ANALYTICAL GUIDE AND UPDATES FOR CARDINAL ARITHMETIC
E-12

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ABSTRACT. Part A: A revised version of the guide in [Sh:g], with corrections and expanded to include later works.
Part B: Corrections to [Sh:g].
Part C: Contains some revised proof and improved theorems.
Part D: Contains a list of relevant references.

CONTENTS

I would like to thank Alice Leonhardt for the beautiful typing.
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Notation: $=^+$ appears in the following context: $\mu =^+ \sup\{\dots\}$ means "both sides are equal, and if in the right side the sup is not obtained, then it is singular."

For a set C of ordinals $\mathrm{acc}(C) = \{\alpha \in C : \alpha = \sup(\alpha \cap C)\}, \mathrm{nacc}(C) = C \backslash \mathrm{acc}(C)$.

The aim of this guide is to help the reader find out what is said in [Sh:g] and related works of the author, what are the theorems and definitions or where to look for them.

Let $[A]^{<\kappa} = \{a \subseteq A : |a| = \kappa\}$, similarly $[A]^{<\kappa}$ we call $[A]^{<\kappa}$ also $\mathscr{S}_{\leq\kappa}[A]$.

§0 $I[\lambda]$ AND PARTIAL SQUARES: SEE [SH 108], [SH 88A]

0.1 Definition. [Sh 345a, 2.3(5)], equivalent forms [Sh 420, 1.2], preservation of stationary subsets by μ-complete forcing [Sh 108, 21], [Sh 88a, 10].

Let $\lambda = \text{cf}(\lambda) > \aleph_0$. For $S \subseteq \lambda$ we have: $S \in I[\lambda]$ iff for some club E of λ and $\langle C_\alpha : \alpha < \lambda \rangle$ we have: C_α is a closed subset of α, $\text{otp}(C_\alpha) < \alpha$,

$$[\beta \in \text{nacc}(C_\alpha) \Rightarrow C_\beta = \beta \cap C_\alpha] \text{ and}$$
$$[\alpha \in E \cap S \Rightarrow \alpha = \sup(C_\alpha)]$$

(and every $\beta \in \text{nacc}(C_\alpha)$ is a successor ordinal); note $C \cap S$ has no inaccessible cardinal as a member. Note that [Sh 420, 1.2] says that the definition just given is equivalent to those used in [Sh 108], [Sh 88a].

We can demand further $\alpha \in E \cap S \Rightarrow \text{otp}(C_\alpha) = \text{cf}(\alpha)$. But we can demand less: for each α we are given $< \lambda$ candidates for C_α, and for C a candidate for α and $\beta < \alpha, C \cap (\beta + 1)$ is a candidate for some $\gamma < \alpha$. $I[\lambda]$ is a normal ideal, and in many cases of the form "non-stationary ideal $+S$" (see [Sh 108]; [Sh 88a]).

0.2 Definition. $I[\lambda]$ is a normal ideal but many times it has the form $\{A \subseteq \lambda : A \cap S \text{ non-stationary}\}$ and then S is the "bad" set of λ. This holds for $I[\lambda] \restriction \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ if $\lambda = \lambda^{<\kappa}$ or less (see [Sh 108], [Sh 88a]).

0.3 Claim. *If λ is regular, then $S = \{\delta < \lambda^+ : cf(\delta) < \lambda\}$ is the union of λ sets on each of which we have a square (see below) hence belongs to $I[\lambda]$, see [Sh 351, 4.1], [Sh:e, III,2.1?]. If $\lambda = \lambda^{<\kappa}$, then $\{\delta < \lambda^+ : cf(\delta) < \kappa\}$ is the union of λ sets on each of which we have a square (see [Sh 237e]), hence the set belongs to $I[\lambda]$. Moreover, if $\lambda > \aleph_0$ is regular and $\alpha < \lambda \Rightarrow cov(|\alpha|, \kappa, \kappa, 2) < \lambda$ then $\{\delta < \lambda : cf(\delta) < \kappa\} \in I[\lambda]$ (see [Sh 420, 2.8]). By Dzamonja, Shelah [DjSh 562] the same assumption gives $\{\delta < \lambda^+ : cf(\delta) < cf(\lambda)\}$ is the union of $\leq \lambda$ sets on each of which we have square. Also in [DjSh 562] there are results on getting squares with λ singular and results with an inaccessible instead of λ^+.*

0.4 Definition. $S \subseteq \mu$ has a square if we have $S^+, S \subseteq S^+ \subseteq \mu$ and $\langle C_\alpha : \alpha \in S^+ \rangle$ such that: C_δ is a closed subset of δ of order type $< \delta$, and $\alpha \in C_\beta \Rightarrow C_\alpha = \alpha \cap C_\beta$ and [α is a limit ordinal iff $\alpha = \sup(C_\alpha)$] for $\alpha \in S$; also if $\alpha \in S \Rightarrow \text{cf}(\alpha) \leq \kappa (< \kappa)$, we can add "$\text{otp}(C_\delta) \leq \kappa (< \kappa)$".

0.5 Related ideals [Sh 345a, 2.3,2.4], [Sh 371, 2.3,2.4,2.5,5.1,5.1A,5.2].

0.6 If $\kappa^+ < \lambda = \text{cf}(\lambda)$, then we can find a stationary

$$S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \text{cf}(\kappa)\}, S \in I[\lambda]$$

[Sh 420, 1.5] (somewhat more [Sh 420, 1.4]).

0.7 Negative consistency results: [Sh 108], ("GCH + the bad set for $\aleph_{\omega+1}$ is stationary") Magidor, Shelah [MgSh 204], Hajnal, Juhasz, Shelah [HJSh 249], consistency of $I[\lambda]$ large but stationary sets reflect [Sh 351].

0.8 On killing stationary sets by forcing [Sh 108], [Sh 88a, 18,19], [Sh 371, 2.4].

0.9 On consequences of pcf structure ([Sh 108], [Sh:g, Ch.VIII,§5?], [Sh 589, 5.17,5.18]), e.g.

(a) (GCH) the bad stationary subsets of $\aleph_{\omega+1}$ does not reflect ([Sh 108] or [Sh 88a]).

§1 Guessing clubs

1.1 Definition. Definition of ideals [Sh 365, 1.3,1.5,3.1]: definition of $g\ell$ [Sh 365, 2.1]: also [Sh 380, 1.8].

For example

Definition. For $\bar{C} = \langle C_\delta : \delta \in S\rangle, S \subseteq \lambda = \text{cf}(\lambda) > \aleph_0, C_\delta$ a club of δ:

$$\text{id}^b(\bar{C}) = \{A \subseteq \lambda : \text{ for some club } E \text{ of } \lambda \text{ for no } \delta \in S \cap A \cap E \text{ is } C_\delta \subseteq E\}$$

$$\text{id}^a(\bar{C}) = \{A \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ for no } \delta \in S \cap A \cap E \text{ is } \sup(C_\delta \backslash E) < \sup C_\delta\}$$

$$\text{id}_p(\bar{C}) = \{A \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ for no } \delta \in S \cap E \text{ is } \delta = \sup(E \cap \text{ nacc}(C_\delta))\}.$$

<u>1.2</u> Easy facts [Sh 365, 1.4,1.6].

<u>1.3</u> For $\lambda, S \subseteq \lambda$ stationary concerning the existence of $\bar{C} = \langle C_\delta : \delta \in S\rangle$ "guessing clubs of λ" [Sh 365, §2] (and [Sh:e, III,7.8A-G]).

- (a) If $\delta \in S \Rightarrow \text{cf}(\delta) < \mu$ for some $\mu < \lambda$, <u>then</u> we can find clubs C_δ for $\delta \in S$ such that $\text{id}^b(\langle C_\delta : \delta \in S\rangle)$ is a proper ideal (i.e. for every club E of λ for some $\delta, C_\delta \subseteq E$) by [Sh 365, 2.3(2)].
- (b) If $\lambda = \mu^+, \mu$ regular, $\delta(*) < \mu$, <u>then</u> for some stationary $S^* \subseteq \lambda$, there is a square $\bar{C} = \langle C_\alpha : \alpha \in S^*\rangle$ (so $\alpha \in S^* \Rightarrow C_\alpha \subseteq S^*, \beta \in C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta$) satisfying $\text{otp}(C_\alpha) \leq \delta(*)$ and $\text{id}^a(\langle C_\delta : \delta \in S^*, \text{otp}(C_\delta) = \delta(*)\rangle)$ is a proper ideal [Sh 365, 2.14(2)] (see part B here).
- (c) If $\lambda = \mu^+, \mu$ regular, $[\delta \in S \Rightarrow \text{cf}(\delta) = \mu], S \subseteq \lambda$ stationary, <u>then</u> we can find $\bar{C} = \langle C_\delta : \delta \in S\rangle, C_\delta$ a club of δ, $\text{otp}(C_\delta) = \mu, [\alpha \in \text{nacc}(C_\delta) \Rightarrow \text{cf}(\alpha) = \mu]$ and $\text{id}_p(\bar{C})$ a proper ideal (i.e. for every club E of λ for some $\delta, \delta = \sup(E \cap \text{ nacc}(C_\delta))$), [Sh 365, 2.3(1)], [Sh 413], [Sh 572, §3].

(d) If $[\lambda = \mu^+, \mu$ singular and: $\delta \in S \Rightarrow \text{cf}(\delta) = \text{cf}(\mu) > \aleph_0]$ or $[\lambda$ inaccessible and: $\delta \in S \Rightarrow \text{cf}(\delta) \in (\aleph_0, \delta)]$, then for some $\bar{C} = \langle C_\alpha : \alpha \in S \rangle$ we have: $\text{id}^a(\bar{C})$ is proper and for each $\delta \in S$ we have: $\langle \text{cf}(\alpha) : \alpha \in \text{nacc}(C_\delta) \rangle$ converges to $|\delta|$ (and is strictly increasing) [Sh 365, 2.6,2.7].

(e) If $S^* \subseteq \lambda$ is stationary and does not reflect outside itself and $S \subseteq \lambda$ is stationary, then for some $\bar{C} = \langle C_\delta : \delta \in S \rangle$ we have $\text{nacc}(C_\delta) \subseteq S^*$, and $\text{id}_p(\bar{C})$ is a proper ideal, [Sh 365, 2.13].

(f) Similar theorems with ideals [Sh 380, 1.7,2.4], [Sh 413, 1.11,1.12] other related ideals [Sh 380, 1.10].

(g) More in the places above and [Sh 413, 2.6,2.8,2.9] and [KjSh 449].

(h) Assume $\lambda = \text{cf}(\lambda), S \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \lambda\}$ is stationary and χ satisfies one of the following: $\lambda = \chi^+$ or $\chi = \min\{\chi < \lambda : (\exists \theta \leq \chi)\chi^\theta \geq \lambda\}$ or λ strongly inaccessible not Mahlo [???].
Then we can find $\langle C_\delta, h_\delta : \delta \in S \rangle$ such that: $C_\delta = \{\alpha_{\delta,\zeta} : \zeta < \lambda\}$ is a club of $\delta, \alpha_{\delta,\zeta}$ increasing in $\zeta, h_\delta : C_\delta \to \chi$ and for every club E of λ^+ for stationarily many $\delta \in S$, for each $i < \chi$

$$\{\zeta < \lambda : \alpha_{\delta,\zeta} \in E, \alpha_{\delta,\zeta+1} \in E \text{ and } h_\delta(\alpha_{\delta,\zeta}) = i\}$$

is a stationary subset of λ (see [Sh 413, §3], [Sh 572, §3]). If λ is a limit of inaccessibles, we can demand $\text{cf}(\alpha_{\delta,\zeta+1}) > \zeta$.

(i) If $\lambda, \bar{C} = \langle C_\delta : \delta \in S^+ \rangle$ is as in 0.1, $S \subseteq S^+, \sup\{|C_\alpha|^+ : \alpha \in S\} < \lambda$ then for some club E of $\lambda, \bar{C}' = \langle g\ell(C_\delta, E) : \delta \in S^+ \cap \text{acc}(E) \rangle$ is as in 0.1 and for every club $E_1 \subseteq E$ of λ for stationarily many $\delta \in S$ we have $\alpha \in C'_\delta \Rightarrow \sup(C'_\delta \cap \alpha) \leq \sup(E \cap \alpha)$.

(j) Assume $\lambda = \text{cf}(\lambda)$ and $S, S_\theta \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \lambda\}$ is stationary. Then we can find an S-club system $\bar{C} = \langle C_\delta : \delta \in S \rangle$ and $h : S \to \lambda$ such that for any club E of λ^+ for stationarily many $\delta \in S$ for every $i < \lambda$ the set $\text{nacc}(C_\delta) \cap h^{-1}(\{i\})$ is unbounded in δ (under reasonable assumption $|\{C_\delta \cap \alpha : \alpha \in \text{nacc}(C_\delta)\}| \subseteq \lambda$), see [Sh 413], 2.3.

1.4 On $\otimes_{\bar{C}}, \otimes^\kappa_{\bar{C}}$ for some S-club system [Sh 365, 2.12,2.12A,4.10] and a colouring theorem [Sh 365, 4.9] (see earlier [Sh 276]). Where, for λ a Mahlo cardinal,

$\otimes_{\bar{C}}$ $\bar{C}$ has the form $\langle C_\delta : \delta \in S \rangle, S \subseteq \lambda$ a set of inaccessibles, C_δ a club of δ such that: for every club E of λ for stationary many $\delta \in S, E \cap \delta \backslash C_\delta$ is unbounded in δ

and for $\kappa < \lambda$:

$\otimes^\kappa_{\bar C}$ $\bar C$ has the form $\langle C_\delta : \delta \in S^\lambda_{\rm in}\rangle, S^\lambda_{\rm in} = \{\mu < \lambda : \mu$ inaccessible$\}$, such that: for every club E of λ, for stationarily many $\delta \in S^\lambda_{\rm in} \cap$ acc(E), for no $\zeta < \kappa$ and $\alpha_\varepsilon \in S^\lambda_{\rm in}(\varepsilon < \zeta)$ is nacc$(E) \cap \delta \setminus \bigcup\limits_{\varepsilon<\zeta} C_{\alpha_\varepsilon}$ bounded in δ.

By [Sh 365, 4.9] if κ is a Mahlo cardinal and $\otimes^\kappa_{\bar C}$, then for some 2-place function c from κ to ω, for every pairwise disjoint $w_i \subseteq \kappa, |w_i| < \kappa$ for $i < \kappa$, and n, for some $i < j$, Rang$(c \restriction w_i \times w_j) \subseteq (n,\omega)$. By [Sh 365, 4.10B], $\otimes^2_{\bar C} \Leftrightarrow \otimes^{\aleph_0}_{\bar C}$, also $\otimes^2_{\bar C}$ is a strengthened form of "κ not weakly compact", which fails under mild conditions ([Sh 365, 4.10A]). See more in [Sh 365, 4.13].

1.5 id$_p(\bar C, \bar I)$ is decomposable [Sh 365, 3.2,3.3].

1.6 If $\kappa^+ < \lambda$, we can find $\langle \mathscr{P}_\alpha : \alpha < \lambda\rangle$ such that:
$\mathscr{P}_\alpha$ is a family of $< \lambda$ closed subsets of α,

$$[\beta \in \text{ nacc}(C) \ \& \ C \in \mathscr{P}_\alpha \Rightarrow C \cap \beta \in \mathscr{P}_\beta] \text{ and}$$

for every club E of λ for stationarily many $\alpha < \lambda$, there is $C \in \mathscr{P}_\alpha, \kappa =$ otp(C), $\alpha = \sup(C)$ and $C \subseteq E$ [Sh 420, 1.3] (we can replace κ by $\delta(*), |\delta(*)| = \kappa$).

1.7 More on 1.3(c) in [Sh 413, §3] and better in [Sh 572, §3].

1.8 If we want to preserve $\alpha \in$ nacc$(C_\beta) \cap$ nacc$(C_\beta) \cap$ nacc$(C_\gamma) \Rightarrow C_\beta \cap \alpha = C_\gamma \cap \alpha$ we can weaken the guessing to: $\forall$ club $E \exists^{\rm stat} \delta$ such that E is not disjoint to any interval of C_α. See the proof of [Sh 430, 6.2], [DjSh 562].

1.9 On ideals related to Jonsson algebras and guessing clubs: [Sh 380], [Sh 413, §1] (used in §8 here).

§2 Existence of lub of $\bar f = \langle f_\alpha : \alpha < \delta\rangle$ mod I where $f_\alpha \in {}^\kappa\mathrm{Ord}$, $\mathrm{cf}(\delta) > \kappa^+$: [Sh 68], [Sh 111], [Sh 282, 14], best [Sh 355, §1]

1.1 Definition. We say "f is a lub of $\langle f_\alpha : \alpha < \delta\rangle$ mod I" where I is an ideal on Dom I, f_α : Dom $I \to$ ordinals if $\bigwedge_{\alpha<\delta} f_\alpha \le f$, and $\bigwedge_{\alpha<\delta} f_\alpha \le f' \Rightarrow f \le f'$ mod I.

We say "f is an eub (exact upper bound) of $\langle f_\alpha : \alpha < \delta\rangle$ mod I" where I is an ideal on Dom(I), f_α : Dom$(I) \to$ ordinals, if $\bigwedge_{\alpha<\delta} f_\alpha \le_I f$ and if $g <_I \max\{f, 1\}$ then for some $\alpha < \delta$ we have $g \le_I f_\alpha$ (see [Sh 345a, 1.4(4)]); usually $\alpha < \beta \Rightarrow f_\alpha \le_I f_\beta$; "$f$ is an eub of $\langle f_\alpha : \alpha < \delta\rangle$ mod I" says more than "f is a lub of $\langle f_\alpha : \alpha < \delta\rangle$ mod I".

1.2 The trichotomy theorem on the existence of eub [Sh 355, 1.2,1.6] (slightly more [Sh 430, 6.1], on eub $\neq$ lub, see example [Sh 430, 6.1A]).
For example for I a maximal ideal on κ, $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \delta$, cf$(\delta) > \kappa^+$,
$\bar f = \langle f_\alpha/I : \alpha < \delta\rangle$ increasing, either $\bar f$ has a $<_I$-eub, or for some sequence
$\bar w = \langle w_i : i < \kappa\rangle$ of sets of ordinals, $|w_i| \le \kappa$ we have:

$$\bigwedge_{\alpha<\delta} \bigvee_{\beta<\delta} (\exists g \in \prod_{i<\kappa} w_i)[f_\alpha/I < g/I < f_\beta/I].$$

The cf$(\delta) > \kappa^+$ is necessary by [KjSh 673].

1.3 Definition Sufficient conditions for the existence of eub [Sh 355, 1.7],
[Sh 345a, 2.6]. For example if

$$gd_I(\bar f) =: \Big\{\alpha < \delta : \mathrm{cf}(\alpha) > \kappa \text{ and there is an unbounded } A \subseteq \delta \text{ and members } s_i \text{ of } I \text{ for } i \in A \text{ such that: } i \in A \ \&\ j \in A \ \&\ i < j \ \&\ \zeta \in \kappa \backslash s_i \backslash s_j \Rightarrow f_i(\zeta) \le f_j(\zeta)\Big\}$$

is a stationary subset of δ.

1.4 What is $Ch^{\mathfrak a}_N$ where for $N \prec (H(\lambda), \in)$: we define $Ch^{\mathfrak a}_N(\theta) = \sup(N \cap \theta)$ for $\theta \in \mathfrak a$ [Sh 345a, 3.5], [Sh 355, 3.4(stationary)], [Sh 371, 1.2,1.3,1.4] more [Sh 400, 3.3A,5.1A] and [Sh 430, §6].

1.5 On the good/bad/chaotic division. For $\bar{f}$ a $<_I$-increasing sequence of functions from κ to ordinals, we have a natural division of $\ell g(\bar{f})$, for example to $gd_I(\bar{f})$ (see 1.3 above),

$$ch(\bar{f}) = \Big\{\delta < \ell g(\bar{f}) : \text{for some ultrafilter } D \text{ on } \ell g(\bar{f}) \text{ disjoint to } I \text{ and } w_i \subseteq \text{ ordinals for } i \in \text{ Dom } I, |w_i| \le |\text{Dom}(I)| \text{ and } \bigwedge_{i<\ell g(\bar{f})} \bigvee_{j<\ell g(\bar{f})} (\exists g \in \Pi w_i)[f_i \le_D g \le_D f_j]\Big\}$$

and $bd_I(\bar{f}) = \ell g(\bar{f}) \backslash gd_I(\bar{f}) \backslash ch_I(\bar{f})$. Note: for every $\delta < \ell g(\bar{f})$ of uncountable cofinality there is a club C of δ such that $\delta \in gd_I(\bar{f})$ & $\alpha \in C$ & $\text{cf}(\alpha) > \kappa \Rightarrow \alpha \in gd_I(\bar{f})$ and $\delta \in ch_I(\bar{f}) \Rightarrow C \subseteq ch_I(\bar{f})$; also for $bd_I(\bar{f})$ to be non-trivial, $\ell g(\bar{f})$ should not be so small among the alephs.
There are connections to NPT (see §12) and $I[\text{cf}(\ell g(\bar{f}))]$ (see §1) (and consistency of the existence of counterexamples; see [Sh 108], [MgSh 204], [Sh 355, 1.6], [Sh 523]).

1.6 Problem: Is the following consistent: $\{\delta < \aleph_{\omega+1} : \text{cf}(\delta) = \aleph_2\} \notin I[\aleph_{\omega+1}]$ or $2^{\aleph_0} < \aleph_\omega$ and $\{\delta < \aleph_{\omega+1} : \text{cf}(\delta) = (2^{\aleph_0})^+\} \notin I[\aleph_{\omega+1}]$ (also for inaccessibles) or $\bar{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle, f_\alpha \in \prod\limits_{n<\omega} \aleph_n, ch_{J^{bd}_\omega}(\bar{f}) \cap \{\delta < \aleph_{\omega+1} : \text{cf}(\delta) = \aleph_2\}$ stationary or $(\forall S)[S \in I[\aleph_2]$ & $\bigwedge\limits_{\delta \in S} \text{cf}(\delta) = \aleph_1 \Rightarrow S$ not stationary]?

1.7 More on §2, see in [Sh 497] (in universes without full choice).

1.8 See more in [Sh 506] for generalization to the case in $\text{cf}(\delta) \le |\text{Dom } I|$.
On existence of eub see [Sh 506, 3.10] and [Sh 589, 6.4].

1.9 Assume $\lambda = \text{cf}(\lambda) \ge \mu > 2^\kappa, f_\alpha \in {}^\kappa\text{Ord}$ for $\alpha < \kappa$. Then for some β^*_i $(i < \kappa)$ and $w \subseteq \kappa$ we have: $i \in w \Rightarrow \text{cf}(\beta^*_i) > 2^\kappa$ and for every $f \in \prod\limits_{i \in w} \beta^*_i$ for unboundedly many $\alpha < \lambda$ we have $i \in w \Rightarrow f(i) < f_\alpha(i) < \beta^*_i$ and $i \in \kappa \backslash w \Rightarrow f_\alpha(i) = \beta^*_i$; [Sh 430, 6.6D] (slightly more general); more detailed proof [Sh 513, 6.1], more variants [Sh 620, §7].

1.10 On decreasing sequences see [Sh 589, 6.1,6.2].

§3 Uncountable cofinality and $\aleph_1$-complete filters and products: [Sh 71], [Sh 111], [Sh 256]

<u>2.1</u> Assume $\langle \lambda_i : i \le \kappa \rangle$ is an increasing continuous sequence of singulars, $\aleph_0 < \kappa = \mathrm{cf}(\kappa) < \lambda_i$. Let $\lambda = \lambda_\kappa$. If $\{i < \kappa : \mathrm{pp}(\lambda_i) = \lambda_i^+\}$ is a stationary subset of κ, <u>then</u> $\mathrm{pp}(\lambda) = \lambda^+$, [Sh 355, 2.4(1)].

Moreover, $\mathrm{pp}(\lambda_\kappa)$ is bounded by $\lambda_\kappa^{+\|h\|}$ where $\mathrm{pp}(\lambda_i) = \lambda_i^{+h(i)}$ hence we have a bounding $\mathrm{pp}(\lambda)$ in many cases [Sh 355, 2.4], [Sh 371, 1.10].

<u>2.2</u> Definition of various ranks and niceness of filters in [Sh 386, 1.1,1.2,1.4,3.12] (more generally on pair (t, D) or for $D \in \mathrm{Fil}(e, y)$ see [Sh 410, §5] and [Sh 420, §3,§4,§5]). For $\kappa = \mathrm{cf}(\kappa) > \aleph_0, D$ a normal filter on κ and $f \in {}^\kappa\mathrm{Ord}$ let $\mathrm{rk}^2(f, D)$ be $\le \alpha$ iff for every $A \in D^+$ and $g <_{D^+ + A} f$ for some normal filter, $D_1 \supseteq D + A$ we have $\mathrm{rk}^2(g, D_1) \le \beta$ for some $\beta < \alpha$.
D is nice if $f \in {}^{\mathrm{Dom}(D)}\mathrm{Ord} \Rightarrow \mathrm{rk}^2(f, D) < \infty$.

<u>2.3</u> If for any $A \subseteq 2^{\aleph_1}$ in $K[A]$, there are Ramsey cardinals (or suitable Erdös cardinals which occurs if cardinal arithmetic is not trivial, essentially by Dodd and Jensen [DJ1]), <u>then</u> every normal filter on ω_1 is nice [Sh 386, 1.7,1.13]; more in [Sh 386, §1], [Sh 420, §3,§4,§5].

<u>2.4</u> [Sh 386, 2.2,2.2A,2.4,2.7], [Sh 420, §4]
$A_e(f)$ [Sh 386, 3.3].

<u>2.5</u> Rank, basic properties:
[Sh 386, 2.3,2.4,2.8,2.9,2.10,2.11,2.12,2.14,2.21,3.4,3.8].

<u>2.6</u> Rank, connection to forcing: [Sh 386, Definition 2.6 (E_p^t),2.6A,2.7A], [Sh 420, §3].

<u>2.7</u> Rank, relation with T_D [Sh 386, 2.15,2.16,2.17,2.18,2.19,2.20,2.22].

<u>2.8</u> Ranks-going down: ranks when we divide ω_1, [Sh 386, 3.2] each f successor [Sh 386, 3.6], each f limit [Sh 386, 3.7].

<u>2.9</u> Rank, getting κ-like reduced products [Sh 386, 3.10,3.11,3.11A].

<u>2.10</u> Generic ultrapower with all $\kappa > \beth_2(\aleph_1)$ represented: [Sh 333, 1.3], just for one [Sh 333, 1.4] (earlier [Sh 111]).

<u>2.11</u> Ranks are $< \infty$ [Sh 386, 3.13-18].

2.12 Preservative pairs (see 2.15), definition and basic properties [Sh 386, 4.15].

2.13 Specific functions are preservative:
[Sh 386, 4.6] (H_s = successor), [Sh 386, 5.8] (H^{ia} = next inaccessible), [Sh 386, 5.9] ($H^{\epsilon-m}$ = next ϵ-Mahlo).

2.14 The class of preservative pairs is closed under:

(1) $H^*(i)$ iterating H i times [Sh 386, 4.7,4.8,4.9]
(2) composition [Sh 386, 4.10]
(3) $\sup_{n<\omega} H^n$ [Sh 386, 4.11]
(4) iterating α times, $\alpha < \omega_1$ [Sh 386, 4.12]
(5) more [Sh 386, 4.13]
(6) induction [Sh 333, §2].

2.15 Preservative pairs are bounds on cardinal exponentiation
[Sh 386, 5.1,5.2,5.3].

2.16 If $\mathrm{rk}^2_E(f) = \mathrm{rk}^3_E(f) = \lambda$ inaccessible, then modulo (fil E) almost every $f(i)$ is inaccessible [Sh 386, 5.7].

2.17 Generalizing normal filters and then ranks [Sh 410, §5],
[Sh 420, §3,§4,§5].

2.18 For set theory with weak choice much remains (see [Sh 497]).

§3A Products, $T_D(f), \mathbf{U}$

We deal with computing $T_D(f), \mathbf{U}_D(f)$ and reduced products $\prod\limits_{i<\kappa} f(i)/D$ from pcf, mainly when $(\forall i)[f(i) > 2^\kappa]$ see [Sh 506, §3], [Sh 589, §1], [Sh 589, §4] on T_D earlier, Galvin Hajnal [GH].

3.1 Definition. 1)$T_D(f) =:$ Min$\{|F| : F \subseteq \prod\limits_i (f(i)+1)$ and $f \neq g \in F \Rightarrow f \neq_D g$ (i.e. $\{i : f(i) \neq g(i)\} \in D$) and F is maximal with respect to those properties$\}$.
$T_\Gamma(f) = \sup\{T_D(f) : D \in \Gamma\}$ for Γ set of filters on Dom(f), similarly for Γ set of ideals and naturally $T_\Gamma(\lambda)$.
2) $\mathbf{U}_D(f, < \theta) = \{Min\{|\mathscr{A}| : \mathscr{A} \subseteq \pi[f(i)]^{<\theta}$, each of cardinality $< \theta$ such that for every $g^i \in {}^\kappa\mathrm{Ord}, g <_D f$ for some $\bar A \in \mathscr{A}$ we have $\{i < \kappa : g(i) \in A_i\} \neq \emptyset$ mod D. If $\theta = \kappa^+$ we may omit it, (note: if cf$(\theta) > \kappa$ we can replace $\bar A$ by $\bigcup\limits_{i<\kappa} A_i$. [Saharon add: [Sh 430], [Sh 552].

<u>3.2</u> If $\lambda > 2^{<\theta}, \theta \geq \sigma =$ cf$(\sigma) > \aleph_0$ and $\Gamma = \Gamma(\theta, \sigma)$ (the set of σ-complete ideals on a cardinal $< \theta$) we have

$$T_\Gamma(\lambda) = \text{ cov}(\lambda, \theta, \theta, \sigma)$$

(the latter can be computed from case of pp$_\Gamma$); [Sh 355, 5.9,p.94].
If $\theta^\kappa < \mathbf{U}_D(\lambda)$, <u>then</u> $T_D(f) = \mathbf{U}_D(f)$.

<u>3.3</u> A pcf characterization when $\lambda \leq T_D(f)$ holds, under $2^{\text{Dom}(D)} < \text{Min}_i f(i)$ and $(\forall \alpha)(\alpha < \lambda \rightarrow |\alpha|^{\aleph_0} < \lambda)$, see [Sh 506, 3.15], (note if $A_n \in D, \bigcap\limits_{n<\omega} A_n = \emptyset$, then $T_D(f) = T_D(f)^{\aleph_0}$).
See more in [Sh 506, §3].

<u>3.4</u> Assume D is a filter on $\kappa, \mu =$ cf$(\mu) > 2^\kappa, f \in {}^\kappa\mathrm{Ord}$ and: D is $\aleph_1$-complete or $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$. Then $(\exists A \in D^+)T_{D+A}(f) \geq \mu$ <u>iff</u> for some $A \in D^+$ and $\bar\lambda \leq_{D+A} f$ we have $\prod\limits_{i<\kappa} \lambda_i/(D+A)$ has true cofinality μ (for approximations see [Sh 506, §3], proof [Sh 589, 1.1], note $\Leftarrow$ is trivial). This is connected to the problem of the depth of products (e.g. ultraproducts) of Boolean Algebra.

<u>3.5</u> If $2^{2^\kappa} \leq \mu < T_D(\bar\lambda)$ and $\mu^{<\theta} - \mu$, <u>then</u> for some θ-complete ideal $E \subseteq D$ we have $\mu < T_E(\bar\lambda)$, [Sh 506, 3.20].

3.6 On $\prod_{i<\kappa} \lambda_i/D$ see [Sh 506, 3.1-3.9B], essentially this gives full pcf characterization when it is $> 2^\kappa$. In particular for an ultrafilter D on κ with regularity θ (i.e. not θ-regular but σ-regular for $\sigma < \lambda$) and $\lambda_i > 2^\kappa$, we have

$$\prod_{i<\kappa} \lambda_i/D = (\sup\{\text{tcf} \prod_{i<\kappa} \lambda'_i/D : 2^\kappa < \lambda'_i = \text{cf}(\lambda'_i) \leq \lambda_i\})^{<\text{reg}(D)}$$

(see mainly [Sh 506, 3.9]).

3.7 Assume D is an ultrafilter on κ and θ is the regularity of D (i.e. minimal θ such that D is not θ-regular). Then every $\lambda = \lambda^\theta > 2^\kappa$ can be represented as $\prod_{i<\kappa} \lambda_i/D$. (Note $\lambda = \lambda^{<\theta}$ is necessary) (see [Sh 589, §6]).

3.8 Assume $\theta < \kappa, J = [\kappa]^{<\theta}$ and $\lambda > \kappa^\theta$ then

$$\begin{aligned} T_J(\lambda) = \sup\Big\{ &\text{tcf}(\prod_{\substack{n<n_i\\ i<\kappa}} \lambda_{i,n}/J) : n_i < \omega, \lambda_{i,n} \text{ regular } \in [\kappa^\theta, \lambda) \\ &\text{and } J = \text{ an ideal on } \bigcup_{i<\kappa} \{i\} \times n_i, \text{ and} \\ &A \subseteq \kappa \ \& \ |A| \geq \theta \Rightarrow \bigcup_{i\in A} \{i\} \times n_i \in J^+ \text{ and} \\ &\prod_{\substack{n<n_i\\ i<\kappa}} \lambda_{i,n}/J \text{ has true cofinality} \Big\}. \end{aligned}$$

This is just a case of the "θ-almost disjoint family $\subseteq [\lambda]^\kappa$" problem as clearly $T_J(\lambda) = \sup\{\mathscr{A} : \mathscr{A} \subseteq [\lambda]^\kappa$ is θ-almost disjoint; i.e. $A \neq B \in \mathscr{A} \Rightarrow |A \cap B| < \kappa\}$. See [Sh 410, §6].

3.9 If $\lambda \geq \kappa > \beth_\omega(\theta)$ then in 3.7, $T_J(\lambda) = \lambda$. (See [Sh 460]).

3.10 ([Sh 430, 1.2]). Assume $\lambda > \mu = \text{cf}(\mu) > \theta > \aleph_0$ and $\text{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$. Then the following are equal

$$\lambda(0) = \text{Min}\Big\{\kappa : \text{if } \mathfrak{a} \subseteq \text{ Reg } \cap \lambda \backslash \mu, |\mathfrak{a}| \le \theta \text{ then we can partition } \mathfrak{a} \text{ to } \langle \mathfrak{a}_n : n < \omega \rangle$$

$$\text{such that } n < \omega \ \& \ \mathfrak{b} \subseteq \mathfrak{a}_n \ \& \ |\mathfrak{b}| \le \aleph_0 \Rightarrow \max \text{pcf}(\mathfrak{b}) \le \kappa$$

$$\text{and } [\mathfrak{a}_n]^{\le \aleph_0} \text{ is included in the ideal generated by}$$

$$\{\mathfrak{b}_\theta[\mathfrak{a}_n] : \theta \in \mathfrak{d}_n\} \text{ for some } \mathfrak{d} \subseteq \kappa^+ \cap \text{ pcf}(\mathfrak{a}_n) \text{ of cardinality } < \mu \Big\}$$

$$\lambda(1) = \text{Min}\Big\{|\mathscr{P}| : \mathscr{P} \subseteq [\lambda]^{<\mu} \text{ and for every } A \in [\lambda]^{\le \theta}$$

$$\text{for some partition } \langle A_n : n < \omega \rangle \text{ of } A \text{ we have :}$$

$$\langle \mathscr{P}_n : n < \omega \rangle, \mathscr{P}_n \subseteq \mathscr{P}, |\mathscr{P}_n| < \mu, \mu > \sup_{B \in \mathscr{P}_n} (B)$$

$$\text{and } n < \omega \ \& \ a \in [A_n]^{\aleph_0} \Rightarrow (\exists A \in \mathscr{P}_n)(a \subseteq A)\Big\}.$$

§4 PCF THEORY: [Sh 68], [Sh:b, Ch.XIII,§5,§6], [Sh 282], [Sh 345]

$\mathfrak{a}$ denotes a set of regulars $> |\mathfrak{a}|$ (except for a generalization in [Sh 371, §3]).

For a partial order P let $\mathrm{cf}(P) = \mathrm{Min}\{|A| : A \subseteq P, \bigwedge_{p\in P} \bigvee_{q\in A} q \le q\}$. We say P has true cofinality if it has a well ordered cofinal subset whose cofinality is called $\mathrm{tcf}(P)$ (equivalently - a linearly ordered cofinal subset).

4.1 $J_{<\lambda}[\mathfrak{a}], J_{\le\lambda}[\mathfrak{a}]$ [Sh 345a, 1.2(2),(3)], also [Sh 345a, 3.1], [Sh 371, 3.1]. For example

$$J_{<\lambda}[\mathfrak{a}] =: \{\mathfrak{b} \subseteq \mathfrak{a} : \text{ for every ultrafilter } D \text{ on } \mathfrak{b},\ \mathrm{cf}(\Pi\, \mathfrak{b}/D) < \lambda\}.$$

$$J_{\lambda}[\mathfrak{a}] =: \{\mathfrak{b} \subseteq \mathfrak{a} : \text{ for every ultrafilter } D \text{ on } \mathfrak{b},\ \mathrm{tcf}(\Pi\, \mathfrak{b}/D) \ne \lambda\}.$$

4.2 Definition of variants of pcf [Sh 345a, 1.2(1),(2)], [Sh 371, 3.1] for example

$$\mathrm{pcf}(\mathfrak{a}) = \{\mathrm{cf}(\Pi\, \mathfrak{a}/D) : D \text{ an ultrafilter on } \mathfrak{a}\}.$$

For given cardinals $\theta > \sigma$ let

$$\mathrm{pcf}_{\Gamma(\theta,\sigma)}(\mathfrak{a}) = \{\mathrm{tcf}(\Pi\, \mathfrak{b}/J : \mathfrak{b} \subseteq \mathfrak{a}, |\mathfrak{b}| < \theta, J \text{ is a } \sigma\text{-complete ideal on } \mathfrak{b} \text{ and } \Pi\, \mathfrak{b}/J \text{ has true cofinality}\},$$

$\Gamma(\theta)$ means $\Gamma(\theta^+, \theta)$.

4.3 Trivial properties [Sh 345a, 1.3,1.4].

4.4 Basic properties [Sh 345a, 1.5,1.8,2.6,2.8,2.10,2.12].

4.5 $|\mathrm{pcf}(\mathfrak{a}) \le 2^{|\mathfrak{a}|}$ [Sh 345a, 1.8(5)], $\mathrm{pcf}(\mathfrak{a})$ has a last member [Sh 345a, 1.9] also if $|\mathfrak{a} \cup \mathfrak{b}| < \min(\mathfrak{a} \cup \mathfrak{b})$ then $(\mathrm{pcf}(\mathfrak{a})) \cap (\mathrm{pcf}(\mathfrak{b}))$ has a last member actually $|\mathfrak{a}| < \min(\mathfrak{a}), |\mathfrak{b}| < \min(\mathfrak{b})$ suffices (by [Sh 430, 6.4A], can take intersections of many $\mathfrak{a}_i$).

4.6 If D, D_i $(i < \kappa < \min(\mathfrak{a}))$ are filters on $\mathfrak{a}, E$ a filter on κ,

$$D = \{\mathfrak{b} \subseteq \mathfrak{a} : \{i : \mathfrak{b} \in D_I\} \in E\} \text{ and } \lambda_i = \mathrm{tcf}(\Pi\, \mathfrak{a}/D_i)$$

(well defined) then
$\mathrm{tcf}(\Pi\,\mathfrak{a}/D_i)$, and $\mathrm{tcf}(\prod\limits_{i<\kappa}\lambda_i/E)$ are equal [Sh 345a, 1.10]. Moreover, $\bigwedge\limits_i \kappa < \lambda_i$ is enough [Sh 345a, 1.11].
(And see more in [Sh 410, 3.3,3.6], generalization [Sh 506, 1.10]).

4.7 (Repeating 1.4) What is $Ch^{\mathfrak{a}}_N$ (where $N \prec (\mathscr{H}(\chi), \in)), CH^{\mathfrak{a}}_N(\theta) = \sup(N \cap \theta)$ for $\theta \in \mathfrak{a}$) [Sh 345a, 3.4], [Sh 355, 3.4], "stationary $F \subseteq \Pi\,\mathfrak{a}$" [Sh 371, 1.2,1.3,1.4], more in [Sh 430, §6]. Mirna?

4.8 $\mathrm{cf}(\Pi\,\mathfrak{a}) = \max \mathrm{pcf}(\mathfrak{a})$ [Sh 355, 3.1,more 3.2], [Sh 345a, 3.4], other representation [Sh 506].

4.9 There is a generating sequence $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$; i.e. $J_{\leq\theta}[\mathfrak{a}] = J_{<\theta}[\mathfrak{a}] + \mathfrak{b}_\theta[\mathfrak{a}]$, so $J_{<\lambda}[\mathfrak{a}]$ is the ideal on $\mathfrak{a}$ generated by $\{\mathfrak{b}_\theta[\mathfrak{a}] : \theta < \lambda\}$ and $\Pi\,\mathfrak{b}_\theta[\mathfrak{a}]/J_{<\theta}[\mathfrak{a}]$ has true cofinality θ and $J_\lambda[\mathfrak{a}]$ is the ideal on $\mathfrak{a}$ generated by $\{\mathfrak{b}_\theta[\mathfrak{a}] : \theta < \lambda\} \cup \{\mathfrak{a}\backslash\mathfrak{b}_\lambda[\mathfrak{a}]\}$; [Sh 371, 2.6] also [Sh 345a, 3.1] + [Sh 420, §1], more in [Sh 400, 4.1A]; nice good cofinal $\bar f$: [Sh 345a, §3], [Sh 355, 3.4A], [Sh 371, 1.2,1.3,1.4], [Sh 400, 4.1A(2)]. Another representation is included in [Sh 506] (see 1.8 on the framework and 4.23); it uses 0.6 from [Sh 420].

4.10 If $\mathfrak{b} \subseteq \mathfrak{a}$ and $\mathfrak{c} = \mathrm{pcf}(\mathfrak{b})$, then for some finite $\mathfrak{d} \subseteq \mathfrak{c}, \mathfrak{b} \subseteq \bigcup\limits_{\theta\in\mathfrak{d}} \mathfrak{b}_\theta[\mathfrak{a}]$ see [Sh 345a, 3.2(5)].

4.11 Cofinality sequence [Sh 345a, 3.3], [Sh 371, 2.1], more in the proof of [Sh 400, 4.1].

4.12 $\bar f$ is x-continuous (nice) [Sh 345a, 3.3,3.5,3.8(1),(2)].

4.13 For a discussion of when $\mathfrak{a}$ has a generating sequence which is smooth and/or closed [Sh 345a, 3.6,3.8(3)], [Sh 400, 4.1A(4)]; *smooth* means $\mu \in \mathfrak{b}_\lambda[\mathfrak{a}] \Rightarrow \mathfrak{b}_\mu[\mathfrak{a}] \subseteq \mathfrak{b}_\lambda[\mathfrak{a}]$, *closed* means $\mathrm{pcf}(\mathfrak{b}_\lambda[\mathfrak{a}]) = \mathfrak{b}_\lambda[\mathfrak{a}]$. If for example $|\mathrm{pcf}(\mathfrak{a})| < \min(\mathfrak{a})$ we can have both [Sh 345a, 3.8] and more, then we can use the "pcf calculus" style of proof. Proof in this style can be carried generally complicated a little, as done in [Sh 430, 6.7-6.7E] (particularly 6.7C(3)), on a generalization see [Sh 506].

4.14 If $\lambda = \max \mathrm{pcf}(\mathfrak{a})$, and $\mu =: \sup(\lambda \cap \mathrm{pcf}(\mathfrak{a}))$ is singular, then for $\mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{a})$ unbounded in $\mu, \mathrm{tcf}(\Pi\,\mathfrak{c}/J^{bd}_\mu) = \lambda$ [Sh 345a, 3.7], [Sh 371, 2.10(2)], where for A a set of ordinals, $J^{bd}_A = \{B \subseteq A : \sup(B) < \sup(A)\}$.

4.15 If $\lambda \in \text{pcf}(\mathfrak{a})$, then for some $\mathfrak{b} \subseteq \mathfrak{a}$ we have: $\lambda = \max \text{pcf}(\mathfrak{b})$ and $\lambda \cap \text{pcf}(\mathfrak{b})$ has no last element and $\lambda \notin \text{pcf}(\mathfrak{a} \backslash \mathfrak{b})$; see [Sh 371, 2.10(1)].

4.16 If $\forall \mu[\mu < \lambda \Rightarrow \mu^{<\kappa} < \lambda]$, then $J_{<\lambda}[\mathfrak{a}]$ is κ-complete [Sh 371, 1.6(1)].

4.17 Localization: if $\lambda \in \text{pcf}(\mathfrak{b}), \mathfrak{b} \subseteq \text{pcf}(\mathfrak{a})$ (so we assume just $|\mathfrak{b}| < \min(\mathfrak{b})$), then for some $\mathfrak{c} \subseteq \mathfrak{b}$ we have $|\mathfrak{c}| \leq |\mathfrak{a}|$ and $\lambda \in \text{pcf}(\mathfrak{c})$, [Sh 371, 3.4].

Also if $\lambda \in \text{pcf}_{\sigma\text{-complete}}(\mathfrak{b}), \mathfrak{b} \subseteq \text{pcf}(\mathfrak{a})$ then for some $\mathfrak{c} \subseteq \mathfrak{b}$, we have $|\mathfrak{c}| \leq |\mathfrak{a}|$ and $\lambda \in \text{pcf}_{\sigma\text{-complete}}(\mathfrak{a})$, see [Sh 430, 6.7F(4),(5)].

4.18

(a) $\text{pcf}(\mathfrak{a})$ cannot contain an interval of Reg (= the class of regulars) of cardinality $|\mathfrak{a}|^{+4}$.

In fact:

(b) for no $\mathfrak{a}$ and χ is $\{i < |\mathfrak{a}|^{+4} : \chi^{+i+1} \in \text{pcf}(\mathfrak{a})\}$ unbounded in $|\mathfrak{a}|^{+4}$.
[Why? If so, there is $\lambda \in \text{pcf}((\chi, \chi^{+|\mathfrak{a}|^{+4}}) \cap \text{pcf}(\mathfrak{a}))$ such that $\lambda > \chi^{|\mathfrak{a}|^{+4}}$, hence by localization for some $\mathfrak{c} \subseteq (\chi, \chi^{+|\mathfrak{a}|^{+4}}) \cap \text{pcf}(\mathfrak{a})$ of cardinality $\leq |\mathfrak{a}|$ we have $\lambda \in \text{pcf}(\mathfrak{c})$, hence for some limit ordinal $\delta < |\mathfrak{a}|^{+4}$, $\text{pp}_{|\mathfrak{a}|}(\chi^{+\delta}) \geq \lambda \geq \chi^{+|\mathfrak{a}|^{+4}}$ and we get a contradiction by [Sh 400, §4].]

4.19 Defining (μ, θ, σ)- inaccessibility [Sh 410, 3.1,3.2].

4.20 On $\text{pcf}(\mathfrak{b})$ for $\mathfrak{b} \subseteq \text{pcf}(\mathfrak{a}), |\mathfrak{b}| < \min(\mathfrak{b})$ or even with no inaccessible accumulation points, see [Sh 345a, 1.12], [Sh 371, §3], mainly: having $b^*_\lambda[\mathfrak{a}] \subseteq \text{pcf}[\mathfrak{a}]$.

4.21 Uniqueness of $\bar{f}$ ($<_J$-increasing cofinal) [Sh 345a, 2.7,2.10].

4.22 If $J = J_{<\lambda}[\mathfrak{a}], \lambda = \text{tcf}(\Pi\, \mathfrak{a}/J), \mathfrak{a} = \bigcup_{i<\alpha} \mathfrak{a}_i$, then for some finite $\mathfrak{b}_i \subseteq \text{pcf}(\mathfrak{a}_i) (i < \alpha)$ we have $\lambda = \max \text{pcf}(\bigcup_{i<\alpha} \mathfrak{b}_i)$, and for $w \subseteq \alpha : \max \text{pcf}(\bigcup_{i\in w} \mathfrak{b}_i) < \lambda \Leftrightarrow (\bigcup_{i \in w} \mathfrak{a}_i) \in J$ [Sh 371, §1], more in [Sh 430, §6].

4.23 If $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle, I^*$ a weakly θ-saturated ideal on κ (see below) $\theta = \text{cf}(\theta) < \lambda_i$, then the pcf analysis, e.g. from 4.9 holds for $\bar{\lambda}$ when we restrict ourselves to ideals on κ extending I^* (see [Sh 506, §1,§2]).
E.g. θ can play the role of $\kappa = \text{Dom}(I)$ if I is weakly θ-saturated, i.e.

$(*)_{I,\theta}$ there is no division of κ to θ set none of which is in I.

4.24 If $|\mathfrak{a}| < \min(\mathfrak{a}), \aleph_0 \le \sigma = \mathrm{cf}(\sigma)$, then for some $\alpha < \sigma$ and $\lambda_\beta, \mathfrak{a}_\beta$ $(\beta < \alpha)$ we have

(i) $\mathfrak{a} = \bigcup_{\beta<\alpha} \mathfrak{a}_\beta$,

(ii) $\lambda_\beta = \max \mathrm{pcf}(\mathfrak{a}_\beta)$

(iii) $\lambda_\beta \notin \mathrm{pcf}(\mathfrak{a} \backslash \mathfrak{a}_\beta)$ and

(iv) $\lambda_\beta \in \mathrm{pcf}_{\sigma-\mathrm{com}}(\mathfrak{a}_\beta)$.

[Why? We prove this by induction on max pcf($\mathfrak{a}$), hence by the induction hypothesis we can ignore (iii) as we can regain it, now let

$$J = \{\mathfrak{b} \subseteq \mathfrak{a} : \text{we can find } \alpha < \sigma, \langle \mathfrak{a}_\beta : \beta < \alpha \rangle \text{ such that } \mathfrak{b} = \bigcup_{\beta<\alpha} \mathfrak{a}_\beta \text{ and } (ii), (iv) \text{ above}\}.$$

Clearly J is a family of subsets of $\mathfrak{a}$, includes the singletons, is closed under subsets and under unions of $< \sigma$ members. If $\mathfrak{a} \in J$ we are done. If not, choose $\mathfrak{c} \subseteq \mathfrak{a}$ such that $\mathfrak{c} \notin J$ and (under these restrictions) $\lambda_{\mathfrak{c}} =: \max \mathrm{pcf}(\mathfrak{c})$ is minimal. Now by the minimality of $\lambda_{\mathfrak{c}}$, $J_{<\lambda_{\mathfrak{c}}}[\mathfrak{a}] \subseteq J$ so $\mathfrak{b}_{\lambda_{\mathfrak{c}}}[\mathfrak{a}]$, satisfies the requirement for $\mathfrak{b} \in J$ (with $\alpha = 1$), contradiction].

4.25 See more in 6.18 and [Sh 497] and particularly [Sh 513].

4.26 If $\lambda = \max \mathrm{pcf}(\mathfrak{b})$ and $\lambda \cap \mathrm{pcf}(\mathfrak{b})$ has no last element (see 4.15) and $\mu < \sup(\lambda \cap \mathrm{pcf}(\mathfrak{b})$, then for some $\mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{b}) \backslash \mu$ of cardinality $\le |\mathfrak{b}|$ we have $\lambda = \max \mathrm{pcf}(\mathfrak{c})$ and $\theta \in \mathfrak{c} \Rightarrow \max \mathrm{pcf}(\theta \cap \mathfrak{c}) < \theta$ (see [Sh 413], **?**, 1.5(2), an ex.).
—> scite{2.4A} undefined

§5 Representation and pp

<u>5.1</u> Definition of pp and variants [Sh 355, 1.1]. For λ singular

$$\begin{aligned}\mathrm{pp}_\theta(\lambda) = \sup\{\mathrm{tcf}(\Pi\,\mathfrak{a}/J) : &\ \mathfrak{a} \text{ is a set of } \leq \theta \text{ regular cardinals,}\\ &\text{unbounded in } \lambda, J \text{ an ideal on } \mathfrak{a} \text{ including } J^{bd}_{\mathfrak{a}}\\ &\text{and } \Pi\,\mathfrak{a}/J \text{ has true cofinality}\},\end{aligned}$$

$\mathrm{pp}^+(\lambda)$ is the first regular without such a representation

$$\mathrm{pp}(\lambda) = \mathrm{pp}_{\mathrm{cf}(\lambda)}(\lambda),$$

$\mathrm{pp}_\Gamma(\lambda)$ means we restrict ourselves to J satisfying Γ,

$$\mathrm{pp}^*_I(\lambda) = \mathrm{pp}_{\{I\}}(\lambda)$$

and

$$\mathrm{pp}_I(\lambda) = \sup\{\mathrm{pp}^*_J(\lambda) : J \text{ an ideal extending } I\},$$

$\lambda =^+ \mathrm{pp}(\lambda)$ means more than equality; the supremum in the right hand side is obtained if it is regular.

<u>5.2</u> Downward closure:

If $\lambda = \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i/I), \lambda_i = \mathrm{cf}(\lambda_i) > \kappa$, and $\kappa < \lambda' = \mathrm{cf}(\lambda') < \lambda$, <u>then</u> for some λ'_i we have $\kappa \leq \lambda'_i = \mathrm{cf}(\lambda'_i) < \lambda_i$ and $\lambda' = \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda'_i/I)$, moreover $\mathrm{tlim}_I\lambda_i = \mu < \lambda' < \lambda \Rightarrow \mathrm{tlim}_I\lambda'_i = \mu$ and $\lambda' = \mathrm{tcf}(\Pi\,\lambda'_i, \leq_I)$ is exemplified by μ^+-free $\bar{f}$ which means: if $w \subseteq \lambda'$ & $|w| \leq \mu$, then for some $\langle s_\alpha : \alpha \in w\rangle, s_\alpha \in I$ and for each $i < \kappa, \langle f_\alpha(i) : \alpha \in w, i \notin s_\alpha\rangle$ is without repetition, in fact we get "strictly increasing." [Sh 355, 1.3,1.4,2.3] more [Sh 400, 4.1] a generalization [Sh 506, 3.12].

<u>5.3</u> If $\lambda > \kappa \geq \mathrm{cf}(\lambda), I$ an ideal on κ, κ is an increasing union of $\mathrm{cf}(\lambda)$ members of I, <u>then</u> $\{\mathrm{tcf}(\prod_{i<\kappa} \lambda_i/I) : \mathrm{tlim}_I \lambda_i = \lambda \text{ and } \lambda_i = \mathrm{cf}(\lambda_i)\}$ is an initial segment of Reg $\backslash\lambda$, so the first member is λ^+, [Sh 355, 1.5,2.3].

<u>5.4</u> If $\lambda > \mathrm{cf}(\lambda) > \aleph_0$, <u>then</u> for some increasing continuous $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ with limit λ, $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i^+/J^{bd}_{\mathrm{cf}(\lambda)}$ has true cofinality λ^+, [Sh 355, 2.1].

<u>5.5</u> $\mathrm{pp}(\lambda) > \lambda^+$ contradicts "large cardinal" type assumptions, for example "every μ-free abelian group is free" [Sh 355, 2.2,2.2B], for the parallel fact on cov see [Sh 355, 6.6].

<u>5.6</u>

(*a*) (inverse monotonicity) If $\mu > \lambda > \kappa \geq \mathrm{cf}(\lambda) + \mathrm{cf}(\mu)$ and $\mathrm{pp}^+_\kappa(\lambda) > \mu$, <u>then</u> $\mathrm{pp}^+_\kappa(\lambda) \geq \mathrm{pp}^+_\kappa(\mu)$

(*b*) so given $\kappa_0 < \kappa_1 < \mu$ if λ is minimal such that $\lambda > \kappa_1 \geq \kappa_0 \geq \mathrm{cf}(\lambda)$, $\mathrm{pp}(\lambda) \geq \mu$, <u>then</u>:
$\mathfrak{a} \subseteq \mathrm{Reg} \cap [\kappa_1, \lambda), |\mathfrak{a}| \leq \kappa_0, \sup(\mathfrak{a}) < \lambda$ implies $\max \mathrm{pcf}(\mathfrak{a}) < \lambda$, equivalently: $\lambda' \in (\kappa_1, \lambda) \ \& \ \mathrm{cf}(\lambda') \leq \kappa_0 \Rightarrow \mathrm{pp}_{\kappa_1}(\lambda') < \lambda$ [Sh 355, 2.3] (with more)

(*c*) assume $\kappa \leq \chi < \mu$, and

$$(\forall \lambda)[\lambda \in (\chi, \mu) \ \& \ \mathrm{cf}(\lambda) \leq \kappa \Rightarrow \mathrm{pp}(\lambda) < \mu]$$

<u>then</u> for every $\mathfrak{a} \subseteq (\chi, \mu)$ of cardinality $\leq \kappa$, $\sup(\mathfrak{a}) < \mu$ we have $\max \mathrm{pcf}(\mathfrak{a}) < \mu$
[by (*d*) below and 5.9 below]

(*d*) $\max \mathrm{pcf}(\mathfrak{a}) \leq \sup\{\mathrm{pp}_{|\mathfrak{a}\cap\mu|}(\mu) : \mu \notin \mathfrak{a}, \mu = \sup(\mathfrak{a} \cap \mu)\}$
[by the definition].

Similar assertion holds for $\mathrm{pp}_\Gamma, \Gamma$ is "nice" enough.

