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ON THE NUMBER OF NONISOMORPHIC MODELS OF AN INFINITARY THEORY WHICH HAS THE INFINITARY ORDER PROPERTY. PART A

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Abstract. Let κ and λ be infinite cardinals such that $\kappa \leq \lambda$ (we have new information for the case when $\kappa < \lambda$). Let T be a theory in $L_{\kappa^+,\omega}$ of cardinality at most κ, let $\varphi(\bar{x}, \bar{y}) \in L_{\lambda^+,\omega}$. Now define

$$\begin{aligned}\mu^*_\varphi(\lambda, T) = \operatorname{Min}\{\mu^*: &\text{If } T \text{ satisfies } (\forall \mu < \mu^*)(\exists M_\mu \models T)(\exists\{\bar{a}_i: i < \mu\} \subseteq M_\mu)\\ &(\forall i,j < \mu)[i < j \Leftrightarrow M_\mu \models \varphi[\bar{a}_i, a_j]]\\ &\text{then } (\exists \varphi \in L_{\kappa^+,\omega})(\forall \chi > \kappa)(\exists M_\chi \models T)(\exists\{a_i: i < \chi\} \subseteq |M_\chi|)\\ &(\forall i,j < \chi)[i < j \Leftrightarrow M_\chi \models \varphi[a_i, a_j]]\}.\end{aligned}$$

Our main concept in this paper is $\mu^*_\varphi(\lambda, \kappa) = \operatorname{Sup}\{\mu^*(\lambda, T)$: T is a theory in $L_{\kappa^+,\omega}$ of cardinality κ at most, and $\varphi(x, y) \in L_{\lambda^+,\omega}\}$. This concept is interesting because of

THEOREM 1. *Let $T \subseteq L_{\kappa^+,\omega}$ of cardinality $\leq \kappa$, and $\varphi(\bar{x}, \bar{y}) \in L_{\lambda^+,\omega}$. If*

$$(\forall \mu < \mu^*(\lambda, \kappa))(\exists M_\mu \models T)(\exists\{\bar{a}_i: i < \mu\})(\forall i,j < \mu)[i < j \Leftrightarrow M_\mu \models \varphi[\bar{a}_i, \bar{a}_j]]$$

then $(\forall \chi > \kappa) I(\chi, T) = 2^\chi$ (where $I(\chi, T)$ stands for the number of isomorphism types of models of T of cardinality χ).

Many years ago the second author proved that $\mu^*(\lambda, \kappa) \leq \beth_{(2^\lambda)^+}$. Here we continue that work by proving

THEOREM 2. $\mu^*(\lambda, \aleph_0) = \beth_{\lambda^+}$.

THEOREM 3. *For every $\kappa \leq \lambda$ we have $\mu^*(\lambda, \kappa) \leq \beth_{(\lambda^\kappa)^+}$.*

For some κ or λ we have better bounds than in Theorem 3, and this is proved via a new two cardinal theorem.

THEOREM 4. *For every $\kappa \leq \lambda$, $T \subseteq L_{\kappa^+,\omega}$, and any set of formulas $\Delta \subseteq L_{\lambda^+,\omega}$ such that $\Delta \supseteq L_{\kappa^+,\omega}$, if T is (Δ, μ)-unstable for μ satisfying $\mu^{\mu^*(\lambda,\kappa)} = \mu$ then T is Δ-unstable (i.e. for every $\chi \geq \lambda$, T is (Δ, χ)-unstable). Moreover, T is $L_{\kappa^+,\omega}$-unstable.*

In the second part of the paper, we show that always in the applications it is possible to replace the function $I(\chi, T)$ by the function $IE(\chi, T)$, and we give an application of the theorems to Boolean powers.

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Contents

§1. Introduction. In [Sh1] Shelah proved

THEOREM 1.1. *Let T be a first order theory. If T has the order property* (*namely, there are a model $M \models T$, a first order formula $\varphi(\bar{x}, \bar{y})$ in the language of T, and there exist $\{\bar{a}_i: i < \omega\} \subseteq |M|$ such that $i < j \Leftrightarrow M \models \varphi[\bar{a}_i, \bar{a}_j]$—this is equivalent to saying that T is unstable*), *then for every cardinal $\chi > |T|$, $I(\chi, T) = 2^\chi$.*

So the next natural question is: Can Theorem 1.1 be generalized to logics stronger than first order logic? Here is one generalization.

THEOREM 1.2. *Let κ be a cardinal and T a theory in $L_{\kappa^+,\omega}$ of cardinality at most κ. Then there exists a cardinal μ^** (*depending on the logic rather than the theory T*) *such that the following implication is true*:

If T has the $(L_{\kappa^+,\omega}, \mu^*)$-order property (*namely, has a model M and a formula $\varphi(\bar{x}, \bar{y}) \in L_{\kappa^+,\omega}$ and there exist $\{\bar{a}_i: i < \mu^*\} \subseteq |M|$ such that $i < j \Leftrightarrow M \models \varphi[\bar{a}_i, \bar{a}_j]$*), *then $(\forall \chi > |T|) I(\chi, T) = 2^\chi$.*

In [Sh2] Shelah proved Theorem 1.2, and in addition he proved that $\mu^* \leq \beth_{(2^\kappa)^+}$. This theorem has found a number of applications in algebra of the following form:

You have a class K of algebraic structures of a certain form (for example a class of certain groups) which is not axiomatizable in first order logic but has an axiomatization in an infinitary logic, for example in $L_{\kappa^+,\omega}$, and the question is: What is the number of nonisomorphic structures in K in a given uncountable cardinality χ?

A way to answer those questions is by proving that K has the $(L_{\kappa^+,\omega}, \mu^*(\kappa, \kappa))$ order property and applying Theorem 1.2.

In fact we can point out two examples of that procedure (there are many more). There are two specific classes of groups for which group theorists asked questions of the form above and the questions were answered for the first time using Theorem 1.2. The two classes are universal locally finite groups (see Macintyre and Shelah [MSh]), and existentially closed groups (see Shelah and Ziegler [ShZ]). For another application to Boolean powers see Part B of the present paper.

Instead of proving Theorem 1.2 directly, Shelah in [Sh2] proved a stronger theorem than Theorem 1.2. Using our notation from the Abstract this can be formulated as $\mu^*(\lambda, \kappa) \leq \beth_{(2^\lambda)^+}$. Namely in his theorem the given formula φ (which defines the order) is not necessarily from $L_{\kappa^+,\omega}$, but may be from $L_{\lambda^+,\omega}$ for some $\lambda > \kappa$. The effort to deal with formulas not in $L_{\kappa^+,\omega}$ was the origin of most of the difficulties in [Sh2], and we shall see similar difficulties in the next section when we attempt to prove Theorem 1.2 for formulas not from $L_{\kappa^+,\omega}$. This is a very natural strengthening of Theorem 1.2. In [Sh3] Shelah applied the stronger form for the first time to prove that if T is a first order theory with the dimensional order property then T has many models. The application is by concluding from the dimensional order property that there exists an infinitary formula in $L_{\lambda^+,\omega}$ defining a long order as required by the theorem. In [Sh4a] Theorem 1.2 is used to show that the omitting type order property of a theory implies existence of the maximal number of models in powers greater than the cardinality of the theory.[2]

Here we continue that work by improving the bound for $\mu^*(\lambda, \kappa)$ and for the case $\kappa = \aleph_0$ we obtain the exact bound (namely, instead of an inequality we get an equality). We do not assume the reader is familiar with [Sh2]; our method of proof of the bounds is even simpler than that used by Shelah in [Sh2].

First we prove the upper bound part of Theorem 1 in the Abstract. Namely,

THEOREM 1.3. $\mu^*(\lambda, \omega) \leq \beth_{\lambda^+}$.

Instead of proving this theorem directly, we first prove Theorem 1.2. The proof can be found in the next section, §2. In §3, using forcing and absoluteness, we reduce Theorem 1.3 to part of the proof of Theorem 1.2.

What do we get for $\kappa \neq \aleph_0$? Well, we shall prove

THEOREM 1.4. *For every pair of cardinals κ, λ satisfying $\kappa \leq \lambda$ we have $\mu^*(\lambda, \kappa) \leq \beth_{(\lambda^\kappa)^+}$.*

In §3, using forcing and absoluteness, we shall also deduce Theorem 1.4 from a part of the proof of Theorem 1.2.

In §4 we shall introduce another cardinal function, denoted by $\bar{\mu}(\aleph, \kappa)$. This function is natural to define if one is interested in the following two cardinal theorem:

THEOREM 1.5. *Let $\kappa \leq \lambda$, and let $T \subseteq L_{\kappa^+,\omega}$ be a theory of cardinality $\leq \kappa$. Assume that $L(T)$ contains a unary predicate P and a binary predicate $<^P$ such that $T \vdash <^P$ linearly orders P. If there exists $M \models T$ such that $(P^M, <^{PM})$ is well ordered of order type $< \lambda^+$ and $\|M\| \geq \bar{\mu}(\lambda, \kappa)$, then $(\forall \chi \geq \kappa)\exists M_\chi \models T$ such that* o.tp.$(P^{M_\chi}, <^{P^{M_\chi}})$ *is an ordinal $< \kappa^+$ and M_χ has power χ.*

REMARKS. (1) Let $\bar{\mu}(\lambda, \kappa)$ be the first cardinal such that Theorem 1.5 holds (for more formal definitions see Definition 4.1(2) below). As we shall see, $\bar{\mu}(\lambda, \kappa) \geq \mu^*(\lambda, \kappa)$. Why is Theorem 1.5 interesting and related to our main theme (which is investigation of the function $\mu^*(\lambda, \kappa)$ and finding upper bounds for it)? Well, it is easy to show that always $\mu^*(\lambda, \kappa) \leq \bar{\mu}(\lambda, \kappa)$. This is done by expanding the theory T by enough set theory to enable one to talk about the rank of the formula defining the order; i.e. rank(φ) is defined by induction on the structure of φ:

$$\begin{aligned} &\text{rank}(\varphi) = 0 \quad \text{if } \varphi \text{ is atomic},\\ &\text{rank}(\exists x\varphi) = \text{rank}(\neg\varphi) = \text{rank}(\varphi) + 1,\\ &\text{rank}(\varphi \wedge \psi) = \text{Max}\{\text{rank}(\varphi), \text{rank}(\psi)\} + 1,\\ &\text{rank}\left(\bigwedge_{\alpha<\kappa} \varphi_\alpha\right) = \bigcup_{\alpha<\kappa} \text{rank}(\varphi_\alpha). \end{aligned}$$

The predicate P is interpreted as the rank of φ, and since T contains enough set theory we can decode φ from P and we may assume that φ orders sequences of the same cardinality as the cardinality of the model.

[2]Shelah has another sufficient condition for constructing many nonisomorphic models for a first order theory. Namely the conclusions of Theorem 1.1 holds when "T is unstable" is replaced by "T is not superstable". In this light it is natural to ask: Does there exist a natural generalization of Theorems 1 and 2 from the Abstract (and the other results of this paper)? The answer is positive and will appear in [GSh4].

(2) Only the fact that $\operatorname{rank}(\varphi) < \lambda^+$ was used in (1) to conclude that $\mu^*(\lambda, \kappa) \leq \bar{\mu}(\lambda, \kappa)$. Hence all our theorems are true when $\varphi(\bar{x}, \bar{y})$ is not necessarily from $L_{\lambda^+,\omega}$; it is enough to assume that $\varphi(\bar{x}, \bar{y}) \in L_{\infty,\omega}$ and $\operatorname{rank}(\varphi) < \lambda^+$.

(3) As almost always in Hanf number computations, while proving Theorem 1.5 we prove a stronger theorem whose assumption is weaker. Instead of assuming existence of a model of power $\geq \bar{\mu}(\lambda, \kappa)$ it is enough to assume that for every $\mu < \bar{\mu}(\lambda, \kappa)$ there exists a model of power μ as in the statement of the theorem.

