FORCING AXIOMS FOR $\lambda$-COMPLETE $\mu^+$-C.C.

SH1036

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Abstract. We consider forcing axioms for suitable families of $\mu$-complete $\mu^+$-c.c. forcing notions. We show that some form of the condition "$p_1, p_2$ have a $\leq Q$-lub in $Q$" is necessary. We also show some versions are really stronger than others.

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References like [Sh:950, Th0.2=Ly5] means the label of Th.0.2 is y5. The reader should note that the version in my website is usually more updated than the one in the mathematical archive. The author thanks Alice Leonhardt for the beautiful typing. First typed July 31, 2012.
§ 0. Introduction


We investigate the relationships between some forcing axioms related to pressing down functions for $\mu^+$-c.c., mainly from Sh:546. This in particular is to answer Kolesnikov's question of having $\mathbb{P}$ satisfying one condition but no $\mathbb{P}'$ equivalent to $\mathbb{P}$ satisfying another. A side issue is clarifying a point in BKSh:927 (a rephrasing is (2) from Sh:80). We intend to continue this considering related axioms in Sh:F1856.

We justify the "well met, having lub" in some forcing axioms, e.g. condition (c) in $^1_{\mu,Q}$ such forcing axiom was proved consistent, for forcing notion satisfying:

- $^1_{\mu,Q}$ a forcing notion such that
  - $\langle q_\alpha : \alpha < \mu \rangle$ is $\leq Q$-increasing for $\ell = 1, 2$ and $p_\alpha \in Q$ is a $\leq Q$-lub of $p_{1,\alpha}, p_{2,\alpha}$ (i.e. $\bigwedge_{\ell=1}^2 p_{\ell,\alpha} \leq Q p_\alpha$ and $(\forall q)(p_{1,\alpha} \leq Q q \land p_{2,\alpha} \leq Q q \rightarrow p_\alpha \leq Q q)$). Then $\langle p_\alpha : \alpha < \epsilon \rangle$ is $\leq Q$-increasing hence if $\{p_\alpha : \alpha < \epsilon\}$ has an upper bound then so does $\{p_{1,\alpha}, p_{2,\alpha} : \alpha < \epsilon\}$.

An obvious fact used is

\[ \langle q_\alpha : \alpha < \epsilon \rangle \text{ is } \leq Q \text{-increasing hence if } \{q_\alpha : \alpha < \epsilon\} \text{ has an upper bound then so does } \{q_{1,\alpha}, q_{2,\alpha} : \alpha < \epsilon\}. \]

Our main conclusions are Sh:80, Sh:93, ShSt:154.

The immediate reason for this paper is that the statement in Baldwin-Kolesnikov-Shelah BKSh:927, 3.6] is misquoting Sh:93, 4.12], we shall show below that the statement is inconsistent because as stated it totally waives the condition "every two compatible members of $\mathbb{P}$ have a lub". Also it is stated that in Sh:80, Sh:93, 4.12] this was claimed but quoting only Sh:80. In Shelah-Stanley ShSt:154, pg.235 it
is said: FILL. In Shelah-Spinas [ShSi:1110] we consider another strengthening of the axioms. FILL.

More fully [Sh:33, 4.12] omit the condition above, but demand the existence of lub's of some pairs of conditions so that it holds in the cases it is actually used, so the proof of [Sh:30] works, and see more in [Sh:316, Def.1.1] which gives an even weaker condition called $*_{\mu}^1$.

Concerning $*_{\mu}^1$, the preservation of a related condition was proved independently by Baumgartner, who instead of (b) use a somewhat strong condition $(b)^{+}$ which says that $Q$ is the union of $\mu$ sets of pairwise compatible elements with lub$^+$, this is represented in Kunen-Tall [KT79], see history in the end of [Sh:30] and see more in [Sh:546]. We thank Mirna Dzamonja for drawing our attention to the problem and Ashutosh Kuman for various corrections.

§ 0(B). Are Some Version of Axioms Equivalent?

To phrase our problem see the Definition below.

Kolesnikov ask:

\[a\]

Question 0.1. Is there a forcing notion $\mathbb{P}$ satisfying $(1)_a, (2)_b, (3)_{b,\omega}$ but not equivalent to a forcing notion $\mathbb{P}'$ satisfying $(1)_a, (2)_b, (3)_a$?

\[b\]

Definition 0.2. Consider the following conditions on a forcing notion $\mathbb{P}$ for a fix $\mu = \mu^{<\mu}$:

- completeness:
  - $(1)_a$ increasing chains of length $< \mu$ has a lub.
  - $(1)_{a,\theta}$ increasing chains of length $< \theta$ has a lub.
  - $(1)_{a,\omega}$ increasing chains of length $\theta$ has an lub.
  - $(1)_b$ increasing chains of length $< \mu$ has a ub.
  - $(1)_{b,\theta}$ increasing chains of length $< \theta$ has an ub.
  - $(1)_{b,\omega}$ increasing chains of length $\theta$ has a ub.
  - $(1)_c$ $\mathbb{P}$ is strategically $\alpha$-complete for every $\alpha < \mu$, see $[z_3]$.
  - $(1)_{c,\alpha}$ $\mathbb{P}$ is strategically $\alpha$-complete where here $\alpha \leq \mu$.
  - $(1)_{c}^{+}$ there is a “stronger” order $<_{st}$ on $\mathbb{P}$ which means:
    - $p_1 <_{st} p_2 \Rightarrow p_1 <_p p_2$
    - $p_1 \leq p \leq p_3 \leq p_4 \Rightarrow p_1 <_{st} p_4$
    - any $<_{st}$-increasing chain of length $< \mu$ has a $\leq_{st}$-ub (hence a $<_{st}$-ub)
  - $(1)_{d,\theta}$ any increasing continuous chains of length $< \theta$ has a lub.
  - $(1)_{d,\omega}$ any increasing continuous chain of length $\theta$ has a lub.

Strong $\mu^{+}$-c.c.: for a stationary $S \subseteq S^\mu_{\mu^+}$, the default value being $S^\mu_{\mu^+}$, see $[z_7]$:10; we may write $(2)_z[S]$ when $S$ not the default value or clear from the context.

- $(2)_a$ Given a sequence $\langle p_i : i < \mu^+ \rangle$ of members of $\mathbb{P}$ there are a club $C$ of $\mu^+$ and a regressive function $h$ on $C \cap S$ such that $\alpha, \beta \in C \cap S \wedge h(\alpha) = h(\beta) \Rightarrow p_\alpha, p_\beta$ have a lub.
- $(2)_b$ like $(2)_a$ but demanding just that $p_\alpha, p_\beta$ have an ub
(2)\textsuperscript{+\textsubscript{a,θ}} \textit{if} p_α \in P \textit{for} \alpha < µ^+ \textit{then} we can find a club \textit{E} of µ^+ and a regressive
\textit{h} : S \cap E \rightarrow µ^+ \textit{such that: if} i(*) < 1 + θ, δ_i \in S \cap E \textit{for} i < i(*) \textit{and}
h\{δ_i : i < i(*)\} is constant \textit{then} \{p_{δ_i} : i < i(*)\} has a lub
\textit{(2)\textsuperscript{+\textsubscript{a,θ}}} \textit{like (2)\textsubscript{a,θ}} \textit{but in the end the set has a lub}
\textit{(2)\textsuperscript{a,θ}} \textit{if} p_α \in P \textit{for} \alpha < µ^+ \textit{then} we can find \bar{q}, E, h \textit{such that}
\begin{itemize}
\item \bar{q} = (q_α : \alpha < µ)
\item p_α \leq_P q_α
\item E \textit{a club of} µ^+
\item h \textit{is a regressive function on} S \cap E
\item if \mathcal{U} \subseteq S \cap E \textit{has cardinality} < θ \textit{and} h\mathcal{U} \textit{is constant, then} \{q_δ : δ \in \mathcal{U}\} has a lub.
\end{itemize}

For ε < µ a limit ordinal, \textit{e.g.} ω:
\begin{itemize}
\item (3)\textsubscript{a} any two compatible \textit{p}_1, \textit{p}_2 \in P \textit{has a lub}.
\item (3)\textsubscript{b,ε} if \textit{p}_{1,ζ} : ζ < ε \textit{is increasing for} ℓ = 1, 2 \textit{and} \textit{p}_{1,ζ}, \textit{p}_{2,ζ} \textit{are compatible for}
\textit{every} ζ < ε \textit{then} \{p_{ζ,ℓ} : ℓ \in \{1, 2\}, ζ < ε\} \textit{has an upper bound; recall } \mathbb{B} \textit{of §(0A)}.
\end{itemize}

\textbf{Definition 0.3.} For \textit{D} a normal filter on µ^+ to which \textit{S}^+ \textit{belongs} (we may omit \textit{D} when it is the club filter on µ^+ + \textit{S}^+, see Definition \textbf{E}\textsuperscript{+\textsubscript{2}}) and \textit{P} a forcing notion and an ordinal ε < µ, a limit ordinal if not said otherwise, we define the following
conditions on \textit{P}
\begin{itemize}
\item (2)\textsubscript{ε,D} in the following game the \textit{COM} player has a winning strategy (we may omit \textit{D} if clear from the context)
\item (a) a play last ε-moves
\item (b) in the ζ-th move (\textit{p}_ζ, h_ζ) is chosen such that:
\begin{itemize}
\item (α) \textit{p}_ζ = (p_{ζ,α} : α \in S_ζ)
\item (β) \textit{p}_{ζ,α} \in P
\item (γ) S_ζ \in \textit{D}
\item (δ) S_ζ \subseteq \bigcap\{S_ξ : ξ < ζ\}
\item (ε) if α \in S_ζ then \textit{p}_{ζ,α} : ξ \leq ζ) is a \textit{≤}_P-increasing sequence
\item (ζ) h_ζ is a pressing down function on S_ζ
\item (η) \textit{COM} chooses \textit{p}_{ζ,1} (\textit{p}_{ζ,1}, h_ζ) when 1 + ζ is even, \textit{INC} chooses it when 1 + ζ
is odd
\item (d) \textit{COM} wins a play when it always could have made a legal move, and
in the end if there is \textit{S}_ζ \in \textit{D} included in \bigcap\{S_ξ : ξ < ζ\}
and \textbf{a,θ} \textit{α}_2 = h_ζ(\textit{α}_2) then \{p_{ξ,ℓ} : ξ < ε, ℓ \in \{1, 2\}\} has an ub
\item (2)\textsubscript{ε,D} is defined as above replacing clause (b)(ε) by
\item (ε') if α \in S_ζ then \textit{p}_{ζ,ε} : ξ < ζ) is \textit{≤}_P-increasing continuous.
\end{itemize}
\end{itemize}

\textsuperscript{1}Why 1 + ζ not, \textit{e.g.} ζ + 1? First, we like the \textit{INC} to have the first move so that if \textit{P} satisfies the condition and \textit{p} \in \textit{P} then \textit{P}\{q : p \leq q\} satisfies the condition. Second, we like the player \textit{COM} to move in limit stages, as this is a weaker demand.
Remark 0.4. 1) So for a forcing notion $Q$, $(2)_{c,D}^\varepsilon$ for $\varepsilon$ limit is $\ast\mu^+_{\mu}D^\varepsilon$ [Sh:546, 1.1] and $(2)_{\mu,Q}^\varepsilon$ is $\ast\mu^+_{\mu}$ from the beginning of §0(A) for $D$ the club filter $+\delta^{\mu,+}$ and $(2)_{\mu,n}^\varepsilon$ is $\ast\mu_{\mu,Q}^\varepsilon$ which too appears in §0(A).

2) Note that “$P$ satisfies $(2)_{c,D}^\varepsilon$” includes a weak version of strategic completeness.

Definition 0.5. 1) Let $Ax\lambda,\mu((1)_x, (2)_y, (3)_z)$ means: if $P$ is a forcing notion satisfying those conditions and $\mathcal{F} \subseteq P$ is dense open for $i < i(*) < \lambda$ then some directed $G \subseteq P$ meet every $\mathcal{F}i$.

2) We may omit $\lambda$ if $\lambda = 2^\mu$, we may write $Ax\lambda,\mu(K)$ for $K$ a property of forcing notion.

3) For an ordinal $\varepsilon < \mu$, a limit ordinal if not said otherwise, let $Ax\mu,\mu(1)_c, (2)_c^\varepsilon$ means: $Ax\lambda,\mu((1)_x, (2)_y, (3)_z)$, we may omit $\lambda = 2^\mu > \lambda^+$.

See on more axioms Roslanowski-Shelah [RoSh:655] parallel to forcing and $[Sh:589]$ and references there. In §1 if we replace $C_3$ by a stationary, co-stationary subset of $\delta$, we can iterate appropriate $\mu^+\text{-c.c.} (< \mu)$-complete forcing notion.

Question 0.6. 1) Is there an example $P$ where $\ast\mu^\theta$ holds but $\ast\mu^\theta$ fail for any $\theta \in RCar\setminus\{\theta\}$ where $\theta = cf(\theta), \gamma = \partial \theta < \mu^+?$. The case $\theta = \aleph_0 < \theta$ is natural.

2) Do we have an example for $Ax\varepsilon((1)\varepsilon + (2)\varepsilon + (3)\varepsilon)$ but not $Ax\mu,\varepsilon = \omega$.

For answers on this question see §3.

Discussion 0.7. 1) Note: if we have $(3)\varepsilon = \text{well met then we have } (2)\varepsilon \equiv (2)\varepsilon$. If in addition we have $(1)\varepsilon$ then we have $(2)\varepsilon$ for every $\varepsilon$. Hence $(2)(1)$ may be the true question.

In §2 we get a result, see there, but not on $(2)(1)$.

2) Suppose we consider a forcing notion as in §1, i.e. for §2 use $\theta = 1$, but as in §3, for $\alpha < \delta \in S^\mu$ no uniformization demand. This makes $\ast\mu^\theta$ holds for this forcing notion, but $\ast\mu^\theta$ fail, so all seems fine.

3) In fact for $(C_\delta, \mathcal{F}_\delta : \delta \in S)$, we may force also the $\mathcal{C}_\delta$ (in $Q$ in §1); we may not ask that $\mathcal{C}_\delta$ is closed in $\delta$ and let $\alpha_{\delta,\xi} = (\alpha_{\delta,\xi} : \xi < \mu)$ list $\mathcal{C}_\delta$ in increasing order so with limit $\delta$, but generically we can have $\alpha_{\delta,\xi} = \alpha_{\delta,\xi}^\ast$, $\mathcal{F}_\delta(\alpha_{\delta,\xi}^\ast) \neq \mathcal{F}_\delta(\alpha_{\delta,\xi}^\ast)$ for $\ast\mu^\delta$, i.e. anyhow seems reasonable.

Observation 0.8. Assume $\mu = \mu^\varepsilon < \mu$ and $\varepsilon < \mu$ limit.

1) If the forcing notion $Q$ satisfies the conditions $(1)\varepsilon, |\varepsilon|, (3)\varepsilon$ and $(2)\varepsilon$ here equivalently $(2)\varepsilon$, then $Q$ satisfies $\ast\mu^\theta$, of $[Sh:546, 1.1]$, i.e. $(2)\varepsilon$ from Definition 0.3, for every limit $\varepsilon < \mu$.

2) If $P$ satisfies $(3)\varepsilon$ then $P$ satisfies $(3)\varepsilon$.

3) If $P$ satisfies $(1)\varepsilon, |\varepsilon|, (3)\varepsilon\varepsilon$ then $P$ satisfies $(2)\varepsilon$.

4) For any $P$ we have: $(1)\varepsilon \Rightarrow (1)\varepsilon \Rightarrow (1)\varepsilon \Rightarrow (1)\varepsilon$ and $(1)\varepsilon \Rightarrow (1)\varepsilon \Rightarrow (1)\varepsilon$. Similarly $(1)\varepsilon, \theta \Rightarrow (1)\varepsilon, \theta \Rightarrow (1)\varepsilon, \theta$ and $(1)\varepsilon, \theta \Rightarrow (1)\varepsilon, \theta$ and $(1)\varepsilon, \theta \Rightarrow (1)\varepsilon, \theta$.

5) For any $P$ we have $(2)\varepsilon, \theta \Rightarrow (2)\varepsilon, \theta \Rightarrow (2)\varepsilon, \theta$.

6) If $P$ satisfies $(2)\varepsilon$ then forcing with $D$ add no new sequence of ordinals of length $\varepsilon$.

Proof. Just read the definitions carefully.

E.g. 2really omitting $(1)\varepsilon$ does not make a real difference but is natural.
3) Recall $\boxtimes$ of §(0A).

**Claim 0.9.** 1) $\text{Ax}_\mu^c$, i.e. $\text{Ax}_\mu((1)_c + (2)_c)$ is equivalent to the axiom in [Sh:546].
2) $\text{Ax}_\mu((1)_b, (2)_a, (3)_a)$ is the axiom from [Sh:80]. If $\theta, \sigma$ are regular cardinals $< \mu$ and $\text{Ax}_\sigma^\theta$ does not imply $\text{Ax}_\sigma^\theta$ then $\text{Ax}_\mu((1)_b, (2)_a, (3)_a)$ so the axiom from [Sh:80], does not imply $\text{Ax}_\mu^\sigma$.

*Proof.* Easy, too.

Many works on forcing for uniformizing see [Sh:64], [Sh:587], [Sh:f, Ch.VIII] and on ZFC results see [DvSh:65], [Sh:f, AP, §1].

§ 0(C). **Preliminaries.**

**Notation 0.10.** Fix regular $\theta < \lambda$ let $S^\theta_\delta = \{ \delta < \lambda : \delta \text{ has cofinality } \theta \}$.

**Definition 0.11.** 1) We say that a forcing notion $P$ is strategically $\alpha$-complete when for each $p \in P$ in the following game $\mathcal{G}_\alpha(p, P)$ between the players COM and INC, the player COM has a winning strategy.

A play lasts $\alpha$ moves; in the $\beta$-th move, first the player COM chooses $p_\beta \in P$ such that $p \leq P p_\beta$ and $\gamma < \beta \Rightarrow q_\gamma \leq P p_\beta$ and second the player INC chooses $q_\beta \in P$ such that $p_\beta \leq P q_\beta$.

The player COM wins a play if he has a legal move for every $\beta < \alpha$.

2) We say that a forcing notion $P$ is $< \lambda$-strategically complete when it is $\alpha$-strategically complete for every $\alpha < \lambda$.

**Definition 0.12.** For a filter $D$ on a set $I$

(a) $D^+ = \{ A \subseteq I : I \setminus A \notin D \}$

(b) for $S \in D^+$ let $D + S = \{ A \subseteq I : A \cup (I \setminus S) \in D \}$. 
§ 1. ON $\mu^+$-REGRESSIVE-C.C.; AN EXAMPLE

The counterexample is by [Sh:f, AP.3.9, pg.990].
First, we shall concentrate on the case $\mu$ is not strongly inaccessible.

**Hypothesis 1.1.** 1) $\mu = \mu^\prec > \aleph_0$.
2) $S = S_{\mu^+} = \{ \delta < \mu^+: \text{cf}(\delta) = \mu \}$.  

**Definition 1.2.** $\bar{C}$ is an $S$-club system when $\bar{C} = (C_\delta : \delta \in S), C_\delta$ a club of $\delta$ of order type $\mu$.

**Definition 1.3.** 1) We say $(\mathcal{W}, \bar{f})$ is a $(S, \bar{C}, \kappa)$-parameter or just $(\bar{C}, \kappa)$-parameter when:

   (a) $S \subseteq S_{\mu^+}$
   (b) $\bar{C}$ is an $S$-club-system so we may omit $S$
   (c) $\kappa \leq \mu$ is $\geq 2$, if $\kappa = 2$ we may omit $\kappa$ and write $\bar{C}$
   (d) $\mathcal{W} \subseteq \mu$; if $\mathcal{W} = \mu$ we may omit $\mathcal{W}$
   (e) $\bar{f} = (f_\delta : \delta \in S)$
   (f) $f_\delta : C_\delta \rightarrow \kappa$.

2) For $(\mathcal{W}, \bar{f})$ an $(S, \bar{C}, \kappa)$-parameter we define a forcing notion $Q = Q_{(\mathcal{W}, \bar{f}, \bar{C})}$ as follows:

   (A) $p \in Q$ iff $p$ consists of
      (a) $u \in [S]^\mu$
      (b) $g$ a function with domain $u$
      (c) if $\delta \in u$ then $g(\delta)$ is a closed bounded subset of $\mu$
      (d) if $\delta_1, \delta_2 \in u$ and $\alpha \in C_{\delta_1} \cap C_{\delta_2}$ (can add $C_{\delta_1} \cap C_{\delta_2} \cap \alpha$) and otp$(\alpha \cap C_{\delta_1}) \in g(\delta_1)$ and otp$(C_{\delta_1} \cap \alpha) \in \mathcal{W}$ for $\ell = 1, 2$ then $f_{\delta_1}(\alpha) = f_{\delta_2}(\alpha)$
      (e) if $\delta_1 \neq \delta_2 \in u$ and $\beta \in C_{\delta_1} \cap C_{\delta_2}$ then $\text{otp}(\beta \cap C_{\delta_1}) \leq \text{max}(g(\delta_1), g(\delta_2))

   (B) $p \leq q$ iff:
      (a) $u_p \subseteq u_q$
      (b) $\delta \in u_p \Rightarrow g_p(\delta) \leq g_q(\delta)$.

**Definition 1.4.** Let $(\mathcal{W}, \bar{f})$ be a $(\bar{C}, \kappa)$-parameter and let $Q = Q_{\mathcal{W}, \bar{f}, \bar{C}}$.
1) For $p \in Q$ let $h_p$ be the function

   (a) with domain

      \{ $\alpha : \text{some } \delta \text{ witness } \alpha \in \text{Dom}(h_p) \text{ which means } \delta \in u_p, \alpha \in C_\delta$

          $\text{otp}(C_\delta \cap \alpha) \in g_p(\delta)$ and otp$(C_\delta \cap \alpha) \in \mathcal{W}$ \}

   (b) for $\alpha \in \text{Dom}(h_p)$ we have:

      $$h_p(\alpha) = f_\delta(\alpha) \text{ for every witness } \delta \text{ for } \alpha \in \text{dom}(h_p).$$

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3So this holds also for $\delta_2$. 

2) Let $h$ be the $\mathbb{Q}$-name $\cup\{h_p : p \in G\}$.
3) Let $E_\delta = E_\delta[Q]$ be the $\mathbb{Q}$-name $\cup\{g_p(\delta) : p \in G, \delta \in u_p\}$ and let $\mathcal{W}_\delta = \{\alpha \in E_\delta : \text{otp}(C_\delta \cap \alpha) \in \mathcal{W}\}$.

**Claim 1.5.** Assume $(\mathcal{W}, \mathcal{F})$ is an $(S, C, \kappa)$-parameter and $Q = Q[(\mathcal{W}, \mathcal{F}, C)]$.

1) $Q$ is $\langle \mu \rangle$-complete, moreover any $\leq \mu$-increasing sequence of length $\mu$ has a $\leq \mu$-club.

2) If $\delta \in S$ and $i < \mu$ then the following subsets of $Q$ are dense and open:

   - $\mathcal{F}_\delta = \{p \in Q : \delta \in u_p\}$
   - $\mathcal{F}_{\delta,i} = \{p \in \mathcal{F}_\delta : i < \text{sup}(g_p(\delta))\}$
   - $\mathcal{F}^+ = \{p \in \mathcal{F} : \text{if } \delta \in u_p \text{ then } i < \text{sup}(g_p(\delta))\}$

3) $h$ almost extends $f_\delta$, i.e. $\models Q " h \supseteq f_\delta^1 \{\alpha \in C_\delta : \text{otp}(\alpha \cap C_\delta) \in W_\delta\} \text{ and } E_\delta \text{ a club of } \mu \text{ and if } \mathcal{W} = \mu \then W_\delta \text{ is a club of } \mu^+.$

**Proof.** 1) Straightforward, see clause (A)(e) of Definition 1.6 in particular.

2) Also easy.

**Claim 1.6.** Let $(\mathcal{W}, \mathcal{F}), (S, C, \kappa), Q$ be as above.

$Q$ satisfies clause (2)_b of Definition 1.6 that is

$\ast_\mu^1$ if $\bar{p} = \langle p_\alpha : \alpha \in S \rangle$ and $\alpha \in S \Rightarrow p_\alpha \in Q$ then there is a club $E$ of $\mu^+$ and pressing down function $f : S \cap E \rightarrow \mu^+$, i.e. $f(\delta) < \delta$, such that: 

$(\delta_1 \neq \delta_2 \in S \cap E) \land (\delta_1 = f(\delta_2) \Rightarrow p_{\delta_1}, p_{\delta_2} \text{ are compatible.}$

**Proof.** First, by 1.12(1)(2), we choose $\langle q_\alpha : \alpha \in S \rangle$ such that:

- $\circ_1 (a)$ $p_\alpha \leq q_\alpha$
- $\circ_2 (a)$ if $\delta \in u_{q_\alpha}$ but $\delta > \alpha$ then $\text{otp}(C_{\delta} \cap \alpha) < \text{sup}(q_{\alpha}(\delta))$
- $\circ_3 (a)$ $\alpha \in S \Rightarrow \alpha \in u_{q_\alpha}$

Second, choose a club $E$ of $\mu^+$ such that $\alpha \in S \cap E \Rightarrow \text{sup}(u_{q_\alpha}) < \text{min}(\langle E \setminus (\alpha + 1) \cap S \rangle$.

Third, choose function $f$ with domain $E \cap S$ such that:

- $\circ_1 (a)$ if $\delta(1) = \delta_1 \neq \delta_2 \neq \delta(2)$ are from $E \cap S$ and $f(\delta_1) = f(\delta_2)$ and $\langle \alpha_{f, j} : i < \text{otp}(u_{q_{\alpha(\delta_i)})} \rangle$ list $u_{q_{\alpha(\delta_i)}}$ in increasing order for $\ell = 1, 2$ then for some $j_s$:
  - $\circ_2 (a)$ $\text{otp}(u_{q_{\alpha(\delta_s)})} = \text{otp}(u_{q_{\alpha(\delta_s)})}$ call it $i^*$
  - $\circ_2 (b)$ $j_s < i^*(*)$ and $\alpha_{1,j_s} = \delta_1, \alpha_{2,j_s} = \delta_2$
  - $\circ_2 (c)$ if $j < j_s$ then $\alpha_{1,j} = \alpha_{2,j}$
  - $\circ_2 (d)$ if $j > j_s$ but $j < i^*(*)$ then $C_{\alpha_{1,j}} \cap \delta_1 = C_{\alpha_{2,j}} \cap \delta_2$
  - $\circ_2 (e)$ $g_{q_{\alpha(\delta_1)}(\alpha_{1,i})} = g_{q_{\alpha(\delta_2)}}(\alpha_{2,i})$ for $i < i^*(*)$
  - $\circ_2 (f)$ if $\varepsilon \in g_{q_{\alpha(\delta_1)}}(\alpha_{1,i})$ then the $\varepsilon$-th member of $C_{\delta_1}$ is equal to the $\varepsilon$-th member of $C_{\delta_2}$.

Now it suffices to prove:

- $\circ_3 (a)$ if $\delta_1 \neq \delta_2 \in S \cap E$ and $f(\delta_1) = f(\delta_2)$ then $q_{\delta_1}, q_{\delta_2}$ are compatible in $Q$.

**Why?** Define $q$ as follows:

- $u_q = u_{q_{\alpha(1)}} \cup u_{q_{\alpha(2)}}$
• \(g(\delta) = g_{\text{rel}}(\delta)\) if \(\ell \in \{1, 2\}, \delta \in u_p \setminus \{\bar{\delta}\}\)

• \(g_{\text{rel}}(\delta) = g_{\text{rel}}(\delta_1) \cup \{\gamma\}\) where \(\gamma < \mu, \gamma > \max\{g_{\text{rel}}(\delta_1) \cup g_{\text{rel}}(\delta_2)\}\) and \(\gamma > \sup\{\alpha \cap C, \ell : \ell \in \{1, 2\}\}\) and \(\alpha \in C, \delta_1 \cap C, \delta_2\).

It is easy to check that \(q \in Q\) and \(q_1 \leq q, q_2 \leq q\), so \(\odot_3\) holds indeed. \(\square\)

**Claim 1.7.** Assume \(\mu\) is a successor cardinal or just not inaccessible. For every \(S\)-club system \(\mathcal{C}\), for some \((S, C, 2)\)-parameter \(f\), so \(W = \mu\) there is no directed \(G \subseteq Q\) meeting \(\mathcal{A}_x, \mathcal{A}_y, \mathcal{A}_z\) for every \(\delta \in S, i < \mu\), see [Sh:7].

**Proof.** Why? By [Sh:3, AP.3.9]. This is the only place “strongly inaccessible” enters. \(\square\)

**Conclusion 1.8.** The condition “have least upper bound” cannot be omitted in\(^4 [Sh:80]\). That is:

1. There are \(Q\) and \(\mathcal{A}(\alpha < \mu^+)\) such that:
   a. \(Q\) is a forcing notion, \((< \mu)\)-complete, in fact every \(\leq_Q\)-increasing sequence of length \(\mu\) has a lub, i.e. satisfies (1)\(_a\)
   b. \(Q\) satisfies (2)\(_b\)
   c. each \(\mathcal{A}\) is a dense open subset of \(Q\)
   d. no directed \(G \subseteq Q\) meet every \(\mathcal{A}_x, \alpha < \mu^+\).

**Proof.** Let \(C\) be an \(S\)-club system. If \(\mu\) is a successor or just not strongly inaccessible, choose \(f\) and \(\mathcal{A} = (\mathcal{A}_x, \mathcal{A}_y, \mathcal{A}_z) : \delta \in S, i < \mu\) as in \(\square\) above, so \(Q = Q(f, C)\). So \(Q\) satisfies clause (a) by \(\square\) above, satisfies clause (b) by \(\square\) above and satisfies clauses (c),(d) by the choice of \(f\) and \(\mathcal{A}\). We are left with the case \(\mu\) is strongly inaccessible, then we use \(\square\) below instead of \(\square\), i.e. instead of quoting. \(\square\)

We shall prove below that in \(\square\) above we may replace \(\mu = 2^\kappa\) by \(\mu = \kappa > \aleph_0\) (in the notation there replace \(\kappa = 2^\theta\) by “\(\kappa = \theta > \aleph_0\)”).

**Theorem 1.9.** If (A) then (B) where:

\(A\)

1. \(\kappa\) is strongly inaccessible
2. \(S = \{\delta < \kappa^+ : \text{cf}(\delta) = \kappa\}\)
3. \(\eta_\delta\) is an increasing continuous sequence of ordinals \(< \delta\) of length \(\kappa\) with limit \(\delta\), for each \(\delta \in S\)
4. \(F\) is a function from \(\{h : h\) is a function \(\text{dom}(h) \in [\kappa^+]^{< \kappa}\) and \(\text{rang}(h) \subseteq \kappa\}\) into \(\kappa\)
5. \(a = \langle a_\delta : \delta \in S, i < \text{cf}(\delta)\rangle\) with \(a_\delta \in \eta_\delta(i) + 1\)

\(B\) we can find \(d = \langle d_\delta : \delta \in S\rangle\) with \(d_\delta \in \kappa^\kappa\) such that for any \(h : \kappa^+ \rightarrow \kappa\) for stationarily many \(\delta \in S\) for stationarily many \(i < \delta\) we have \(d_\delta(i) \neq F(h[a^i])\).

**Proof.** As there. \(\square\)

In \(\square\) above we get failure when we waive in \([Sh:80]\) the “well met condition”.

**Conclusion 1.10.** In \(\square\) above we may replace (a) by (a)’ and add (e) where:

\(\vdash\) and the related works
Remark 1.11. 1) In \[\mathsf{KT79}\] and \[\mathsf{KT80}\] we can moreover find \(\mathcal{I}_\mathcal{E} : \varepsilon < \mu\) such that \(\mathcal{I} = \bigcup_{\varepsilon < \mu} \mathcal{I}_\mathcal{E}\) is dense and \(p, q \in \mathcal{I}_\mathcal{E} \Rightarrow p, q\) are compatible (as in \[\mathsf{KT79}\]).

Proof. We use the forcing from Definition \ref{1.51} below so a variant of the forcing above. This forcing notion satisfies clause \((\alpha)'\) by \ref{1.52}(1),(2) below clause \((\beta)\), i.e. \((\beta)_b\), by \ref{1.53}(3) below. As for clauses \((c),(d)\) we choose \(f\) by \ref{1.54} or \ref{1.55} above. \(\Box\)

(a)' \(\mathcal{Q}\) is a forcing notion strategically \((< \mu)\)-complete (i.e. \((2)_c\)), in fact some partial order \(\leq^*_\mathcal{Q}\) witness it in a strong way (i.e. \((1)^*_\mathcal{Q}\)) , meaning

\((\alpha)\) any \(< \mu\)-increasing sequence of length \(< \mu\) has an upper bound

\((\beta)\) \(p_1 \leq p_2 \leq p_3 \leq p_4 \rightarrow p_1 \leq^*_\mathcal{Q} p_4\)

\((\gamma)\) \((\forall p)(\exists q)(p <^*_\mathcal{Q} q)\).

\((e)\) (well met) if \(p, q \in \mathcal{Q}\) are compatible then they have a lub.
§ 2. Forcing axiom - non equivalence

We use Definitions 2.2, 2.3 freely.

Theorem 2.1. Assume \( \theta = \text{cf}(\theta) < \mu = \mu^{<\mu} \) and \( Q \) is adding \( \mu^+, \mu \text{-Cohen} \).

Then, in \( V^Q \) we have:

- \( \exists \mu, \varepsilon \) for some \( P \)
  - (a) \( P \) is a forcing notion
  - (b) \( P \) is not equivalent to any forcing notion satisfying \( (1)_b + (2)_{a, \theta} \) or even just \( (2)_{b, \theta} \), see Definition 2.2

Remark 2.2. Hence the relevant forcing axioms are not equivalent!

Proof. By \( \text{2.2}, \text{2.3}, \text{2.4} \) below.

In details: let \( f \) be from \( \text{2.7}(0) \), i.e. after the preliminary forcing \( Q \), in \( V^Q \) and \( P = P_f \).

Clause (a)(a), \( P \) a forcing notion holds, by Definition 2.5, i.e. first statement of \( \text{2.1} \).

Clause (a)(b), i.e. \( (2)_{b, \theta} \) holds by \( \text{2.7}(5) \).

Clause (a)(c), "P of cardinality \( \mu^+ \)", holds by \( \text{2.7}(1) \).

Clause (a)(d), \( (1)_b^+ \) and so \( P \) satisfies \( \text{2.1}, \text{2.2} \), "P complete, \( \mu \text{-complete} \), by \( \text{2.7}(1),(2) \).

Clause (a)\( (e) \), means \( (2)_{b, \theta}^+ \) which holds by \( \text{2.7}(5) \).

Clause (a)(f), "if \( p, q \) are compatible then they have a lub", holds by \( \text{2.7}(3) \).

Also Clause (b)(a), "\( P \) not equivalent to a forcing satisfying \( (1)_b + (2)_{a, \theta} \) or even just \( (2)_{b, \theta} \), see Definition 2.2"

Clause (b)(b), \( P \) not equivalent to any forcing notion satisfying \( (1)_b + (2)_{a, \theta} \) or even just \( (2)_{b, \theta} \) holds, by Clause (b)(\( b \)).

Conclusion 2.3. If \( \theta = \text{cf}(\theta) < \mu = \mu^{<\mu} \) then \( \text{Ax}_{\mu^+}(1)_a + (2)_b + (3)_{b, \theta} \) does not imply \( \text{Ax}_{\mu}^{\text{cf}(\theta)} \) from \( \text{2.5}(3) \).
Proof. Let the forcing notion \( \mathbb{R} \) forces \( \Delta \) of \((1) + (2)a + (3)b,\theta \) and satisfies those conditions, we know such \( \mathbb{R} \) exists by [Sh:546]. Now \( \mathbb{R} \) satisfies \((1)a\) by \((1)a\) and it satisfies \((2)\) by \((2)\). We get an equivalent forcing, we lose some nice properties but have a lub. \( \square \)

For this section (clearly if \( \mu = \mu^{<\mu} > \aleph_0 \) then there are so)

**Hypothesis 2.4.** 1) \( \mu = \mu^{<\mu} > \theta = \text{cf}(\theta) \geq \aleph_0 \).

2) \( S = S^\mu_\mu = \{ \delta < \mu^+: \text{cf}(\delta) = \mu \} \) or \( S \) just a stationary subset of \( S^\mu_\mu \).

3) \( C \) is an \( S \)-club system, see Definition 1.3.

**Definition 2.5.** For \( \bar{f} \) is an \((S,C,\mu)\)-parameter, see Definition 1.3, we define a forcing notion \( \mathbb{P} = \mathbb{P}_\bar{f} \) as follows:

\( \bar{f} \) is a forcing notion \( \mathbb{P} = \mathbb{P}_\bar{f} \) as follows:

(A) \( p \in \mathbb{P} \) if \( p \) consists of (so \( u_p = u \), etc.)

(a) \( u \in [\mu^+]_<^\mu \)

(b) \( g : u \to [\mu]_<^\theta \), (can use \( g : u \to \theta \) when \( \bigwedge \text{Rang}(f) \subseteq \theta \))

(c) \( v \subseteq S \) of cardinality \( < \mu \)

(d) \( h \) a function with domain \( v \)

(e) if \( \delta \in v \) then

(\( \alpha \)) \( h(\delta) \) is a closed bounded non-empty subset of \( C_\delta \)

(\( \beta \)) \( h(\delta) \subseteq u \)

(\( \gamma \)) \( \text{if } \beta \in h(\delta) \text{ then } f(\beta) = g(\delta) \)

(B) \( p \leq q \), i.e. \( \mathbb{P}_\bar{f} \models "p \leq q" \) if \( p \subseteq q \)

(a) \( u_p \subseteq u_q \) and \( g_p \subseteq g_q \)

(b) \( v_p \subseteq v_q \)

(c) if \( \delta \in v_p \) then \( h_p(\delta) \) is an initial segment of \( h_q(\delta) \)

(C) we define \( \leq_{st} = \leq_{st}^\mu \), the strong order by: \( p \leq_{st} q \) if \( p \leq q \) and

(d) \( p \in v_p \) and \( h_p(\delta) \neq h_q(\delta) \) then \( \sup(h_q(\delta)) > \sup(\cup\{\delta \in C_{\gamma} : \gamma \in v_p \setminus \{\delta\}\}) \).

Remark 2.6. If in clause (A)(c)(\( \alpha \)) we demand only \( h(\delta) \) is only closed in its supremum then we get an equivalent forcing, we lose some nice properties but gain others. Mainly we gain in having more cases of having a lub, in particular for increasing sequence which has an upper bound, really any set of \( < \mu \) members which has an upper bound; but we lose for \( \Delta \)-systems, i.e. \( \bar{f}(6) \). Also we have to be more careful in \( \bar{f}(13) \). We shall use this version in \( \bar{f}(3) \).

**Claim 2.7.** 1) \( \mathbb{P}_\bar{f} \) is a forcing notion of cardinality \( \mu^+ \), also \( \leq_{st} \) is a partial order \( \subseteq \leq_{\text{st}} \) and \( p_1 \leq_{\text{st}} p_2 \leq_{\text{st}} p_3 \leq_{\text{st}} p_4 \Rightarrow p_1 \leq_{\text{st}} p_4 \) and \( (\forall p)(\exists q)(p \leq_{\text{st}} q) \), i.e. it exemplifies \((1)^{\mu}_+ \).

2) Any \( \leq_{\text{st}} \)-increasing sequence in \( \mathbb{P}_\bar{f} \) of length \( < \mu \) has an upper bound (this is a strong/no memory of strategic \( \mu \)-completeness).

3) If \( p_1, p_2 \in \mathbb{P}_\bar{f} \) are compatible then they have a lub.

4) \( \{ p_i : i < i(*) \} \) has a \( \leq_{\text{st}} \)-lub in \( \mathbb{P}_\bar{f} \) when \( \bigwedge_{i < i(*)} (p_i, p_j \text{ are compatible}) \) and \( i(*) \) is finite or \( i(*) < \mu \) and for every \( \delta \), the set \( \{ h_p(\delta) : i < i(*) \text{ is such that } \delta \in v_p \} \) is finite or at least has a maximal member. Note this set is linearly ordered by being an initial segment.
4A) \{p_i : i < i(\ast)\} has an \textit{ub when} \ i(\ast) < \mu \ and \ \{p_i : i < i(\ast)\} \ is \ a \ set \ of \ pairwise 
compatible \ members \ of \ \mathbb{P}_\lambda \ and \ i(\ast) \ is \ finite \ or \ i(\ast) < \theta \ or \ at \ least \ i(\ast) < \mu \ and \ for \ every \ limit \ ordinal \ \alpha \ the \ following \ set \ has \ cardinality < \theta:

- \ \{\delta \in \bigcup_i v_{p_i} : \alpha = \sup\{h_{p_i}(\delta) + 1 : i < i(\ast) \text{ and } \delta \in v_{p_i}\}\}.

5) \mathbb{P}_\lambda \ satisfies \ clause \ (2)_b, (2)_b^+ \ of \ Definition \ \mathbb{P}_\lambda \ satisfies \ provided \ that \ \varepsilon < \mu \ is \ a \ limit 
ordinal \ of \ cofinality \ \neq \varepsilon.

6) \mathbb{P}_\lambda \ satisfies \ clauses \ (2)_b, (2)_b^+ \ of \ Definition \ \mathbb{P}_\lambda \ satisfies \ provided \ that \ \varepsilon < \mu \ is \ a \ limit 
ordinal \ of \ cofinality \ \neq \theta.

\textit{Proof.} 1) Recall \ \mu = \mu^{<\mu} \ hence \ (\mu^+) = \mu^{<\mu} \ and \ easily \ |\mathbb{P}| = \mu^+ \. \ Also \ the \ statements \ on \ <_{st} \ are \ obvious.

2) Let \ \gamma < \mu \ be \ a \ limit \ ordinal \ and \ \bar{p} = \{p_i : i < \gamma\} \ be \ a \ <_{st}-increasing \ sequence \ of 
members \ of \ \mathbb{P}_\lambda.

Let

\begin{itemize}
  \item \((*) \ (a) \ v_\ast = \bigcup_i \{v_{p_i} : i < \gamma\}
  \item \((*) \ (b) \ \text{let } i : v_\ast \to \gamma \text{ be } i(\delta) = \min\{i : \gamma < \delta \in v_{p_i}\}
  \item \((*) \ (c) \ v_\ast^+ = \{\delta \in v_\ast : \text{the sequence } (h_{p_i}(\delta) : i \in [i(\delta), \gamma)) \text{ is not}
  \item \((*) \ (d) \ \text{eventually constant}\}
  \item \((*) \ (e) \ \text{for } \delta \in v_\ast^+ \text{ let } \zeta_\delta = \sup(\bigcup\{h_{p_i}(\delta) : i \in [\bar{i}(\delta), \gamma]\}),
  \item \((*) \ (f) \ \text{let } v_\ast^* = v_\ast \setminus v_\ast^2.
\end{itemize}

We try naturally to define \(p = (u_p, v_p, g_p, h_p)\) as \(\bigcup_{i < \gamma} p_i\), that is

\begin{itemize}
  \item \((*) \ (a) \ v_p = v_\ast := \bigcup_i \{v_{p_i} : i < \gamma\}
  \item \((*) \ (b) \ u_p = \bigcup_i \{u_{p_i} : i < \gamma\} \cup \{(\zeta_\delta : \delta \in v_\ast^+\}
  \item \((*) \ (c) \ g_p = \bigcup_i \{g_{p_i} : i < \gamma\} \cup \{(\zeta_\delta, \{\bar{f}_p(\zeta_\delta)\}) : \delta \in v_\ast^+\}
  \item \((*) \ (d) \ h_p \text{ is a function with domain } v \text{ such that}
  \item \((*) \ (e) \ \text{if } \delta \in v_\ast^+ \text{ then } h_p(\delta) = p_i(\delta) \text{ for } i < \delta \text{ large enough}
  \item \((*) \ (f) \ \text{if } \delta \in v_\ast^* \text{ then } h_p(\delta) = \bigcup\{h_{p_i}(\delta) : i \in [\bar{i}(\delta), \gamma]\} \cup \{\zeta_\delta\}.
\end{itemize}

The point is to check that \(p \in \mathbb{P}_\lambda\), as for \(i < \delta \Rightarrow p_i \leq p\) it is immediate:

\begin{itemize}
  \item \(u_p \in [\mu^+]^{<\mu}\) because \(u_{p_i} \in [\mu^+]^{<\mu}\) and \(\delta < \mu = \text{cf}(\mu)\)
  \item \(v_p \in [\mu^+]^{<\mu}\) because \(v_{p_i} \in [\mu^+]^{<\mu}\) and \(\delta < \mu = \text{cf}(\mu)\) and \(|v_\ast^+| \leq \Sigma\{|v_{p_i}| : i < \gamma\} < \mu\)
  \item \(h_p \text{ is a function with domain } v \text{ such that } \delta \in v \Rightarrow h_p(\delta) \text{ is a bounded closed}
  \item \(\text{subset of } C_\delta \text{ (check the two cases)}\)
  \item \(g_p \text{ is a function from } u_p \text{ to } \lambda \text{ as each } g_{p_i} \text{ is a function from } v_{p_i} \text{ to } \lambda \text{ and } \bar{p}
  \item \(\text{is } <_{st}-increasing \text{ and:}
  \item \((*) \ \text{if } \delta \in v_\ast^* \text{ then } \zeta_\delta \notin \bigcup_i u_{p_i}\)
  \item \((***) \ \text{if } \delta_1 \neq \delta_2 \in v_\ast^* \text{ then } \zeta_{\delta_1} \neq \zeta_{\delta_2}\)
\end{itemize}
3) By (4).
4) Take the union as in the proof of part (2), only now it is easier.
4A) Similar to the proof of part (2).
5) Let \( A_\alpha = \{ p \in \mathbb{P}_\mathfrak{f} : u_p, v_p \subseteq \alpha \} \) so \( \langle A_\alpha : \alpha < \mu^+ \rangle \) is \( \subseteq \)-increasing continuous. Given \( (p_\alpha : \alpha < \mu^+) \) for \( \delta \in S \) let \( h(\delta) = (u_\delta, v_\delta, g_\delta) \) \((u_\delta \cap \delta, v_\delta \cap \delta, g_\delta)(u_\delta \cap \delta), h_\delta)(v_\delta \cap \delta)\)
and let \( E \) be a club of \( \mu^+ \) such that \( \delta \in E \land \alpha < \delta \Rightarrow u_{p_\alpha} \cup v_{p_\alpha} \subseteq \delta \). Now if \( (\delta_1, \delta_2) \subseteq E \cap S \land (h(\delta_1) = h(\delta_2)) \) then \( p_{\delta_1}, p_{\delta_2} \) are compatible and hence have a lub by part (3).
6) Like (5) using part (4).

\[ \text{Claim 2.8.} \quad \mathcal{F}_{\mathfrak{f}, \alpha} \text{ is a dense open subset of } Q_{\mathfrak{f}} \text{ where} \]
\begin{itemize}
\item \( \mathcal{F}_{\mathfrak{f}, \alpha} = \{ p \in \mathbb{P}_\mathfrak{f} : \alpha \in u_p \text{ and } \alpha \in S \Rightarrow \alpha \in v_p \}. \)
\end{itemize}

Proof. Assume \( p \in \mathbb{P}_\mathfrak{f} \) and we shall find \( q \in \mathcal{F}_{\mathfrak{f}, \alpha} \) such that \( p \leq q \).

Case 1: If \( (\alpha \notin S \lor \alpha \in v_p) \) and \( \alpha \in u_p \)
Let \( q = p \).

Case 2: \( \alpha \notin u_p \)
Define \( q \) by:
\begin{itemize}
\item \( u_q = u_p \cup \{ \alpha \} \)
\item \( v_q = v_p \)
\item \( g_q = g_p \cup \{ (\alpha, \emptyset) \} \)
\item \( f_q = f_p \)
\end{itemize}
Now check that \( p \leq q \land \alpha \in u_q \). If \( \alpha \in v_p \) we are done, if not apply case 3.

Case 3: \( \alpha \in S, \alpha \in u_p \) and \( \alpha \notin v_p \)
Let \( \beta \in C_\alpha \) be such that \( \delta \in v_p \Rightarrow \beta > \sup(C_\delta \cap \alpha) \) and define \( q \in \mathbb{P}_\mathfrak{f} \) by:
\begin{itemize}
\item \( u_q = u_p \cup \{ \beta \} \)
\item \( v_q = v_p \cup \{ \alpha \} \)
\item \( g_q = g_p \cup \{ (\beta, \{ f_\alpha(\beta) \}) \} \)
\item \( h_q = h_p \cup \{ (\alpha, \{ \beta \}) \} \).
\end{itemize}
Clearly \( p \leq q \in \mathcal{F}_{\mathfrak{f}, \alpha} \).

\[ \text{Definition 2.9.} \quad \text{We say that } \mathfrak{f} = (\kappa, \delta)\text{-generic enough when } (A) \Rightarrow (B) \text{ and, of course, } \mathfrak{f} = (\mathfrak{f}_\delta : \delta \in S^\mu_{\mathfrak{f}}), \mathfrak{f}_\delta : C_\delta \rightarrow \mu \text{ where } (\kappa \text{ is a cardinal } < \mu, \delta \text{ is a regular cardinality } < \mu) \text{ and}: \]
\begin{itemize}
\item \( (A) \) \( (a) \) \( E \) is a club of \( \mu^+ \)
\item \( (b) \) \( \langle \alpha_{\delta, \zeta} : \zeta < \mu \rangle \) is an increasing continuous sequence of members of \( C_\delta \)
\item \( (c) \) \( h_\zeta \) is a pressing down function from \( E \cap S \) for \( \zeta < \mu \)
\end{itemize}
\begin{itemize}
\item \( (B) \) we can find \( \xi < \mu \) of cofinality \( \delta \) and a sequence \( (\delta_i : i < \kappa) \) of ordinals from \( E \cap S \) such that:
\item \( 1 \) if \( \zeta < \xi \) then \( h_\zeta \upharpoonright \{ \delta_i : i < \kappa \} \) is constant
\item \( 2 \) \( \langle \alpha_{\delta_i, \zeta} : \zeta < \xi \rangle \) does not depend on \( i < \kappa \) hence also \( \alpha_{\delta_i, \xi} \) by continuity
\item \( 3 \) \( \{ \mathfrak{f}_{\delta_i}(\alpha_{\delta_i, \xi}) : i < \kappa \} \) has cardinality \( \kappa \).
\end{itemize}
Claim 2.11. \(\alpha p\) 

1) This is used when \(\theta_\alpha\) say: any \(< \theta\) has lub inside the proof of \(\alpha p\).

2) For \(\theta = 2\) we need a stronger version - with the game, see §3.

3) In \(\alpha p\) we add:

- \(\{\alpha \in C_\delta : \text{otp}(\alpha < \alpha_{\delta, \zeta})\text{ for some } \zeta < \xi\text{ does not depend on } i\}
- \text{the } \xi\text{'s agree on this, see?}

Now we arrive to the main point.

Claim 2.12. \(\alpha p\) 

1) Assume \(\mathbb{R}\) is the forcing notion for adding \(\mu^+\) many \(\mu\)-Cohen's.

2) For\(\delta \in \mathbb{R}\) there is an \((S, C, \mu)\)-parameter \(f\) which is \((\kappa, \partial)\)-generic enough for every cardinal \(\kappa, \partial \in [\mathbb{N}_\alpha, \mu]\), \(\partial\) regular.

Proof. 1) Now (modulo equivalence, so without loss of generality) \(\mathbb{R}\) can be described as follows:

\[(*)_1 \quad (a) \quad p \in \mathbb{R} \iff p\text{ is a function, } \text{dom}(p) \subseteq [S^\mu]^{<\alpha}\text{ for every } \delta \in \text{dom}(p), p(\delta)\text{ is a function}\]
\n2) \(\mathbb{R} \models \text{ if } \exists p \in \mathbb{R} \text{ such that:} \)

\(\{\alpha \in C_\delta : \text{otp}(\alpha < \alpha_{\delta, \zeta})\text{ for some } \zeta < \xi\text{ does not depend on } i\}
- \text{the } \xi\text{'s agree on this, see?}

Then in \(\mathbb{V}^\mathbb{R}\), there is an \((S, C, \mu)\)-parameter \(f\) which is \((\kappa, \partial)\)-generic enough for every cardinal \(\kappa, \partial \in [\mathbb{N}_\alpha, \mu]\), \(\partial\) regular.

2) If \(\diamond_\delta\) then there is \(\mathbb{C}\) as above.

It suffices to prove \(\mathbb{V}^\mathbb{R} \models \langle f_\delta : \delta \in S^\mu \rangle\text{ is as required}\). So assume

\(\forall \zeta < \xi\) \(\mathbb{R} \models \text{ if } \langle \delta : \alpha \in \text{dom}(\delta) \wedge (p(\alpha) = q(\alpha)) \rangle\text{ then } \bar{m} \text{ for } \delta \in S^\mu\)

It suffices to find a condition \(q\) above \(p_\ast\) forcing \(\langle \delta_i : i < \kappa\rangle\) and \(\xi\) as in clause (B) of Definition \(\alpha p\). For each \(\delta \in S^\mu\) we choose \(p_{\delta, \varepsilon}, \xi_{\delta, \varepsilon}, \alpha_{\delta, \varepsilon}\) by induction on \(\varepsilon < \delta\) such that:

\((*)_{\delta, \varepsilon}\) \(\alpha_{\delta, \varepsilon}\) is a pressing down function on \(S^\mu\) for \(\zeta < \mu\) and \(\langle \alpha_{\delta, \zeta} : \zeta < \mu\rangle\) is increasing continuous sequence of members of \(C_\delta\) for \(\delta \in S^\mu\).

There is no problem to carry the induction. Let \(\xi_{\delta} = \bigcup\{\xi_{\delta, \varepsilon} : \varepsilon < \delta\} < \mu, \alpha_{\delta}^* = \sup\{\text{dom}(p_{\delta, \varepsilon}(\delta)) : \varepsilon < \delta\}, p_{\delta} = \bigcup\{p_{\delta, \varepsilon}, \varepsilon < \delta\}\)

Now we can define a pressing down function \(h\) on \(S^\mu\) such that:

- \(h(1) = h(2), \varepsilon < \partial\) and \(\varepsilon < \partial\) then \(\alpha_{\delta, \varepsilon} = \alpha_{\delta, \varepsilon}\) and for every \(\alpha \in \text{Rng}(\alpha_{\delta, \varepsilon})\) we have: \(\text{Sup}(\alpha_{\delta, \varepsilon} \cap \alpha) = \text{Sup}(\alpha_{\delta, \varepsilon} \cap \alpha) = p_{\delta}(\delta)\) if \(\text{Sup}(\alpha_{\delta, \varepsilon} \cap \alpha) = p_{\delta}(\delta)\) and \(h_{\varepsilon}(1) = h_{\varepsilon}(2)\) then \(\xi_{\delta} = \xi_{\delta}\) and \(p_{\delta} \mid \delta_1 = p_{\delta} \mid \delta_2\).
Next choose an increasing sequence \( \langle \delta_i : i < \kappa \rangle \) of members of \( S^\omega \) such that as constant on \( \{ \delta_i : i < \kappa \} \) and \( i < j \Rightarrow \text{dom}(p_{\delta_i}) \subseteq \delta_j \).

Define \( q \in \mathbb{R} \):

\[
\begin{align*}
(\ast) \quad & (a) \quad \text{dom}(q) = \cup\{\text{dom}(p_{\delta_i, \varepsilon}) : i < \kappa, \varepsilon < \kappa\} \\
& (b) \quad q(\delta_i) = \cup\{p_{\delta_i, \varepsilon}(\delta_i) : \varepsilon < \theta\} \cup \{\langle \alpha^*_i, i \rangle \}, q(\alpha) = \cup\{p_{\delta_i, \varepsilon}(\alpha) : \alpha \in \text{dom}(p_{\delta_i, \varepsilon})\}.
\end{align*}
\]

2) