{\kappa_n} = \kappa_n^{+(n+1)}$ and $\kappa_n = \kappa_{n-1}^{+\omega_n}$. It is possible to define an ordering $\leqq^*$ on P_n so that $\langle P_n, \leqq, \leqq^* \rangle$ will be κ_{n-1}^+-weakly closed satisfying the Prikry condition (or more precisely this will hold above some condition). So the strongness of κ_i's $(i<n)$ would not be effected in $V_1^{P_n}$. Choose P_i's for $i<n$ in the same fashion. $V_2 = V^{P_n * P_{n-1} * \cdots * P_0}$ will be as required. □

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