In §4 we shall introduce yet another function $\bar{\delta}(\lambda, \kappa)$, and give an upper bound to $\bar{\mu}(\lambda, \kappa)$ by proving in Theorem 4.2 that $\bar{\mu}(\lambda, \kappa) = \beth_{\bar{\delta}(\lambda,\kappa)}$. Computing bounds for $\bar{\delta}(\lambda, \kappa)$ for certain cases gives us improvements of Theorem 1.4. For example using $\bar{\mu}(\lambda, \kappa)$ and $\bar{\delta}(\lambda, \kappa)$ we have the following finer bounds

THEOREM 1.6. *For every* $\kappa \leq \lambda$:

(1) *If* cf $\lambda = \aleph_0$ *then* $\mu^*(\lambda, \kappa) \leq \beth_{(\sum_{\mu<\lambda}\mu^\kappa)^+}$.

(2) *If* cf κ = cf $\lambda = \aleph_0$ *then* $\mu^*(\lambda, \kappa) \leq \beth_{(\sum_{\chi<\kappa}\lambda^\chi)^+}$.

(3) *If* cf κ = cf $\lambda = \aleph_0$ *then* $\mu^*(\lambda, \kappa) \leq \beth_{(\sum_{\mu<\lambda,\chi<\kappa}\mu^\chi)^+}$.

(4) *If* $\lambda = (\beth_\omega)^{+\alpha}$ *and* $\kappa \leq \beth_\omega$ *for some* $\alpha < \omega_1$, *then* $\bar{\mu}(\lambda, \kappa) \leq \beth_{\lambda^+}$.

In Part **B** we shall continue §4 and also continue Theorem 1.6. Namely, we shall improve the bound obtained in Theorem 1.4; for details see the remark after Corollary 4.6. Another related subject is the stability spectrum theorem. In [Sh4, Chapter III, Theorem 5.15] Shelah solved the stability spectrum for first order theories. That is, he gave a complete characterization of the cardinals for which a first order theory T is stable. It is natural to try to solve the same problem for infinitary logics. Namely, given a theory $T \subseteq L_{\kappa^+,\omega}$ and a set of formulas $\Delta \subseteq L_{\kappa^+,\omega}$, characterize the set of cardinals in which the theory T is stable in Δ. For theories which have large homogeneous models the problem is essentially solved (for cardinals $\geq \beth_{(2^\kappa)^+}$ it is completely solved) in [Sh0], but here we are concentrating on the general problem.

Since we want to keep our promise from a previous paragraph and not require familiarity with [Sh2], we repeat some definitions from that paper.

DEFINITION 1.7. Let Δ be a set of formulas in $L_{\lambda^+,\omega}$.

(0) Let M be a model, $|A| \subseteq M$, and $m < \omega$. For a sequence of length m of elements $\bar{b}$ from M let

(i) $\operatorname{tp}_\Delta(\bar{b}, A, M) = \{\varphi(\bar{x}, \bar{a}) : \bar{a} \in A, \varphi \in \Delta, M \models \varphi[\bar{b}, \bar{a}]\}$,

(ii) $S^m_\Delta(A, M) = \{\operatorname{tp}_\Delta(\bar{b}, A, M) : \bar{b} \in {}^m|M|\}$.

(1) A model M is (Δ, χ)-*stable* if for each $A \subseteq |M|$, $m < \omega$, $|A| \leq \chi$,

$$|S^m_\Delta(A)| \leq \chi.$$

(2) A theory $T \subseteq L_{\kappa^+,\omega}$ is (Δ, χ)-*stable* if every model of T is (Δ, χ)-stable.

(3) A theory $T \subseteq L_{\kappa^+,\omega}$ is Δ-*stable* if there exists a cardinality χ such that T is (Δ, χ)-stable.

(4) We say $\chi \in \operatorname{Od}_\Delta(M)$ if there exist a formula $\varphi(\bar{x}, \bar{y}) \in \Delta$ and a set $\{\bar{a}_i : i < \chi\} \subseteq M$ such that for every $i, j < \chi$

$$i < j \Leftrightarrow M \models \varphi[\bar{a}_i, \bar{a}_j].$$

(5) A theory T has the (Δ, χ)-*order property* if there exists a model $M \models T$ such that $\chi \in \operatorname{Od}_\Delta(M)$.

Shelah [Sh4, Theorem II.2.13(2)] proved that if a first order theory T is unstable in a cardinal λ satisfying $\lambda^{|T|} = \lambda$ then T is unstable (= for every cardinal χ, T is unstable in χ). So it is natural to ask whether the direct generalization of the above is true for $L_{\kappa^+,\omega}$. In [Sh2] he proved the following:

THEOREM. *Let $T \subseteq L_{\kappa^+,\omega}$, $|T| \leq \kappa$, and let $\Delta \subseteq L_{\lambda^+,\omega}$ for $\kappa \leq \lambda$. If T is (Δ,μ)-unstable for μ satisfying $\mu^{\mu(\lambda)} = \mu$ (where $\mu(\lambda)$ is the Hanf number of the logic $L_{\lambda^+,\omega}$), then T is Δ-stable.*

Notice that for the case $\kappa = \aleph_0$ and $\lambda = \aleph_{17}$ by the above theorem it is enough to check stability in μ satisfying $\mu = \mu^{\beth_\rho}$, where $\rho = (2^{\aleph_{17}})^+$. Below we shall improve this by proving that it is enough to check stability in μ satisfying $\mu = \mu^{<\beth_{\aleph_{17}}}$. Namely, we shall prove

THEOREM 1.8. *Let $\kappa \leq \lambda$, and let $T \subseteq L_{\kappa^+,\omega}$; let $\Delta \subseteq L_{\lambda^+,\omega}$ be such that $\Delta \supseteq L_{\kappa^+,\omega}$. If T is (Δ,μ)-unstable for μ satisfying $\mu = \mu^{\mu^*(\lambda,\kappa)}$, then T is Δ-unstable.*

Note that what we said in the above paragraph follows from Theorem 1.8 together with Theorem 1.3. Theorem 1.8 will be proved in §5. We shall not prove Theorem 1.7 directly; rather in §5 we shall deduce it from the stronger theorem stated below.

THEOREM 1.9. *For every $\lambda \geq \kappa$, $T \subseteq L_{\kappa^+,\omega}$, and any set of formulas $\Delta \subseteq L_{\lambda^*,\omega}$ such that $\Delta \supseteq L_{\kappa^+,\omega}$, if T is (Δ,μ)-unstable for μ satisfying $\mu^{<\mu^*(\lambda,\kappa)} = \mu$ and $\mu = \mu^{|\Delta|}$, then T is Δ-unstable. Moreover, T is $L_{\kappa^+,\omega}$-unstable.*

REMARK. As in Remark (2) after Theorem 1.5, also Theorem 1.9 can be strengthened to: $\Delta \subseteq L_{\infty,\omega}$, $\Delta \supseteq L_{\kappa^+,\omega}$ and $(\forall\varphi \in \Delta)[\mathrm{rank}(\varphi) < \lambda^+]$.

Let us return to the problem of the number of nonisomorphic models. Assume K is a class of algebraic structures all of the same similarity type. Frequently after proving $(\forall\chi > |L(K)|)I(\chi,K) = 2^\chi$, the next question is what about the function IE?

DEFINITION 1.10. Let K be as in the above paragraph and let χ and μ be infinite cardinals. We denote by $IE(\chi,K) \geq \mu$ the following statement: $\exists\{M_i : i < \mu\}$ such that

(1) $i < \mu \Rightarrow \|M_i\| = \chi$ and $M_i \in K$, and

(2) $i \neq j \Rightarrow$ there is no K-embedding from M_i into M_j.

Clearly $0 \leq IE(\chi,K) \leq 2^\chi$. If the reader wants an example of the above procedure he is invited to read §1 of Wilfrid Hodges [Ho]. In [GSh2] we shall prove

THEOREM 1.11. *Let $T \subseteq L_{\kappa^+,\omega}$ of cardinality $\leq \kappa$, and let $\varphi(\bar{x},\bar{y}) \in L_{\kappa^+,\omega}$ be quantifier free. If*

$$(\forall\mu < \mu^*(\lambda,\kappa))(\exists M_\mu \models T)(\exists\{\bar{a}_i : i < \mu\})(\forall i,j < \mu)[i < j \Leftrightarrow M_\mu \models \varphi[\bar{a}_i,\bar{a}_j]]$$

then

$$(\forall\chi > \kappa)[[\chi \text{ is regular} \vee (\exists\mu < \chi)[\kappa < \mu < \chi \wedge 2^\mu = 2^\chi] \vee (\chi^{\aleph_0} = \chi > \kappa^{\aleph_0}) \vee 0^\# \notin V] \rightarrow IE(\chi,T) = 2^\chi].$$

This answers Hodges' question and all other similar questions simultaneously.

NOTATION. α, β, γ, δ, ξ, i, j stand for ordinals; λ, κ, χ, μ stand for infinite cardinal numbers; M, N stand for models. All the above letters may be used with indices. φ and ψ stand for formulas in a certain logic. Be careful: sometimes they are used as first order formulas and sometimes as formulas from an infinitary logic. L is a similarity type and also the collection of first order formulas in the similarity type L. Sent(L) is the collection of sentences in L. T and T_1 are theories, sometimes first order theories, and in other cases infinitary. By $L(T)$ we denote the similarity type corresponding to the theory T. Let k, l, m, n stand for natural numbers. Let T be a theory, Γ a set of types (in $L(T)$, namely without parameters) and T_1 an expansion of T (we shall always use expansions whose power is $\aleph_0|T|$). Then we use the following notation:

$$PC(T_1, \Gamma, T) = \{M \upharpoonright L(T): M \models T_1, (\forall p \in \Gamma) M \text{ omits the type } p\}.$$

We always assume that the members of $PC(T_1, \Gamma, T)$ are infinite and of cardinality greater than or equal to $|T_1|$. If $T_1 = T$ then instead of writing $PC(T, \Gamma, T)$ we write $EC(T, \Gamma)$. When working with a first order theory T we assume that $\mathfrak{C}$ is a $\bar{\kappa}$-saturated model of T ($\bar{\kappa}$ is bigger than the biggest cardinal which appears in our proofs); when we work with an expansion T_1 of T, assume $\mathfrak{C}$ is a model of T_1. The collection of all finite subsets of the set X is denoted by $\mathscr{S}_{<\aleph_0}(X)$, and $\mathscr{S}_\kappa(X)$ is the collection of subsets of power κ. As usual, $\lambda_2 = \{f \mid f: \lambda \to 2\}$, and $2^\lambda = |{}^\lambda 2|$. The cardinal $\lambda^{+\alpha}$ is the αth successor of λ, so for example $\lambda^{+1} = \lambda^+$, and if $\aleph_\delta = \beth_\omega$ then $(\beth_\omega)^{+\alpha} = \aleph_{\delta+\alpha}$. Also ${}^\lambda 2 = \{f \mid f: \lambda \to 2\}$, and $2^\lambda = |{}^\lambda 2|$.

Prerequisites. We assume very little knowledge; the minimal knowledge is familiarity with the contents of the first three chapters in Chang and Keisler's book [CK], the very basic part of Keisler's [Ke1] or Dickmann's [Di] books, and familiarity with the definition "T is stable in χ". We assume only a little about forcing (forcing with κ^+-complete poset does not add subsets of cardinality κ). When we shall use something not in the above minimal knowledge we state it explicitly (as a fact) and give an exact reference to it in the literature.

The results of this paper were announced in [GSh1] and [GSh3].

§2. Proof of Theorem 1.2 in the case when $\lambda = \kappa$. We shall prove that μ^* as in the statement of the theorem exists and is the Hanf number of the logic $L_{\kappa^+,\omega}$. So from now on we assume that μ^* is a given cardinal and is equal to the Hanf number of the appropriate logic, and we prove in this section that μ^* satisfies the statement of Theorem 1.2.