<u>5.7</u>

(*A*) If λ is singular, $\mu < \lambda$, <u>then</u> for some $\delta \leq \mathrm{cf}(\lambda)$ and increasing sequence $\langle \lambda_i : i < \delta\rangle$ of regular cardinals in (μ, λ) and $\delta = \mathrm{cf}(\delta) \vee \delta < \omega_1$ we have: $\lambda_i > \max \mathrm{pcf}\{\lambda_j : j < i\}$ and $\lambda^+ = \mathrm{tcf}(\Pi)\lambda_i/J^{bd}_\delta$, [Sh 355, 3.3]

(*B*) If λ is singular and $\aleph_0 < \mathrm{cf}(\lambda) = \kappa$ and $\bigwedge_{\mu<\lambda} \mu^\kappa < \lambda$ and $\lambda < \theta = \mathrm{cf}(\theta) \leq \lambda^\kappa$, <u>then</u> for some increasing sequence $\langle \lambda_i : i < \kappa\rangle$ of regulars $< \lambda$,

$\lambda = \sum_{i<\kappa} \lambda_i$ and $\prod_{i<\kappa} \lambda_i / J^{bd}_\kappa$ has true cofinality θ (see [Sh 371, 1.6(2)]). Moreover, we can demand $i < \delta \Rightarrow \max \mathrm{pcf}\{\lambda_j : j < i\} < \lambda_i$. We can weaken the hypothesis to $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda_0 < \lambda$ and $(\forall \mu)[\lambda_0 < \mu < \lambda$ & $\mathrm{cf}(\mu) \le \kappa \to \mathrm{pp}(\mu) < \lambda]$ (see [Sh 371, 1.6(2)]. If we allow $\mathrm{cf}(\lambda) = \kappa = \aleph_0$ we still get this, but for possibly larger J, see [Sh 430, 6.5]; see 5.12 below.

5.8 $\mathrm{pp}_{\Gamma(\theta,\sigma)}$ can be reduced to finitely many $\mathrm{pp}_{\Gamma(\theta)}$, see [Sh 355, 5.8].

5.9 If $\mu > \theta \ge \mathrm{cf}(\mu)$ and for every large enough $\mu' < \mu$:

$$[\mathrm{cf}(\mu') \le \theta \Rightarrow \mathrm{pp}_\theta(\mu') < \mu]$$

then

$$\mathrm{pp}(\mu) =^+ \mathrm{pp}_\theta(\mu) =^+ \mathrm{pp}_{\Gamma(\mathrm{cf}(\mu))}(\mu)$$

[Sh 371, 1.6(3)(5),1.6(2)(4)(6),1.6A]; see 5.12 below.

5.10 If $\langle \mathfrak{b}_\zeta : \zeta < \kappa \rangle$ is increasing, $\lambda \in \mathrm{pcf}(\bigcup_{\zeta<\kappa} \mathfrak{b}_\zeta) \setminus \bigcup_{\zeta<\kappa} \mathrm{pcf}(\mathfrak{b}_\zeta)$, then:

(1) for some $\mathfrak{c} \subseteq \bigcup_\zeta \mathrm{pcf}(\mathfrak{c}_\zeta)$, $|\mathfrak{c}| \le \kappa$, we have $\lambda \in \mathrm{pcf}(\mathfrak{c})$

(2) if $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, then for some club $C \subseteq \kappa$ and $\lambda_\zeta \in \mathrm{pcf}(\bigcup_{\xi<\zeta} \mathfrak{b}_\xi)$ for $\zeta \in C$, we have $\lambda = \prod_{\zeta \in C} \lambda_\zeta / J^{bd}_\kappa$, $\lambda_\zeta (\zeta \in C)$ is increasing and $\zeta \in C \Rightarrow \lambda_{\zeta+1} > \max \mathrm{pcf}\{\lambda_\xi : \xi \in C$ and $\xi \le \zeta\}$, [Sh 371, 1.5].

5.11 If $\mu > \theta \ge \mathrm{cf}(\mu) \ge \sigma = \mathrm{cf}(\sigma)$, and for every large enough $\mu' < \mu$:

$$[\sigma \le \mathrm{cf}(\mu') \le \theta \Rightarrow \mathrm{pp}_{\Gamma(\theta^+,\sigma)}(\mu') < \mu]$$

then

$$\mathrm{pp}_{\Gamma(\mathrm{cf}(\mu)^+,\sigma)}(\mu) = \mathrm{pp}_{\Gamma(\mathrm{cf}(\mu))}(\mu)$$

[Sh 420, 1.2] and more there.

5.12 If $\mu > \kappa = \mathrm{cf}(\mu) > \aleph_0$ and for every large enough $\mu' < \mu$

$$(\mu')^\kappa < \mu \text{ or just } [\mathrm{cf}(\mu') \leq \mathrm{cf}(\mu) \Rightarrow \mathrm{pp}_\kappa(\mu') < \lambda],$$

then $\mathrm{pp}^+(\mu) = \mathrm{pp}^+_{J^{bd}_\kappa}(\mu)$ and we can get the conclusion in 5.7(B) above [Sh 371, 1.8]. Generalization for $\Gamma(\theta, \sigma)$ in [Sh 410, 1.2].

5.13 If $\lambda > \kappa = \mathrm{cf}(\lambda) > \aleph_0, \lambda > \theta$ then for some increasing continuous sequence $\langle \lambda_i : i < \kappa \rangle$ with limit λ:

(a) for every $i < \kappa, \lambda_i < \mu < \lambda_{i+1}$ & $\mathrm{cf}(\mu) \leq \theta \Rightarrow \mathrm{pp}_\theta(\mu) < \lambda_{i+1}$

or

(b) for every $i < \kappa$, $\mathrm{pp}_{\theta + \mathrm{cf}(i)}(\lambda_i) \geq \lambda$ [Sh 371, 1.9;more 1.9A].

5.14 If $\sigma \leq \mathrm{cf}(\mu) \leq \theta < \kappa < \mu$ then:

$$\mathrm{pp}_\theta(\mu) < \mu^{+\theta^+} \Rightarrow \mathrm{pp}_\kappa(\mu) = \mathrm{pp}_\theta(\mu);$$

and

$$\mathrm{pp}_{\Gamma(\theta^+,\sigma)}(\mu) < \mu^{+\theta^+} \Rightarrow \mathrm{pp}_{\Gamma(\kappa^+,\sigma)}(\mu) = \mathrm{pp}_\theta(\mu)$$

[Sh 371, 3.6; more 3.7,3.8].

5.15 If $\langle \mu_i : i \leq \kappa \rangle$ is increasing continuous, $\mu_0 > \kappa^{\aleph_0} > \kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $\mathrm{cov}(\mu_i, \mu_i, \kappa^+, 2) < \mu_{i+1}$, then for some club E of κ we have: $\delta \in E \cup \{\kappa\} \Rightarrow \mathrm{pp}_{J^{bd}_{\mathrm{cf}(\delta)}}(\mu_\delta) = \mathrm{cov}(\mu_\delta, \mu_\delta, \kappa^+, 2)$; so e.g. for most limit $\delta < \omega_1$, $\mathrm{pp}_{J^{bd}_\omega}(\beth_\delta) =^+ \beth_{\delta+1}$ (see [Sh:E12, part C,remark to X,§5,p.412].

5.16 If $pp^+_\sigma(\mu) > \lambda = \mathrm{cf}(\lambda)$, (so $\mathrm{cf}(\mu) \leq \sigma$) then

(a) for some $\mathfrak{a}$ an unbounded subset of μ, $|\mathfrak{a}| \leq \sigma, \lambda = \mathrm{tcf}(\pi(\mathfrak{a})/J^{bd}_{\mathfrak{a}}) = \max \mathrm{pcf}(\mathfrak{a})$ or

(b) for some $\mathfrak{b} \subseteq (\mu, \lambda)$ of cardinality $\leq \sigma, \lambda = \max \mathrm{pcf}(\mathfrak{a})$ and $\theta \in \mathfrak{a} \Rightarrow \max \mathrm{pcf}(\theta \cap \mathfrak{a}) < \theta$ (see [Sh 413, 2.4A,2.4(2)]).

§6 cov:

6.1 Definition. [Sh 355, 5.1]

$$\mathrm{cov}(\lambda,\mu,\theta,\sigma)=\min\Big\{|\mathscr{P}|:\mathscr{P}\text{ a family of subsets of }\lambda\text{ each of cardinality }<\mu\text{ such that: for every }a\subseteq\lambda,|a|<\theta\text{ for some }\alpha<\sigma\text{ and }A_i\in\mathscr{P}\text{ for }i<\alpha\text{ we have }a\subseteq\bigcup_{i<\alpha}A_i\Big\}.$$

So $\mathrm{cov}(\lambda,\kappa^+,\kappa^+,2)=\mathrm{cf}([\lambda]^{\leq\kappa},\subseteq)$.

6.2 Basic properties [Sh 355, 5.2,5.3] see also [Sh 355, 3.6]; for example if $\lambda>\theta>\mathrm{cf}(\lambda)\geq\sigma$, then for some $\mu<\lambda$ we have

$$\mathrm{cov}(\lambda,\lambda,\theta,\sigma)=\mathrm{cov}(\lambda,\mu,\theta,\sigma).$$

6.3 cov and cardinal arithmetic and $T_\Gamma(\lambda)$ see e.g. [Sh 355, 5.10], [Sh 355, 5.6,5.7,5.9,5.10, Definition of T_Γ] for example

$$\lambda^\kappa=\mathrm{cov}(\lambda,\kappa^+,\kappa^+,2)+2^\kappa.$$

By this and 6.4, 6.5 we shall use assumptions on cases of pp rather than conventional cardinal arithmetic.

6.4 On cov = pp: if $\lambda\geq\mu\geq\theta>\sigma=\mathrm{cf}(\sigma)>\aleph_0,\lambda>\mu+\sigma\vee\mathrm{cf}(\mu)\in[\sigma,\theta)$, then $\mathrm{cov}(\lambda,\mu,\theta,\sigma)=\sup\{\mathrm{pp}_{\Gamma(\theta,\sigma)}(\chi):\chi\in[\mu,\lambda],\mathrm{cf}(\chi)\in[\sigma,\theta)\}$, we have $=^+$ if $\mu=\theta$; [Sh 355, 5.4].

Assuming for simplicity $\lambda=\mu$, if $=^+$ fails, then for some $\mathfrak{a}\subseteq\mathrm{Reg}\cap\mu$ we have $|\mathfrak{a}|<\mu,\sup(\mathfrak{a})=\mu$ and

$$\mathrm{cov}(\lambda,\mu,\theta,\sigma)=\sup\{\mathrm{tcf}(\Pi\,\mathfrak{b}/J:\mathfrak{b}\subseteq\mathfrak{a},|\mathfrak{b}|<\theta,\mu=\sup(\mathfrak{b}),\ J\text{ and ideal on }\mathfrak{b}\text{ extending }J^{bd}_{\mathfrak{b}}\};$$

see [Sh 513, 6.12].

6.5 The parallel of 6.4 for $\sigma = \aleph_0$ "usually holds", i.e. :

(a) for λ singular, $\mathrm{cov}(\lambda, \lambda, \mathrm{cf}(\lambda)^+, 2) = \mathrm{pp}(\lambda)$ if for every singular $\chi < \lambda$, $\mathrm{pp}(\chi) = \chi^+$; [Sh 400, §1] (and weaker assumptions and intermediate stages there)

(b) if $\mathrm{cf}(\lambda) = \aleph_0$, $\bigwedge_{\mu<\lambda} \mu^{\aleph_0} < \lambda$ and $\mathrm{pp}(\lambda) < \mathrm{cov}(\lambda, \lambda, \aleph_1, 2)$, then $\{\mu : \lambda < \mu = \aleph_\mu < \mathrm{pp}(\lambda)\}$ is uncountable [Sh 400, 5.9], more in [Sh 420, 6.4], if λ is a strong limit, then the set has cardinality $> \lambda$;

(c) few exceptions: if $\langle \lambda_i : i \le \kappa \rangle$ is increasing continuous and $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\bigwedge_{i<\kappa} \mathrm{cov}(\lambda_i, \lambda_i, \kappa^+, 2) < \lambda_\kappa$, then for some club C of $\kappa, \delta \in C \cup \{\kappa\} \Rightarrow$ equality, i.e.

$$\mathrm{cov}(\lambda_\delta, \lambda_\delta, \aleph_1, 2) = \mathrm{pp}(\lambda_\delta)$$

[Sh 400, 5.10]

(d) for example for a club of $\delta < \omega_1, 2^{\beth_\delta}$ [Sh 400, 5.13]

(e) if on λ there is a $\aleph_1$-saturated λ- complete ideal (extending J^{bd}_λ) for example λ a real valued measurable, then $\mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2) \le \lambda$ [Sh 430, §3] and more there

(f) in clause (c), if $\kappa^{\aleph_0} < \lambda_0$ we can add $\mathrm{pp}(\lambda_\delta) =^+ \mathrm{pp}_{J^{bd}_\omega}(\lambda_\delta)$; of course, there $\mathrm{cov}(\lambda_\delta, \lambda_\delta, \aleph_1, 2) = \mathrm{cov}(\lambda_\delta, \lambda_\delta, \kappa^+, 2)$.

6.6 cov = minimal cardinality of a stationary S [Sh 355, 3.6,5.12], [Sh 400, 3.6,3.8,3.8A,5.11,5.2A], [Sh:g, Ch.VII,§1,§4], [Sh 410, 2.6 (using 2.2), 3.7], finally [Sh 420, 3.6]; for example

$$\mathrm{cf}(\mathscr{S}_{\le\kappa}(\lambda), \subseteq) = \min\{|S| : S \subseteq [\lambda]^{\le\kappa} \text{ is stationary}\}.$$

Moreover, we got a measure one set of this cardinality for an appropriate filter; for another filter see [Sh 580].

6.7 Covering by normal filters (prc) [Sh 371, §4], [Sh 410, §1], generalization [Sh 410, §5], essentially [Sh 430, proof of 4.2 second case]. To quote [Sh 410, §1]. Saharon?

6.8 On $\mathrm{cf}_J(\Pi\,\mathfrak{a}, <_I)$, a generalization, see [Sh 400, 3.1].

6.9 Computing $\mathrm{cf}^\sigma_{<\theta}(\Pi\,\mathfrak{a})$ [Sh 400, 3.2]; computing from it $\mathrm{pp}(\lambda)$ for non-fixed point λ by it [Sh 400, 3.3].

6.10 cov is $\mathrm{cf}^{\sigma}_{<\theta}\big(\Pi(\mathrm{Reg}\cap\lambda), <_{J^{bd}_\lambda}\big)$ is $\mathrm{pp}_{\Gamma(\theta,\sigma)}$, when $\mathrm{cf}(\sigma) > \aleph_0$ [Sh 400, 3.3,3.4,3.5].

6.11 $\mathrm{cov}(\lambda,\lambda,\mathrm{cf}(\lambda)^+,2) =^+ \mathrm{pp}(\lambda)$ when λ is singular non-fixed point
[Sh 400, 3.7(1),and more 3.7(1)-(5),3.8].

6.12 Computing $\mathrm{cov}(\lambda,\theta,\theta,2)$ by using $\mathrm{cf}_{<\theta}$ when $\theta > \mathrm{cf}(\lambda) = \aleph_0$
[Sh 400, 5.1,5.2,5.3,5.4,5.4A,5.5,restriction to subset of $\lambda\cap$ Reg is 5.5A; more (for 6.5 here), 5.7,5.8].

6.13 Finding a family $\mathscr{P}$ of subsets of λ covering many of the countable subsets of λ, for example, if $a \in [\lambda]^{\aleph_1}$ we can find $H : a \to \omega$ such that each countable subset of $H^{-1}(\{0,\dots,n\})$ is included in a member of $\mathscr{P}$. I.e. we characterize the minimal cardinality of such $\mathscr{P}$ by pcf [Sh 410, 2.1-2.4], [Sh 430, 1.2] more in [Sh 513].

6.14 Characterizing the existence of $\mathscr{P} \subseteq [\lambda]^{\aleph_1}, |\mathscr{P}| > \lambda$ with pairwise finite intersection [Sh 410, §6] more in [Sh 430, 1.2], [Sh 513].

6.15 If $\lambda \ge \mu > \sigma = \mathrm{cf}(\sigma) > \aleph_0$, then $\{\mathrm{cov}(\lambda,\mu,\theta,\sigma) : \mu \ge \theta > \sigma\}$ is finite ([GiSh 412]).

6.16 Let $\lambda > \kappa > \aleph_0$ be regular, then: $\bigwedge_{\mu<\lambda} \mathrm{cov}(\mu,\kappa,\kappa,2) < \lambda$ iff for every $\mu < \lambda$ and $\langle a_\alpha : \alpha < \lambda\rangle, a_\alpha \subseteq \mu, |a_\alpha| < \kappa$ for some unbounded $s \subseteq \lambda, |\bigcup_{a\in s} a_\alpha| < \kappa$ (a problem of Rubin Shelah [RuSh 117], see [Sh 371, 6.1], [Sh 430, 3.1]). For λ successor of regular, a stronger theorem: see [Sh 371, §6]; more [Sh 513, 6.13,6.14].

6.17 If $\mu > \lambda \ge \kappa, \theta = \mathrm{cov}(\mu,\lambda^+,\lambda^+,\kappa)$ and $\mathrm{cov}(\lambda,\kappa,\kappa,2) \le \mu$ (or at least $\le \theta$), then $\mathrm{cov}(\mu,\lambda^+,\lambda^+,2) = \mathrm{cov}(\theta,\kappa,\kappa,2)$, [Sh 430, 2.1].

6.18 If $\lambda \ge \beth_\omega$, then for some $\kappa < \beth_\omega$, $\mathrm{cov}(\lambda,\beth^+_\omega,\beth^+_\omega,\kappa) = \lambda$, [Sh 460, 1.1]; any strong limit singular can serve instead of $\beth_\omega$. For a singular limit cardinal μ (for example $\mu = \aleph_\omega$) sufficient conditions (for replacing $\beth_\omega$ by μ) are given in [Sh 460, 2.1,4.1]. For example such a condition is

$$(*)_{\kappa,\mu} \quad \mathfrak{a} \subseteq \mathrm{Reg}\backslash\mu \ \& \ |\mathfrak{a}| < \mu \Rightarrow |\mathrm{pcf}_{\kappa\text{-complete}}(\mathfrak{a})| < \mu.$$

So for every $\lambda \ge \beth_\omega$ for some n and $\mathscr{P} \subseteq [\lambda]^{<\beth_\omega}$ of cardinality λ, every $X \in [\lambda]^{\le\beth_\omega}$ is the union of $\le \beth_n$ sets from $\mathscr{P}$; ([Sh 460, 2.5]) and the inverse [Sh 460, 4.2] (see [Sh 513]).
Also if the statement above holds for e.g. $\aleph_\omega$ then $(*)_{\aleph_n,\aleph_{\omega+1}}$ holds (by [Sh 460, 2.6]).

§7 Bounds in cardinal arithmetic

7.1 If $\langle \lambda_i : i \le \kappa \rangle$ is increasing continuous, J a normal ideal on κ and $\mathrm{pp}_J(\lambda_i) \le \lambda_i^{+h(i)}$, then $\mathrm{pp}_J(\lambda_\kappa) \le \lambda_\kappa^{\|h\|}$ [Sh 355, 2.4], [Sh 371, 1.10,1.11] where $\|h\|$ is Galvin Hajnal rank, i.e.

$$\|h\| = \sup\{\|f\| + 1 : f <_{D_\kappa} h\},$$

D_κ the club filter on κ.

7.2 If C_0 is the class of infinite cardinals,

$$C_\xi =: \{\lambda : \text{ for every } \xi < \zeta, \lambda \text{ is a fixed point of } C_\zeta, \text{ i.e., } \lambda = \text{ otp}(C_\zeta \cap \lambda)\},$$

then for example

$$\mathrm{pp}(\omega_1\text{-th member of } C_1 \backslash \beth_2(\aleph_1)) < \beth_2(\aleph_1)^+\text{-th member of } C_1 \backslash \beth_2(\aleph_1)$$

[Sh 386, 5.6].

7.3 For $\zeta < \omega_1$ we have

$$\mathrm{pp}_{\mathrm{nor}}(\aleph^\zeta_\omega(\beth_2(\aleph_1))) < \aleph^\zeta_{(\beth_2(\aleph_1))^+}(\beth_2(\aleph_1))$$

and more on $\aleph^\zeta_\delta$, see [Sh 386, 5.4,5.5], where

$$\aleph^0_\alpha(\lambda) = \lambda^{+\alpha}, \aleph^{\zeta+1}_0(\lambda) = \lambda, \aleph^{\zeta+1}_{\alpha+1}(\lambda) = \aleph^i_\zeta(\aleph_0)$$

where

$$\zeta = \aleph^{i+1}_\alpha(\lambda) + 1 \text{ and } \aleph^{\zeta+1}_\delta(\lambda) = \bigcup_{\alpha<\delta} \aleph^{i+1}_\alpha(\lambda),$$

and for i limit,

$$\aleph^i_0(\lambda) = \lambda, \aleph^i_{\alpha+1}(\lambda) = \bigcup_{j<i} \aleph^j_{\alpha+1}(\aleph^i_\alpha(\lambda)) \text{ and } \aleph^i_\delta(\lambda) = \bigcup_{\alpha<\delta} \aleph^i_\alpha(\lambda).$$

7.4 If there are no [there are $\leq \aleph_1$] inaccessibles before $\lambda, \lambda > 2^{\aleph_1}$, $\mathrm{cf}(\lambda) = \aleph_1$, then there are no [there are $\leq 2^{\aleph_1}$] inaccessibles $<$ $\mathrm{pp}(\lambda)$ [Sh 386, 5.10], similarly for Mahlo, ϵ-Mahlo.

7.5 If $\bigwedge\limits_{\delta<\omega_1} \mathrm{pp}(\aleph_\delta) < \aleph_{\omega_1}$, $\mathrm{pp}(\aleph_{\omega_1}) = \aleph_{\alpha^*}$, then are are $|\alpha^*|$ subsets of ω_1 with pairwise countable intersection [Sh 371, 1.7(1), more (2)] getting Kurepa trees [Sh 371, 2.8.2.9].

7.6 The minimal counterexample to Tarski statement is simple, Jech-Shelah [JeSh 385].
In [Ta1] Tarski showed that for every limit ordinal β, $\prod\limits_{\xi<\beta} \aleph_\xi = \aleph_\beta^{|\beta|}$, and conjectured that

$$\prod_{\xi<\beta} \aleph_{\sigma_\xi} = \aleph_\alpha^{|\beta|}$$

holds for every ordinal β and every increasing sequence $\{\sigma_\xi\}_{\xi<\beta}$ such that $\lim_{\xi<\beta} \sigma_\xi = \aleph_\alpha$.
Now: if a counterexample exists, then there exists one of length $\omega_1 + \omega$ (Jech and Shelah [JeSh 385]).

7.7 $\mathrm{pp}(\aleph_{\alpha+\delta}) < \aleph_{\alpha+|\delta|^{+4}}$ [Sh 400, 2.1,2.2, more 2.3-2.8].

7.8 If $\delta < \aleph_4$, $\mathrm{cf}(\delta) = \aleph_0$ then $\mathrm{pp}(\aleph_\delta) < \aleph_{\omega_4}$. If $|\delta| +\ \mathrm{cf}(\delta)^{+3} < \kappa$, then $\mathrm{pp}(\aleph_{\alpha+\delta}) < \aleph_{\alpha+\kappa}$ [Sh 400, 4.2,4.3,4.4], more [Sh 410, 3.3-3.6].

7.9 More on the number of inaccessibles: [Sh 430, §4].

7.10 Gitik and Shelah [GiSh 412]:

(a) if μ is a Jonsson limit cardinal not strong limit, then $\langle 2^\sigma : \sigma < \mu \rangle$ is eventually constant.

(b) If μ is a limit cardinal, $\mu_0 < \mu$ and $\bigwedge\limits_{\theta \in (\mu_0, \mu)} \mu \to [\theta]^{<\omega}_{\theta, \mu_0}$, then $\langle 2^\theta : \mu_0 < \sigma < \mu \rangle$ has finitely many values.

(c) If on μ there is an μ_0^+-saturated, uniform μ-complete ideal for example μ a real value measurable $\leq 2^{\aleph_0}$, then the assumption of (b) holds, hence its conclusion.

§8 Jonsson algebras

8.1 Definition and previously known results: [Sh 355, 4.3,4.4]. A Jonsson algebra is one with no proper subalgebra with the same cardinality. A Jonsson cardinal is λ such that there is no Jonsson algebra with countable vocabulary and cardinality λ.

8.2 Definition of $\mathrm{id}_j(\bar{C})$, $\mathrm{id}^j_\theta(\bar{C})$ see [Sh 380, 1.8], $\mathrm{id}^j_J(\bar{C})$ see [Sh 380, 1.16] (also with k instead of j).

8.3 Jonsson games: Definition [Sh 380, 2.1], connection to [Sh 380, 2.3] (for example $\lambda = \aleph_{\omega+1}$).

8.4 λ^+ (for a singular λ) is not a Jonsson cardinal when:

(a) λ is not an accumulation point of inaccessible Jonsson cardinals [Sh 355, 4.5 more 4.6]

(b) weaker hypothesis (for $\lambda^+ \to [\lambda^+]^{<\omega}_\kappa$) [Sh 413, 2.5]

(c) $\lambda = \beth^+_\omega$ (see [Sh 413], [Sh:535] more there) ? Sh:535 ?

(d) on every large enough regular $\mu < \lambda$, there is an algebra M on μ which has no proper subalgebra with set of elements and is a stationary subset of μ, see [Sh 572, 3.3].

8.5 Sufficient condition for "λ not Jonsson" [Sh 365, 1.8,1.9] for $\lambda \nrightarrow [\lambda]^{<\omega}_\sigma$ [Sh 365, 1.10,3.5,3.6,3.7].

8.6 λ inaccessible is not Jonsson when: λ not Mahlo [Sh 365, 3.8], λ has a stationary subset S not reflecting in inaccessibles [Sh 365, 3.9], λ not λ-Mahlo [Sh 380], λ not $\lambda\times\omega$-Mahlo [Sh 413, 1.14], there is a set S of singulars satisfying, $\mathrm{rk}_\lambda(S) > \mathrm{rk}_\lambda(S^+)$ where $S^+ = \{\kappa < \lambda : \kappa \text{ inaccessible}, S \cap \kappa \text{ stationary}\}$, [Sh 413, 1.15].

8.7 If μ^+ is a Jonsson cardinal, $\mu > \mathrm{cf}(\mu) > \aleph_0$, then $\mathrm{cf}(\mu)$ is "almost" μ^+-supercompact [Sh 413, 2.8] other [Sh 413, 2.10].

8.8 If λ is regular, and for every regular large enough $\mu < \lambda$, for some $f : \mu \to \lambda$ we have $\|f\|_{J^{bd}_\mu} \geq \lambda$ (or at least this holds for "enough" μ's), then on λ there is a Jonsson algebra, [Sh 380, 2.12+2.12A]. More sufficient conditions there.

8.9 See more [Sh 413], [Sh:535]. ? Sh:535 ?

§9 Colouring = negative partition: ([Sh 282], [Sh 280], [Sh 327])

9.1 Definition of $Pr_\ell : Pr_0$, see [Sh:g, AP,1.1], $Pr_1^{(-)}$, see [Sh:g, AP,1.2], $Pr_2^{(y)}$, see [Sh:g, AP,1.3], $Pr_3^{(y)}$, see [Sh:g, AP,1.4], Pr_4, see [Sh 365, 4.3]. For example: $Pr_1(\lambda,\mu,\theta,\kappa)$ means: there is a 2-coloring of λ by θ colours (= symmetric 2-place function from λ to θ) such that: if $\langle w_i : i < \mu\rangle$ is a sequence of pairwise disjoint subsets of $\lambda, \bigwedge_i |w_i| < \kappa$ and $\zeta < \theta$, then for some $i < j$, on $w_i \times w_j$ the coloring c is constant. In $Pr_0(\lambda,\mu,\theta,\kappa)$ we replace ζ by $h : \kappa\times\kappa \to \theta$ and demand $a \in w_i$ & $\beta \in w_j \Rightarrow c(\alpha,\beta) = h(\text{otp}(w\cap\alpha), \text{otp}(w_j \cap \beta))$. If $\mu = \lambda$ we may omit it, if $\kappa = \aleph_0$ we may omit it. (See [Sh:g, AP,1.2]).

9.2 Trivial implications [Sh:g, AP,1.6,1.6A,1.7] and $Pr_1 \Rightarrow Pr_0$ [Sh 365, 4.5(3)], $Pr_4 \Rightarrow Pr_1$ [Sh 365, 4.5(1)]. For example if $Pr_1(\lambda,\mu,\theta,\sigma), \chi = \chi^{<\sigma} + 2^\theta \le \mu \le \lambda < 2^\chi$ then $Pr_1(\lambda,\mu,\theta,\sigma)$. Other such Pr and implications [Sh 572, §2,§4].

9.3 Colouring for successor of singular: [Sh 355, 4.1,4.7], [Sh 413, §2] for example $Pr_1(\lambda^+,\lambda^+,(\text{cf}(\lambda))^+,2)$ for λ singular.

9.4 Combining Pr_ℓ's [Sh 355, 4.8,4.8A].

9.5 Using pcf:

(a) if $\lambda = \text{tcf}(\Pi\,\mathfrak{c}/J^{bd}_{\mathfrak{c}})$ and $[\theta \in \mathfrak{c} \Rightarrow |\mathfrak{c}\backslash\theta| = |\mathfrak{c}|]$, then $Pr_1(\lambda,\lambda,2^{|\mathfrak{c}|},\text{cf}(\mathfrak{c}))$, see [Sh 355, 4.1B]

(b) getting colouring on $\lambda \in \text{pcf}(\mathfrak{a})$ from colourings on every $\theta \in \mathfrak{a}$, see [Sh 355, 4.1D].

9.6 Using guessing of clubs: Definition and basic properties of for example $(Dx)^\lambda_{\kappa,\sigma,\theta,\tau}$ [Sh 365, 4.1].

9.7 Proof of such properties [Sh 365, 4.2], [Sh 413, 2.6]

(a) if λ is a regular $\lambda > \sigma > \kappa$ then $Pr_1(\lambda^+,\lambda^+,\sigma,\kappa)$, [Sh 365, §4]

(b) if λ is inaccessible with a stationary subset S not reflecting in inaccessibles and $\sigma \le \min_{\delta\in S}\text{cf}(\delta)$ and $\kappa < \lambda$ then $Pr_1(\lambda,\lambda,\kappa,\sigma)$, [Sh 365, 4.1+4.7]

(c) if $\lambda = \mu^+, \mu > 2^{\text{cf}(\mu)}, \kappa < \mu$, then $Pr_1(\lambda,\lambda,\text{cf}(\mu),\kappa)$, [Sh 413, 2.7]

(d) if $\lambda = \mu^+, \mu > \text{cf}(\mu)$ then $Pr_1(\lambda,\lambda,\text{cf}(\mu),\text{cf}(\mu))$, [Sh 355, 4.1]

(e) by [Sh:535] we get such properties for e.g. $\lambda = \beth^+_\omega$? Sh:535 ?

(f) if $\lambda = \aleph_2$ & $\mu = \aleph_0$ or if $\lambda = \mu^{++}$, μ regular then $Pr_1(\lambda,\lambda,\lambda,\mu)$ ([Sh 572, §1]).

9.8 $(E2)$ implies Pr_4 [Sh 365, 4.4].

9.9 $(D2) \Rightarrow Pr_1$ [Sh 365, 4.7].

9.10 Concerning the results in [Sh:95] on partition relations restriction of the kind appearing there are necessary (we use 10.11) see, some day [Sh:F50].

9.11 Galvin conjecture:

(a) $\aleph_n \nrightarrow [\aleph_1]^{n+1}_{\aleph_0}$ ([Sh 288, 5.8(1)], more there), but

(b) for the naturally defined $h : \omega \to \omega$ if CON(ZFC $+\lambda \to (\aleph_1)^{\omega}_2$) then it is consistent with ZFC that: $2^{\aleph_0} = \lambda \to [\aleph_1]^n_{h(n)}$, (we can even get $X \in [\lambda]^{\aleph_1}$ which exhibits the conclusion simultaneously for all $n, \lambda \to [\aleph_1]^n_{h_1(n)}$, if $h_1(n) \geq n, h_1(n)/h(n) \to \infty$), [Sh 288, 3.1]

(c) if κ is measurable indestructible by adding (even many) Cohen subsets to κ, then a generalization of Halpern Lauchli theorem holds to ${}^{\kappa>}2$ (but using some $\langle <_\alpha : \alpha < \kappa \rangle, <_\alpha$ a well order of ${}^{\alpha}2$) ([Sh 288, 4.1,4.2 + §2]). See more in [Sh 481], [Sh 546], [RbSh 585].

9.12 More on colouring (improving results on Jonssonness from [Sh 413] to colouring) see [Sh:535], e.g. for $\lambda = \beth^+_\omega$ we have $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \aleph_0)$. ? Sh:535 ?

§10 Trees, linear order and Boolean algebras

<u>10.1</u> If $\lambda = \max \mathrm{pcf}\{\lambda_i : i < \delta\}$ and $\lambda_i > \max \mathrm{pcf}\{\lambda_j : j < i\}$, <u>then</u> we can find in $\Pi\{\lambda_i : i < \delta\}$ a $<_{J_{<\lambda}[\{\lambda_i : i<\delta\}]}$- increasing cofinal sequence $\langle f_\alpha : \alpha < \lambda\rangle$ such that $\{f_\alpha \restriction \{\lambda_j : j < i\} : i < \delta, \alpha < \lambda\}$ form a tree with δ levels, level i of cardinality $\max \mathrm{pcf}\{\lambda_j : j < i\} < \lambda_i$ and $\geq \lambda$ δ-branches [Sh 355, 3.5].
Note

- (a) the lexicographic order on $F = \{f_\alpha : \alpha < \lambda\}$ has density $\sum\limits_{i<\delta} \lambda_i$
- (b) if $\Pi\, \lambda_i/I$ is as in [Sh 355, 1.4(1) (see 1.3)], F is $(\Sigma\lambda_i)^+$-free (see 5.3), hence any set of cardinality $\leq \Sigma\lambda_i$ is the union of $\leq \mathrm{gen}(I)$ sets F' each satisfying for some $s \in I$ we have $\langle f_\alpha \restriction (\delta \backslash s) : f_\alpha \in F'\rangle$ is increasing i.e. $\alpha < \beta, f_\alpha \in F', f_\beta \in F', i \in \delta\backslash s \Rightarrow f_\alpha(i) < f_\beta(i)$
  where $\mathrm{gen}(I)$ is $\min\{|\mathscr{P}| : \mathscr{P} \subseteq I$ generates the ideal $I\}$, [Sh 355, 1.4(3)]
- (c) if $\lambda > 2^{|\delta|}$, <u>then</u> we can have such trees with exactly λ branches [Sh 276]; somewhat more: [Sh 430, 6.6B].

<u>10.2</u> There are quite many $\langle \lambda_i : i < \delta\rangle, \lambda$ as in 10.1: for example if $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu_0 < \mu < \lambda = \mathrm{cf}(\lambda) < \mathrm{pp}_\kappa(\mu)$, <u>then</u> we can find such $\langle \lambda_i : i < \kappa\rangle$ with limit μ with $\mu_0 < \lambda_i < \mu$, if $\bigwedge\limits_{\alpha<\mu} |\alpha|^\kappa < \mu$ or at least $(\forall \mu' < \mu)[\mathrm{pp}_\kappa(\mu') < \mu]$, see [Sh 371, 1.6(2),(4)]. Also "$\mathrm{pp}(\aleph_{\alpha+\delta}) < \alpha_{\alpha+|\delta|^+}$ help to get such examples, see [Sh 462, §5], [RoSh 534].

<u>10.3</u> For $\lambda > \kappa = \mathrm{cf}(\kappa)$ the following cardinals are equal:

$$\sup\{\mu : \text{some tree with } \lambda \text{ nodes has } \geq \mu\, \kappa\text{-branches}\}$$

and

$$\sup\{\mathrm{pcf}(\mathfrak{a}) : |\mathfrak{a}| < \min(\mathfrak{a})\mathrm{cf}(\mathrm{otp}(\mathfrak{a})) = \kappa, \mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \kappa \text{ and } \theta \in \mathfrak{a} \Rightarrow \max \mathrm{pcf}(\mathfrak{a} \cap \theta) < \theta\}$$

see [Sh 589, 2.2].

<u>10.4</u> Definition of Ens, entanged linear order and basic facts see for example [Sh:g, AP,2.1,2.2 more 2.3].

A linear order $\mathscr{I}$ is λ-entangled if given any $n < \omega$ and pairwise distinct $x^e_\zeta \in \mathscr{I}$ $(e < n, \zeta < \lambda)$ and $w \subseteq \{0, 1, \ldots, n-1\}$ there are $\zeta < \xi$ such that for $e < n$ we have: $x^e_\zeta < x^e_\xi \leftrightarrow e \in w$. We say $\mathscr{I}$ is entangled if it is $|\mathscr{I}|$-entangled; $\mathrm{Ens}(\lambda, \mu)$ means there are μ linear orders $\mathscr{I}_\zeta (\zeta < \mu)$ each of cardinality λ and if $n < \omega, \zeta_e < \mu$ distinct $(e < n)$ and $w \subseteq n$ and if $x^e_\zeta \in \mathscr{I}_\zeta$ are distinct then for some $\alpha < \beta < \mu$ we have $\mathscr{I}_{\zeta_e} \models x^e_\alpha < x^e_\beta \Leftrightarrow e \in w$.

More on σ-entangled linear orders see [Sh 462].

10.5 $\mathrm{Ens}(\lambda^+, \mathrm{cf}(\lambda))$ for λ singular [Sh 355, 4.9 more 4.11,4.14] more [Sh 371, 5.3].

10.6 For μ regular uncountable and a linear order $\mathscr{I}$ of power $\mu, \mathscr{I}$ is entangled iff the interval Boolean algebra of $\mathscr{I}$ is λ-narrow (see [Sh 345b, 2.3] or [Sh 462, §1]).

10.7 A sufficient condition for existence of entangled linear order of cardinality λ is: $\lambda = \max \mathrm{pcf}(\mathfrak{a}), \kappa = |\mathfrak{a}|, [\theta \in \mathfrak{a} \Rightarrow \theta > \max \mathrm{pcf}(\theta \cap \mathfrak{a})], 2^\kappa \geq \sup(\mathfrak{a}), \mathfrak{a}$ divisible to κ sets not in $J_{<\lambda}[\mathfrak{a}]$, [Sh 355, 4.12]; if we omit "$2^\kappa \geq \lambda$" we can still prove $\mathrm{Ens}(\lambda, \kappa)$; [Sh 355, 4.10A] more in [Sh 355, 4.10F,4.10G], [Sh 371, 5.4,5.5,5.5A].

10.8 If $\mathrm{cf}(\lambda) < \lambda \leq 2^{\aleph_0}$, then there is an entangled order in λ^+, [Sh 355, 4.13].

10.9 If $\lambda \in \mathrm{pcf}(\mathfrak{a})$ and $[\theta \in \mathfrak{a} \Rightarrow \theta > \max \mathrm{pcf}(\theta \cap \mathfrak{a})]$ and for each $\theta \in \mathfrak{a}$ there is an entangled linear order or just $\mathrm{Ens}(\theta, \max \mathrm{pcf}(\theta \cap \mathfrak{a}))$, then on λ there is one, [Sh 355, 4.10C].

10.10

(a) If $\kappa^{+4} \leq \mathrm{cf}(\lambda) < \lambda < 2^\kappa$, then there is an entangled linear order in λ^+, [Sh 410, 4.1 more 4.2,4.3].

(b) There is a class of cardinals λ for which there is an entangled linear order of cardinality λ^+, [Sh 371, §5].
It is not clear if we can demand e.g. $\lambda = \lambda^{\aleph_0}$, but if this fails, then for κ large enough, $\kappa^{\aleph_0} = \kappa \Rightarrow 2^\kappa < \aleph_{\kappa^{+4}}$ (see (a), more in [Sh 462]).

(c) There is a class of cardinals λ for which there is a Boolean algebra B of cardinality λ^+ with neither chain nor antichain of cardinality λ^+, i.e. if $Y \subseteq B, |Y| = |B|$ then $(\exists x, y \in Y)[x < y]$ and $(\exists x, y \in Y)[x \nleq y \wedge y \nleq x]$ (in fact for any sequence $\langle x_\alpha : \alpha < \lambda^+ \rangle$ of distinct members of B:

(i) $(\exists \alpha < \beta)(x_\alpha < x_\beta)$,

(ii) $(\exists \alpha < \beta)(x_\alpha > x_\beta)$ and

(iii) $(\exists \alpha < \beta)[x_\alpha \nleq x_\beta \wedge x_\beta \nleq x_\alpha]$; see [Sh 462, 4.3].

(*d*) Moreover, in part (c), for any given λ_0, letting μ be the minimal $\mu = \aleph_\mu > \lambda_0$ then we can find B as there with density μ (everywhere); similarly in (b).

(*e*) Moreover in (c) (and (b)) if the density character is $\mu, \ell \in \{0,1,2\}, \theta = \mathrm{cf}(\theta) < \mu$ and $x_\alpha \in B$ (for $\alpha < \lambda$) are distinct then for some $w \subseteq \lambda, |w| = \theta$ we have for any $\alpha, \beta \in w, \alpha < \beta$:

$$\ell = 0 \Rightarrow x_\alpha < x_\beta$$
$$\ell = 1 \Rightarrow x_\alpha > x_\beta$$
$$\ell = 2 \Rightarrow x_\alpha \not\leq x_\beta \wedge x_\beta \not\leq x_\alpha$$

Similarly in part (b).

(*f*) If 2^λ is singular, then there is an entangled linear order of cardinality $(2^\lambda)^+$ (the assumption implies $(\forall\mu)[\lambda < \mathrm{cf}(\mu) < \mu \leq 2^\lambda < \mathrm{pp}(\mu)]$ (i.e. $\mu = 2^\lambda$), this suffices as we can use 5.6(b), 5.12, 10.7; see [Sh 462, §5]).

<u>10.11</u> Universal linear orders: see Section 13, Model Theory.

<u>10.12</u> For every λ there is $\mu, \lambda \leq \mu < 2^\lambda$ such that (A) or (B):

(A) $\mu = \lambda$ and for every regular $\chi \leq 2^\lambda$ there is a tree T of cardinality λ with $\geq \chi$ branches (so a linear order of cardinality $\geq \chi$ and density $\leq \lambda$)

(B) $\mu > \lambda$, and:

- (α) $\mathrm{pp}(\mu) = 2^\lambda, \mathrm{cf}(\mu) \leq \lambda, (\forall\theta)[\mathrm{cf}(\theta) \leq \lambda < \theta < \mu \Rightarrow \mathrm{pp}_\lambda(\theta) < \mu]$. Hence, by [Sh 371, §1] for every regular $\chi \leq 2^\lambda$ there is a tree from [Sh 355, 3.5]: $\mathrm{cf}(\mu)$ levels, every level of cardinality $< \mu$ and χ $(\mathrm{cf}(\mu))$-branches
- (β) for every $\chi \in (\lambda, \mu)$, there is a tree T of cardinality λ with $\geq \chi$-branches of the same height
- (γ) $\mathrm{cf}(\mu) = \mathrm{cf}(\lambda_0)$ for $\lambda_0 = \min\{\theta : 2^\theta = 2^\lambda\}$ and even $\mathrm{pp}_{\Gamma(\mathrm{cf}(\mu))}(\mu) = 2^\lambda$ see [Sh 355, 5.11], [Sh 410, 4.3] and [Sh 430, 3.3]; see more in [Sh 600, 2.10].

<u>10.13</u> If $\theta_{n+1} = \min\{\theta : 2^\theta > 2^{\theta_n}\}, \sum\limits_{n<\omega} \theta_n < 2^{\theta_0}$, <u>then</u> for some $n > 0$ and regular $\mu \in [\theta_n, \theta_{n+1})$ for every regular $\chi \leq 2^{\theta_n}$, there is a tree with μ nodes and $\geq \chi$ μ-branches [Sh 430, 3.4].

<u>10.14</u> Kurepa trees: there are two contexts that arise

(*a*) we can get Kurepa trees of singular cardinality: if $\delta < \lambda_i = \mathrm{cf}(\lambda_i)$, $\lambda_i >$ max pcf$\{\lambda_j : j < i\}$ <u>then</u> there is a tree with δ levels, the i-th level of cardinality $< \lambda_i$, and at least max pcf$\{\lambda_i : i < \delta\}$ δ-branches, see [Sh 355, 3.5], hence can derive consequences from conventional cardinal arithmetic assumptions

(*b*) if for example $\mathrm{pp}(\aleph_{\omega_1}) > \aleph_{\omega_2}$ and for a club of $\delta < \omega_1$, $\mathrm{pp}(\aleph_\delta) < \aleph_{\omega_1}$, <u>then</u> there is an $(\aleph_1)$-Kurepa tree (see [Sh 371, 2.8] for more). We get a large family of sets with small intersection in more general circumstances [Sh 371, 1.7].

(*c*) If $\bigwedge\limits_{\alpha<\mu} |\alpha|^\kappa < \mu$, $\mathrm{cf}(\mu) = \kappa > \aleph_0$ and $\mu \le \lambda < \mu^\kappa$, <u>then</u> there is a tree with μ nodes, κ levels and exactly λ branches, λ of them of height κ. We can derive results on linear orders (really they are the same problems). If we speak on the number of κ-branches (or for linear order number of Dedekind cuts of cofinality from at least one side κ), instead of "$\bigwedge\limits_{\alpha<\mu} |\alpha|^\kappa < \mu$" it suffices that

$$2^\kappa < \mu_0 < \mu + (\forall \chi)[\mu_0 < \chi < \mu \ \ \& \ \ \mathrm{cf}(\chi) \le \chi \Rightarrow \ \mathrm{pp}(\mu) < \mu'].$$

See [Sh 262] or [Sh 430, 6.6(1)]. (Similarly, other results can be translated between trees and linear orders).

<u>10.15</u> Boolean algebras and topology. λ-c.c. is not productive and $\lambda - L$-spaces exists and $\lambda - S$-spaces exist and more follows from $Pr_1^-(\lambda, 2)$ (or appropriate colouring) see [Sh 282a, 1.6A] so [Sh 355, 4.2] is a conclusion of this. This is translated to results on cellularity of topological spaces (cellularity $\le \lambda \Leftrightarrow \lambda^+$-c.c.). We have

(*a*) if $\lambda \ge \aleph_1$, for some λ^+-c.c. Boolean algebras B_1, B_2 we have: $B_1 \times B_2$ is not λ^+-c.c. (why? now $\mathrm{Pr}_1(\lambda^+, \lambda^+, 2, \aleph_0)$ suffice [Sh:g, AP,1.6A] and it holds by [Sh 327] or [Sh 365, 4.8(1),p.177] if λ regular $> \aleph_1$, [Sh 355, 4.1,p.67] if λ is singular and lastly by [Sh 572, §1] if $\lambda = \aleph_1$)

(*b*) if λ is inaccessible and has a stationary subset not reflecting in any accessible, <u>then</u> for some λ-c.c., Boolean algebras B_1, B_2 we have: $B_1 \times B_2$ is not λ-c.c. (see [Sh 365, 4.8])

(*c*) if λ is Mahlo, $\otimes_\lambda^{\aleph_0}$ (see 1.4) <u>then</u> for some λ-c.c. Boolean algebras B_n, for any proper filter I on ω extending J_ω^{bd} we have $\Pi B_n/I$ fails the λ-c.c., [Sh 365, 4.11].

10.16 Topology: characterizing by pp when there are $f_\alpha \in {}^\kappa\sigma$ for $\alpha < \theta$ such that $\alpha < \beta \Rightarrow \bigvee_{i<\kappa} f_\alpha(i) < f_\beta(i)$, see [Sh 410, 3.7], is needed for Gerlits, Hajnal and Szentmiklossy [GHS]. The condition is (when θ is regular for simplicity)

$$2^\kappa \geq \theta \text{ or } (\exists\mu)[\mathrm{cf}(\mu) \leq \kappa < \mu \;\&\; \mu \leq \sigma \;\&\; \mathrm{pp}^+(\mu) > \theta]$$

(for θ singular just ask if for every regular $\theta_1 < \theta$).
(Why not just $\theta \leq \sigma^\kappa$? Because if e.g. $\kappa = \beth_\omega, \theta = \beth_{\omega+1}, \sigma = \aleph_0$ we do not know whether $\mathrm{pp}^+(\kappa) = \theta^+$).

10.17 Topology: let X be a topological space, B a basis of the topology (not assuming the space to be Hausdorf or even T_0). If $\lambda = \aleph_0$ or λ is strong limit of cofinality $\aleph_0$, and the number of open sets is $> |B| + \lambda$, then it is $\geq \lambda^{\aleph_0}$; see for $\lambda = \aleph_0$ [Sh 454], for $\lambda > \aleph_0$, [Sh 454a] relaying on [Sh 460].

10.18 Topology: densities of box products: for example if μ is strong limit singular, $\mu = \sum_{i<\,\mathrm{cf}(\mu)} \lambda_i$, $\mathrm{cf}(\lambda_i) = \aleph_0, 2^{\lambda_i} = \lambda_i^+, \lambda_i$ are strong limit cardinals,
$\max \mathrm{pcf}\{\lambda_i : i < \mathrm{cf}(\mu)\} < 2^\mu$, $\mathrm{cf}(\mu) < \theta < \mu$, then the density of the $(\mathrm{cf}(\mu))^+$-box product ${}^\theta\mu$ is 2^μ [Sh 430, §5].

Gitik Shelah [GiSh 597] prove consistency results.

10.19

(a) the results in 10.18 come from starting to analyze the following: given μ_i-complete filter D_i on λ_i: for $i < \kappa$, what is

$$\min\{|A| : A \subseteq \prod_{i<\kappa} \lambda_i \text{ such that for every } \langle A_i : i < \kappa\rangle \in \prod_{i<\kappa} D_i \text{ we have } A \cap \prod_{i<\kappa} A_i \neq \emptyset\}$$

continued in [Sh 575] and then [Sh 620].

(b) This is applied also to the problem of λ-Gross spaces (vector space V over a field F with an inner product such that for $U \subseteq V$ of dimension λ,

$$\dim\{x \in V : \bigwedge_{y\in u} (x, y) = 0\} < \dim V),$$

in Shelah Spinas [ShSi 468].

10.20 A well known problem in general topology is whether every Hausdorff space can be divided to two sets each not containing a homeomorphic copy of Cantor's discontinuum. In [Sh 460] we have a sufficient condition for this (e.g. $|\mathfrak{a}| \le \aleph_0 \Rightarrow |\text{pcf}(\mathfrak{a})| \le \aleph_0 + 2^{\aleph_0} \ge \aleph_\omega$, by [Sh 460, 3.6(2)], the $(*)_1$ version relying on [Sh 460, Th.2.6]). But we can prove: if $c\ell$ is a closure operation on $\mathscr{P}(X)$ (i.e. $a \subseteq c\ell(a) = c\ell(c\ell(a)), a \subseteq b \Rightarrow c\ell(a) \subseteq c\ell(b)$) and $|a| \ge \aleph_0 \Rightarrow |c\ell(a)| > \beth_\omega$, then we can partition X to two sets, each not containing any infinite $a = c\ell(a)$. (Can prove more).

Related weaker problem is to find large $A \subseteq {}^\omega\lambda$ containing no large closed subsets, λ strong limit of cofinality $\aleph_0$, if $\text{pp}(\lambda) = 2^\lambda$ is easy (hence for higher cofinalities this holds and e.g. for many $\beth_\delta, \delta < \omega_1$). See [Sh 355, 6.9], more in [Sh 430, 3.3,3.4]. See more in [Sh 460], [Sh 668].

10.21 If $\text{pp}(\lambda) > \lambda^+$, $\text{cf}(\lambda) = \aleph_0$ (or just a consequence from [Sh 355, §1], see 5.2 here), then there is first countable λ-collectionwise Hausdorff (and even λ-metrizable), not λ^+-collectionwise Hausdorff space (see [Sh:E9]; when we assume just $\text{cov}(\lambda, \lambda, \aleph_1, 2) > \lambda^+$ use [Sh 355, §6]).

10.22 If $\lambda < \lambda^{<\lambda}$, then there is a regular $\kappa < \lambda$ and tree T with κ levels, for each $\alpha < \kappa, T$ has $< \lambda$ members of level $\le \alpha$, and T has $> \lambda$ κ-branches. If $\lambda < \lambda^{<\lambda}$ and $\neg(\exists\mu)[\mu \text{ strong limit } \ \& \ \mu \le \lambda < 2^\mu]$, then above $2^\kappa > \lambda$; see [Sh 430, 6.3].

10.23 Depth of homomorphic images of ultraproducts of Boolean algebras, [Sh 506, §3] and resolved for $\lambda_i > 2^{|\text{Dom } D|}$ in [Sh 589, §3].

10.24 If λ is strong limit singular, $\kappa = \text{cf}(\lambda)$ and e.g. $2^\lambda = \lambda^+$, then for some Boolean algebras B_1, B_2 we have: B_1 is λ^+-c.c., B_2 is $(2^\kappa)^+$-c.c. but $B_1 \times B_2$ is not λ^+-c.c. (see for more [Sh 575]). More constructions in [Sh 620].

10.25 If $\lambda = \lambda^{\beth_\omega}$, B a $\beth_\omega$-c.c. Boolean algebra of cardinality $\le 2^\lambda$ then B is λ-linked (that is $B \backslash \{0\}$ is the union of $\le \lambda$ sets of pairwise non-disjoint elements), see [Sh 575, §8].

10.26 On the measure algebra, [Sh 620].

10.27 On independent sets in Boolean Algebra, [Sh 620].

10.28 On ultraproducts of Boolean Algebra: $s(B)$, spread, i.e. constructing examples of $\text{inv}(\prod\limits_{i<\kappa} B_i/D) > \prod\limits_{i<\kappa} \text{inv}(B_i)/D$, see:

(a) for inv being s, (spread), Roslanowski Shelah [RoSh 534], [Sh 620] (*b*) similarly hd (hereditarily density)

(*c*) similarly hL (hereditarily Lindelof)

(*d*) for inv being Depth, [Sh 641]

(*e*) for inv being Length, [Sh 641].

Saharon - end?

Saharon - other things?

§11 Strong covering, forcing, choiceless universes and partition calculus

11.1 Preservation under forcing: essentially pcf and pp are preserved except for forcing notion involving large cardinals. Specifically if (the pair of universes) (V, W) satisfies κ-covering [i.e. $V \subseteq W$ and if $a \subseteq \text{Ord}, W \models |a| < \kappa$ then for some $b \in V, a \subseteq b \subseteq \text{Ord}$ and $W \models |b| < \kappa$] and $\mathfrak{a} \subseteq \text{Ord} \backslash \kappa$ is a set from W of cardinality $< \kappa$ of regulars of W then

$$\text{pcf}^V\{\text{cf}^V(\theta) : \theta \in \text{pcf}^V(\mathfrak{a})\} = \{\text{cf}^V(\lambda) : \lambda \in \text{pcf}^W(\{\text{cf}^W(\theta) : \theta \in \mathfrak{a}\})\}$$

(this applies for example to (K, V) if there is no inner model with measurable by Dodd and Jensen [DJ1]).

11.2 The strong covering lemma: see [Sh:f, Ch.XIII,§1,§2] or better [Sh:g, Ch.VII,§1,§2]; see more in [Sh 410, 2.6,p.407] and [Sh 580], each can be read independently.

Suppose $W \subseteq V$ is a transitive class of V including all the ordinals and is a model of ZFC, let $\lambda > \kappa$ be cardinals of V.

We say (W, V) satisfies the strong (λ, κ)-covering property if for every model $M \in V$ with universe λ and $<$ predicates and function symbols there is $N \prec M$ of cardinality $< \kappa, N \cap \kappa \in \kappa, N \in V$ but the universe of N belongs to W; we also use stronger versions (like the set of such N's is positive of even equal to $[\lambda]^{\leq \kappa}$ modulo some ideal, or weaker versions like union of few sets from W).
Those papers do this without using fine structure assumptions, just that (W, V) satisfies (λ, κ)-covering and related properties.
Uri? Saharon!

11.3 Application of ranks (see 2.2) to partition calculus: Shelah Stanley [ShSt 419]. If there is a nice filter of κ (see 2.2) and λ,cf$(\lambda) > \kappa =$ cf(κ), $(\forall \mu < \lambda)\, \mu^\kappa < \lambda$ then $\lambda \to (\lambda, \omega + 1)^2$.

11.4 Polarized Partition Relations
If λ is strong limit singular and $\kappa <$ cf(λ) and $2^\lambda > \lambda^+$, then $\binom{\lambda^+}{\lambda} \to \binom{\lambda}{\lambda}^{1,1}_2$, see [Sh 586].

11.5 See [Sh 497] - Saharon.

§12 Transversals and (λ, I, J)-sequences

See [Sh 161] (and [Sh 52]), a transversal is a one to one choice function.

12.1 If I is an ideal on $\kappa, \lambda >$ cf(λ) and pp$_I(\lambda) > \mu$, then we can find a family of functions $f_\alpha(\alpha < \mu)$ from κ to λ, which is λ^+-free for I i.e. any λ of them are strictly increasing on each $x \in$ Dom(I) if for each α we ignore a set $s_\alpha \in I$ such that $i \in \kappa \backslash s_\alpha \backslash s_\beta \Rightarrow f_\alpha(i) < f_\beta(i)$ (so $\{\text{Rang}(f_\alpha) : \alpha \in u\}$ has a transversal when $u \subseteq \mu, |u| \leq \lambda$) [Sh 355, 1.5A] (the case μ singular changes nothing for this purpose). So NPT(λ^+, κ) (see Definition below). On weakening "pp$_I(\lambda) > \mu$" to "pp$^+_I(\lambda) > \mu$" for μ successor of regular see [Sh 371, §6] (μ singular-easy). On weakening pp$_I(\lambda) > \mu$ to cov$(\lambda, \lambda, \kappa^+, 2) > \mu$, see [Sh 355, §6] for some variants; in particular NPT$_{J^{bd}_\omega}(\lambda^+, \aleph_0)$ when cov$(\lambda, \lambda, \kappa^+, 2) > \lambda$ by [Sh 355, 6.3,p.99].

12.2 Definitions of variants of NPT, [Sh 355, 6.1], [Sh 371, 6.3] for example NPT(λ, κ) means that there is a family $\{A_i : i < \lambda\}$ of sets each of cardinality $\leq \kappa$, and $< \lambda$ of them has a transversal, but not all. Similarly for NPT$_J(\lambda, \kappa)$ we have f_α : Dom$(J) \to$ ordinals as in 12.1.

12.3 Trivial and easy facts [Sh 355, 6.2,6.7], why concentrating on "NPT$(\lambda^+, \aleph_1)$, cf$(\lambda) = \aleph_0$" [Sh 355, 6.4].

12.4 If $\lambda >$ cf$(\lambda) = \aleph_0$ and cov$(\lambda, \lambda, \aleph_1, 2) > \lambda^+$, then NPT$_{J^{bd}_\omega}(\lambda^+, \aleph_1)$,
[Sh 355, 6.3] more in [Sh 355, 6.5,6.8], [Sh 371, 6.1], [Sh 371, 6.2] application to [RuSh 117],
[Sh 371, 6.4,6.5].

12.5 When λ is a strong limit of cofinality $\aleph_0$, there is $T \subseteq {}^\omega\lambda, |T| = 2^\lambda$ with no large dense subset, [Sh 355, 6.9] (there is a subclaim with more information).

12.6 If I is an ideal on $\kappa, \mu > \kappa \geq$ cf(μ), pp$^+_I(\lambda) > \lambda =$ cf$(\lambda) > \mu, \lambda_i =$ cf$(\lambda_i) > \kappa$ (for $i < \kappa$), tlim$_I \lambda_i = \mu, \langle f_i : i < \lambda\rangle$ is $<_I$-increasing cofinal in $\prod_{i<\kappa} \lambda_i/I$, then for some $A \subseteq \lambda, |A| = \lambda$ for every $B \subseteq A$ of cardinality λ and $\delta < \mu^+$ there is $B' \subseteq A$ of order type δ and $\langle s_\alpha : \alpha \in B'\rangle$ such that: $s_\alpha \in I, \alpha < \beta$ & $\zeta \in \kappa \backslash s_\alpha \backslash s_\beta$ & $\alpha \in B'$ & $B \in B' \Rightarrow f_\alpha(\zeta) < f_\beta(\zeta)$ (so $\langle \text{Rang}(f_\alpha \restriction (\kappa \backslash s_\alpha) : \alpha \in B'\rangle)$ is a sequence of pairwise disjoint sets (for somewhat more [Sh 430, 6.2,6.2A(3)]).

12.7 (κ-MAD families)

Let $\kappa =$ cf$(\kappa) > \aleph_0$. For any $\mu \geq 2^\kappa$ letting $\chi = \chi^\kappa_\mu = \sup\{\text{pp}_{J^{bd}_\kappa}(\mu') : 2^\lambda \leq \mu' \subseteq \mu, \text{cf}(\mu') = \kappa\}$ we have:

(a) every κ-almost disjoint subfamily of $[\mu]^\kappa$ (i.e. intersection if two has cardinality $< \kappa$) has cardinality $\leq \chi$; also $\chi^\kappa_\mu = T_{J^{bd}_\kappa}(i)$

(b) trivially there is maximal κ-almost disjoint family $\subseteq [\mu]^\kappa$ and all such families have the same cardinality which is in χ

(c) if $\chi_0 = \chi, \chi_{n+1} = \chi^\kappa_{(\chi_n)}, \chi_\omega = \sum\limits_{n<\omega} \chi_n$ <u>then</u>

(α) $\chi_n = \sup\{\text{pp}_{J_n}(\mu') : 2^\lambda \leq \mu' \leq \mu, \text{cf}(\mu') = \kappa\}$ where
$J_n = \{A \subseteq \kappa^n : (\exists^{<\kappa}\alpha_0)(\exists^{<\kappa}\alpha_1)\ldots(\exists^{<\kappa}\alpha_{n-1}(\langle\alpha_0,\ldots,\alpha_{n-1}\rangle \in A\}$

(β) $\chi^\kappa_{(\chi_\omega)} = \chi_\omega$
(hence is doubtful if it is consistent to have $\chi_n \neq \chi_{n+1}$).

<u>12.8</u> $\bar\eta = \langle \eta_\alpha : \alpha < \lambda\rangle$ is a (λ, I, J)-sequence for $\bar I = \langle I_i : i < \delta\rangle$ iff each $\eta_\alpha \in \prod\limits_{i<\delta} \text{Dom}(I_i)$, J is an ideal on δ, I is an ideal on λ, each I_i is an ideal on $\text{Dom}(I_i)$, and

$$X \in I^+ \Rightarrow \{i < \delta : \{\eta_\alpha(i) : \alpha \in X\} \in I_i\} \in J.$$

The definition was introduced in [Sh 575] and considered again in [Sh 620]. In [Sh 620] first the case of the Erdös-Rado ideal defined there was considered. For the case $\bar I = \langle J^{bd}_{\lambda_i} : i < \delta\rangle$ and $\lambda = \text{tcf}(\prod\limits_{i\leq\delta} \lambda_i / J^{bd}_\delta)$ and $J = J^{bd}_\lambda, I = J^{bd}_\delta$, the existence of a (λ, I, J)-sequence comes from pcf theory. Also the case $I_i = \prod\limits_{\ell<n_i} J^{bd}_{\lambda_{i,\ell}}$ for $\langle\lambda_{i,\ell} : \ell < n_i\rangle$ increasing a sufficient pcf condition for the existence of a (λ, I, J)-sequence was given in [Sh 620] which holds sometimes (for any given $\langle n_i : i < \delta\rangle$. Also in [Sh 620] the case $I_i = J^{bd}_{\langle\lambda_{i,\ell}:\ell<n\rangle}$, for $\langle\lambda_{i,\ell} : \ell < n\rangle$ a decreasing sequence of regulars was considered, giving a sufficient condition which requires pcf to be reasonably complicated. A most case $I_i = \prod\limits_{\ell<n} J^{\text{nst},\theta}_{\lambda_{i,\ell}}, \lambda_{i,\ell}$ regular decreasing, $J^{\text{nst},\theta}_\lambda$ is the ideal of non-stationary sets $+\{\delta < \lambda : \text{cf}(\delta) \neq \theta\}$, when e.g. $\delta < \kappa < \lambda_{i,\ell}$ and we prove existence for some $\langle\lambda_{i,\ell} : \ell < n, i < \delta\rangle$. Many applications for Boolean algebras can be found in [Sh 620].

<u>12.9</u> The family $\{\kappa : NPT(\kappa, \aleph_1)\}$ is not too small, see [Sh 108], Magidor Shelah [MgSh 204], [Sh 523].

§13 Model Theory and algebra

13.1 $L_{\infty,\lambda}$-equivalent non-isomorphic models in λ: if $\lambda > \mathrm{cf}(\lambda) > \aleph_1$ there are such models of cardinality λ (if $\mathrm{cf}(\lambda) = \aleph_1$, it suffices to have: there is $(\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ increasing sequence of regulars with limit $\lambda, \{\delta < \mathrm{cf}(\lambda)$: there is an unbounded $a \subseteq \delta$ with $\lambda > \max \mathrm{pcf}\{\lambda_i : i \in a\}\}$ is stationary); not known if this fails in some universe of set theory, see [Sh 355, §7].

13.2 Universal Models for example the class of linear orders. If λ is regular and $\exists \mu(\mu^+ < \lambda < 2^\mu)$, then there is in λ no universal linear order, not even a universal model (for elementary embeddings) for T in λ where T is a first order theory with the strict order property. For almost all singular λ we have those results, more specifically if λ is not a fixed point of the second order the result holds; and if it fails for λ the consequences for pp are not known to be consistent, see [KjSh 409] which rely on guessing clubs.

13.3 A much weaker demand on the first order T suffices in 13.2: NSOP_4, see [Sh 500, §2] on the remaining cardinals see some information in [Sh 457, §3]; on complimentary consistency (only for $\lambda = \aleph_1$) see [Sh 100, §4].

13.4 Universal models for $(\omega+1)$-trees with $(\omega+1)$-levels and or stable unsuperstable T:

Similar results: if λ regular $(\exists \mu)[\mu^+ < \lambda < \mu^{\aleph_0}]$ then there is no universal member; also for most singular [KjSh 447].
Similarly if $\kappa = \mathrm{cf}(\kappa) < \kappa(T), (\exists \mu)(\mu^+ < \lambda < \mu^\kappa)$.

13.5 Universal abelian groups have similar results for pure embedding (under reasonable restrictions (mainly the groups are reduced, because there are divisible universal abelian groups the interesting cardinals are $\lambda^{\aleph_0} > \lambda > 2^{\aleph_0}$). For torsion free reduced abelian groups, $\mathfrak{K}^{rtf}$, or reduced separable p-groups, $\mathfrak{K}^{rs(p)}$ if $2^{\aleph_0} + \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$, then there is no universal. For "most" λ, λ regular can be omitted.

(This and more [KjSh 455]).

13.6 We can use the usual embedding but restrict the class of abelian groups. The natural classes: $\mathfrak{K}^{rtf}$ (torsion free, reduced i.e. has no divisible subgroups) and $\mathfrak{K}^{rs(p)}$ (reduced separable p-groups). But in addition we restrict ourselves to the abelian groups which are $(< \lambda)$-stable (see [Sh 456], club guessing is used).

13.7 For classes $\mathfrak{K}^{rtf}, \mathfrak{K}^{rs(p)}$ from 13.6 of abelian groups under embeddings see [Sh 552]: mainly if $\lambda^{\aleph_0} > \lambda > 2^{\aleph_0}$ there are negative results except when some pcf phenomena not known to be consistent (also club guessing is used). Below the continuum there are independence results. More on the existence of universals see [Sh 457] on metric spaces see [Sh 552] and on normed spaces [DjSh 614].

13.8 For cardinals $\geq \beth_\omega$, for the classes $\mathfrak{K}^{rtf}, \mathfrak{K}^{rs(p)}$ the results in 13.7 are improved to have demands on the cardinals like 13.5, see [Sh 622].

13.9 Diamonds and Omitting Types In the omitting type theorem for $L(Q)$ in the λ^+ interpretation, not only $\lambda = \lambda^{<\lambda}$ (needed even for the completeness) was used in [Sh 82] but $(D\ell)_\lambda$ [for λ successor this is $\diamondsuit_\lambda$, generally it means: there is $\langle \mathscr{P}_\alpha : \alpha < \lambda \rangle, \mathscr{P}_\alpha$ a family of $< \lambda$ subsets of λ such that for every $A \subseteq \lambda$ for stationarily many $\delta < \lambda, A \cap \delta \in \mathscr{P}_\delta$]. Now by [Sh 460]: if $\lambda > \beth_\omega$ then $\lambda = \lambda^{<\lambda} \Leftrightarrow (D\ell)_\lambda$. In fact: if $\lambda = \lambda^{<\lambda}$ & $(\forall \mu < \lambda)(\mu^{tr,\kappa} < \lambda) \Rightarrow (D\ell)_{S^\lambda_\kappa}$ (where $(D\ell)_{S^\lambda_\kappa}$ is defined as above but for $\alpha \in S^\lambda_\kappa =: \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$, and $\mu^{tr,\kappa} = \sup\{\lambda :$ there is a tree with μ-nodes and λ κ-branches$\}$.

13.10 There are uses for proving Black Boxes (see [Sh:e, Ch.III,§6]), those are construction principles provable in ZFC, and have quite many uses, see there for references.

13.11 On tiny models: On tiny models see Laskovski, Pilley and Rothmaler [LaPiRo], M is tiny if $\mu = \|M\| < |T|, T$ categorical in $|T|^+$, where $|T|$ is the number of formulas up to equivalence. Assume further that for T not every regular type is trivial, then existence of such T for given μ is equivalent to the existence of $A_i \in [\mu]^\mu$ for $i < \mu^+$ such that $\bigwedge_{i<j} |A_i \cap A_j| < \aleph_0$, hence necessarily $\mu < \beth_\omega$. (Proved in the appendix of [Sh 460]).

13.12 On cofinalities of symmetric group: Let Sp be the family of regular λ such that the permutation group of ω is the union of a strictly increasing chain of subgroups. Now Sp has closure properties under pcf, say if $n < \omega \Rightarrow \lambda_n \in Sp$ then $\mathrm{pcf}\{\lambda_n : n < \omega\} \subseteq Sp$ (Shelah and Thomas [ShTh 524]).

13.13 Hanf number

On application to Hanf numbers see Grossberg Shelah [GrSh 238].

13.14 On the number of non-isomorphic models: see [Sh 600, §2].

§14 Discussion

14.1 Artificially/naturality thesis.

Probably you will agree that for a polyhedron v (number of vertices) e (number of edges) and f (number of faces) are natural measures, whereas $e+v+f$ is not, but from a deeper point of view $v-e+f$ runs deeper than all. In this vein we claim: for λ regular 2^λ is the right measure of $\mathscr{P}(\lambda)$, and λ^κ is a good measure of $[\lambda]^{\le\kappa}$. However, the various cofinalities are better measures. λ^κ is an artificial combination of more basic things of two kinds: the function $\lambda \mapsto 2^\lambda$ (λ regular which is easily manipulated) and the various cofinalities we discuss (which are not). For example $\text{pp}(\aleph_\omega) < \aleph_{\omega_4}$ is the right theorem, not $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4} + (2^{\aleph_0})^+$ (not to say: $2^{\aleph_\omega} < \aleph_{\omega_4}$ when $\aleph_\omega$ is strong limit). Also the equivalence of the different definitions which give apparently weak and strong measures, show naturality:

(a) $\text{cf}([\aleph_\omega]^{\aleph_0}) = \text{pp}(\aleph_\omega)$

(b) $\min\{|S| : S \subseteq [\lambda]^{\le\kappa} \text{ stationary}\} = \text{cf}([\lambda]^{\le\kappa}, \subseteq)$ for $\kappa < \lambda$

(c) if $\lambda \ge \mu > \theta > \sigma = \text{cf}(\sigma) > \aleph_0$ then

$$\text{cov}(\lambda,\mu,\theta,\sigma) = \sup\{\text{pp}_{\Gamma(\text{cf}(\chi),\theta,\sigma)}(\chi) : \mu \le \chi \le \lambda, \sigma \le \text{cf}(\chi) < \theta\}.$$

Note, $\chi \le \text{pp}_\theta(\lambda)$ says $[\lambda]^{\le\kappa}$ is at least as large as χ in a strong sense, whereas $\chi \ge \min\{|S| : S \subseteq [\lambda]^{\le\kappa} \text{ stationary}\}$ says that $[\lambda]^{\le\kappa}$ can be exhausted very well by χ "points" (for the right filters: measure 1).

We tend to think the pp's are enough, but there is a gap is our understanding concerning cofinality $\aleph_0$, mainly: is it true that

$$(*)\quad \lambda > \text{cf}(\lambda) = \aleph_0 \Rightarrow \text{cf}([\lambda]^{\le\kappa}, \subseteq) = \text{pp}_{\aleph_0}(\lambda).$$

We have many approximations saying that this holds in many cases (see 6.5).

More generally, we should replace power by products, and cardinality by cofinality, and therefore deal with $\text{pcf}(\mathfrak{a})$.

14.2 The Cardinal Arithmetic below the continuum thesis:

We should better investigate our various cofinalities without assuming anything on powers (for example, the difference between the old result $\text{pp}(\aleph_\omega) < \aleph_{(2^{\aleph_0})^+}$ and the latter result $\text{pp}(\aleph_\omega) < \aleph_{\omega_4}$ is substantial); as

(a) you should try to get the most general result (when it has substance of course) (*b*) if we add many Cohen reals, all non-trivial products are $\geq 2^{\aleph_0}$, but our various cofinalities do not change, so we should not ignore this phenomenon

(*c*) even if we want to bound 2^λ for λ strong limit singular, we need to investigate what occurs in the interval $[\lambda, 2^\lambda]$ which is a problem of the form indicated above; this is central concerning the problem (see [Sh 430]): if λ is the ω_1-fixed point then 2^λ is $<$ the ω_4-th fixed point

(*d*) looking at cardinal arithmetic without assumptions on the function $\lambda \to 2^\lambda$, makes induction on cardinality more useful.

<u>14.3</u> <u>Thesis</u>

(*A*) $\text{pp}(\lambda)$ is the right power set operation.

$\lambda \mapsto 2^\lambda$ (λ regular) is very elastic, you can easily manipulate it, but $\text{pp}(\lambda)$ (λ singular) and $\text{cov}(\lambda, \mu, \theta, \sigma)$ are not; it is hard to manipulate them, and we can prove theorems about them in ZFC.

(*B*) ?

1) Consider $[\lambda]^{\leq\kappa} = [\lambda]^{\leq\kappa}$, the family of subsets of λ of cardinality $\leq \kappa$, when $\lambda > \kappa$ (see 14.1).
2) λ^κ is the crude measure of $[\lambda]^{\leq\kappa}$.

It is very interesting to measure it, and cardinality is generally a very crude measure; $\text{pp}_\kappa(\lambda)$ is a fine measure; and we have intermediate ones: $\text{cf}([\lambda]^{\leq\kappa}, \subseteq)$, $\min\{|S| : S \subseteq [\lambda]^{\leq\kappa} \text{ stationary}\}$ and more. The best is when we can compute cruder numbers from finer ones; particularly when they are equal, so we could use different definitions for the same cardinal depending on what we want to prove. So we want to show that the $\text{pp}_{\Gamma(\text{cf}(\lambda))}(\lambda)$ (λ singular) is enough.

<u>14.4</u> $\text{pp}_{\Gamma(\theta,\sigma)}(\lambda)$ is the finest we have for what we want; they are like the skeleton of set theory; you can change easily your dress and even can manage to change how much flesh you have; but changing your bones is harder. You may take hypermeasurable λ, blow up 2^λ and make it singular; this does not affect for example $\text{pp}_{\Gamma(\aleph_1)}(\lambda^*)$ when $\lambda^* > \lambda$, $\text{cf}(\lambda^*) = \aleph_1$ (even if $\lambda^* <$ new 2^λ), nor $\text{cov}(\lambda^*, \lambda^*, \aleph_1, \sigma)(\sigma = 2, \aleph_1)$; they measure really how many subsets of λ^* of cardinality $\aleph_1$ there are - not through some $\lambda' < \lambda^*$ having many subsets of cardinality $\leq \aleph_1$.

<u>14.5</u> Subconscious remnants of GCH have continued to influence the research: concentration on strong limit cardinals; but from our point of view, even if $2^{\aleph_0}$ is large and $\mu < 2^{\aleph_0} \Rightarrow 2^\mu = 2^{\aleph_0}$, the cardinal arithmetic below $2^{\aleph_0}$ does not become simpler.

Also GCH was used as an additional assumption (or semi-axiom), but rarely wsa the negation of CH used like this: simply because one didn't know to prove interesting theorems form $\neg CH$. But now we know that violations of GCH have interesting consequences (see below).

14.6 Up to now we have many consequences of GCH (or instances of it) and few of the negations of such statements. We now begin to have consequences of the negation, for example see here 10.10; so we can hope to have proofs by division to cases. For example, let λ be a strong limit singular; if $\mathrm{pp}(\lambda) > \lambda^+$ then $\mathrm{NPT}(\lambda^+, \mathrm{cf}(\lambda))$ and if $\mathrm{pp}(\lambda) \leq \lambda^+$ then $2^\lambda = \lambda^+$ (and $\diamondsuit^*_{\{\delta<\lambda^+ : \mathrm{cf}(\delta) \neq \mathrm{cf}(\lambda)\}}$) and so various constructions are possible (see here 10.10(b) and [Sh 462] on more, also [Sh:E9], [RoSh 534]).

14.7 The right problems.

An outside viewer may say that the main problem,

$$(\aleph_\omega = \beth_\omega \Rightarrow 2^{\aleph_\omega} < \aleph_{\omega_1})$$

was not solved. As an argument we may accuse others: maybe $\aleph_{\omega_4}$ is the right bound. But more to the point is our feeling that this is not the right problem; right problems are:

(α) Does $\mathrm{pcf}(\mathfrak{a})$ always have cardinality $\leq |\mathfrak{a}|$?

(β) Is $\mathrm{cov}(\lambda, \lambda, \aleph_1, 2) =^+ \mathrm{pp}(\lambda)$ when $\mathrm{cf}(\lambda) = \aleph_0$?

Now (α) is just a member of a family of problems quite linearly ordered by implication discussed in [Sh 420, §6], [Sh 460], which seem unattackable both by the forcing methods and ZFC methods. The borderline between chaos and order seems

(α)$^-$ can $\mathrm{pcf}(\mathfrak{a})$ has an accumulation point which is an inaccessible cardinal (hopefully not).

Similarly (β) is the remnant of the conjecture that all $\mathrm{cov}(\lambda, \mu, \theta, \sigma)$ can be expressed by the values of $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda')$ and even $\mathrm{pp}_{\Gamma(\mathrm{cf}(\lambda'))}(\lambda')$; this has been proved in many cases (see 6.5). On an advance see [Sh 460].

Also though (α), (β) have not been solved, much of what we want to derive from them has been proved.

Another problem on which no light was shed is:

(γ) if λ is the first fix point, find a bound on $\mathrm{pp}(\lambda)$ (or better $\mathrm{cov}(\lambda, \lambda, \aleph_1, 2)$).

We can hope for the ω_4-th fixed point, to serve as a bound but will be glad to have the first inaccessible as a bound. Even getting a bound assuming GCH below λ would open our eyes. This becomes a problem after [Sh 111], [Sh:b, Ch.XII,§5,§6].

(δ) Generalize [Sh 355, §1] to deal with what occurs above $\text{tlim}_I \lambda_i$ (for example 2.1, (λ, σ)-entangled linear order).

More accurately, assume $\prod_{i<\delta} \lambda_i/J$ has true cofinality $\lambda, \mu = \text{tlim}_I(\lambda_i) = \sup(\lambda_i), \lambda_i$ regular $> \delta$, and $\sup_{i<\delta} \lambda_i < \theta = \text{cf}(\theta) < \lambda$. We can find regular $\lambda'_i < \lambda_i$ such that $\text{tcf}(\Pi \lambda'_i/J) = \theta)$ as exemplified by $\bar{f}$, which is μ^+-free (hence $\text{tlim}(\lambda'_i) = \lambda_i$) in addition: if $\delta < \theta$, $\text{cf}(\delta) < \theta$ and $\text{cf}(\delta) > 2^{|\delta|}$ (or just $\bar{f} \restriction \delta$ has a $<_J$-lub) then without loss of generality f_δ/J is the $<_J$-lub of $\bar{f} \restriction \delta$, we want to know something on $\langle \text{cf}(f_\delta(\alpha)) : \alpha < \delta \rangle$. For more information see [Sh 400, 4.1,4.1A].
Note that we also do not know, for example

(ε) if $\text{cf}(\lambda) \leq \kappa < \lambda$, is $\text{cf}(\text{pp}_\kappa(\lambda)) > \lambda$? (we know that it is $> \kappa$)

(ζ) we believe pcf considerations will eventually have impact on cardinal invariants of the continuum, but this has not materialized so far.

<u>14.8</u> The perspective here led to phrasing some hypothesis, akin to GCH or SCH.

The "strong hypothesis" says $\text{pp}(\lambda) = \lambda^+$ for (every) singular λ; note it is like GCH but is not affected say by c.c.c. forcing, it follows from $\neg 0^\#$ and from GCH; its negation is known to be consistent and I feel it is a natural axiom. Other hypothesis may still follow from ZFC for example, the medium hypothesis says $|\text{pcf}(\mathfrak{a})| \leq |\mathfrak{a}|$, and the weak say $\{\mu : \text{pp}(\mu) \geq \lambda, \mu < \lambda, \text{cf}(\mu) = \aleph_0 [> \aleph_0]\}$ is countable [finite], there are intermediate ones, such hypothesis and consequences are dealt with in [Sh 420, §6], see more in [Sh 460], [Sh 513]. Particularly concerning the connection of the medium and weak ones, (see 11.3, 6.18).

* * *

Part B - Corrections to the book [Sh:g]

page 50,line 22: see more in Part C.

page 51,line 12: replace λ^+ by μ.

page 51,line 13: see more in Part C.

page 66,Theorem 3.6: second line of theorem:
replace $\lambda^{\beta+1}$ by $\lambda_0^{\beta+1}$

add after the second line of Remark 3.6A:
2) This is essentially the proof from [Sh:b, Ch.XIII,§6] and more appears in Ch.IX

first line of the proof:
replace $\lambda > \aleph_0$ by "$\lambda_0 > |\alpha|^+$ (why? as we can replace λ_0 by λ_0^+ and deduce the result on the original λ_0 from the result on λ_0^+)"

replace fifth line of the proof:

$$N_h = \cap\{\text{Skolem Hull}_M(\lambda_0 \cup \bigcup_{\beta<\alpha} C_\beta) : C_\beta \text{ a club of } f(\lambda^{+\beta+1}) \text{ for } \beta < \alpha\}$$

add in the end of the proof:

Clearly this family is a family of subsets of λ each of cardinality at most λ_0 of the right cardinality. So we have to prove just that it is cofinal. So let X be a subset of λ of cardinality at most λ_0, and we shall find a member of the family which includes it. Let χ be large enough. By 3.4 we can find an elementary submodel N_i of $(\mathscr{H}(\chi), \in, <^*_\chi)$, for $i \leq \delta =: |\alpha|^+$ each of cardinality such that $\{F, \lambda_0, \alpha^*, \lambda, X, f, g\} \in N_i$ and $i < j \to N_i \in N_j$ increasing continuous with i and condition (b) form 3.4 holds for $f \in F$.

It is enough to prove that

(*) N_f includes $N_\delta \cap \lambda$

for this it is enough to prove

(**) if C_β is a club of $\lambda_0^{\beta+1}$ for each $\beta < \alpha^*$ and M' is the Skolem Hull in M of $\lambda_0 \cup \bigcup\{C_\beta : \beta < \alpha^*\}$ then M' include $N_\delta \cap \lambda$.

For this we prove by induction on $\gamma \leq \alpha$ that

$(**)_\gamma$ M' includes $\lambda \cap \lambda_0^{+\gamma}$.

Case 1: $\gamma = 0$.

In this case as M includes λ_0 this is trivial.

Case 2: γ a limit cardinal ordinal.

In this case the induction hypothesis implies the conclusion trivially.

Case 3: $\gamma = \beta + 1$.

Use the induction hypothesis and the choice of the functions f and g.

(See more Ch.IX,3.3)

page 136,lines 21,22,23:

replace by:

No problem to define. We define B_i^α (for $i < \lambda, \alpha \in S$) by induction on α:

$$B_i^\alpha = \begin{cases} \{\beta : \mathrm{cf}(\beta) \neq \lambda \text{ and } \beta \in A_i^\alpha \vee \beta = \sup(\beta \cap A_i^\alpha)\} & \underline{\text{if}}\ \mathrm{cf}(\alpha) \neq \aleph_1 \\ \bigcap\{\bigcup_{\beta \in C} B_i^\beta : C \text{ a club of } \alpha \text{ such that } \bigwedge_{\beta \in C} \mathrm{cf}(\beta) = \aleph_0\} & \underline{\text{if}}\ \mathrm{cf}(\alpha) = \aleph_1 \end{cases}$$

(or see [Sh 351, 4.1]).

page 210, line 15:

add:

or λ is not Mahlo and we can use Ch.III.

page 222,line 24:

replace by:

Definition 1.4. 1) We say D is strongly nice if it is strongly nice to every

page 224,line 8:

replace by:

$$\sup\left\{\prod_{i<\omega_1} f(i)/D : D \text{ is a normal filter extending } D^*\right\}$$

page 228,line 1:

replace D^* by $D^* \in V^*$.

pages 334-337: see a rewriting in [Sh:E11]

page 334,line -4

replace by:

(2) The first phrase follows from part 1 and check the second

page 335,line 4:
replace "$f \restriction \mathfrak{b}\mu[\mathfrak{a}] \le f^{\mu}_{\alpha}$" by "$f \restriction \mathfrak{b}_{\mu}[\mathfrak{a}] \le f^{\mu}_{\alpha}$"

page 335,line 18:
space after $\emptyset$; replace $\bigcap\limits_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}]$ by $\bigcup\limits_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}]$

page 336,line 3:
replace $\mathfrak{b}$ by $\mathfrak{c}$

page 336,line -7:
replace $\square_{3.3}$ by $\square_{3.2}$

page 381,lemma3.5 and page 383,line 21:
No! But see [Sh 400, 5.12] and [Sh 513, §6]

page 410,line -1:
replace by: $\{\delta < \sigma : \mathrm{cov}(\lambda_\delta, \lambda_\delta, \theta^+, 2) < \mu_\delta\}$ contains a club of σ, where

$(*)(i)$ let μ_δ be $\mathrm{pp}^{cr}_{\theta}(\lambda_\delta)$ the first regular $\mu > \lambda_\delta$ such that:
if $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda_\delta \backslash |\mathfrak{a}|^+$, then
$\sup\{\max \mathrm{pcf}(\mathfrak{b}) : \mathfrak{b} \subseteq \mathfrak{a}, |\mathfrak{b}| \le \theta$ and $(\forall \chi < \lambda_\delta) \max \mathrm{pcf}(\mathfrak{b} \cap \chi) < \lambda_\delta\}$
(so normally this means $\mathrm{cov}(\lambda_\delta, \lambda_\delta, \theta^+, 2) =^+ \mathrm{pp}_\theta(\lambda_\delta)$).

page 411,line 1:
replace by:

(ii) $\mathrm{cov}(\lambda, \lambda, \theta^+, 2) < \mathrm{pp}^{cr}_{\theta}(\lambda)$ which normally means $\mathrm{cov}(\lambda, \lambda, \theta^+, 2) =^+ \mathrm{pp}_\theta(\lambda)$, e.g. if $\mathrm{cov}(\lambda_i, \theta^+, \theta^+, 2) < \lambda$ for a club of $i < \sigma$

(iii) if e.g. $\sigma^{\aleph_0} < \lambda$, then we can add
$\{\delta < \sigma : \text{if } \mathrm{cf}(\delta) = \aleph_0 \text{ then } \mathrm{pp}^{cr}_{J^{bd}_\omega}(\lambda_\delta) > \mathrm{cov}(\lambda_\delta, \lambda_\delta, \theta, 2)\}$ contains a club
(for the changes needed for the proof see below, Part C).

page 417,line 11:
add:
Here examples are constructed for λ singular and in [Sh 572] for $\lambda = \aleph_1$ which was the last case.

page 418,line 20:
sequence of not sequence of

Part C - Expansions for [Sh:g]

page 50,line 22:
add: [this is the proof of II,1.4(3)].

Case 1: $\mathrm{otp}(A)$ is zero.
Trivial.

Case 2: $\mathrm{otp}(A)$ is a successor ordinal.

Let α be the last member of A and let A' by $A \backslash \{\alpha\}$. Clearly the order type of A' is (strictly smaller than that of A) hence by the induction hypothesis we can find $s'_\beta \in i$ for $\beta \in A'$ as required. Define s_β for $\beta \in A$ as follows:

if $\beta = \alpha$, then $s_\beta = \emptyset$ and if $\beta \in A'$ then $s_\beta =: \{i < \kappa : i \in s'_\beta$ or $f_\alpha(i) \leq f_\beta(i)\}$.
Now s_β is a subset of κ and if $\beta = \alpha$ is the union of two sets: s'_β and $\{i < \kappa : f_\alpha(i) \leq f_\beta(i)\}$, now the first belongs to I by its choice and the second as we know $f_\beta <_I f_\alpha$ (because $\beta < \alpha$). So S_β, their union is in I, too.
This holds also in the case $\beta = \alpha$. So $s_\beta \in I$ for $\beta \in A$, and it is easy to check the requirements.

Case 3: $\mathrm{otp}(A)$ is a limit ordinal.

Let δ be $\sup(A)$, so is a limit ordinal. So by 1.3(ii)(δ) there is a closed unbounded subset C of δ and sets $\tau_\alpha \in I$ for $\alpha \in C$ such that $i \in \kappa \backslash \tau_\alpha \backslash s_\beta$ and $\alpha < \beta$ implies $f(i) < f_\beta(i)$.
Without loss of generality $0 \in C$ (let $t_0 =: \{i < \kappa : f_0(i) \geq f_{\mathrm{Min}(A)}(i)\}$).

Now for every $\alpha \in C$ let $A_\alpha =: A \cap [\alpha, \mathrm{Min}(A \backslash (\alpha+1))$. Clearly $\mathrm{otp}(A_\alpha) < \mathrm{otp}(A)$, let $A'_\alpha =: A_\alpha \cup \{\alpha\}$. So $\mathrm{otp}(A'_\alpha) = 1 + \mathrm{otp}(A_\alpha) < \mathrm{otp}(A)$ (as the latter is a limit ordinal). So we can apply the induction hypothesis, getting s'_β for $\beta \in A'_\alpha$ as guaranteed there.

Now we define s_β for $\beta \in A$ as follows: let $\alpha_\beta =: \sup(C \cap \beta)$ and
$\gamma_\beta =: \mathrm{Min}(A \backslash (\alpha+1))$. So $\beta \in A_{\alpha_\beta}$ hence s'_β is well defined and let
$s_\beta =: s'_\beta \cap \{i < \kappa$: it is not true that $f_{\alpha_\beta}(i) \leq f_\beta(i)\}$.
Now check.

* * *

page 51,line 13:
add to the end of line (this is line 7 of the proof of II,1.5A).

Of course, we do not have knowledge on the relation between $f_\alpha(i)$ and $f_\beta(j)$, so we just e.g. use f'_α defined by $f'_\alpha(i) =: \kappa f_\alpha(i) + i$ (so f'_α is a function from κ to λ (as $\kappa < \lambda$)). Now $\langle f'_\alpha : \alpha < \mu \rangle$ is as required (note that $\langle \{f_\alpha(i) : i < \mu\} : i < \kappa \rangle$ is a sequence of pairwise disjoint subsets of λ).

MORE ON II,3.5

17.1 Claim. *Assume*

(a) $\mathfrak{a} = \{\lambda_i : i < \delta\}$ *is an increasing sequence of regular cardinals* $> \delta$

(b) $\lambda = \ tcf(\mathfrak{a}/J^{bd}_\delta)$

(c) $\lambda_0 > 2^{|i|}$ *for* $i < \delta$ *o rjust* $\lambda_0 > |pcf(\mathfrak{a})|$

(d) $cf(\delta) > \aleph_0$

(e) $S =: \{i < \delta :$ *for some* $i_0 < i_1, pcf\{\lambda_j : i_0 < j < i\} \setminus \sum\limits_{j<i} \lambda_j$ *is a singleton cardinal* $< \sup\limits_{j<\delta} \lambda_j\}$ *is stationary.*

Then we can find $\langle f_\alpha : \alpha < \lambda\rangle$ *such that*

(a) $f_\alpha \in \prod\limits_{i<\delta} \lambda_i$ *is* $<_{J^{bd}_0}$*-increasing and cofinal*

(b) *if* $f \in \prod\limits_{i<\delta} \lambda_i$ *and* $(\forall i < \delta)(\exists \alpha < \lambda)(f \restriction i = f_\alpha \restriction i)$ *then* $f \in \{f_\alpha : \alpha < \lambda\}$.

Remark. This is just the proof of [Sh:g, II,3.5], just we use more of it.

Proof. Let $\mathfrak{a} = \{\lambda_i : i < \delta\}$, so $\min(\mathfrak{a}) > |\text{pcf}(\mathfrak{a})|$. Let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta : \theta \in \text{ pcf}(\mathfrak{a})\rangle$ be a generating sequence for pcf($\mathfrak{a}$). Choose $\langle \bar{f}^\theta : \theta \in \text{ pcf}(\mathfrak{a})\rangle$ as in claim 17.3 below. Now we let $\mathscr{F} = \{f \in \prod\limits_{i<\delta} \lambda_i$: for every $\theta \in$ pcf($\mathfrak{a}$) for some $n < \omega$ and $\theta_0 < \ldots \ldots < \theta_{n-1}$ from pcf($\mathfrak{b}_\theta$) and $\alpha_0 < \theta_0, \ldots, \alpha_{n-1} < \theta_{n-1}$ we have $f \restriction \mathfrak{b}_\theta = \max\{f^{\theta_\ell}_{\alpha_\ell} : \ell < n\}\}$.
First clearly

$(*)_2$ $\alpha < \lambda \Rightarrow f_\alpha \in \mathscr{F}$.

Secondly, the main point is

$(*)_2$ if $f', f'' \in \mathscr{F}$ then $f' <_{J^{bd}_{\mathfrak{a}\delta}} f''$ or $f' =_{J^{bd}_\delta} f''$ or $f'' <_{J^{bd}_{\mathfrak{a}}} f'$.

Why $(*)_2$ holds? gives $f', f'' \in \mathscr{F}$ let $\mathfrak{c}_1 = \{\theta \in \mathfrak{a} : f'(\theta) < f''(\theta)\}, \mathfrak{c}_2 = \{\theta \in \mathfrak{a} : f'(\theta) = f''(\theta)\}$ and $\mathfrak{c}_3 = \{\theta \in \mathfrak{a} : f'(\theta) > f''(\theta)\}$, so $\langle \mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3\rangle$ is a partition of $\mathfrak{a}$. Let $E = \{i < \delta :$ for $\ell = 1, 2, 3$ if $\sup(\mathfrak{c}_\ell) = \sup(\mathfrak{a})$ then $\sup(\mathfrak{c}_\ell \cap \lambda_i) = \sup(\mathfrak{a} \cap \lambda_i)$ and if $\sup(\mathfrak{c}_\ell) < \sup(\mathfrak{a})$ then $\sup(\mathfrak{c}_\ell) < \lambda_j$ for some $j < i\}$.

Clearly E is a club of δ; by clause (c) of the assumption, $S \subseteq \delta$ is stationary hence $S \cap E \neq \emptyset$, so let $i \in S \cap E$ and let θ_i be the single member of $\mathrm{pcf}(\mathfrak{a} \cap \lambda_i) \backslash \bigcup_{j<i} \lambda_j = \mathrm{pcf}(\{\lambda_j : j < i\}) \backslash \sum_{j<i} \lambda_j$ (recall the definition of S). So $\mathfrak{b}_{\theta_i}$ is $\subseteq \mathfrak{a} \cap \lambda_i$ and contains an end-segment of it - say $\mathfrak{b}'$. By 17.3(β) and the choice of θ_i, we know that for some end segment $\mathfrak{b}''$ of $\mathfrak{b}'$, $f' \restriction \mathfrak{b}'' \in \{f^{\theta_i}_\alpha \restriction \mathfrak{b}'' : \alpha < \theta_i\}$ and without loss of generality also $f'' \restriction \mathfrak{b}'' \in \{f^{\theta_i}_\alpha \restriction \mathfrak{b}'' : \alpha < \theta_i\}$. So for some $\beta', \beta'' < \theta_i$ we have $f' \restriction \mathfrak{b}'' = f^{\theta_i}_{\beta'} \restriction \mathfrak{b}''$ and $f'' \restriction \mathfrak{b}'' = f^{\theta_i}_{\beta''} \restriction \mathfrak{b}''$.

Now $\beta' < \beta'' \vee \beta' = \beta'' \vee \beta' > \beta''$ and accordingly we get one of the three possibilities in $(*)_2$.

Now clearly we are done.

17.2 Claim. *1) In 17.1 we can weaken assumption (d) to $(d)^-_{\mathfrak{a}}$ letting $\langle \mu_i : i < \sigma \rangle$ be increasing continuous with limit sup$(\mathfrak{a})$ so $\sigma =$ cf$(\sup(\mathfrak{a}))$ for some normal filter D on cf$(\sup(\mathfrak{a}))$*

- $(d)^-_D$ *if $\mathfrak{a}' \subseteq \mathfrak{a}(=: \{\lambda_i : i < \delta\}), \sup(\mathfrak{a}') = \sup(\mathfrak{a}')$ then $\{i < \sigma : \max(pcf(\mathfrak{a}' \cap \mu_i)) = \max(pcf(\mathfrak{a} \cap \mu_i))$.*

2) Assume $\mathfrak{a}$ has no last element and $\lambda =$ tcf$(\pi\mathfrak{a}/J^{bd}_{\mathfrak{a}})$ and $\mu < \sup(\mathfrak{a}) \Rightarrow$ max pcf$(\mathfrak{a} \cap \mu) < \sup(\mathfrak{a})$, (e.g. $\mathfrak{a} = \{\lambda_i : i < \delta\}$ from 17.1 assuming clauses (a)-(d) of 17.1.

<u>Then</u> for some unbounded $\mathfrak{a}^* \subseteq \mathfrak{a}$, we have (clause (a), (b), (c) of 17.1 and) clause $(d)^-_{\mathfrak{a}^*}$ of part (1) holds (hence the conclusion of 17.1).

Proof. 1) Let $\mathfrak{a} = \{\lambda_i : i < \delta\}, \lambda_i$ increasing.

We repeat the proof of 17.1. So our problem is that $f', f'' \in \mathscr{F}$ and in the partition $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3$ of $\mathfrak{a}$, at least two parts are unbounded in $\mathfrak{a}$ say $\mathfrak{c}_{\ell_1}, \mathfrak{c}_{\ell_2}$. We define the club E of σ as there, clearly

$\boxtimes$ if $\theta \in \mathrm{pcf}(\mathfrak{c}_\ell) \backslash \{\lambda\}$ then $\mathfrak{b}_\theta \backslash \mathfrak{c}_\ell \in \mathbf{J}_\theta[\mathfrak{a}]$.

Now $\sup(\mathfrak{c}_\ell) = \sup(\mathfrak{a}) \Rightarrow \{j < \sigma : \max \mathrm{pcf}(\mathfrak{c}_\ell \cap \mu_\ell) = \max \mathrm{pcf}(\mathfrak{a} \cap \mu_\ell)\} = \mathrm{cf}(\sup(\mathfrak{a}))\}$ mod D.
So for the D-majority of $j < \sigma$ we have $\sup(\mathfrak{c}_\ell \cap \mu_j) = \mu_0 = \sup(\mathfrak{c}_{\ell_2} \cap \mu)$ and max $\mathrm{pcf}(\mathfrak{c}_{\ell_j} \cap \mu_j) =$ max $\mathrm{pcf}(\mathfrak{a} \cap \mu_j) =$ max $\mathrm{pcf}(\mathfrak{c}_{\ell_2} \cap \mu_j)$ and we get contradiction by $\boxtimes$.
2) We try to choose $\langle \mathfrak{a}_\eta : \eta \in {}^n\sigma \rangle$ by induction on $< \omega$ such that

- (i) $\mathfrak{a}_{<>} = \mathfrak{a}$
- (ii) $\mathfrak{a}_\eta \subseteq \mathfrak{a}_{\eta \restriction n}$ for $\eta \in {}^{n+1}\sigma$
- (iii) $\sup(\mathfrak{a}_\eta) = \sup(\mathfrak{a})$

(*iv*) for every $\eta \in {}^n\sigma$ for some club E_η of σ we have: for every $j \in E_\eta$ there is $i < j$ such that $\max \mathrm{pcf}(\mathfrak{a}_{\eta\hat{}<i>} \cap \mu_j) < \max \mathrm{pcf}(\mathfrak{a}_n \cap \mu)$.

Now for $n = 0$ there is no problem and if $\mathfrak{a}_n, \langle \mathfrak{a}_\eta : \eta \in {}^n\sigma\rangle$ has been chosen but there is no suitable $\langle \mathfrak{a}_\eta : \eta \in {}^{n+1}\sigma\rangle$ then for some $\eta \in {}^n\sigma$ letting $\mathscr{P}_\eta = \{\{i < \sigma : \max \mathrm{pcf}(\mathfrak{b} \cap i) < \max \mathrm{pcf}(\mathfrak{a}_\eta \cap i), \mathfrak{b} \subseteq \mathfrak{a}_\eta, \sup(\mathfrak{b}) = \sup(\mathfrak{a}_\eta)\}$, the normal ideal D_η (on σ) which $\mathscr{P}_\eta$ generates satisfies $\emptyset \in D_\eta$ so $\mathfrak{a}_\eta, D_\eta$ are as required. Lastly, not all the $\mathfrak{a}_\eta$'s ard defined as then we let $E = \{i < \sigma : i$ a limit ordinal such that $\eta \in {}^{\omega>}i \Rightarrow i \in E_\eta\}$, clearly E is a club of σ_i. Now for any $i \in E$, we choose by induction on $n < \omega$, a sequence $\eta_n \in {}^n i$ such that $\eta_n \triangleleft \eta_{n+1}$ and $\max \mathrm{pcf}(\mathfrak{a}_{\eta_n}) > \max \mathrm{pcf}(\mathfrak{a}_{\eta_{n+1}})$. We let $\eta_0 =<>$ and η_{n+1} exists by clause (iv). So $\langle \max \mathrm{pcf}(\mathfrak{a}_{\eta_n}) : n < \omega\rangle$ is a strictly decreasing sequence of cardinals, a contradiction. So we are done.

17.3 Claim. *Assume*

(*a*) $|pcf(\mathfrak{a})| < min(\mathfrak{a}), \mathfrak{a}$ *as usual a set of regular cardinals*

(*b*) $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta : \theta \in [\mathfrak{a}]\rangle$ *a generating sequence for* $pcf(\mathfrak{a})$ *(exists by x.x) which is closed (i.e.* $\mu \in \mathfrak{b}_\theta \Rightarrow \mathfrak{b}_\mu \subseteq b_\theta$*) and normal (?)* $(\mathfrak{a} \cap pcf(\mathfrak{b}_\mu) \cap \mathfrak{a} = \mathfrak{b}_\mu$. *We can choose by induction on* $\theta \in pcf(\mathfrak{a}), \bar{f}^\theta = \langle f^\theta_\alpha : \alpha < \lambda\rangle$ *such that*

  (α) $f^\theta_\alpha \in \Pi\, \mathfrak{b}_\theta$ *is* $<_{J_{<\theta}[\mathfrak{b}_\theta]}$*-increasing and cofinal*

  (β) *if* $\theta \in pcf(\mathfrak{a}), \alpha < \theta$ *and* $\mu \in \mathfrak{b}_\theta$ *then for some* $n < \omega, \mu_0, \ldots, \mu_{n-1} \in pcf(\mathfrak{b}_\mu)$ *and* $\beta_0 < \mu_0, \ldots, \beta_{n-1} < \mu_{n-1}$ *and* $\beta < \mu$
  $f \restriction \mathfrak{b}_\mu = Max[\{f^{\mu_\ell}_{\beta_\ell} : \ell < n\}]$.

Proof. This is a restatement of x.x.

More on III,4.10: Densely running away from Colours

18.1 Question [Hajnal]: Let $\lambda = (2^{\aleph_0})^+$. Is there $c : [\lambda]^2 \to \omega$ such that

$$(\forall A \in [\lambda]^\lambda)(\forall n < \omega)(\exists B \in [A]^\lambda)(n \notin \text{ Rang}(c \restriction [B]^2)?$$

Answer: yes.

Clearly it is equivalent to the property $P_7(\lambda, \aleph_0, 2)$ defined below for $\lambda = (2^{\aleph_0})^+$. Now Claim 18.3 covers the case $\lambda = (2^{\aleph_0})^+$ and then we have more. We look again at [Sh:e, Ch.III,4.9-4.10C,pp.177-181].

18.2 Definition. $\text{Pr}_7(\lambda, \sigma, \theta)$ where $\lambda \geq \theta \geq 1, \lambda \geq \sigma = \text{ cf}(\sigma)$ means that there is $c : [\lambda]^2 \to \sigma$ such that

$$(\forall A \in [\lambda]^\lambda)(\forall \alpha < \sigma)(\exists B \in [A]^\lambda)(\text{Min Rang}(c \restriction [B]^2) > \alpha)$$

(So far, θ is redundant). Moreover, if $w_\alpha \in [\lambda]^{<1+\theta}$ for $\alpha < \lambda$ are pairwise disjoint and $\zeta < \sigma$ then for some $X \in [\lambda]^\lambda$ we have

(∗) if $\alpha < \beta$ are from X then
$(\forall i \in w_\alpha)(\forall j \in w_\beta)(c\{i,j\} \geq \zeta)$.

18.3 Claim. *Assume λ is a regular uncountable cardinal, $2 \leq \kappa < \lambda$ and $\otimes^\kappa_\lambda$ holds or just $\oplus^\kappa_\lambda$ (see below).*
Then there is a symmetric 2-place function c from λ to $\aleph_0$ such that:

(∗) *if $\langle w_i : i < \lambda\rangle$ is a sequence of pairwise disjoint non-empty subsets of $\lambda, |w_i| < \kappa$ and $n < \omega$, then for $Y \in [\lambda]^\lambda$ for every $i < j$ from Y we have:*

$$max(w_i) < \ min(w_j)$$

$$\bigwedge_{\alpha \in w_i} \bigwedge_{\beta \in w_j} c(\alpha, \beta) > n.$$

(i.e. $Pr_7(\lambda, \aleph_0, \kappa)$).
Note that Definition 18.4(1) is from [Sh:e, Ch.III,4.10,p.178].

18.4 Definition. 1) For a Mahlo (inaccessible) cardinal λ and $\kappa < \lambda$ let

$\otimes^\kappa_\lambda$ there is $\bar{C} = \langle C_\delta : \delta \in S^\lambda_{in}\rangle$, where $S^\lambda_{in} =: \{\delta < \lambda : \delta \text{ is inaccessible}\}, C_\delta$ a club of δ, such that: for every club E of λ for some $\delta \in \text{acc}(E) \cap S^\lambda_{\text{in}}$ of cofinality $\geq \kappa$, for no $\zeta < \kappa$ and $\alpha_\varepsilon \in S^\lambda_{in}$ (for $\varepsilon < \zeta$) do we have

(*) $\text{nacc}(E) \cap \delta \setminus \bigcup\limits_{\varepsilon<\zeta} C_{\alpha_\varepsilon}$ is bounded in δ.

2) For λ regular $> \kappa = \text{cf}(\kappa) \geq \aleph_0$, let

$\oplus^\kappa_\lambda$ there is $\bar{C} = \langle C_\delta : \delta \in S\rangle, S = \{\delta < \lambda : \delta \text{ limit}\}, C_\delta$ a club of δ such that: for every club E of λ for some $\delta \in \text{acc}(E)$ of cofinality $\geq \kappa$, for no $\zeta < \kappa$ and $\alpha_\varepsilon \in S$ (for $\varepsilon < \zeta$) do we have

(*)′ $S^\lambda_{\geq\kappa} \cap E \setminus \bigcup\limits_{\varepsilon<\zeta} C_{\alpha_\varepsilon}$ is bounded in δ where

$S^\lambda_{\geq\kappa} = \{\delta < \lambda : \text{cf}(\delta) \geq \kappa\}$.

18.5 Remark. 1) For λ Mahlo, the property $\otimes^2_\lambda$ holds if there are stationary subsets S_i of λ for $i < \lambda$ such that for no $\delta < \lambda$, $\bigwedge\limits_{i<\delta}[S_i \cap \delta$ a stationary in $\delta]$ (we can consider only δ inaccessible).
[Why? Choose C_δ a club of δ disjoint to S_i for some $i(\delta) < \delta$, such that $\min(C_\delta) > i(\delta)$].
2) This is close to [Sh 276, §3], see [Sh:g, Ch.III,2.12]. As in [Sh 276, §3], the proof is done such that from appropriate failures of Chang conjectures or existence of colourings we can get stronger colourings here. For the result as stated also $c(\beta,\alpha) = \ell g[\rho(\beta,\alpha)]$ is O.K., but the proof as stated is good for utilizing failure of Chang conjecture (as in [Sh 276, §3]).
3) Note that $\otimes^2_\lambda$ is closely related to $\otimes_{\bar{C}}$ from [Sh:g, Ch.III,2.12]. Also if $\kappa \leq \aleph_0$, then in $\otimes^\kappa_\lambda$ we can replace $\text{nacc}(E)$ by E.
4) Note that λ weakly compact fails even $\otimes^2_\lambda$ and forcing notion P which is θ-c.c. for some $\theta < \lambda$ preserves this.

18.6 Observation: In Definition **?** in (*) and (*)′ if $\kappa \leq \aleph_0$ it does not matter
—> scite{17.4} undefined
whether we write E or $\text{nacc}(E)$.

18.7 Observation: 1) $\otimes^2_\lambda$ implies $\otimes^{\aleph_0}_\lambda$.
2) $\oplus^2_\lambda$ implies $\oplus^{\aleph_0}_\lambda$.

3) If $\kappa_1 < \kappa_2 < \lambda$ then $\otimes_\lambda^{\kappa_2} \Rightarrow \otimes_\lambda^{\kappa_1}$ and $\oplus_\lambda^{\kappa_2} \Rightarrow \oplus_\lambda^{\kappa_1}$.
4) $\otimes_\lambda^\kappa \Rightarrow \oplus_\lambda^\kappa$ if λ is inaccessible $> \aleph_0$.

Proof. 1) Let $\bar{C}$ exemplify $\otimes_\lambda^2$ and we shall show that it exemplifies $\otimes_\lambda^{\aleph_0}$, assume not and let E be a club of λ which exemplifies this. We choose by induction on $k < \omega$ a club E_k of $\lambda : E_0 = E$, if E_k is defined let

$$A_k =: \{\delta < \lambda : \delta \in \operatorname{acc}(E_k) \cap S_{\text{in}}^\lambda \text{ and for no } \alpha \in S_{in}^\lambda \text{ is } E_k \cap \delta \backslash C_\alpha \text{ bounded in } \delta\}.$$

As $\bar{C}$ exemplifies $\otimes_\lambda^2$, clearly A_k is a stationary subset of λ and let

$$E_{k+1} = \{\delta \in E_k : \delta = \sup(A_k \cap \delta)\}.$$

Let $\delta(*) \in \bigcap\limits_{k<\omega} E_k$ which necessarily belong $\subseteq E$. By the choice of E we can find $n < \omega = \kappa$ and $\alpha_\ell \in S_{\text{in}}^\lambda$ for $\ell < n$ such that $\operatorname{nacc}(E) \cap \delta(*) \backslash \bigcup\limits_{\ell<n} C_{\alpha_\ell}$ is bounded in $\delta(*)$. Now we choose by induction on $k \le n, \delta_k \in \operatorname{acc}(E_{n+1-k})$ such that $\delta_k < \delta(*)$ and $\operatorname{nacc}(E_{n+1-k}) \cap \delta_k \backslash \bigcup\limits_{\ell<n-k} C_{\alpha_\ell}$ is bounded in δ_k. For $k = 0$ any large enough $\delta \in \delta(*) \cap E_{n+1}$ is O.K. For $k+1$ use the definition of E_{n+1-k}. For $k = n, \delta_n$ gives a contradiction to the choice of E.
2) Same proof replacing S_{in}^λ by $S_{\ge\kappa}^\lambda$.
3) The same $\bar{C}$ witnesses it.
4) Here λ is inaccessible. That is, we have to show that:
<u>the version with $(*) \Rightarrow$ the version with $(*)'$</u>
Let $\bar{C}' = \langle C'_\delta : \delta \in S_{in}^\lambda \rangle$ exemplifies $\otimes_\delta^\kappa$. We define $S = \{\delta < \lambda : \delta \text{ limit}\}$ and $\bar{C} = \langle C_\delta : \delta \in S \rangle$ as follows: if $\delta \in S_{in}^\lambda (\subseteq S)$ we let $C_\delta = C'_\delta$ and if $\delta \in S \backslash S_{in}^\lambda$ let C_δ be a club of δ of order type $\operatorname{cf}(\delta)$ with $\operatorname{cf}(\delta) < \delta \Rightarrow \operatorname{Min}(C_\delta) > \operatorname{cf}(\delta)$ and if δ is a successor cardinal, say θ^+ then $\operatorname{Min}(C_\delta) > \theta$ (possible as $\delta \notin S_{in}^\lambda \Rightarrow \operatorname{cf}(\delta) < \delta \vee (\exists \theta < \delta)(\delta = \theta^+)$). We shall show that $\langle C_\delta : \delta \in S \rangle$ exemplify $\oplus_\lambda^\kappa$.

Given a club E of λ let $E_0 = \{\delta \in E : \delta \text{ a limit cardinal } \operatorname{otp}(\delta \cap E) = \delta \text{ and } \delta > \kappa\}$ and $E_1 = \{\delta \in E_0 : \operatorname{otp}(\delta \cap E_0) \text{ is divisible by } \kappa^+\}$, so E_1 is a club of λ so by the version with $(*)$ there is $\delta \in \operatorname{acc}(E_1) \cap S_{\text{in}}^\lambda$ hence $\operatorname{cf}(\delta) > \kappa$ satisfying $(*)$, i.e. the requirement in 18.4(1); we shall show that it satisfies the requirement in 18.4(2) thus finishing.

So let $\zeta < \kappa$ and $\alpha_\varepsilon \in S$ for $\varepsilon < \zeta$ and we should prove that $Y =: S_{\ge\kappa}^\lambda \cap E \backslash \bigcup\limits_{\varepsilon<\zeta} C_{\alpha_\varepsilon}$ is unbounded in δ, so fix $\beta^* < \delta$ and we shall prove that $Y \cap (\beta^*, \delta) \ne \emptyset$ thus finishing.

Let ζ be the disjoint union of u_0, u_1, u_2 where $u_0 = \{\varepsilon < \zeta : \alpha_\varepsilon < \delta\}, u_1 = \{\varepsilon < \zeta : \alpha_\varepsilon \geq \delta$ and $\alpha_\varepsilon \in S \backslash S^\lambda_{in}\}$ and $u_2 = \{\varepsilon < \zeta : \alpha_\varepsilon \geq \delta$ and $\alpha_\varepsilon \in S^\lambda_{in}\}$.

By the choice of δ we know that $Y_2 = \text{nacc}(E_1) \cap \delta \backslash \bigcup\limits_{\varepsilon \in u_2} C_{\alpha_\varepsilon}$ is unbounded in δ. As $\text{cf}(\delta) \geq \kappa$ (see its choice, i.e. $\delta \in \text{acc}(E) \cap S^\lambda_{\text{in}} \wedge \text{Min}(E) > \kappa$), we can find $\beta \in Y_2$ such that $\beta < \delta, \beta > \beta^*$ and $\beta > \alpha_\varepsilon$ for $\varepsilon \in u_0$. Now $\text{cf}(\beta) = \kappa^+$ as $\beta \in Y_2 \subseteq \text{nacc}(E_1)$ and the choice of E_1. Also $\varepsilon \in u_0 \Rightarrow \sup(C_{\alpha_\varepsilon}) < \beta$ and $\varepsilon \in u_2 \Rightarrow \sup(C_{\alpha_\varepsilon} \cap \beta) < \beta$ (as otherwise $\beta \in C_{\alpha_\varepsilon}$ contradicting $\beta \in Y_2$), so we can find $\beta_0 < \beta$ such that $\varepsilon \in u_0 \cup u_2 \Rightarrow \sup(C_{\alpha_\varepsilon} \cap \beta) < \beta_0$. Now for $\varepsilon < \zeta$, if $C_{\alpha_\varepsilon} \cap (\beta_0, \beta) \neq \emptyset$ then $\varepsilon \in u_1$, so by the choice of C_{α_ε} we know $|C_{\alpha_\varepsilon}| = \text{cf}(\alpha_\varepsilon) < \text{Min}(C_{\alpha_\varepsilon}) < \beta$, noting that β is a cardinal as E_0 is a set of cardinals. By the definition of E_0, E_1 we know that $E \cap S^\lambda_{\geq\kappa} \cap \beta$ has cardinality β hence $E \cap S^\lambda_{\geq\kappa} \backslash \beta_0$ has cardinality β, so we finish. $\square_{18.6}$

Proof of 18.3. By 18.11(4) without loss of generality $\oplus^\kappa_\lambda$, so let $\bar{C}$ be as required in $\oplus^\kappa_\lambda$. We define e_α for every ordinal $\alpha < \lambda$ as follows

(a) if $\alpha = 0, e_\alpha = \emptyset$

(b) if $\alpha = \beta + 1, e_\alpha = \{0, \beta\}$

(c) if α is a limit ordinal, then we let $e_\delta = C_\delta \cup \{0\}$.

Let S be the set of limit ordinals $< \lambda$. For $\alpha < \beta$ we define by induction on $\ell < \omega$ the ordinals $\gamma^+_\ell(\beta, \alpha), \gamma^-_\ell(\beta, \alpha)$.

<u>$\ell = 0$</u>: $\gamma^+_\ell(\beta, \alpha) = \beta, \gamma^-_\ell(\beta, \alpha) = 0$

<u>$\ell = k + 1$</u>: $\gamma^+_\ell(\beta, \alpha) = \min(e_{\gamma^+_k(\beta,\alpha)} \backslash \alpha)$ if $\alpha < \gamma^+_k(\beta, \alpha)$ and $\gamma^-_\ell(\beta, \alpha) = \sup(e_{\gamma^+_k(\beta,\alpha)} \cap \alpha)$ <u>if</u> $\alpha < \gamma^+_k(\beta, \alpha)$ and $\alpha \notin \text{acc}(e_{\gamma^+_k(\beta,\alpha)})$.

Note that $\gamma^-_\ell(\beta, \alpha) < \alpha \leq \gamma^+_\ell(\beta, \alpha)$ if they are defined and then $\ell > 0 \Rightarrow \gamma^+_\ell(\beta, \alpha) < \gamma^+_{\ell-1}(\beta, \alpha)$ (prove by induction). So if $\alpha < \beta < \lambda$ for some $k = k(\beta, \alpha) < \omega$ we have: $\gamma^+_\ell(\beta, \alpha)$ is defined iff $\ell \leq k$ and: $\gamma^-_\ell(\beta, \alpha)$ is defined iff $\ell < k \vee [\ell = k \ \ \& \ \ \gamma^+_k(\beta, \alpha) = \alpha]$ and: $\gamma^+_k(\beta, \alpha) = \alpha$ or $\alpha \in \text{acc}(e_{(\gamma^+_k(\beta,\alpha))})$. Let $\rho(\beta, \alpha) = \langle \gamma^+_\ell(\beta, \alpha) : \ell \leq k(\beta, \alpha) \rangle$. Note (we shall use it freely):

$\otimes_1$ if $\gamma < \alpha < \beta, k \leq k(\beta, \alpha)$ and $\gamma^-_k(\beta, \alpha)$ is defined and $\bigwedge\limits_{\ell \leq k} \gamma^-_\ell(\beta, \alpha) < \gamma$ <u>then</u>

(α) $\ell \leq k \Rightarrow \gamma^+_\ell(\beta, \alpha) = \gamma^+_\ell(\beta, \gamma)$

(β) $\ell \leq k \Rightarrow \gamma^-_\ell(\beta, \alpha) = \gamma^-_\ell(\beta, \gamma)$

(γ) $k(\beta, \gamma) \geq k(\beta, \alpha)$ and $\rho(\beta, \alpha) \trianglelefteq \rho(\beta, \gamma)$.

Now we define $c\{\alpha,\beta\} = c(\beta,\alpha) = c(\alpha,\beta)$ for $\alpha < \beta < \lambda$ as follows:

$$c(\beta,\alpha) = k(\beta,\alpha) + 1$$

So assume $\bar{w} = \langle w_i : i < \lambda\rangle$ is a sequence of pairwise disjoint subsets of $\lambda, |w_i| < \kappa$ and $n(*) < \omega$. Without loss of generality for some $\kappa^* < 1 + \kappa$, $\bigwedge_{i<\lambda} |w_i| = \kappa^*$ and $i < \min(w_i)$ and $[i < j \Rightarrow \sup(w_i) < \min(w_j)]$. Let $w_i = \{\alpha^i_\varepsilon : \varepsilon < \kappa^*\}$. Let $\chi \geq (2^\lambda)^+$ and we choose by induction on $n < \omega$ and for each n by induction on $i < \lambda, N^n_i \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ such that $\|N^n_i\| < \lambda, \{\langle N^n_\varepsilon : \varepsilon \leq j\rangle : j < i\} \subseteq N^n_i, \bar{w} \in N^0_i, N^n_i$ increasing continuous in i and $\langle N^m_i : i < \lambda\rangle \in N^n_0$ for $m < n$.

Let us define for $\ell < \omega$

$$E^\ell = \{\delta < \lambda : N^\ell_\delta \cap \lambda = \delta\}$$

$$S^\ell = \{\delta \in S^\lambda_{\geq\kappa} \cap \text{ acc}(E^\ell) : \text{for } \underline{\text{no}}\ \zeta < \kappa \text{ and } \alpha_\varepsilon < \lambda \text{ for } \varepsilon < \zeta \text{ do we have } \delta > \sup[S^\lambda_{\geq\kappa} \cap E^\ell \cap \delta \setminus \bigcup_{\varepsilon<\zeta} C_{\alpha_\varepsilon}]\}.$$

Note that $\alpha < \lambda \Rightarrow (E^\ell, S^\ell) \in N^{\ell+1}_\alpha$ hence $\delta \in E^{\ell+1} \Rightarrow \delta = \sup(\delta \cap S^\ell)$.

We know that S^ℓ is a stationary subset of λ as E^ℓ is a club of λ because $\oplus^\lambda_\kappa$ is exemplified by $\bar{C}$.

Choose $\delta_{n(*)} \in E^{2(n(*)+1)} \cap S^{2(n(*)+1)}$ and then choose $\alpha(*) < \lambda$ such that $\alpha(*) > \delta_{n(*)}$. We now choose by downward induction on $m < n(*)$ ordinals δ_m, ζ^*_m such that:

$(*)(i)$ $\delta_m < \delta_{m+1}$

(ii) $\delta_m \in E^{2m} \cap S^{2m}$

(iii) $\delta_m > \sup\{\gamma^-_\ell(\beta,\delta_{m+1}) : \ell \leq k(\beta,\delta_{m+1})$ and $\gamma^-_\ell(\beta,\delta_{m+1})$ is well defined and $\beta \in w_{\alpha(*)}\}$

(iv) $\delta_m \notin \cup\{C_\gamma : \gamma = \gamma^+_{k(\beta,\delta_{m+1})}(\beta,\delta_{m+1})$ for some $\beta \in w_{\alpha(*)}\}$

(v) $\zeta^*_m < \delta_m, \zeta^*_m < \zeta^*_{m+1}$ if $m+1 < n(*)$

(vi) if $\alpha \in [\zeta^*_m, \delta_m)$ then $(\forall\beta' \in w_\alpha)(\forall\beta'' \in w_{\alpha(*)})(\rho(\beta'',\delta_m) \triangleleft \rho(\beta'',\beta'))$

[Why can we do it? Assume $\delta_{m+1} \in S^{2(m+1)}$ has already been defined and we shall find δ_m, ζ_m as required. Let $Y_m = \{\gamma^-_\ell(\beta, \delta_{m+1}) : \ell \leq k(\beta, \delta_{m+1})$ and $\gamma^-_\ell(\beta, \delta_{m+1})$ is well defined and $\beta \in w_{\alpha(*)}\}$, so Y_m is a subset of δ_{m+1} of cardinality $< \kappa$, but $\delta_{m+1} \in S^{2(m+1)}$ (if $m = n(*) - 1$ by the choice of δ_n, if $m < n-1$ by the induction hypothesis). But $S^{2(m+1)} \subseteq S^\lambda_{\geq\kappa}$ hence $(\forall \delta \in S^{2(m+1)})[\mathrm{cf}(\delta) \geq \kappa]$, hence $\sup(Y_m) < \delta_{m+1}$. Also as $\delta_{m+1} \in E^{2(m+1)} \cap S^{2(m+1)}$ by the definition of $S^{2(m+1)}$, there is $\xi^*_m \in S^\lambda_{\geq\kappa} \cap E^{2(m+1)} \cap \delta \setminus \cup \{e_\gamma :$ for some $\beta \in w_{\alpha(*)}$ we have $\gamma = \gamma^+_{k(\beta,\delta_{m+1})}(\beta, \delta_{m+1})\} \setminus \sup(Y_m)$. As each e_γ is closed and there are $< \kappa$ of them $\zeta^*_m = \sup[\{\sup Y_m\} \cup \{\sup(e_\gamma \cap \xi^*_m) :$ for some $\beta \in w_{\alpha(*)}$ we have $\gamma = \gamma^+_{k(\beta,\delta_{m+1})}(\beta, \delta_{m+1})\}]$ is $< \xi^*_m$. So we can find $\delta_m \in (\zeta^*_m, \xi^*_m) \cap S^\lambda_{\geq\kappa} \cap E^{2m} \cap S^{2m}$ as required and choose $\zeta_m < \delta_m$ large enough.]

(**) For every $\alpha \in [\zeta^*_0, \delta_0)$ we have

$$(\forall \beta' \in w_\alpha)(\forall \beta'' \in w_\alpha)[c\{\beta', \beta''\} \geq n].$$

[Why? By clause (vi) above.]

Let

$$W = \big\{\delta < \lambda : \delta > \zeta^*_0 \text{ and for some } \alpha'' \geq \delta \text{ we have for every } \alpha' \in (\zeta^*_0, \delta) \text{ we have } (\forall \beta' \in w_{\alpha'})(\forall \beta'' \in w_{\alpha''})[c\{\beta', \beta''\} \geq n]\big\}.$$

As $\delta_0 \in E_0$ (see $(*)(ii)$) so by E_0's definition, $\delta_0 = N^0_{\delta_0} \cap \lambda$ hence $\zeta_0 \in N^0_{\delta_0}$. Now $\bar{w} \in N^0_{\delta_0}$ (read definition) hence $W \in N^0_{\delta_0}$ and by $(*) + (**)$ and W's definition $\delta_0 \in W$, hence W is a stationary subset of λ. For $\delta \in W$, let $\alpha''(\delta)$ be as in the definition of W. So $E = \{\delta^* : (\forall \delta \in W \cap \delta^*)(\alpha''(\delta) < \delta^*\}$, it is a club of λ hence $W' = W \cap E$ is a stationary subset of λ and $\{\alpha''(\delta) : \delta \in W'\}$ is as required. $\square_{18.3}$

<u>18.8 Conclusion</u>: If $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ is not Mahlo (or is Mahlo as in 18.4(1) or 18.4(2)), κ <u>then</u> $\mathrm{Pr}_7(\lambda, \aleph_0, \aleph_0)$.

Proof. By 18.3 it suffices to prove $\oplus^{\aleph_0}_\lambda$. This holds by 18.9, 18.11 and 18.12 below.

18.9 Claim. *1) If $\lambda = \mu^+$ then $\oplus^{\mathrm{cf}(\mu)}_\lambda$.*
2) If λ is (weakly) inaccessible, not Mahlo or Mahlo as in 18.4(1), e.g. as in 18.5(1), and $\aleph_0 \le \kappa < \lambda$ then $\oplus^\kappa_\lambda$.

Proof. 1) Choose C_δ a club of δ of order type $\mathrm{cf}(\delta)$.
Repeat the proof of 18.6(2), using $E_0 = \{\delta < \lambda : \delta > \mu$ and $\mathrm{otp}(E \cap \delta) = \delta$ is divisible by $\mu^2\}$.
The only point slightly different is $|C_{\alpha_\varepsilon} \cap (\beta_0, \beta)| < |\beta|$ (now β is not a cardinal). For μ singular, $|C_{\alpha_\varepsilon}| < \mu = |\beta| = |\beta \cap S^\lambda_{\ge\kappa} \cap E \backslash \beta_0|$, and for μ regular we choose δ of cofinality μ and everything is easy.
2) Now $\oplus^\lambda_\kappa$ holds trivially (choose a club E^*_0 of λ with no inaccessible member and choose C_δ a club of δ of order type $\mathrm{cf}(\delta)$ such that $\mathrm{cf}(\delta) < \delta \Rightarrow \mathrm{Min}(C_\delta) > \mathrm{cf}(\delta)$ and $\delta \notin E^* \Rightarrow \mathrm{Min}(C_\delta) > \sup(E^* \cap \delta)$, now for any club E choose $\delta \in \mathrm{acc}(E \cap E^*)$)
So we can apply 18.6(2). $\square_{18.9}$

18.10 Definition. $\mathrm{Pr}_8(\lambda, \mu, \sigma, \theta)$ means:

> there is $c : [\lambda]^2 \to [\sigma]^{<\aleph_0} \backslash \{\emptyset\}$ such that if $w_\alpha \in [\lambda]^{<\theta}$ for $\alpha < \lambda$ are pairwise disjoint and $\zeta < \sigma$ then for some $Y \in [\lambda]^\mu$ we have $\alpha' \in Y$ & $\alpha'' \in Y$ & $\alpha' < \alpha'' \Rightarrow \forall \beta' \in w_{\alpha'} \forall \beta'' \in w_{\alpha''} [\zeta \in c\{\beta', \beta''\}]$.

18.11 Observation. Note that $\mathrm{Pr}_8(\lambda, \lambda, \sigma, \theta) \Rightarrow \mathrm{Pr}_7(\lambda, \sigma, \theta)$ because we can use $c'\{\alpha, \beta\} = \max[c\{\alpha, \beta\}]$.

18.12 Claim. *1) If λ is regular and $\aleph_0 \le \sigma \le \lambda$ then $Pr_8(\lambda^+, \lambda^+, \sigma, \lambda)$.*
2) If μ is singular, $\lambda = \mu^+$ and $\aleph_0 \le \sigma \le cf(\mu)$ then $Pr_8(\lambda, \lambda, \sigma, cf(\mu))$.
3) If λ is inaccessible $> \aleph_0, S \subseteq \lambda$ stationary not reflecting in inaccessibles and $\sigma < \lambda, \theta = Min\{cf(\delta) : \delta \in S\}$ then $Pr_8(\lambda, \lambda, \sigma, \theta)$.

Proof. The proofs in [Sh:g, Ch.III,§4] gives this - in fact this is easier. E.g.
1) Follows by Claim 18.3 (and [Sh:g, Ch.III,4.2(2),p.162]) but let us give some details.

Let $\bar{e}$ be as there (i.e. $\bar{e} = \langle e_\alpha : \alpha < \lambda^+ \rangle, e_0 = \emptyset, e_{\alpha+1} = \{\alpha\}, e_\delta$ a club of δ of order type $\mathrm{cf}(\delta)$). Let $h : \lambda^+ \to \sigma$ be such that $\forall \zeta < \sigma (\exists^{\mathrm{stat}} \delta < \lambda^+)(\mathrm{cf}(\delta) = \lambda$ & $h(\delta) = \zeta), h_\alpha = h \restriction e_\alpha, \bar{h} = \langle h_\alpha : \alpha < \lambda^+ \rangle$. Let $\gamma(\beta, \alpha), \gamma_e(\beta, \alpha), \rho_{\bar{h}}$ be as there (Stage A,p.164) and also the colouring d: for $\alpha < \beta < \lambda^+$

$$d(\beta, \alpha) = \mathrm{Max}\{h(\gamma_{\ell+1}(\beta, \alpha) : \gamma_{\ell+1}(\beta, \alpha) \text{ well defined}\}.$$

By Stage B there the result should be clear. $\square_{18.12}$

Hajnal has shown the following

Theorem. *Assume* $\lambda = (2^{<\kappa})^+, \kappa = cf(\kappa) > \omega, I$ *is a normal ideal concentrating on* $S_{\kappa,\lambda} = \{\alpha < \lambda : cf(\alpha) = \kappa\}, \mathfrak{G} \subset [\lambda]^2$ *is such that* $\mathfrak{G} \cap [B]^2 \neq \emptyset$ *for all* $B \in I^+$ *and* $\mathfrak{G} = \bigcup_{\eta<\xi} \mathfrak{G}_\eta$ *for some* $\xi < \kappa$.

Then there exist I *and* T *such that* $I \subset J, T \subset \xi, J$ *is a normal ideal and for all* $\eta \in T$ *and* $B \in J^+$ *we have*

$$[B]^2 \cap \mathfrak{G}_\eta \neq \emptyset \text{ and } G \cap [B]^2 \subset \bigcup\{\mathfrak{G}_\eta : \eta \in T\}.$$

This comes from the following

Lemma. *Assume* $\lambda = (2^{<\kappa})^+, \kappa = cf(\kappa) > \omega$.
I *is a normal ideal concentrating on* $S_{\kappa,\lambda}, P$ *is a partial order not containing decreasing sets of type* κ.
Assume further that

$$p : \mathscr{P}(\lambda) \to P \text{ and}$$

$$p(A) \leq_p p(B) \text{ for } A \subset B.$$

Then there is an $A \in I^+$ *and a normal ideal* $J \supset I$ *satisfying* $B \in J$ *iff* $B \in I$ *or* $p(B) \prec_P p(A)$ *for* $B \subset A$.

<u>page 412</u>: add at the end

The following improves [Sh:g, Ch.IX,5.12,p.410].

18.13 Lemma. *1) Assume*

(a) $\sigma = cf(\sigma) > \aleph_0$
(b) $\langle \lambda_i : i < \sigma \rangle$ *increasing continuous,* $\lambda = \sup\{\lambda_i : i < \sigma\}$
(c) $\sigma \leq \theta < \lambda$ *and* $\sigma^{\aleph_0} < \lambda$
(d) $cov(\lambda_i, \lambda_i, \theta^+, 2) < \lambda$ *for* $i < \sigma$.

Then

(α) $pp_\theta(\lambda) =^+ cov(\lambda,\lambda,\theta^+,2)$ *and* $pp^{cr}_{J^{bd}_\sigma}(\lambda) = cov(\lambda,\lambda,\theta^+,2)^+$ *(on* pp^{cr} *see below).*

(β) $S^* = \{\delta < \sigma : cov(\lambda_\delta,\lambda_\delta,\theta^+,2)^+ = pp^{cr}_{J^{bd}_{\text{cf}(\delta)}}(\lambda_\delta)\}$ *contains a club of* σ.

2) Instead of "$\sigma^{\aleph_0} < \lambda$*" it suffices*

⊗ *for some club* C *of* σ, *if* $i < \delta \in C, \delta$ *of cofinality* $\aleph_0$ *and set* $\mathfrak{a} \subseteq \lambda_\delta$ *of cardinality* $\le \lambda_i$ *and* $\mathfrak{a}$ *is a set of regular cardinals, then* $\lambda > |\{tcf(\Pi\mathfrak{b}/J^{bd}_\mathfrak{b}) : \mathfrak{b} \subseteq \mathfrak{a}, \sup(\mathfrak{b}) = \lambda_\delta, \ otp(\mathfrak{b}) = \omega, \Pi\mathfrak{b}/J^{bd}_\mathfrak{b}$ *has true cofinality*$\}|$. *(So without loss of generality* $\lambda_{\delta+1}$ *is above this cardinality.)*

18.14 Definition. Let J be an ideal on some ordinal Dom(J). We let $\text{pp}^{cr}_J(\lambda) = \text{Min}\{\mu : \mu \text{ regular } > \lambda, \text{ and } \sup\{\text{tcf} \prod_t \lambda_t/J : \bar\lambda = \langle \lambda_t : t \in \text{Dom}(J)\rangle,$ $\bar\lambda \text{ is strictly increasing with limit } \lambda\} < \mu\}$.

Proof. 1) Similar to the proof of [Sh:g, Ch.IX,5.12]. We assume toward contradiction that the desired conclusion fails.

Without loss of generality

$(*)_0(a)$ each λ_i is singular of cofinality $< \sigma$

(b) $\theta^{+3} < \lambda_0$ and $\sigma^{\aleph_0} < \lambda_0$

(c) $\text{cov}(\lambda_i,\lambda_i,\theta^+,2) < \lambda_{i+1}$

(d) $\mu \in (\lambda_0,\lambda_{i+1}) \Rightarrow pp_\theta(\mu) < \lambda_{i+1}$

[why? clearly we can replace $\langle \lambda_i : i < \sigma\rangle$ by $\bar\lambda \restriction C = \langle \lambda_i : i \in C\rangle$ for any club C of σ, hence it is enough to show that each of the demands holds for $\bar\lambda \restriction C$ for any small enough club C of σ. Now (a) holds whenever $C \subseteq \{i < \lambda : i \text{ limit}\}$, clause (b) holds for $C \subseteq [i_0,\sigma)$ when $\theta^{+3} < \lambda_{i_0}$ and clause (c) holds as $\text{cov}(\lambda_i,\lambda_i,\theta^+,2) < \lambda$ and use [Sh:g, Ch.II,5.3,10] + Fodour's lemma and monotonicity of cov. Lastly, clause (d) holds as if $\{\mu < \lambda : pp_\theta(\mu) \ge \lambda \text{ and } \text{cf}(\mu) \le \theta\}$ is unbounded in λ, we get a contradiction by [Sh:g, Ch.II,2.3(4)].]

Let $\lambda_\sigma =: \lambda$. By [Sh:g, Ch.VIII,1.6(3)] we have (but shall not use)

$(*)_1$ if $\delta \le \sigma$ and $\text{cf}(\delta) > \aleph_0$ <u>then</u> $\text{pp}^+_\theta(\lambda_\delta) = pp^{cr}_{J^{bd}_{\text{cf}(\lambda_\delta)}}(\lambda_\delta)$ (and $\text{cf}(\lambda_\delta) = \text{cf}(\delta)$).

Now by clause (d)

$(*)_2$ $\mathfrak{a} \subseteq \text{Reg} \cap \lambda_i \backslash \lambda_0, |\mathfrak{a}| \le \theta$ and $\sup(\mathfrak{a}) \le \lambda_i$ <u>implies</u> max pcf$(\mathfrak{a}) < pp^+_\theta(\lambda_i)$.

Let

$$S =: \{i \le \sigma : \ \mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2) \ge pp^{\mathrm{cr}}_{J^{\mathrm{bd}}_{\mathrm{cf}(\lambda_i)}}(\lambda_i)\}.$$

So it is enough to prove that S is not stationary.
Let for $i \le \sigma, \mu_i =: pp^{\mathrm{cr}}_{J^{\mathrm{bd}}_{\mathrm{cf}(\lambda_i)}}(\lambda_i)$, so $\lambda_{i+1} > \mu_i > \lambda_i, \mu_i$ is regular. Note that $\mu_\sigma = pp_\theta(\lambda_\sigma) = pp^{\mathrm{cr}}_{J^{\mathrm{bd}_\sigma}}(\lambda_i)$ by [Sh:g, Ch.VIII,1.6(3)].
Clearly

$(*)_3$ $\lambda_i < \mu_i = \ \mathrm{cf}(\mu_i) \le \ \mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2)^+$.

We can find $\bar{A} = \langle A_\zeta : \zeta < \lambda \rangle$ such that:

$(*)_4(a)$ $\zeta < \lambda_0 \Rightarrow A_\zeta = \emptyset$

(b) $\lambda_i \le \zeta < \lambda_{i+1} \Rightarrow A_\zeta \subseteq \lambda_i$ & $|A_\zeta| < \lambda_i$

(c) for every $A \subseteq \lambda_i$ of cardinality $\le \theta$, for some $\zeta, \lambda_i < \zeta <$ $\mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2)$ (which is $< \lambda_{i+1}$) we have $A \subseteq A_\zeta$.

Choose χ regular large enough, now choose by induction on $i \le \sigma$ an elementary submodel M^*_i of $(\mathscr{H}(\chi), \in, <^*_\chi), \|M^*_i\| < \mu_i, M^*_i \cap \mu_i$ is an ordinal such that

$(*)_5$ if $i \le \sigma$, <u>then</u>

$$\bigcup_{j<i} M^*_j \cup \{\zeta : \zeta \le \lambda_i\} \cup \{\langle \lambda_i : i < \sigma \rangle, \bar{A}, \langle M^*_j : j < i \rangle\} \subseteq M^*_i.$$

Let $\mathscr{P}_i = M^*_i \cap [\lambda_i]^{<\lambda_i}$. It is enough to show that

$$S_1 = \{i \le \sigma : \text{for some } Y \subseteq \lambda_i, |Y| \le \theta \text{ and } Y \text{ is not a subset of any member of } \mathscr{P}_i\}$$

is not stationary and $\sigma \notin S_1$ (in fact S, S_1 they are equal).
[Why? As clearly $S \subseteq S_1$.]
We assume S_1 is a stationary subset of σ or $\sigma \in S_1$ and eventually will finish by getting a contradiction.

For each $i \in S_1$ choose $Y_i \subseteq \lambda_i$ of cardinality $\le \theta$ which is not a subset of any member of $\mathscr{P}_i$. Let $Y = \bigcup_{i \in S_1} Y_i$, so $Y \subseteq \lambda, |Y| \le \theta$; and for each $i < \sigma$ we can find an ordinal $\zeta(i)$ such that $\lambda_i \le \zeta(i) <$ $\mathrm{cov}(\lambda_i, \lambda_i, \theta, 2)$ (which is $< \lambda_{i+1}$) and

$Y \cap \lambda_i \subseteq A_{\zeta(i)}$. Now $|A_{\zeta(i)}| < \lambda_i$, hence by Fodor's Lemma there is $i(*) < \sigma$ such that

$$S_2 =: \{i < \sigma : |A_{\zeta(i)}| < \lambda_{i(*)}\}.$$

is a stationary subset of σ. Let $Z =: \{\zeta(i) : i \in S_2\}$. Now if $\sigma \in S_1$, then by [Sh:g, Ch.IX,II,5.4] and [Sh:g, Ch.II,§1] we have $pp^{\rm cr}_{J^{\rm bd}_\sigma}(\lambda) = \mathrm{cov}(\lambda, \lambda, \sigma^+, \sigma) =^+ pp_{\Gamma(\sigma^+,\sigma)}(\lambda)$; so there are $j^* < \sigma$ and $B_j \in \mathscr{P}_\sigma = M_\sigma \cap [\lambda]^{<\lambda}$ for $j < j^*$ such that $Z \subseteq \bigcup_{j<j^*} B_j$. So for some $j < j^*$ we have $|Z \cap B_j| = \sigma$. Now the set

$$A^* = \bigcup\{A_\gamma : \gamma \in B_j, |A_j| \le \lambda_{i(*)}\}$$

belongs to M_σ, has cardinality $\le \lambda_{i(*)} \times |B_j| < \lambda$ and

$$\begin{aligned} Y = & \bigcup\{Y \cap \lambda_i : i \in S_2 \text{ and } \zeta(i) \in B_j\} \subseteq \\ & \bigcup\{A_{\zeta(i)} : i \in S_2 \text{ and } \zeta(i) \in B_j\} \subseteq A^* \in \mathscr{P}_\sigma \end{aligned}$$

contradiction. So we have finished the case $\sigma \in S_1$ and from now on we shall deal with the case $\sigma \notin S_1$ hence S_1 is a stationary subset of σ, hence without loss of generality $S_2 \subseteq S_1$. Note that if $\delta < \sigma$ & $\mathrm{cf}(\delta) > \aleph_0$, we can apply this proof to $\lambda_\delta, \langle \lambda_i : i < \delta \rangle$ (for $\sigma' = \mathrm{cf}(\delta)$) hence

$(*)_6$ $i \in S_2 \Rightarrow \mathrm{cf}(i) = \aleph_0$.
Clearly

$(*)_7$ for no $i \in S_2$ and $Z' \subseteq Z \cap \lambda_i$ is Z' unbounded in λ_i and is contained in a member of M^*_i of cardinality $< \lambda_i$.

Now we want to work as in the proof of [Sh:g, CH.IX,3.5], but for σ places at once with "nice" behavior on a club of σ, in the end the model is the Skolem Hull of the union of $\aleph_0$ sets, so one "catches" an unbounded subsets of Z. Let $\bar\lambda = \langle \lambda_i : i \le \sigma \rangle$.

We shall choose by induction on $k < \omega$,

$$N^a_k, N^b_k, g_k, \langle A^k_\ell : \ell < \omega \rangle, \langle \langle A^k_{\ell,i} : i \le \sigma \rangle : \ell < \omega \rangle$$

such that:

(a) for $x \in \{a, b\}, N^x_k$ is an elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi, \sigma, \bar\lambda)$ of cardinality $\le \sigma$ and N^x_k is the Skolem Hull of $N^x_k \cap \lambda$ and $N^a_k \prec N^b_k$

(b) $N_0^a[N_0^b]$ is the Skolem Hull of $\{i : i \leq \sigma\}$ [of $Z \cup \{i : i \leq \sigma\}$] in $(\mathscr{H}(\chi), \in, <^*_\chi, \sigma, \bar{\lambda})$

(c) $g_k \in \Pi(\text{Reg } \cap N_k^a \cap \lambda \backslash \lambda_0^+)$

(d) for $x \in \{a, b\} : N_{k+1}^x$ is the Skolem Hull of

$$N_k^x \cup \{g_k(\kappa) : \kappa \in \text{ Dom}(g_k)\} \cup (N_k^b \cap \lambda_0)$$

(e) $N_k^a \cap \lambda = \bigcup_{\ell < \omega} A_\ell^k$

(f) $A_\ell^k = \bigcup_{i<\sigma} A_{\ell,i}^k$ and $\langle A_{\ell,i}^k : i < \sigma \rangle$ is continuous increasing (in i) and $A_{\ell,i}^k \subseteq \lambda_i$ and $|A_{\ell,i}^k| < \sigma$

(g) if $\kappa \in$ Reg $\cap \lambda \cap N_k^a \backslash \lambda_0^+$ then $\sup(N_k^b \cap \kappa) < g_k(\kappa) < \kappa$

(h) if $\mathfrak{a} \subseteq A_\ell^k$ has order type ω and $\sup(\mathfrak{a}) = \lambda_i$ and $\mathfrak{a}$ is a subset of some $\mathfrak{b} \in M_i^*$ of cardinality $\leq \lambda_0$, <u>then</u> for some infinite $\mathfrak{b} \subseteq \mathfrak{a}, g_k \restriction \mathfrak{b}$ is included in some function $h_{\mathfrak{a}}^k \in M_i^*$ such that $|\text{Dom}(h_{\mathfrak{a}}^k)| \leq \lambda_0$.

For $X \in \mathscr{H}(\chi)$ and a function F we let

$$A(X, F) =: \{F(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X\}.$$

Let us carry the induction for $k = 0$; we define N_0^a, N_0^b by clause (b) and define $\{A_\ell^0 : \ell < \omega\}$ as

$$\{A(\sigma + 1, F) : F \text{ a definable function in } (\mathscr{H}(\chi), \in, <^*_\chi, \sigma, \bar{\lambda})\}.$$

For $k+1$, let $g'_k \in \Pi(\text{Reg } \cap \lambda \cap N_k^a \backslash \lambda_0^+)$ be defined by $g'_k(\kappa) = \sup(N_k^b \cap \kappa)$ (note: the domain of g'_k is determined by N_κ^a, the values - by N_k^b).

We now shall find g_k satisfying:

(α) $\text{Dom}(g_k) = \text{ Dom}(g'_k), g_k \in \Pi(\text{Dom}(g'_k))$

(β) $g'_k < g_k$

(γ) if $i < \sigma, \ell < \omega$ and $\mathfrak{a} \subseteq$ Reg $\cap A_\ell^k \backslash \lambda_0^+$ is unbounded in λ_i and is a subset of some $\mathfrak{b} \in M_i^*$ of cardinality $\leq \lambda_0$ and is of order type ω, <u>then</u> for some infinite $\mathfrak{b} \subseteq \mathfrak{a}$ we have $g_k \restriction \mathfrak{b}$ is included in some $h_{\mathfrak{b}} \in M_i^*$ such that $|\text{Dom}(h_{\mathfrak{b}})| \leq \lambda_0$

(δ) if $\mathfrak{a} \subseteq \lambda_i \cap$ Reg $\cap A_{\ell,i}^k \backslash \lambda_0^+$ and $\mathfrak{a} \in M_{i+1}^*$ <u>then</u> $g_k \restriction \mathfrak{a} \subseteq h$ for some function from M_{i+1}^*.

Note: a function choosing $\langle \bar{f}^{\mathfrak{a},\mu} : \mu \in \text{pcf}(\mathfrak{a})\rangle$ satisfying $(*)_{\mathfrak{a}}$ below for each $\mathfrak{a} \subseteq \text{Reg} \cap \lambda \backslash \theta^+, |\mathfrak{a}| \leq \theta$ is definable in $(\mathscr{H}(\chi), \in, <^*_\chi)$, so each M^*_i is closed under it where

$(*)_{\mathfrak{a}}$ $\bar{f}^{\mathfrak{a},\mu} = \langle f^{\mathfrak{a},\mu}_\alpha : \alpha < \mu\rangle$ satisfies

- (α) $f^{\mathfrak{a},\mu}_\alpha \in \Pi\mathfrak{a}$,
- (β) $\alpha < \beta \Rightarrow f^{\mathfrak{a},\mu}_\alpha <_{J_{<\mu}[\mathfrak{a}]} f^{\mathfrak{a},\mu}_\beta$
- (γ) if $\theta <$ cf$(\alpha) <$ Min$(\mathfrak{a})$ then $f^{\mathfrak{a},\mu}_\alpha(\kappa) = \text{Min}\{\bigcup_{\beta \in C} f^{\mathfrak{a},\mu}_\beta(\kappa) : C \text{ a club of } \alpha\}$
- (δ) if $f \in \Pi\mathfrak{a}$ then for some $\alpha < \mu$ we have $f < f^{\mathfrak{a},\mu}_\alpha$ mod $J_\mu[\mathfrak{a}]$.

Let $\langle \mathfrak{a}_{i,\zeta} : \zeta < \zeta_i \leq \sigma^{\aleph_0}\rangle$ list the $\mathfrak{a}$ such that tcf$(\Pi\mathfrak{a}/J^{\text{bd}}_{\mathfrak{a}})$ is well defined and for some $n < \omega, \mathfrak{a} \subseteq A^k_n, \mathfrak{a} \subseteq \text{Reg} \cap \lambda_i \backslash \lambda^+_0$, otp$(\mathfrak{a}) = \omega, \lambda_i = \sup(\mathfrak{a})$ and there is $\mathfrak{b} \subseteq \text{Reg} \cap \lambda_i \backslash \lambda^+_0, \mathfrak{b} \in M^*_i, |\mathfrak{b}| \leq \lambda_0$ such that $\mathfrak{a} \subseteq \mathfrak{b}$, note that the number of such $\mathfrak{a}$'s is $\leq \sigma^{\aleph_0}$. Let $\{\mathfrak{b}_{i,\zeta} : \zeta < \zeta_i \leq \sigma^{\aleph_0}\}$ be such that $\mathfrak{b}_{i,\zeta} \subseteq \text{Reg} \cap \lambda_i \backslash \lambda^+_0, \mathfrak{b}_{i,\zeta} \in M^*_i, |\mathfrak{b}_{i,\zeta}| \leq \lambda_0$ and $\mathfrak{a}_{i,\zeta} \subseteq \mathfrak{b}_{i,\zeta}$.

So apply [Sh:g, CH.VIII,§1]; i.e. let $\theta_1 = \theta + \sigma^{\aleph_0}$ choose $\langle M^k_\zeta : \zeta < \theta^{++}_1\rangle$ increasing continuous, $M^k_\zeta \prec (\mathscr{H}(\chi), \in, <^*_\chi), \langle M^k_\zeta : \zeta \leq \xi\rangle \in M^k_{\xi+1}, \|M^k_\zeta\| \leq \lambda_0$ and $g'_k, \langle A^k_{\ell,i} : i < \sigma, \ell < \omega\rangle, Z, \langle \mathfrak{b}_{i,\zeta} : i \in S_2, \zeta < \zeta_i\rangle$ belong to M^k_0; and the function $g_k(\kappa) =: \sup(\kappa \cap \bigcup_{\zeta < \theta^{++}_1} M^k_\zeta)$ satisfies clauses $(\alpha), (\beta), (\gamma), (\delta)$ above. Now N^a_{k+1}, N^b_{k+1} are defined by clause (d). Note that by the definition of μ'_i we have: for every $i < \sigma, \zeta < \zeta_i$ for some infinite $\mathfrak{a} = \mathfrak{a}^*_{i,\zeta} \subseteq \mathfrak{a}_{i,\zeta}$ we have $\mu_{i,\zeta}, k = \max \text{pcf}(\mathfrak{a}) < \mu_i$. Moreover $\Pi\mathfrak{a}/J^{\text{bd}}_{\mathfrak{a}}$ has true cofinality. So our main demand on g_k is: $g_{k+1} \restriction \mathfrak{a}^*_{i,\zeta} = f^{\mathfrak{b}_{i,\zeta},\mu_{i,\zeta,k}}_\delta$ mod $J^{\text{bd}}_{\mathfrak{a}^*_{i,\zeta}}$ for a suitable δ, so $\delta = \sup(\mu_{i,\zeta,k} \cap M^k_{\theta^{++}}$ is O.K.) (For clause (γ) use (b) + (c) above.

Now let $\{A^{k+1}_\ell : \ell < \omega\}$ be a list of:

$$\Big\{\lambda \cap A(\bigcup_{m \leq n} A^k_m \cup \text{ Rang}[g_k \restriction \bigcup_{m<n} A^k_m], F) : n < \omega \text{ and } F \text{ a definable function in } (\mathscr{H}(\chi), \in, <^*_\chi, \theta, \bar{\lambda})\Big\}$$

and if

$$A^{k+1}_\ell = \lambda \cap A(\bigcup_{m<n} A^k_m \cup \text{ Rang}[g_k \restriction \bigcup_{m<n} A^k_m], F^{k+1}_\ell) \text{ and } i < \sigma$$

$g_k \restriction (\bigcup_{m<n} A^k_{m,i} \cap \lambda_i \backslash \lambda^+_0)$ is included in some function: $\text{Dom}(h^k_{\ell,i}) = \mathfrak{b}^k_{\ell,i}, h^k_{\ell,i}(\kappa) = \sup(\kappa \cap M^k_{\theta++})$.

Having finished the inductive definition note that

$(*)_8$ $\bigcup_k N^a_k \prec \bigcup_k N^b_k \prec (\mathscr{H}(\chi), \in, <^*_\chi, \theta, \bar\lambda)$

[why? as $N^a_k \prec N^b_k \prec (\mathscr{H}(\chi), \in, <^*_\chi, \theta, \bar\lambda)$ by clause (a) and clause (d)]

$(*)_9$ $\bigcup_k N^a_k \cap \lambda_0 = \bigcup_k N^b_k \cap \lambda_0$

[why? $N^b_k \cap \lambda_0 \subseteq N^a_{k+1} \cap \lambda_0$ (see clause (d))]

$(*)_{10}$ if $\mu \in \text{ Reg } \cap \lambda^+ \backslash \lambda^+_0$ and $\mu \in \bigcup_k N^a_k$ then $\bigcup_{k<\omega} N^a_k$ contains an unbounded subset of $\mu \cap \bigcup_{k<\omega} N^b_k$

[why? by clauses (d) + (g).]

So clearly (as usual)

$$\bigcup_k N^a_k \cap \lambda = \bigcup_k N^b_k \cap \lambda.$$

but $Z \subseteq N^b_0 \subseteq \bigcup_{k<\omega} N^b_k$ and $Z \subseteq \lambda$ hence $Z \subseteq \bigcup_{k<\omega} N^a_k \cap \lambda$. So for each $i \in S_2$, we can find $\langle (\bar a^{i,k}, w^{i,k}, u^{i,k}, \bar F^{i,k}) : k \le k(i) \rangle$ such that:

(a) $\bar a^{i,k(i)} = \langle \zeta(i) \rangle$

(b) $\bar a^{i,k} = \langle a^{i,k}_n : n < n^{i,k} \rangle$

(c) each $a^{i,k}_n$ belongs to $N^a_k \cup (\lambda_0 \cap N^b_{k+1})$

(d) $w^{i,k} = \{n < n^{i,k} : a^{i,k}_n \in \lambda_0 \cap N^b_{k+1}\}$

(e) $u^{i,k} = \{n < n^{i,k} : a^{i,k}_n \in N^a_k \cap \text{ Reg } \cap \lambda \backslash \lambda^+_0\}$

(f) $\bar F^{i,k} = \langle F^{i,k}_n : n \in n^{i,k} \backslash w^{i,k} \rangle$, and
$F^{i,k}_n$ is a definable function in $(\mathscr{H}(\chi), \in, <^*_\chi, \theta, \bar\lambda)$

(g) if $k > 0$, then $a^{i,k}_n = F^{i,k}_n(\ldots, a^{i,k-1}_m, \ldots, g_{k-1}(a^{i,k-1}_{m'}), \ldots)_{\substack{m<n^{i,k-1} \\ m' \in u^{i,k-1}}}$.

Let $a^{i,k}_n \in A^k_{\ell(i,k,n)}$. Note $(*)$
We can find stationary $S_3 \subseteq S_2$ such that:

$(*)$ if $i \in S_3$ then $k(i) = k(*)$ and for $k \le k(*)$ we have $n^{i,k} = n^k, w^{i,k} = w^k, u^{i,k} = u^k, \bar F^{i,k} = \bar F^k, \ell(i,k,n) = \ell(k,n)$.

We can also find a stationary $S_4 \subseteq S_3$ such that:

$(*)$ if $i_1 < i_2$ are in S_4 then $a^{i_1,k}_n \in A^k_{\ell(k,n),i_2}$

$(**)$ if $k < k(*), n \in u^k$ then $\langle a^{i,k}_n : i \in S_4\rangle$ is constant or strictly increasing and if it is strictly increasing and its limit is $\ne \lambda$ (hence is $< \lambda$) then it is $< \lambda_{\mathrm{Min}(S_4)}$.

Let $E = \{\delta < \sigma : \delta = \sup(\delta \cap S_4)$ and if $n \in u^k$, and $\langle a^{i,k}_n : i \in S_4\rangle$ is strictly increasing with limit λ then $\langle a^{i,k}_n : i \in S_4 \cap \delta\rangle$ is strictly increasing with limit $\lambda_\delta\}$.

Now choose $\delta(*) \in E \cap S_1$, and choose b, a subset of $\delta(*) \cap S_4$ of order type ω with limit $\delta(*)$. We can choose $b^{k,n} \in [b]^{\aleph_0}$ for $k \le k(*), n \le n^k$ such that: $b^{0,0} = b, b^{k,n+1} \subseteq b^{k,n}, b^{k+1,0} = b^{k,n^k}$, and if $n \in u^k$, $\langle a^{i,k}_n : i \in S_4\rangle$ strictly increasing with limit λ then $\Pi\{a^{i,k}_n : i \in b^{k,n+1}\}/J^{\mathrm{bd}}_{b^{k,n+1}}$ has true cofinality which necessarily is $< \mu_{\delta(*)}$.

So (recall $n^{k(*)} = 1$) $b^* = b^{k(*),1}$ is a subset of $S_4 \cap \delta(*)$ of order type ω with limit $\delta(*)$ and $b^* \subseteq b^{k,n}$ for $k \le k(*), n \le n^k$ and $b^* \subseteq b^{k,n+1}$ hence $n \in u^k$ & $(\langle a^{i,k}_n : i \in S_4\rangle$ strictly increasing$) \Rightarrow \mu_{\delta(*)} > \max \mathrm{pcf}\{a^{i,k}_n : i \in b^*\}$.

Now we prove by induction on $k \le k(*)$ that for each $n < n^k$ for some $\mathfrak{B}^*_{k,n} \in M^*_{\delta(*)}, \|\mathfrak{B}^*_{k,n}\| \le \lambda_0$ we have $\{a^{i,k}_n : i \in b^*\} \subseteq \mathfrak{B}^*_{k,n}$. For $k = 0$ clearly $A^0_{\ell(k,n)} \in M^*_{\delta(*)}$ has cardinality $\le \sigma$. For $k > 0$, for each $n < n^k$ we use the "$b^{k,n+1} \subseteq b$ and the choice of g_{k-1} and clause $(*)_0(c)$. So we get a contradiction to $(*)_7$ so we are done.
2) A variant of the proof of part (1). First, it is enough to prove, for each $i(*) < \sigma$ restrict ourselves to $S^* \cup \{\delta < \sigma$: the cardinal appearing in $\otimes$ is $\ge \lambda_{i(*)}\}$, then without loss of generality $i(*) = 0$ and see that $\zeta_i \le \lambda_0$ is O.K. $\square_{18.14}$

18.15 Remark. 1) Note that if we just omit "$\sigma^{\aleph_0} < \lambda$" we still get that for a club of $\delta < \sigma$, $\mathrm{cf}(\delta) > \aleph_0$ or $\mathrm{cf}(\delta) = \aleph_0$ and $pp^{\mathrm{cr}}_{J^{\mathrm{bd}}_\omega}(\lambda_\delta)$, if $<$ $\mathrm{cov}(\lambda_i, \lambda_i, \theta^+, 2)$ is still $\ge \lambda^{+\lambda_\delta}_\delta$.

<u>18.16 Conclusion</u>: If μ is strong limit singular of uncountable cofinality <u>then</u> for a club of $\mu' < \mu$ we have $(2^{\mu'})^+ =^+ \mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_{\mathrm{cf}(\mu')}}(\mu')$.

18.17 Conclusion. If $\beth_\delta$ is a singular cardinal of uncountable cofinality, then for a club of $\alpha < \delta$, if $\mathrm{cf}(\alpha) = \aleph_0$ then

$(*)_1$ $2^{\beth_\alpha} =^+ pp(\lambda)$

$(*)_2$ there is $S \subseteq {}^\omega(i_\alpha)$ of cardinality $2^{\beth_\alpha}$ containing no perfect subset (and more - see [Sh 355, §6]).

§19 Guessing[1] clubs by countable C's

Recently Zapletal [Zapox] proved a beautiful theorem ? Zapox ?

Theorem. *If I is a "nice" (definition) of a σ-complete ideal on $\mathscr{P}(\mathbb{R})$ for suitable LC if ZFC + LC $\vdash \mathrm{cov}(I) = 2^{\aleph_0}$ then ZFC + LC $\models \mathrm{Unif}(I) < \aleph_4$.*

He also showed that $\aleph_4$ cannot be replaced by $\aleph_2$. The $< \aleph_4$ comes from quoting guessing clubs. The following shows we can replace $\aleph_4$ by $\aleph_3$.

19.1 Claim. *Assume $\delta^* < \omega_1$ is a limit ordinal and $S \subseteq S^{\aleph_2}_{\aleph_0}$ is stationary. Then we can find $\bar{C} = \langle C_\alpha : \alpha \in S \rangle$ such that*

- *(a) $C_\alpha \subseteq S$*
- *(b) $C_\alpha \subseteq \alpha$*
- *(c) $\beta \in C_\alpha \Rightarrow \beta \in S \ \& \ C_\beta = C_\alpha \cap \beta$*
- *(d) $otp(C_\alpha) \leq \delta^*$*
- *(e) for every club E of ω_2 the set $\{\delta \in S : \delta = \sup(C_\delta), \delta^* = otp(C_\delta)$ and $C_\delta \subseteq E\}$ is a stationary subset of ω_2.*

Proof. For each $\alpha < \omega_2$ choose $\langle a^\alpha_i : i < \omega_1 \rangle$, be an increasing continuous sequence of countable subsets of α with union α. For each $\alpha < \omega_2$ let $C^0_\alpha = \{i < \omega_1 : i$ is a limit ordinal such that $(\forall \beta \in a^\alpha_i)(a^\beta_i = a^\alpha_i \cap \beta)$ and $\alpha < \omega_1 \Rightarrow \alpha \subseteq a^\alpha_i$ and $j < i \Rightarrow$ (the closure of a^α_j is $\subseteq a^\alpha_i \cup \{\alpha\}\}$.

Clearly

$(*)_1$(a) each C^0_α is a club of ω_1

(b) if $i \in C^0_\alpha$ and $\beta \in a^\alpha_i$ then $i \in C^0_\beta$.

Now

$(*)_2$ for some $\zeta = \zeta^* < \omega_1$, for every club E of ω_1 the following set $F_{\zeta^*}(E) \cap S$ is non empty where

$\boxtimes$ $F_\zeta(E)$ is the set of $\delta < \omega_2$ such that:

- (a) $\delta = \mathrm{otp}(\delta \cap E \cap S)$
- (b) $\delta = \sup(E \cap \delta \cap S) = \sup(a^\delta_\zeta \cap E \cap S)$
- (c) $\mathrm{otp}(a^\delta_\zeta \cap E \cap S)$ is divisible by δ^*
- (d) $\zeta \in C^0_\delta$ hence $\beta \in a^\delta_\zeta \Rightarrow \zeta \in C^0_\beta$.

[1] added Fall 2002

[Why $(*)_2$ holds? Otherwise for each $\zeta < \omega_1$ there is a club E_ζ of ω_2 such that $F_\zeta(E_\zeta) = \emptyset$. Let $E^* = \cap\{E_\zeta : \zeta < \omega_1\}\backslash\omega_1$, clearly E^* is a club of ω_2, and so is $E' = \{\delta < \omega_2 : \delta = \text{otp}(E^* \cap \delta \cap S)\}$ and choose $\delta \in E' \cap S$, exists as E' is a club of ω_2 and $S \subseteq S^2_0$ is stationary. Easily the set $C^* = \{\zeta < \omega_1 : \zeta \text{ limit, otp}(a^\delta_\zeta \cap E \cap S) \text{ is divisible by } \delta^*\}$ is a club of ω_1. So there is $\zeta^* \in C^* \cap C^0_\delta$, clearly $\delta \in F_{\zeta^*}(E^*)$ hence $\delta \in F_{\zeta^*}(E_\zeta)$, contradiction.]

$(*)_3$ if $E_1 \subseteq E_0$ are clubs of ω_2 then $F_{\zeta^*}(E_1) \subseteq F_{\zeta^*}(E_0)$.
[Why? Note that $a \subseteq b \subseteq \delta = \sup(a)$ & $\delta^*|\text{otp}(a)| \Rightarrow \delta^*\text{otp}(b)$.]

$(*)_4$ for some club E_0 of ω_2 for every club E_1 of ω_2 the set $F_{\zeta^*}(E_1, E_0) \neq \emptyset$ where $F_{\zeta^*}(E_1, E_0) = \{\delta : \delta \in F_{\zeta^*}(E_0) \text{ and } a^\delta_{\zeta^*} \cap E_0 \cap E_1 = a^\delta_{\zeta^*} \cap E_0\}$.
[Why? If not we choose by induction on $\varepsilon < \omega_1$ a club E_ε of ω_2 such that $i < \varepsilon \Rightarrow E_\varepsilon \subseteq E_i$ and $F_{\zeta^*}(E_{\varepsilon+1}, E_\varepsilon) = \emptyset$. So $E^* = \cap\{E_\varepsilon : \varepsilon < \omega_1\}$ is a club of ω_2 so we can find $\delta \in F_{\zeta^*}(E^*)$, hence $\delta \in \cap\{F_{\zeta^*}(E_\varepsilon) : \varepsilon < \omega_1\}$ by $(*)_3$. Now trivially $\langle a^\delta_{\zeta^*} \cap E_\varepsilon : \varepsilon < \omega_1\rangle$ is a decreasing sequence of subsets of $a^\delta_{\zeta^*}$ which is countable and $\varepsilon < \omega_1 \Rightarrow a^\delta_{\zeta^*} \cap E_\varepsilon \neq a^\delta_{\zeta^*} \cap E_{\varepsilon+1}$ as $F_{\zeta^*}(E_{\varepsilon+1}, E_\varepsilon) = \emptyset$, contradiction.]

We fix E_0 as in $(*)_4$,

$(*)_5$ for some $\xi_\zeta < \omega_1$ we have:
for every club E_1 of ω_2 for some $\delta \in F_{\zeta^*}(E_1, E_0)$ we have $\text{otp}(a^\delta_{\zeta^*} \cap S \cap E_0) = \xi$.
[Why? As in the proof of $(*)_4$.]

So necessarily ξ is divisible by δ^*. Choose $b \subseteq \xi = \sup(b)$, $\text{otp}(b) = \delta^*$. Let

$$S' = \{\alpha < \omega_1 : \text{otp}(a^\alpha_{\zeta^*} \cap S \cap E_0) \in b \cup \{\xi\}\}.$$

Now we define $\bar{C} = \langle C_\alpha : \alpha \in S\rangle$ as follows: if $\alpha \in S\backslash S'$ we let $C_\alpha = \emptyset$ and if $\alpha \in S'$ we let $C_\alpha = \{\beta : \beta \in a^\alpha_{\zeta^*} \cap S \cap E_0 \text{ and } \text{otp}(\beta \cap a^\alpha_{\zeta^*} \cap S \cap E_0) \in b\}$.

Now you can check that $\bar{C} = \langle C_\alpha : \alpha \in S\rangle$ is as required. (Noting that is clause (e), "stationarily many" "at least one" are equivalent demands.) $\square_{19.1}$

Remark. Can we demand above that if C_α has no last element then C_α is a closed subset of α?

Not clear to me, but we can find

⊛ there is $\langle \mathscr{C}_\alpha : \alpha \in S\rangle$ such that
  (*a*) $\mathscr{C}_\alpha$ is a countable family of countable subsets of $\alpha \cap S$, each of order type $\leq \delta(*)$

(b) if $C \in \mathscr{C}_\alpha$ then C is closed as a subset of α

(c) if $\beta \in C \in \mathscr{C}_\alpha$ then $C \cap \beta \in \mathscr{C}_\beta$

(d) if E is a club of ω_2 then for stationarily many $\alpha \in S$ for some $C \in \mathscr{C}_\alpha$ we have $\delta(*) = \text{otp}(C)$ & $C \subseteq E$.

In some cases Zapletal [Zapox] uses $\mathfrak{d} \le \mathfrak{b}^{+n}$ we can replace this by $\text{cf}([\mathfrak{d}]^{\aleph_0}, \subseteq) = \mathfrak{d}$? Zapox ? because

Claim. *Assume κ is regular uncountable.*

If $\lambda > \kappa$ and $cf([\lambda]^{<\kappa}, \subseteq) = \lambda$, then for any $\alpha < \kappa$ there is $Y \subseteq {}^\alpha([\lambda]^{<\kappa})$ which is $E_{\kappa,\alpha}(\lambda)$-positive, i.e.,

⊛ *if $\chi > \lambda$ and $\xi \in \mathscr{H}(\chi)$ then there is $\bar N = \langle N_i : i \le \alpha \rangle$ such that $x \in N_i \prec (\mathscr{H}(\chi) \in <^*_\chi)$*

$$\|N_i\| < \kappa$$

$$N_i \cap \kappa \in \kappa$$

$$N_i \text{ increasing continuous}$$

$$\bar N \restriction (i+1) \in N_{i+1}.$$

Proof. As in [Sh 420, §2] (fill!)

Let $W_0 = \{\alpha \in E_0 : \{\xi^*, \zeta^*\} \subseteq C^0_\alpha$ and $\text{otp}(a^\alpha_{\xi^*} \cap E_0) \le \zeta$ and for $\alpha \in W_0$ let $b_\alpha = a^\alpha_{\xi^*} \cap E_0 \cap S$. Clearly

⊛(a) $\alpha \in W_0 \Rightarrow b_\alpha \subseteq W_0$ & $\text{otp}(b_\alpha) \le \zeta$

(b) $\alpha \in b_\beta \cap \beta \in W \Rightarrow b^*_\alpha = b_\beta \cap \alpha$

(c) if E_1 is a club of ω_2 then for stationarily many $\alpha \in W_0 \cap S$ we have $b_\alpha \subseteq E_1$ and $\text{otp}(b_\alpha) = \xi$.

Part D - list of additional papers

[Sh 410]

[Sh 413]

[Sh 420]

[Sh 430]

[Sh 460]

[Sh 462] (applications to entangled linear order)

[Sh 497] (pcf without choice)

[Sh 506]

[Sh 513]

[ShTh 524] (application to cofinality spectrum of permutation groups)

[RoSh 534] (applications to Boolean Algebras)

[Sh:535] (colourings) ? Sh:535 ?

[Sh 552] (applications to existence of universals in classes of abelian groups)

[Sh 572] (colourings + guessing clubs)

[Sh 575] (applications to Boolean Algebra)

[GiSh 597]

[Sh 580] (strong covering)

[Sh 589] (basic + applications to Boolean Algebras, independence in stability theory)

[Sh 620] (existence of complicated $F \subseteq \prod_{i<\delta} \mathrm{Dom}(I_i)$, applications to Boolean Algebras)

[Sh 622] (application to existence universal in classes of abelian groups)

[KjSh 609] (application to general topology)

[Sh 641] (applications to Boolean Algebras)

[Sh 652] (applications to Boolean Algebras)

[Sh 666] (§1, on open questions)

[Sh 668] (Arhangelskii's problem, essential equiconsistency)

[KjSh:676?] (in the trichotomy, $\mathrm{cf}(\delta) > \kappa^+$ is necessary) ? KjSh:676? ?

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