We offer two different proofs of that theorem, depending on the taste of the reader. The first is more related to infinitary logic than the second one, in which we translate the problem into a problem about classes of models of a first order theory which omit a first order type (or a set of κ types, which is an equivalent problem).

Let us point out what is common to both of the proofs: Namely, how to conclude that the number of isomorphism types of models is maximal? In [Sh1] Shelah proved Theorem 1.1 as we mentioned above. We shall rely here on his proof of Theorem 1.1 (not the statement of the theorem). So we shall quote below what we use, without proving it.

Fact 2.1. *Let T and T_1 be first order theories such that $T \subseteq T_1$, whose languages are L and L_1 respectively. Suppose T_1 has Skolem functions, and $|T| = |T_1| \leq \kappa$. Let there exist* a) *a linearly ordered set I such that $|I| = \kappa$,* b) *a formula $\varphi(\bar{x}, \bar{y})$ with $l(\bar{y}) = m$ in $L_{\kappa^+,\omega}$ (involving relation and function symbols from L only), and* c) *a model $M^1 = EM^1(I)$ such that $M^1 \models T_1$, and M^1 is the Skolem hull of $\{\bar{a}_i: i \in I\}$ under the functions of T_1 when the $\bar{a}_i$'s are m-sequences and the main requirement*

$$i <_I j \Leftrightarrow M^1 \models \varphi[\bar{a}_i, \bar{a}_j]$$

holds. Then for every $\chi > \kappa$ we have $I(\chi, T) = 2^\chi$ exemplified by a family

$\{EM^1(I_i) \restriction L: i < 2^\chi\}$, where $\{I_i: i < 2^\chi\}$ are linearly ordered sets of cardinality χ such that $EM^1(I_i) \restriction L$ is not isomorphic to $EM^1(I_j) \restriction L$ for all $i < j < 2^\chi$.

2.1. REMARK. We can prove a theorem slightly stronger than Theorem 1 of the Abstract by changing the conclusion from $(\forall \chi > \kappa) I(\chi, T) = 2^\chi$ to

$$(\forall \chi \geq (\kappa + \aleph_1)) I(\chi, T) = 2^\chi.$$

Namely, when κ is uncountable we can find many models of power κ. This improvement is done by replacing Fact 2.1 by an improvement, and its proof does not follow from [Sh1] but is patterned on the proof of Theorem 3.3 in Chapter VIII of [Sh4]. But since the improvement of Fact 2.1 does not follow directly from Theorem VIII.3.3 as Fact 2.1 follows from [Sh1], we prefer to formulate Theorem 1 as it is.

Now we shall state another fact we shall use. It is a very basic result in infinitary logic for $L_{\omega_1,\omega}$ (for the logic $L_{\kappa^+,\omega}$ it is the same). See [Ke1, Theorem 19(ii) (the stretching theorem)], or for classes of models for first order theories omitting a given type see [CK, Theorem 3.3.11(e)].

Fact 2.2. (1) *Let $T \subseteq L_{\kappa^+,\omega}$ be a theory of cardinality at most κ which has built-in Skolem functions. Suppose there exists a model $M \models T$ of $L_{\kappa^+,\omega}$ of cardinality κ and let L_T be a fragment of cardinality at most κ containing T. Assume in addition that the model M has an infinite sequence of indiscernibles J. Then for every infinite ordered set I there exists a model N such that I is an infinite set of indiscernibles and* (*writing N for the Skolem hull of I*) $M \equiv_{L_T} N$.

(2) *Let T be a first order theory of cardinality at most κ with built-in Skolem functions. If T has a model M which is the Skolem hull of J, then for every linearly ordered set I there is a model N, formed as the Skolem hull of I, such that N omits exactly the types realized or omitted by M.*

REMARK. Indeed, we can see immediately a variant of Fact 2.2 which we shall call Fact 2.2(A). In both cases J can be a set of m-sequences and N is generated by an indiscernible sequence of the same type.

Now by combining Facts 2.1, 2.2(1) and 2.2(A) we can easily derive

Conclusion 2.3. *Let T be a theory in $L_{\kappa^+,\omega}$, and assume that there exists a formula $\varphi(\bar{x}, \bar{y}) \in L_{\kappa^+,\omega}$ for some κ and T_1 is an expansion of T by Skolem functions* (*in $L(T_1)_{\kappa^+,\omega}$*). *If there exists a model $M \models T_1$ with an infinite sequence of indiscernibles which is ordered by the formula $\varphi(x, y)$, then $I(\chi, T) = 2^\chi$ for every $\chi > \kappa$.*

Or the version for the first order theories is

Conclusion 2.4. *Let T be a first order theory of cardinality at most κ, let $\varphi(\bar{x}, \bar{y})$ be a first order formula in $L(T)$, $T_1 \supseteq T$ an expansion of T of cardinality κ by Skolem functions, and Γ a set of finitary types* (*i.e. with finitely many variables*) *with $|\Gamma| \leq \kappa$. If there exists $M \in PC(T_1, \Gamma, T)$ with an infinite indiscernible sequence ordered by $\varphi(\bar{x}, \bar{y})$, then $I(\chi, PC(T_1, \Gamma, T)) = 2^\chi$ for every $\chi > |T_1|$.*

REMARK. If you are familiar with computations of Hanf numbers you can jump to the last theorem of the present section.

Since the case when $\kappa = \aleph_0$ is in some sense the most useful instance of our theorems, we shall present two proofs for this case (remember that $\mu(\aleph_0) = \beth_{\omega_1}$). So it is enough to prove that the hypothesis of one of the last conclusions holds (or in other words to prove Theorem 1 from the Abstract).

PROOF OF THE ASSUMPTION OF CONCLUSION 2.3. So we have a fixed countable T

$\subseteq L_{\omega_1,\omega}$ and $\varphi(x, y) \in L_{\omega_1,\omega}$ such that T has the $(\{\varphi\}, \beth_{\omega_1})$-order property. Without loss of generality we may assume that T has built-in Skolem functions. Our goal now is to construct a countable model $M \models T$ which is the Skolem hull of an infinite indiscernible sequence ordered by the formula $\varphi(x, y)$.

Recall the basic way to construct models for $L_{\omega_1,\omega}$ namely the extended model existence theorem (see page 14 of [Ke1]):

Fact. Let S be a consistency property, and Γ a countable set of sentences in the language $L_{\omega_1,\omega}$. If $s \cup \{\psi\} \in S$, for all $s \in S$ and $\psi \in \Gamma$, then for all $s \in S$ the set $s \cup \Gamma$ has a countable model.

Let L^* be the expansion of L obtained by adding two countable sets of constants $D = \{d_n : n < \omega\}$ and $C = \{c_n : n < \omega\}$. Let $L_{\mathscr{A}^*}$ be a countable fragment containing $T \cup \{\varphi(x, y)\}$. Let

$$\begin{aligned}\Gamma = \{\psi(d_{i_1}, \ldots, d_{i_p}) \equiv \psi(d_{j_1}, \ldots, d_{j_p}) &: i_1 < \cdots < i_p < \omega,\\ &j_1 < \cdots < j_p < \omega, \psi(x_1, \ldots, x_p) \in L_{\mathscr{A}^*}\}\\ \cup \{\neg\, d_i = d_j &: i, j < \omega, i \neq j\}\\ \cup \{\varphi(d_i, d_j) &: i < j < \omega\} \cup \{\neg\, \varphi(d_j, d_i) : i \leq j < \omega\}.\end{aligned}$$

Clearly if we show that $T \cup \Gamma$ has a model then we are done, since its reduct to L will be as required (the interpretation of the constants $\{d_n : n < \omega\}$ is the sequence of indiscernibles we are seeking).

To prove that $T \cup \Gamma$ has a model we use the above Fact. Namely, it suffices to find a nonempty consistency property S (in $L_{\mathfrak{A}^*}$) such that

$$(\forall \psi \in T \cup \Gamma)(\forall s \in S)[s \cup \{\psi\} \in S].$$

We define:

$$\begin{aligned}S \overset{\text{def}}{=} \{s = s(c_1, \ldots, c_m, d_1, \ldots, d_n) \in \mathscr{S}_{<\aleph_0}(\mathrm{Sent}(L_{\mathscr{A}^*}))&:\\ \text{only finitely many } c\text{'s and } d\text{'s appear in } s, \text{ and}&\\ (\forall \alpha < \omega_1)(\exists M \models T)(\exists \{a_i^\alpha : i < \beth_\alpha\} \subseteq |M|)(\forall i_1 < \cdots < i_n < \beth_\alpha)&\\ [\langle M, a_{i_1}^\alpha, \ldots, a_{i_n}^\alpha\rangle \models (\exists x_1, \ldots, x_m) \textstyle\bigwedge s(x_1, \ldots, x_m, d_1, \ldots, d_n)]\}.&\end{aligned}$$

We mean that the elements $a_{i_1}^\alpha, \ldots, a_{i_n}^\alpha$ $(\in M)$ are interpretations of the constants $d_1, \ldots, d_n$ respectively.

S is not empty since, by the definition of μ^*, $\{(\exists x)[x = x]\} \in S$. Why is S a consistency property? Everything is easy to check except the condition

$$\text{if } \textstyle\bigvee \Theta \in s \in S \text{ then } (\exists \vartheta \in \Theta)[s \cup \{\vartheta\} \in S].$$

Denote $s = s(c_1, \ldots, c_m, d_1, \ldots, d_n)$, and for all $\vartheta \in \Theta$ let $\vartheta = \vartheta(c_1, \ldots, c_m, d_1, \ldots, d_n)$.

Given $\alpha < \omega_1$ let $\beta = \alpha + n$ (which is $< \omega_1$). By the definition of S there exist a model $M_\beta \models T$ and a sequence $\{a_i^\beta : i < \beth_\beta\} \subseteq |M_\beta|$ such that for every $i_1 < \cdots < i_n < \beth_\beta$

$$M_\beta \models (\exists x_1, \ldots, x_m) \textstyle\bigwedge s(x_1, \ldots, x_m, d_1, \ldots, d_n).$$

Since $\bigvee \Theta \in s$ we have for all $i_1 < \cdots < i_n < \beth_\beta$

$$M_\beta \models \bigvee_{\vartheta \in \Theta} (\exists x_1, \ldots, x_m)[\textstyle\bigwedge s(x_1, \ldots, x_m, d_1, \ldots, d_n) \wedge \vartheta(x_1, \ldots, x_m, d_1, \ldots, d_n)].$$

Namely, when $A_\beta = \{a_i^\beta : i < \beth_\beta\}$ we may define a function $f: [A_\beta]^n \to \Theta$ such that $i_1 < \cdots < i_n < \beth_\beta$ implies

$$M_\beta \models (\exists x_1, \ldots, x_m)[\bigwedge s(x_1, \ldots, x_m, d_1, \ldots, d_m) \wedge f(a_{i_1}, \ldots, a_{i_n})(x_1, \ldots, x_m, d_1, \ldots, d_n)].$$

Using the Erdös-Rado partition theorem $(\beth_n(\chi)^+ \to (\chi^+)^{n+1}_\chi)$, we have $\beth_\beta \to (\beth_\alpha)^n_{\aleph_0}$. Applying this to the function f, we get the existence of $B_\alpha \subseteq A_\beta$ such that $|B_\alpha| = \beth_\alpha$, and B_α is homogeneous for f. Namely there exist $\vartheta \in \Theta$ such that for all $i_1 < \cdots < i_n \in B_\alpha$ we have $f(a_{i_1}, \ldots, a_{i_n}) = \vartheta$. Now choose an elementary submodel N_α of M_β containing B_α such that N_α, B_α and ϑ exemplify that $s \cup \{\vartheta\} \in S$.

To finish the proof, it is enough to prove that

$$(\forall \vartheta \in T \cup \Gamma)(\forall s \in S)[s \cup \{\vartheta\} \in S].$$

According to the definition of 1′ there are 5 possibilities (with respect to the location of ϑ).

(i) If $\vartheta \in T$, then $s \cup \{\psi\} \in S$ by the assumption of the theorem.

(ii) If ϑ belongs to the first union of Γ, i.e. if ϑ is $\psi(d_{i_1}, \ldots, d_{i_p}) \equiv \psi(d_{j_1}, \ldots, d_{j_p})$, let $\alpha < \omega_1$ be given. By the definition of S there exists $M_\beta \models T$ (for $\beta = \alpha + p$) with a subset $A_\beta = \{a_i^\beta : i < \beth_\beta\}$ exemplifying $s \in S$. Define a function $g: [A_\beta]^p \to \{0, 1\}$ by $g(a_{i_1}, \ldots, a_{i_p}) = 1 \Leftrightarrow M_\beta \models \psi[a_{i_1}, \ldots, a_{i_p}]$. Now apply the Erdös-Rado partition theorem as before and get a homogeneous set of cardinality $\beth_\alpha$ which exemplifies $s \cup \{\vartheta\} \in S$.

(iii) If ϑ is $\neg(d_i = d_j)$, then clearly $s \cup \{\vartheta\} \in S$.

(iv) If $\vartheta = \varphi(d_i, d_j)$ (for $i < j$), again $s \cup \{\vartheta\} \in S$ by the assumption of the theorem (by the definition of μ^*, choose in stage $\alpha < \omega_1$ the set $\{a_i^\alpha : i < \beth_\alpha\}$ ordered by the formula $\varphi(x, y)$).

(v) If $\vartheta = \neg\varphi(d_j, d_i)$ (for $i \leq j$), proceed exactly as in (iv).

Now to the first order version. First let us translate the problem from infinitary logic to a problem about classes of models of first order theories omitting a set of countable types.

We use the following basic fact due to C. C. Chang [Ch]. For the $L_{\omega_1,\omega}$ version see [Ke1, Theorem 14], or for $L_{\kappa^+,\omega}$ see [Di, Theorem 4.1.1] (which has the same proof as for the case of $L_{\omega_1,\omega}$).

Fact 2.5. *Let $T \subseteq L_{\kappa^+,\omega}$ be the theory of cardinality at most κ, and let $L_{\mathscr{A}}$ be a fragment of $L_{\kappa^+,\omega}$ containing T. There are two first order theories $T^* \subseteq T_1$ (T_1 an expansion of T^*) of cardinality κ, $L(T^*) = L(T)$, and a set of first order types Γ (in $L(T^*)$) of power κ such that*

(1) $\mathrm{Mod}(T) = PC(T_1, \Gamma, T^*)$, *and*

(2) *for every formula $\psi(\bar{x}) \in L_{\mathscr{A}}$ there exists a first order formula $\psi^*(x) \in L(T_1)$ such that $T \vdash (\forall \bar{x})[\psi^*(\bar{x}) \equiv \psi(\bar{x})]$.*

Converting to the situation in Conclusion 2.4 (for $\kappa = \aleph_0$),[3] let $T \subseteq L_{\omega_1,\omega}$ and $\varphi(x, y) \in L_{\omega_1,\omega}$, as before. Let T^*, T_1 and Γ be as in Fact 2.5, when the fact is applied to a countable fragment $L_{\mathscr{A}}$ containing $T \cup \{\varphi(x, y)\}$. By part (2) of the fact there

[3] This is based on Morley's ideas in [Mo1].

exists a first order formula $\varphi^*(x, y) \in L(T_1)$ such that

$$T \vdash \forall x \forall y[\varphi^*(x, y) \equiv \varphi(x, y)].$$

Let T_1^* be an expansion of T_1 by adding built-in Skolem functions so that $PC(T_1^*, \Gamma, T) = PC(T_1, \Gamma, T)$.

Assume we have an enumeration of Γ by $\{p_k: k < \omega\}$. For every $\kappa < \omega$ let $p_k = \{\sigma_n^k(x): n < \omega\}$. Let $C = \{c_n: n < \omega\}$ be a set of new constant symbols. Without loss of generality we may assume $L(T_1^*) \supseteq C$. Let $\{\tau_n: n < \omega\}$ be an enumeration of all terms in $L(T_1^*)$, $\{\psi_n: n < \omega\}$ an enumeration of $L(T_1^*)$.

To prove what we want it is enough to show that there exists $\{f_k: \omega \to \omega \mid \kappa < \omega\}$ such that the theory

$$\begin{aligned} \bar{T}^* = T_1^* &\cup \{\neg c_i = c_j: i < j < \omega\} \\ &\cup \{\neg \sigma^k_{f_k(n)}(\tau_n(c_{i_1}, \dots, c_{i_{l(n)}})): k, n < \omega, i_1 < \cdots < i_{l(n)} < \omega\} \\ &\cup \{\psi_n(c_{i_1}, \dots, c_{i_{l(\psi_n)}}) \equiv \psi_n(c_{j_1}, \dots, c_{j_{l(\psi_n)}}): \\ &\qquad i_1 < \cdots < i_{l(n)} < \omega, j_1 < \cdots < j_{l(\psi_n)} < \omega, n < \omega\} \\ &\cup \{\varphi^*(c_i, c_j): i < j < \omega\} \end{aligned}$$

is consistent.

Let us fix $\{M_\alpha \models T: \alpha < \omega_1\}$ exemplifying the definition of μ^*. Namely each M_α has a subset $\{a_i^\alpha: i < \beth_\alpha\}$ ordered by $\varphi(x, y)$.

By Fact 2.5 we may assume that each M_α omits the types in the set Γ, and since we can expand every first order theory by Skolem functions, we may assume $M_\alpha \models T_1^*$.

By the compactness theorem it is enough to prove by double induction on $\kappa < \omega$ and $n < \omega$ that:

$(*)_{k,n}$ There are $\sigma^k_{i_0}(x), \dots, \sigma^k_{i_{n-1}}(x) \in p^k$ and an unbounded $F_n^k \subseteq \omega_1$ such that for $\alpha \in F_n^k$ there exists $A_\alpha \subseteq |M_\alpha|$ ordered by $\varphi^*(x, y)$ such that for every $\beta \in F_n^k$:

(i) if β is the γth element of F_n^k then $|A_\beta| > \beth_\gamma$,

(ii) $m < n$ and $l \leq k$ imply that for every $a_{i_1}, \dots, a_{i_{l(\tau_m)}}$ which is φ^*-increasing [i.e. $1 \leq k < j \leq l(\tau_m) \Rightarrow \varphi[a_k, a_j]$] we have

$$M_\beta \models \neg \sigma_{i_m}[\tau_m(a_{i_1}, \dots, a_{i_{l(\tau_m)}})],$$

(iii) $M_\beta \models \psi_m[a_{i_1}, \dots, a_{i_{l(\psi_m)}}] \equiv \psi_m[a_{j_1}, \dots, a_{j_{l(\psi_m)}}]$ for every $m < n$ and every $a_{i_1} < \cdots < a_{i_{l(\psi_m)}}$ and $a_{j_1} < \cdots < a_{j_{l(\psi_m)}}$ (both φ^*-increasing) in A_β.

The proof of $(*)_{k,n}$ is (as we said) by induction: for $n = k = 0$ it is trivial, and for the induction step it is by double application of the Erdös-Rado partition theorem (once to fulfill requirement (ii) and once to fulfill requirement (iii)). The choice of $\sigma^k_{i_m}$ is done by the pigeonhole principle (since $|F_n^k| = \aleph_1$ and there are only countably many formulas). The verification is left as an exercise to the reader (hint: see Theorem 7.2.2 in [CK]).

So we have finished the case when $\kappa = \aleph_0$.

Now we give a proof to show that the hypothesis of Conclusion 2.4 holds for general κ. Assume $T \subseteq L_{\kappa^+, \omega}$ is a theory of cardinality κ, $\varphi(x, y) \in L_{\kappa^+, \omega}$ and T has the $(\{\varphi\}, \mu(\kappa))$-order property.

Let us quote from the presentation in [Sh4] (if you want to know the credits you must consult [Di] or the historical appendix to [Sh4a]).

DEFINITION 2.6. For $\kappa \geq \aleph_0$ let $\delta(\kappa)$ be an ordinal such that

$$\delta(\kappa) = \operatorname{Min}\{\delta: \text{for every similarity type } L_0 \text{ with a binary relation } <, \text{ every first order theory } T_0, \text{ and every set } \Gamma \text{ of finitary first order types in } L_0 \text{ of power} \leq \kappa, \textbf{ if } \text{there exists } M \in EC(T_0, \Gamma) \text{ such that o.tp.}(|M|, <^M) \geq \delta \textbf{ then } \text{there exists } N \in EC(T_0, \Gamma) \text{ such that o.tp.}(|N|, <^N) \text{ is not well ordered}\}.$$

By [Sh4, Theorem VII.5.5] (for historical credit see [Di]) we have

Fact 2.7. (1) $\delta(\kappa) \leq (2^\kappa)^+$.

(2) $\delta(\kappa) \geq \kappa^+$.

By the easy half of Theorem VII.5.4 from [Sh4] we have

Fact 2.8. $\mu(\kappa) \geq \beth_{\delta(\kappa)}$.

By Fact 2.5 let T_1 and Γ be appropriate for a fragment (of power κ) of $L_{\kappa^+,\omega}$ containing $T \cup \{\varphi(x,y)\}$, and let $\varphi^*(x,y) \in L(T_1)$ be a first order formula such that

$$T \vdash \forall x \forall y(\varphi^*(x,y) \equiv \varphi(x,y)).$$

As we indicated above, we shall prove $\mu^* \leq \mu(\kappa)$; so assume for the sake of contradiction that $\mu^* > \mu(\kappa)$ (by our notation we do not imply that μ^* exists at all; μ^* may be ∞).

Hence, by Fact 2.8, $\mu^* > \beth_{\delta(\kappa)}$. Therefore by our assumption we have for every $\alpha < \delta(\kappa)$ a model $M_\alpha \models T$ and a sequence $\{a_i^\alpha: i < \beth_\alpha\} \subseteq |M_\alpha|$ ordered by $\varphi(x,y)$. Hence, by Fact 2.5, $M_\alpha \in PC(T_1, \Gamma, T^*)$. We may assume $T_1^* \subseteq T_1$ is an expansion of T_1 by Skolem functions and $M_\alpha \models T_1^*$.

To complete the proof of what we want, it is enough to prove the following: Given a set $C = \{c_\alpha: \alpha < \kappa\}$, let $\Gamma = \{p_i: i < \kappa\}$, where $p_i = \{\sigma_j^i(x): j < \kappa\}$. Let $\{\tau_i: i < \kappa\}$ be an enumeration of the terms in $L(T_i^*)$, and let $\{\psi_i(x): i < \kappa\} = L(T_1^*)$.

There exist $\{f^j: \kappa \to \kappa \mid j < \kappa\}$ such that the theory

$$\begin{aligned}\bar{T}^* = T_1^* &\cup \{\neg(c_i = c_f): i < j < \kappa\} \\ &\cup \{\neg\sigma^j_{f^j(i)}(\tau_i(c_{\alpha_1}, \dots, c_{\alpha_{l(\tau_i)}})): i, j < \kappa, \alpha_1 < \cdots < \alpha_{l(i)} < \kappa\} \\ &\cup \{\psi_i(c_{\alpha_1}, \dots, c_{\alpha_{l(\psi_i)}}) \equiv \psi_i(c_{\beta_1}, \dots, c_{\beta_{l(\psi_i)}}): \\ &\qquad \alpha_1 < \cdots < \alpha_{l(\psi_i)} < \kappa, \beta_1 < \cdots < \beta_{l(\psi_i)} < \kappa\} \\ &\cup \{\varphi^*(c_i, c_j): i < j < \kappa\}\end{aligned}$$

is consistent.

By the choice of $\{M_\alpha: \alpha < \delta(\kappa)\}$ there exists a function (in the universe of set theory in which we are working at present) such that for $\alpha < \delta(\kappa)$, $F: \beth_\alpha \to (M_\alpha, <_\varphi)$ is order preserving. Let $G: \delta(\kappa) \to \{M_\alpha: \alpha < \delta(\kappa)\}$ be such that $G(\alpha) = M_\alpha$. For every $\psi(\bar{x}) \in L(T^*)$ (first order) let us define a formula in the language of set theory,

$$\psi(y, \bar{x}) \stackrel{\text{def}}{=} [y \text{ is an ordinal} < \delta(\kappa) \text{ and } G(y) \models \psi(\bar{x})].$$

For every $p \in \Gamma$ in the variables $\bar{x}$ let $p^{\mathrm{ZF}}(y; \bar{x}) = \{\psi(y, \bar{x}): \psi(\bar{x}) \in p\}$. Since, for every $\alpha < \delta(\kappa)$, $M_\alpha \models T_1^*$ and M_α omits all types from Γ, our present universe of set theory omits all types from $\Gamma^{\mathrm{ZF}} \stackrel{\text{def}}{=} \{p^{\mathrm{ZF}}: p \in \Gamma\}$.

Remember that from the Erdös-Rado theorem it follows that for all α satisfying $\alpha < \delta(\kappa)$ and every $n < \omega$ we have $(\beth_{\alpha+n})^+ \to (\beth_\alpha^+)^{n+1}_{2^\kappa}$.

By the reflection principle (this is due to A. Levy; see Theorem 29(B) in [Je]) there exists an ordinal ξ such that the model $\langle V_\xi, \in\rangle$ reflects all sentences mentioned in the last paragraph.

Let $\mathfrak{B} = \langle V_\xi, \in, T_1^*, P, F, G, \varphi\rangle_{\varphi\in L(T^*)}$, where P is a unary relation symbol such that $P = \{i : i < \delta(\kappa)\}$. By the downward Löwenheim Skolem theorem there is a model $\mathfrak{B}^1 \prec \mathfrak{B}$ of cardinality $|\delta(\kappa)|$ such that $P^{\mathfrak{B}^1} = \delta(\kappa)$. Hence there exists a function $H: \mathfrak{B}^1 \to P^{\mathfrak{B}^1}$ such that H is one-to-one and onto. The function H induces a well-ordering of the universe of the model $\mathfrak{B}^1$; namely, let

$$a <^* b \overset{\text{def}}{\Leftrightarrow} H(a) \in H(b).$$

Let $\mathfrak{B}^2 = \langle \mathfrak{B}^1, H, <^*\rangle$ and $T^{\mathfrak{B}} = \mathrm{Th}(\mathfrak{B}^2)$. Since $\mathfrak{B}^2 \in EC(T^{\mathfrak{B}}, \Gamma^{\mathrm{ZF}})$ (notice that $|T^{\mathfrak{B}}| = |\Gamma^{\mathrm{ZF}}| = \kappa$) and $(|\mathfrak{B}^2|, <^*)$ has order type $\delta(\kappa)$, by Definition 2.6 there exists a model $\mathfrak{B}^3 \models T^{\mathfrak{B}}$ which omits the set Γ^{ZF} and is not well-ordered by $<^{*\mathfrak{B}^3}$. Since H is order preserving, $\langle P^{\mathfrak{B}^3}, \in\rangle$ is not well-ordered; hence there exists a sequence $\{\alpha_n \in P^{\mathfrak{B}^3} : n < \omega\}$ such that for every $n < \omega$, $\mathfrak{B}^3 \models \alpha_n > \alpha_{n+1} + n + 1$.

Now to complete the proof of our theorem it is enough to prove that the theory $\bar{T}^*$ is finitely satisfiable in $\mathfrak{B}$. But T_1^* holds, there exist functions $\{f^j: \kappa \to \kappa \mid j < \kappa\}$ satisfying the first two lines in the definition of $\bar{T}_1^*$, and the last line of the definition holds since, for each $\alpha < \delta(\kappa)$, $G(\alpha) \models \bar{T}_1^*$ omits Γ and has order of length $F(\alpha)$ (by $<_\varphi$). The indiscernibility requirement follows by repeated application of the Erdös-Rado theorem (possible since $\{\alpha_n : n < \omega\}$ is descending). Namely, define by induction on $n < \omega$ subsets $X_n \subseteq |G(\alpha_0)|$ such that $|X_n| = \beth_{\alpha_n}$, the elements of X_n form a $<_\varphi$-increasing sequence, and any two $<_\varphi$-increasing n-tuples from X_n realize the same type. Also $\mathfrak{B}^3 \models X_{n+1} \subseteq X_n$.

THEOREM 2.9. *Let κ be an infinite cardinal. Then $\mu^*(\kappa, \kappa) = \mu(\kappa)$ ($\mu(\kappa)$ is the Hanf number of the logic $L_{\kappa^+,\omega}$).*

PROOF. Clearly $\mu^*(\kappa,\kappa) \le \mu(\kappa)$. Why do we have $\mu^*(\kappa,\kappa) \ge \mu(\kappa)$? Let L, T, φ be respectively a similarity type, a theory in $L_{\kappa^+,\omega}$, and a formula, as in the definition of $\mu^*(\kappa,\kappa)$. Since for every $\mu < \mu(\kappa)(\exists M_\mu \models T)(\exists\{\bar{\alpha}_i : i < \mu\} \subseteq |M_\mu|$ ordered in M_μ by $\varphi)$, we expand $L(T)$ by a unary predicate D_φ and a function symbol F; then we add the axioms $D_\varphi = \{\bar{y} : (\exists \bar{x})\varphi(\bar{x}, \bar{y})\}$, $\varphi(\bar{x}, \bar{y})$ *linearly orders* D_φ and F *is one-to-one from the universe into* D_φ. By the definition of Hanf number the above expanded theory has a model in every cardinality, and the reducts of those models to $L(T)$ exemplify what we wanted to prove.

§3. Proof of Theorems 2 and 3 from the Abstract.

PROOF OF THEOREM 2. We prove here only the upper bound part, namely that $\mu^*(\lambda, \aleph_0) \le \beth_{\lambda^+}$. The lower bound will be considered in [GSh2].

So $\kappa = \aleph_0$. Let $T \subseteq L_{\omega_1,\omega}$ and $\varphi(\bar{x}, \bar{y}) \in L_{\lambda^+,\omega}$ be as in the statement of the theorem, and let T^* be an expansion of T by Skolem functions. We concentrate on proving that $\mu^*(\lambda, \aleph_0) \le \beth_{\lambda^+}$, and in [GSh2] we prove the exact bound. If $\lambda = \aleph_0$, then by the previous section clearly $\mu^*(\aleph_0, \aleph_0) \le \beth_{\omega_1}$.

So from now on assume $\lambda > \aleph_0$.

For the sake of contradiction assume $\mu^*(\lambda, \aleph_0) > \beth_{\lambda^+}$. Define $P = \{f \mid (\exists D \in \mathscr{S}_{<\aleph_0}(\lambda)) f: D \to \omega\}$. Force with P; let V^P be the extension of V after forcing with P. Obviously λ is collapsed to $\aleph_0$; since P satisfies the λ^+-chain condition, $\beth_{\lambda^+}^V$ becomes $\beth_{\omega_1}$ in V^P.

Since λ is a countable ordinal in V^P, we have $V^P \models \varphi^V(\bar{x}, \bar{y}) \in L^{V^P}_{\omega_1,\omega}$. By the above remark $V^P \models \beth_{\omega_1} \leq (\beth_{\lambda^+})^V$. Combining the last two, we have

$$(*)_1 \qquad V^P \models (\forall \mu < \mu^*(\aleph_0, \aleph_0))(\exists M_\mu \models T^*)(\exists \{\bar{a}_i : i < \mu\}) \\ (\forall i, j < \kappa)[i < j \Leftrightarrow M_\mu \models \varphi[\bar{a}_i, \bar{a}_j]].$$

Since $V^P \models T^* \cup \{\varphi\} \subseteq L_{\omega_1,\omega}$ and T^* is countable, we may assume that

$$(*)_2 \qquad V^P \models \{\bigwedge T^*, \varphi\} \in H(\aleph_1)$$

where $H(\aleph_1)$ stands for the set of all sets which have transitive closure of power $\leq \aleph_0$.

By $(*)_1$ and the definition of $\mu^*(\aleph_0, \aleph_0)$,

$$(*)_3 \qquad V^P \models (\exists M \models T^*)(\exists \{\bar{a}_i : i < \omega\} \subseteq |M|) \\ [M \text{ is the Skolem hull of } \{\bar{a}_i : i < \omega\} \text{ and } i < j \Leftrightarrow M \models \varphi[\bar{a}_i, \bar{a}_j]].$$

This property of the formula φ is expressible by a bounded formula ψ of ZFC with T^* as a parameter. Namely, there exists a formula $\psi(x, y)$ in the language of set theory such that if T is a countable theory with Skolem functions and τ is a formula in $L(T)$, then

$$\psi(\tau, T) \Leftrightarrow \tau \text{ has two free variables } \bar{x}, \bar{y}, \\ T \text{ has a model } M \text{ and there exists } A = \{\bar{a}_n : n < \omega\} \subseteq M \text{ such that} \\ M \text{ is the Skolem closure of } A \text{ and } n < \kappa \Leftrightarrow \tau[\bar{a}_n, \bar{a}_k].$$

By $(*)_2$ and $(*)_3$ we have that

$$V^P \models (\exists \tau \in H(\aleph_1))[H(\aleph_1) \models \psi[\tau, \tau^*]].$$

Now, by $(*)_2$, $\bigwedge T^* \in H(\aleph_1)^V$ by Levy's absoluteness theorem[4] (see [Je, Theorem 3.6, or, for the form we use, Exercise 14.18]) we have

$$V \models (\exists \tau \in H(\aleph_1))[H(\aleph_1) \models \psi[\tau, T^*]].$$

Here there exists a formula $\varphi'(\bar{x}, \bar{y}) \in L_{\omega_1,\omega}$ (in our ground model V) such that $(*)_1$ holds for φ' instead of $\varphi(\bar{x}, \bar{y})$. Apply Theorem 1.2 (for $\kappa = \aleph_0$) for T^* and $\varphi'(\bar{x}, \bar{y})$. This completes the proof of Theorem 1.3.

Proof of Theorem 3. Let $T \subseteq L_{\kappa^+,\omega}$ and $\varphi(\bar{x}, \bar{y}) \in L_{\lambda^+,\omega}$ be given as in the hypothesis of the theorem. Let us define a forcing notion

$$P = \{f \mid (\exists x \in \mathscr{S}_\kappa(\lambda)) f : x \to \kappa^+\}.$$

As before, we have

$$(*)_1 \ V \models (\forall \mu < \beth_{(\lambda^\kappa)^+})(\exists M_\mu \models T)(\exists \{\bar{a}_i : i < \mu\} \subseteq |M_\mu|)[i < j \Leftrightarrow M_\mu \models \varphi[\bar{a}_i, \bar{a}_j]],$$

$$(*)_2 \qquad V^P \models (\forall \mu < \beth_{(2^\kappa)^+})(\exists M_\mu \models T)(\exists \{\bar{a}_i : i < \mu\} \subseteq |M_\mu|) \\ [i < j \Leftrightarrow M_\mu \models \varphi[\bar{a}_i, \bar{a}_j]] \wedge [\bar{\varphi}(\bar{x}, \bar{y}) \in L_{\kappa^+,\omega}].$$

Since the properties of φ (in $(*)_2$) are expressible by a bounded formula ψ in the parameters T and κ, by a suitable coding we can view φ^{V^P} as a subset of κ.

[4] Sometimes also called the Levy-Shoenfield absoluteness lemma.

To complete the proof of the theorem it is enough to show that all the subsets of κ of V^P exist already in the ground universe V (this will imply the existence of $\varphi'(\bar{x}, \bar{y}) \in L_{\kappa^+,\omega}$ with the required properties, and an application of Theorem 1.2 will give what we want). But forcing with P does not add subsets of κ since P is κ^+-complete.

§4. The two cardinal theorem and a better bound. We shall use a two cardinal theorem. Let us first state the necessary definitions to formulate it.

DEFINITION 4.1. Let T be a first order theory, $|T| \leq \kappa$, and let Γ be a set of types, $|\Gamma| \leq \kappa$. Suppose $\{P, <, <^P\} \subseteq L(T)$ is such that P is a unary predicate, and $<$ and $<^P$ are binary predicates such that

$$T \vdash [\text{“}<^P\text{” linearly orders } P \wedge \text{“}<\text{” is a linear order}].$$

(1) Let us define an ordinal

$$\begin{aligned}\bar{\delta}(\lambda, \kappa) = \operatorname{Min}\{\bar{\delta}\colon & \textbf{If } \exists M \in EC(T, \Gamma) \text{ such that o.tp.}(P^M, <^{PM}) \text{ is an ordinal } < \lambda^+ \\ & \text{and o.tp.}(|M| - P^M, <^M) \geq \bar{\delta}, \\ & \textbf{then } \exists N \in EC(T, \Gamma) \text{ such that o.tp.}(P^N, <^{PN}) \text{ is an} \\ & \text{ordinal } < \kappa^+ \text{ and } (|N| - P^N, <^N) \text{ is not well-ordered}\}.\end{aligned}$$

(2) Now define a cardinal

$$\begin{aligned}\bar{\mu}(\lambda, \kappa) = \operatorname{Min}\{\bar{\mu}\colon & \textbf{If } T \text{ satisfies } (\forall \mu < \bar{\mu})(\exists M_\mu \in EC(T, \Gamma)) \text{ such that} \\ & \text{o.tp.}(P^{M_\mu}, <^{PM_\mu}) \text{ is an ordinal } < \lambda^+ \text{ and } \|M_\mu\| \geq \mu \\ & \textbf{then } (\forall \chi > \kappa)(\exists M_\chi \in EC(T, \Gamma))[\|M_\chi\| \geq \chi \\ & \text{and o.tp.}(P^{M_\chi}, <^{M_\chi}) \text{ is an ordinal } < \kappa^+]\}.\end{aligned}$$

This is important since there is a relation between $\bar{\delta}(\lambda, \kappa)$ and $\bar{\mu}(\lambda, \kappa)$ that is parallel to the relation between $\delta(\kappa)$ and $\mu(\kappa)$—see the end of §2, or [Sh4, Chapter VII, §5]. Namely, we have

THEOREM 4.2. *For every κ and λ, always $\bar{\mu}(\lambda, \kappa) \leq \beth_{\bar{\delta}(\kappa, \lambda)}$.* (See Remark (6), below.)

REMARK 4.2A. (1) In the definition of $\bar{\delta}(\lambda, \kappa)$ we can replace “o.tp.$(P^N, <^N)$ *is an ordinal less than* κ^+” by requiring that $(P^N, <^N)$ be well-ordered.

(2) It is enough to say that $<$ orders only part of the universe.

(3) In the definition of $\bar{\mu}$ we could add a predicate Q and speak of the power of the predicate Q instead of the power of the universe. This gives us a generalization of Morley's two cardinal theorem (see [Mo1]).

(4) Instead of defining the functions $\bar{\mu}$ and $\bar{\delta}$ with two parameters it is possible to define a stronger notion with three parameters (the new parameter is the power of the set of types Γ), as in [Sh4, Chapter VII, §5].

(5) Everything could be proved for PC classes, but since here we are interested in results for infinitary logic rather than in PC classes we decided to present our results here in the weaker (and less complicated) form.

(6) By repeating an argument similar to that in Theorem 5.4 of Chapter VII in [Sh4] we can prove that $\bar{\mu}(\lambda, \kappa) \geq \beth_{\bar{\delta}(\lambda, \kappa)}$, so that $\bar{\mu}(\lambda, \kappa) = \beth_{\bar{\delta}(\lambda, \kappa)}$.

Theorem 4.2 will be proved below, and it is important because it implies

THEOREM 1.5. *Let $\kappa \leq \lambda$, and let $T \subseteq L_{\kappa^+,\omega}$ be a theory of cardinality $\leq \kappa$. Assume that $L(T)$ contains a unary predicate P and a binary predicate $<^P$ such that $T \vdash <^P$ linearly orders P'. If there exists $M \models T$ such that $(P^M, <^{PM})$ is well-ordered of order* *type* $< \lambda^+$ *and* $\|M\| \geq \bar{\mu}(\lambda, \kappa)$, *then*

$$(\forall \chi \geq \kappa) \exists M_\chi \models T \text{ such that o.tp.}(P^{M_\chi}, <^{P^{M_\chi}}) \text{ is an ordinal } < \kappa^+ \text{ and } \|M_\chi\| \geq \chi.$$

REMARK 4.3A. In the proof of Theorem 1.6 we always get in addition that $N \upharpoonright P^N \prec M \upharpoonright P^M$. Namely we prove a combination of Keisler's two cardinal theorem together with Morley's two cardinal theorem (see [Ke2] and [Mo2]).

Using Remark (1) which appears after Theorem 1.5 in the Introduction, this gives us a way to find upper bounds for $\mu^*(\lambda, \kappa)$.

DEFINITION 4.3. Let $\lambda \geq \kappa$. If cf $\kappa = \aleph_0$, then let $\kappa^* = \kappa$; if cf $\kappa > \aleph_0$, let $\kappa^* = \kappa^+$. We define

$$\operatorname{cov}(\lambda, \kappa) = \operatorname{Min}\left\{|\mathscr{P}| : \mathscr{P} \subseteq \mathscr{S}_{<\kappa^*}(\lambda) \text{ such that } (\forall X \in \mathscr{S}_{<\kappa^*}(\lambda)) \right.$$
$$\left. (\exists \{\omega_l \in \mathscr{P} : l < \omega\}) \left[X \subseteq \bigcup_{l<\omega} w_l \right] \right\}.$$

THEOREM 4.4. $\bar{\delta}(\lambda, \kappa)$ *exists. Moreover*:

(i) *If* $\kappa = \aleph_0$ *then* $\bar{\delta}(\lambda, \aleph_0) = \lambda^+$.

(ii) *If* cf $\kappa > \aleph_0$ *then* $\bar{\delta}(\lambda, \kappa) \leq \chi(\lambda, \kappa) \overset{\text{def}}{=} (\operatorname{cov}(\lambda, \kappa) + 2^\kappa)^+$.

(iii) *If* cf $\kappa = \aleph_0$ *then* $\bar{\delta}(\lambda, \kappa) \leq \chi(\lambda, \kappa) \overset{\text{def}}{=} (\operatorname{cov}(\lambda, \kappa) + 2^{<\kappa} + \aleph_0)^+$.

Hence by applying Theorem 4.4 to the result of Theorem 4.3 we shall have the upper bounds we are seeking. From now on we shall concentrate on the proof of Theorem 4.4

PROOF. (i) is proved exactly as Theorem 2 in §3.

We prove cases (ii) and (iii) simultaneously. Assume there exists a model $M \in EC(T, \Gamma)$ such that o.tp.$(P^M, <^{P^M})$ is an ordinal $< \lambda^+$, and o.tp.$(|M| - P^M, <^M) \geq \chi(\lambda, \kappa)$. Let us fix $\mathscr{P} \subseteq \mathscr{S}_{<\kappa^*}(\lambda)$ as in Definition 4.3 exemplifying that $\operatorname{cov}(\lambda, \kappa) = |\mathscr{P}|$.

For a while we shall give the theory T, its language, and the set of types Γ a cosmetic treatment:

(1) Expand $L(T)$ by κ individual constants $\{c_i : i < \kappa\}$ and a unary predicate Q_ω such that $Q_\omega^M = \kappa$ and $c_i^M = i$. Extend the set Γ by the type $q = \{Q_\omega(x) \wedge x \neq c_i : i < \kappa\}$. Clearly, our assumptions are still satisfied after these changes.

(2) Add to $L(T)$ $\aleph_0$ unary predicates $\{Q_n : n < \omega\}$ as follows. If cf $\kappa > \aleph_0$, let $Q_n^M = Q_\omega^M$ for all $n < \omega$. Otherwise (when cf $\kappa = \aleph_0$) choose $Q_n^M \subseteq Q_{n+1}^M \subseteq Q_\omega^M$, $|Q_n^M| < \kappa$, such that $Q_\omega^M = \bigcup_{n<\omega} Q_n^M$. Clearly we may add to T the axioms

$$\{(\forall x)[Q_n(x) \to Q_{n+1}(x) \wedge Q_\omega(x)] : n < \omega\}$$

and we may assume Γ contains also the type $\{Q_\omega(x) \wedge \neg Q_n(x) : n < \omega\}$.

(3) We may expand $L(T)$ so that T will have Skolem functions.

(4) By adding κ individual constants and $\aleph_0$ new relation symbols, we may assume that $L(T)$ consists of a countable set of relation symbols L_0 and a set of κ individual constants $\{d_i : i < \kappa\}$ (see §2). Let $L = L_0 \cup \{d_i : i < \kappa\}$, and $M_0 = M \upharpoonright L_0$.

Fix $\{a_i \in M - P^M : i < \chi(\lambda, \kappa)\}$ such that $i < j \Rightarrow a_i <^M a_j$.

By induction on $n < \omega$ define sets

$$S_n \subseteq \{\bar{\alpha} = \langle \bar{\alpha}(0), \ldots, \bar{\alpha}(n-1) \rangle : \chi(\lambda, \kappa) > \bar{\alpha}(0) > \cdots > \bar{\alpha}(n-1)\}$$

such that $|S_n| = \chi(\lambda, \kappa)$. For $\bar{\alpha} \in S_n$ define sets

$$\{W^n_{\bar{\alpha},l} \in \mathscr{P} : l < \omega\}, \qquad \{u^m_l \in \mathscr{P} : l, m < n\}$$

such that the following conditions hold:

(1) $(\forall i < \chi)(\exists \bar{\alpha} \in S_n)[\bar{\alpha}(n-1) > i]$.

(2) If $N^n_{\bar{\alpha}}$ is (the Skolem hull of $a_{\bar{\alpha}}$) $\cap P^{M_0}$ then $N^n_{\bar{\alpha}} \subseteq \bigcup_{l<\omega} W^n_{\bar{\alpha},l}$ (when $a_{\bar{\alpha}}$ is the sequence $\langle a_{\bar{\alpha}(0)}, \ldots, a_{\bar{\alpha}(n-1)} \rangle$).

(3) $\bar{\alpha} \in S_n$ and $l, m < n \Rightarrow W^m_{\bar{\alpha},l} = u^m_l$.

(4) $\bar{\alpha}, \bar{\beta} \in S_n \Rightarrow$

$$p_n(y_0, \ldots, y_{n-1}) \stackrel{\text{def}}{=} \operatorname{tp}\left(a_{\bar{\alpha}}, Q^{M_0}_n \cup \bigcup_{l,m<n} u^m_l, M_0\right) = \operatorname{tp}\left(a_{\bar{\beta}}, Q^{M_0}_n \cup \bigcup_{l,m<n} u^m_l, M_0\right).$$

(5) $(\forall \bar{c} \in Q_n \cup \bigcup_{l,m<n} u^m_l)(\forall \tau(y_0, \ldots, y_{n-1}, \bar{c}) \in L(T))$

$$\bar{\alpha}, \bar{\beta} \in S_n \Rightarrow (\forall l < \omega)[\tau(a_{\bar{\alpha}}, \bar{c}) \in W_{\bar{\alpha},l} \Leftrightarrow \tau(a_{\bar{\beta}}, \bar{c}) \in W_{\bar{\beta},l}]].$$

(6) $n \geq m, \bar{\alpha} \in S_n \Rightarrow \bar{\alpha} \restriction m = \bar{\alpha}$.

Why does the construction give what we want? Let $A = Q^{M_0}_\omega \cup \bigcup_{m,l<\omega} u^m_l$. Clearly $A \subseteq P$ and $|A| = \kappa$.

To define a model N, as in Definition 4.1 let

$$p^*(y_0, \ldots, y_n, y_{n+1}, \ldots) = \{\varphi(y_0, \ldots, y_{n-1}, \bar{c}) : \bar{c} \in A, n < \omega, (\exists k \geq n)\varphi(\bar{y}, \bar{c}) \in p_k\}.$$

By (6) and (4), p^* is an infinitary consistent type; so choose $\bar{b} = \{b_n : n < \omega\} \subseteq |\mathfrak{C}|$ elements realizing it. Let N be the Skolem hull of $\bar{b} \cup A$.

Why is $N \in EC(T, \Gamma)$? Let $\bar{c} \in N$. Since N is the Skolem closure of $A \cup \bar{b}$, there exist a sequence of terms $\bar{\tau}$, a sequence $\bar{a}$ of elements from A, and a natural number n such that $\bar{c} = \bar{\tau}(b_0, \ldots, b_{n-1}, \bar{a})$. By requirement (4) there exists in M_0 a sequence $\bar{c}'$ such that $\operatorname{tp}(\bar{c}, \varnothing, N) = \operatorname{tp}(\bar{c}', \varnothing, M_0)$, and hence the model N omits the types from Γ.

By (1), $(|N| - P^N, <^N)$ is not well-founded.

So we have just to show that $P^N = A$ and $(P^N, <^{PN})$ is well-ordered. For this it suffices to show that $P^N = A$ and that $\langle P^N, <^{PN} \rangle \restriction A$ is an elementary submodel of $\langle P^M, <^{PM} \rangle$.

Assume $\bar{c} \in A$ and $N \models P(\tau(b_{i_1}, \ldots, b_{i_n}, \bar{c}))$, choosing n such that

$$\bar{c} \in Q_n \cup \bigcup_{l,m<n} u^m_l, \qquad \tau = \tau(y_0, \ldots, y_{n-1}, \bar{c}).$$

By (5) there exists an $l(*) < \omega$ such that for all $\bar{\alpha} \in S_n$, $\tau(a_{\bar{\alpha}}, \bar{c}) \in W_{\bar{\alpha},l(*)}$. Let $\kappa > \operatorname{Max}\{n, l(*)\}$.

By (3) and (6), for all $\bar{\alpha} \in S_k$,

$$\tau(a_{\bar{\alpha}\restriction n}, \bar{c}) \in Q_n \cup \bigcup_{l,m<k} u^m_l.$$

Hence by the definition of p_k there exists $e \in Q_n \cup \bigcup_{l,m<k} u^m_l$ such that $\{\tau(a_{\bar{\alpha}^k_t\, n}, \bar{c}) = e\} \subseteq p_k$. Therefore $\tau(b_0, \ldots, b_{n-1}, \bar{c}) \in A$. So we have proved that $P^N \subseteq A$.

To finish we just have to show that $\langle P^N, <^{PN} \rangle \prec \langle P^M, <^{PM} \rangle$; and this follows since everything is an elementary submodel of $\mathfrak{C}$.

The construction. For $n = 0$ there are no problems.

For $n+1$, given $\bar{\alpha} \in S_n$ choose $\bar{\beta} \in S_n$ such that $\bar{\beta}(n-1) > \bar{\alpha}(n-1)$. Let $\bar{\gamma}(l) = \bar{\beta}(l)$ for $0 \leq l < n$, and let $\bar{\gamma}(n) = j$. Clearly $\bar{\gamma} \in S_{n+1}$. By the definition of $\mathscr{P}$ there exists $\{v_{\bar{\alpha},l} \in \mathscr{P}: l < \omega\}$ such that

$$\text{Skolem Hull}(a_{\bar{\alpha}} \cap P) \subseteq \bigcup_{l<\omega} v_{\bar{\alpha},l}.$$

Let $q_{\bar{\alpha}} = \text{tp}(a_{\bar{\alpha}}, Q_n \cup \bigcup_{l,m<n} u_l^m, M_0)$. If cf $\kappa > \aleph_0$, then always $|Q_n \cup \bigcup_{l,m<n} u_l^m| \leq \kappa$ and by the definition of $\chi(\lambda, \kappa)$ we have

$$|\{q_{\bar{\alpha}}: \bar{\alpha} \in S_n\}| \leq 2^\kappa < \chi(\lambda, \kappa).$$

Since $\chi(\lambda, \kappa)$ is regular, there exists a set of sequences S such that $|S| = \chi(\lambda, \kappa)$ and $\bar{\alpha}, \bar{\beta} \in S \Rightarrow q_{\bar{\alpha}} = q_{\bar{\beta}}$. If cf $\kappa = \aleph_0$, then $|Q_n| < \kappa$ and $u_l^m \in S_{<\kappa}(\lambda)$. Hence

$$\text{the number of distinct types is } \leq 2^{\aleph_0 + |Q_n \cup \bigcup_{l,m<n} u_l^m|} \leq 2^{\leq \kappa} < \chi(\lambda, \kappa).$$

Again by the regularity of $\chi(\lambda, \kappa)$ we may choose S as above with $|S| = \chi(\lambda, \kappa)$ *unbounded, such that* $\bar{\alpha}, \bar{\beta} \in S \Rightarrow q_{\bar{\alpha}} = q_{\bar{\beta}}$. Finally let $p_{n+1} = q_{\bar{\alpha}}$ for $\bar{\alpha} \in S$.

For every term τ, each sequence $\bar{\alpha} \in S_{n+1}$ and $\bar{c} \in Q_n \cup \bigcup_{l,m<n} u_l^m$ let us define a function into ${}^\omega 2$ (= the infinite 0-1 sequences):

$$\langle \tau, \bar{\alpha}, \bar{c} \rangle \to \langle \tau(a_{\bar{\alpha}}, \bar{c}\rangle \in v_{\bar{\alpha},0}, \tau(\cdots) \in v_{\bar{\alpha},1}, \ldots \rangle.$$

Since

$$\left| \left\{ \langle \tau, \bar{\alpha}, \bar{c} \rangle | : \tau \text{ a term } \bar{c} \in Q_n \cup \bigcup_{l,m<n} u_l^m \right\} \right| = \chi(\lambda, \kappa) > 2^{\aleph_0},$$

there exists $S^* \subseteq S$ with $|S^*| = \chi(\lambda, \kappa)$ such that for $\bar{\alpha}, \bar{\beta} \in S^*$ the statement of (5) holds.

For $\bar{\alpha} \in S^*$ let $W_{\bar{\alpha},l} = v_{\bar{\alpha},l}$. Again take $S^{**} \subseteq S^*$ so that $\bar{\alpha}, \bar{\beta} \in S^{**} \Rightarrow W_{\bar{\beta},l} = W_{\bar{\alpha},l} = u_l^m$ for $l, m \leq n$.

PROOF OF THEOREM 4.2. This proof is similar to the proof for κ uncountable in §2 (after Definition 2.6).

So we have to compute cov(λ, κ). In order to compute cov(λ, κ) it is better to introduce a generalization cov(α, κ) in which λ is replaced by an ordinal α. If $\mathscr{P} \subseteq \mathscr{S}_{<\kappa^*}(\alpha)$ as in Definition 4.3, we say $\mathscr{P}$ is an (α, κ) *cover*. If $\mathscr{P}$ is an (α, κ) cover such that cov$(\alpha, \kappa) = |\mathscr{P}|$, we say that $\mathscr{P}$ exemplifies cov(α, κ).

LEMMA 4.5. (0) cov$(\lambda, \kappa) \leq \lambda^\kappa$.

(1) $\alpha \leq \beta \Rightarrow$ cov$(\alpha, \kappa) \leq$ cov(β, κ).

(2) cov$(\alpha, \kappa) =$ cov$(|\alpha|, \kappa)$.

(3) *We can bound the value of* cov(λ, κ) *according to the following inequalities*:

(i) cov$(\lambda, \lambda) = 1$.

(ii) $\kappa \leq \lambda \Rightarrow$ cov$(\lambda^+, \kappa) \leq$ cov$(\lambda, \kappa) + \lambda^+$.

(iii) *If λ is a limit cardinal and $\kappa < \lambda$, choose $\{\lambda_i: i < \text{cf } \lambda\}$ increasing unbounded in λ, such that $\lambda_0 > \kappa$. Then*

$$\text{cov}(\lambda, \kappa) \leq \prod_{i<\text{cf } \lambda} \text{cov}(\lambda_i, \kappa).$$

(iv) *If κ is a limit cardinal then the analog of* (iii) *holds.*

(v) *If* cf $\lambda = \aleph_0$, *where* $\{\lambda_n < \lambda : \lambda : n > \omega\}$ *is increasing and unbounded in* λ, *and* $\lambda_0 > \kappa$, *then*

$$\operatorname{cov}(\lambda, \kappa) \leq \sum_{n<\omega} \operatorname{cov}(\lambda, \kappa_n).$$

(vi) *If* cf $\kappa = \aleph_0$, *where* $\{\kappa_n < \kappa : n < \omega\}$ *is unbounded and increasing in* κ, *then*

$$\operatorname{cov}(\lambda, \kappa) \leq \sum_{n<\omega} \operatorname{cov}(\lambda, \kappa_n).$$

(vii) *If* cf $\lambda =$ cf $\kappa = \aleph_0$ *and* $\{\lambda_n : n < \omega\}$ *and* $\{\kappa_n : n < \omega\}$ *are unbounded and increasing in* λ *and* κ *respectively, then*

$$\operatorname{cov}(\lambda, \kappa) \leq \sum_{n,\kappa<\omega} \operatorname{cov}(\lambda_\kappa, \kappa_n).$$

PROOF. (0) Obvious.

(1) Let $\mathscr{P}_{\beta,\kappa}$ be a cover exemplifying $\operatorname{cov}(\beta, \kappa)$. Then clearly $\mathscr{P}_{\alpha,\kappa} \stackrel{\text{def}}{=} \{x \cap \alpha : x \in \mathscr{P}_{\beta,\kappa}\}$ exemplifies what we want.

(2) By (1), $\operatorname{cov}(|\alpha|, \kappa) \leq \operatorname{cov}(\alpha, \kappa)$. The inequality is proved as follows: Let $f : \alpha \to |\alpha|$ be one-to-one and onto, and let $\mathscr{P}_{|\alpha|,\kappa}$ be a cover such that $\operatorname{cov}(|\alpha|, \kappa) = |\mathscr{P}_{|\alpha|,\kappa}|$. It is easy to verify that $\mathscr{P}_{\alpha,\kappa} \stackrel{\text{def}}{=} \{f^{-1}(x) : x \in \mathscr{P}_{|\alpha|,\kappa}\}$ is a cover. Hence $\operatorname{cov}(\alpha, \kappa) \leq \operatorname{cov}(|\alpha|, \kappa)$.

(3) (i) is immediate.

(ii) By (2), for every α such that $\lambda \leq \alpha < \lambda^+$ we have $\operatorname{cov}(\alpha, \kappa) = \operatorname{cov}(|\alpha|, \kappa)$ (i.e. $\operatorname{cov}(\lambda, \kappa)$). Fix a cover $\mathscr{P}_{\alpha,\kappa}$ exemplifying $\operatorname{cov}(\alpha, \kappa)$. Let $\mathscr{P}_{\lambda^+,\kappa} = \bigcup_{\alpha<\lambda^+} \mathscr{P}_{\alpha,\kappa}$. Let $x \in \mathscr{S}_{<\kappa^*}(\lambda^+)$; since λ^+ is regular and $\lambda^+ > \kappa^+$ there exists $\alpha < \lambda^+$ such that $x \in \mathscr{S}_{<\kappa^*}(\alpha)$. Hence, by the choice of $\mathscr{P}_{\alpha,\kappa}$ there exists $\{W_n \in \mathscr{P}_{\alpha,\kappa} : n < \omega\}$ such that $x \subseteq \bigcup_{n<\omega} W_n$ but $\mathscr{P}_{\alpha,\kappa} \subseteq \mathscr{P}_{\lambda^+,\kappa}$. Hence $\mathscr{P}_{\lambda^+,\kappa}$ is a (λ^+, κ) cover, and

$$\operatorname{cov}(\lambda^+, \kappa) \leq |\mathscr{P}_{\lambda^+,\kappa}| = \left| \bigcup_{\alpha<\lambda^+} \mathscr{P}_{\alpha,\kappa} \right| \leq |\mathscr{P}_{\lambda,\kappa}| \lambda^+ = \operatorname{cov}(\lambda, \kappa) + \lambda^+.$$

(iii) For every $i <$ cf λ let $\mathscr{P}_i$ be a (λ, κ) cover. We claim that $\mathscr{P}_{\lambda,\kappa} = \{\bigcup_{i\in S} s_i : s_i \in \mathscr{P}_i, S \subseteq \text{cf } \lambda, |S| < \kappa^*\}$ is a (λ, κ) cover.

(iv) Similar to (iii).

(v) Let $\{\kappa_n < \kappa : n < \omega\}$ be increasing unbounded and let $\mathscr{P}_n$ exemplify $\operatorname{cov}(\lambda, \kappa_n)$. Clearly $\mathscr{P}_\kappa \stackrel{\text{def}}{=} \{x_n : n < \omega, x_n \in \mathscr{P}_n\}$ is a cover as required, and

$$\operatorname{cov}(\lambda, \kappa_n) \leq |\mathscr{P}_\kappa| \leq \sum_{n<\omega} |\mathscr{P}_n| = \sum_{n<\omega} \operatorname{cov}(\lambda, \kappa_n).$$

(vi) Similar to (v).

Let $\{\lambda_n < \lambda, \kappa_\kappa < \kappa : n, k < \omega\}$ be increasing unbounded, and let $\mathscr{P}_{k,n}$ be a cover exemplifying $\operatorname{cov}(\lambda_n, \kappa_k)$. Clearly $\mathscr{P}_{\lambda,\kappa} = \{x_{k,n} : k, n < \omega, x_{k,n} \in \mathscr{P}_{k,n}\}$ is a cover as required. Hence

$$\operatorname{cov}(\lambda, \kappa) \leq |\mathscr{P}_{\lambda,\kappa}| \leq \sum_{n,k<\omega} \operatorname{cov}(\lambda_k, \kappa_n).$$

COROLLARY 4.6. *Let* $\kappa \leq \lambda$. *Then we have the following bounds*:

(1) *If* cf $\lambda = \aleph_0$ *then* $\operatorname{cov}(\lambda, \kappa) \leq \sum_{\mu<\lambda} \mu^\kappa$.

(2) *If* cf $\kappa = \aleph_0$ *then* $\operatorname{cov}(\lambda, \kappa) \le \sum_{\chi<\kappa} \lambda^\kappa$.

(3) *If* cf $\kappa = \lambda = \aleph_0$ *then* $\operatorname{cov}(\lambda, \kappa) \le \sum_{\mu<\lambda, \chi<\kappa} \mu^\chi$.

(4) *For* $\alpha < \omega_1$, $\kappa \le \beth_\omega$ *if* $\lambda = (\beth_\omega)^{+\alpha} \Rightarrow \bar{\delta}(\lambda, \kappa) \le \lambda^+$.

PROOF. (1), (2) and (3) follow from (v), (vi), and (vii) of Lemma 4.5 (respectively) using Lemma 4.5(0).

(4) By induction on α: for α successor use (3)(ii); if α is a limit, use (3)(iv).

In [GSh2] we shall deal with a more general notion of covering number, and there we shall prove that there are improved bounds also in other cases (e.g. $\bar{\delta}(\lambda, \kappa) \le \beth_\omega^{+(2^{\aleph_1})+}$

Now we can prove

THEOREM 1.6. *For every* $\kappa \le \lambda$,

(1) *if* cf $\lambda = \aleph_0$ *then* $\mu^*(\lambda, \kappa) \le \beth_{(\sum_{\mu<\lambda}\mu^\kappa)^+}$;

(2) *if* cf $\kappa = \aleph_0$ *then* $\mu^*(\lambda, \kappa) \le \beth_{(\sum_{\chi<\kappa}\lambda^\chi)^+}$;

(3) *if* cf κ = cf $\lambda = \aleph_0$ *then* $\mu^*(\lambda, \kappa) \le \beth_{(\sum_{\mu<\lambda, \chi<\kappa}\mu^\chi)^+}$; *and*

(4) *if* $\lambda = (\beth_\omega)^{+\alpha}$ *and* $\kappa \le \beth_\omega$ *for some* $\alpha \le \omega_1$, *then* $\bar{\mu}(\lambda, \kappa) \le \beth_{\lambda^+}$.

PROOF. Combine Corollary 4.6, Theorem 4.4, Theorem 4.2, Theorem 1.5 and the remark after the statement of Theorem 1.5 in the Introduction. (4) follows from the corresponding part of Corollary 4.6.

§5. Proof of Theorem 4 of the Abstract. As we said in the Introduction, we shall prove the following stronger theorem.

THEOREM 1.9. *For every* $\lambda \ge \kappa$, *any* $T \subseteq L_{\kappa^+,\omega}$, *and any set of formulas* $\Delta \subseteq L_{\lambda^+,\omega}$ *such that* $\Delta \supseteq L_{\kappa^+,\omega}$, *if* T *is* (Δ, μ)*-unstable for* μ *satisfying* $\mu^{<\mu^*(\lambda,\kappa)} = \mu$ *and* $\mu = \mu^{|\Delta|}$, *then* T *is* Δ*-unstable.*

But instead of proving Theorem 1.9 directly we prefer to prove an even slightly stronger result:

THEOREM 5.1. *For every* $\lambda \ge \kappa$ *and* $T \subseteq L_{\kappa^+,\omega}$, *if* T *is* (Δ, μ)*-unstable for* μ *satisfying* $\mu = \mu^{<\mu^*(\lambda,\kappa)}$ *and* $\mu^{|\Delta|} = \mu$, *then there exists* $\varphi'(\bar{x}, \bar{y}) \in L_{\kappa^+,\omega}$ *such that* T *is* $\varphi'(\bar{x}, \bar{y})$*-unstable.*

By the proof of Theorem 2.1 in [Sh2] we have

Fact 5.2. *Let* χ *be a given infinite cardinal. If* M *is* $(\{\varphi\}, \mu)$*-unstable as witnessed by* $A \subseteq |M|$ *(i.e.* $|A| \le \mu < S^1_{\{\varphi\}}(A, M)|$*) for* μ *satisfying* $\mu = \sum_{\chi^*<\chi}(\mu^{\chi^*} + 2^{2^{\chi^*}})$, *then there exist* $\{\bar{a}_i{:}\, \bar{b}_i{:}\, i < \chi\} \subseteq A$ *and* $\{c_i{:}\, i < \chi\} \subseteq |M|$ *such that for every* $i, j < \chi$

$$M \vDash \varphi[c_i, \bar{a}_j] \equiv \varphi[c_i, \bar{b}_j] \Leftrightarrow i < j.$$

PROOF OF THEOREM 5.1. Let μ, $A \subseteq M$, $|A| \le \mu < \|M\|$, and let Δ exemplify the assumption. Fixing an enumeration of Δ, let $\Delta = \{\varphi_i(x, \bar{y}_i){:}\, i < |\Delta|\}$. Define a function

$$g{:}\, S^1_\Delta(A, M) \to \mathop{\times}_{i<|\Delta|} S^1_{\{\varphi_i(x,\bar{y}_i)\}}(A, M)$$

by letting

$$g(p(x)) = \langle p \restriction \{\varphi_0\}, \ldots, p \restriction \{\varphi_i\}, \ldots \rangle$$

when $p \restriction \varphi = \{\varphi(x, \bar{a}){:}\, \bar{a} \in A,\ \varphi(x, \bar{a}) \in p(x)\}$. Clearly g is one-to-one. Hence there exists an $i_0 < |\Delta|$ such that $|S^1_{\{\varphi_{i_0}\}}(A, M)| > \mu$.

⟦Why? Well, it is easy cardinal arithmetic: Assume for the sake of contradiction that, for every $i < |\Delta|$, $|S^1_{\{\varphi_i\}}(A, M)| \leq \mu$. Then, using the fact that g is one-to-one, we have

$$\mu = \mu^{|\Delta|} \geq \prod_{i<|\Delta|} |S^1_{\{\varphi_i\}}(A, M)| \geq |S^1_\Delta(A, M)|;$$

but this is a contradiction to the choice of Δ, A, M and μ.⟧

Since $\mu^{<\mu^*(\lambda,\kappa)} = \mu$ we have $\mu = \sum_{\chi^* < \mu^*(\lambda,\kappa)}(\mu^{\chi^*} + 2^{2^{\chi^*}})$. Hence by Fact 4.1 there exist $\{\bar{a}_i, \bar{b}_i: i < \mu^*(\lambda, \kappa)\} \subseteq A$ and $\{c_i: i < \mu^*(\lambda, \kappa)\} \subseteq |M|$ such that for $i, j < \mu^*(\lambda, \kappa)$

$$i < j \Leftrightarrow M \models \varphi_{i_0}[c_i, \bar{a}_j] \equiv \varphi_{i_0}[c_i, \bar{b}_j].$$

In the same way as Theorem 4.2 was deduced from Theorem 4.4, we have that $\exists\varphi \in L_{\kappa^+,\omega}$ (the collapse of φ_{i_0}) such that for every ordered set I there exists $\{\bar{a}_s, \bar{b}_s, c_s: s \in I\} \subseteq EM(I)$ such that

$$(*) \qquad s <_I t \Leftrightarrow EM(I) \models \varphi[c_s, \bar{a}_t] \equiv \varphi[c_s, \bar{b}_t].$$

Let χ be a given cardinal. We want to show that T is $(\{\varphi\}, \chi)$-unstable. Choose $\chi_1 = \mathrm{Min}\{\chi_1 \leq \chi: 2^{\chi_1} > \chi\}$; let $I = {}^{\chi_1 \geq}2$ and $J = {}^{\chi_1 >}2$. By the choice of χ_1, $|I| = 2^{\chi_1} > \chi \geq |J|$. There is a natural linear order on I—the lexicographic order. We claim that $M = EM(I)$ and $A = \{\bar{a}_s, \bar{b}_s: s \in J\}$ exemplify that T is $(\{\varphi\}, \chi)$-unstable. This is so since, for $t_1, t_2 \in I$,

$$t_1 \neq t_2 \Rightarrow \mathrm{tp}_{\{\varphi\}}(c_{t_1}, A, M) \neq \mathrm{tp}_{\{\varphi\}}(c_{t_2}, A, M).$$

⟦Why? Assume without loss of generality that $t_1 <_I t_2$. Since J is dense in I, there exists $s \in J$ such that $t_1 <_I s <_I t_2$. By $(*)$,

$$M \models [\varphi[c_{t_1}, \bar{a}_s] \equiv \varphi[c_{t_1}, \bar{b}_s]] \wedge \neg[\varphi[c_{t_2}, \bar{a}_s] \equiv \varphi[c_{t_2}, \bar{b}_s]].$$

Hence

$$\varphi(x, \bar{a}_s) \in \mathrm{tp}_{\{\varphi\}}(c_{t_1}, A, M) \Rightarrow \varphi(x, \bar{b}_s) \in \mathrm{tp}_{\{\varphi\}}(c_{t_1}, A, M)$$

and

$$\varphi(x, \bar{a}_s) \in \mathrm{tp}_{\{\varphi\}}(c_{t_2}, A, M) \Rightarrow \varphi(x, \bar{b}_s) \notin \mathrm{tp}_{\{\varphi\}}(c_{t_2}, A, M).$$

By the last two lines together we are done.⟧

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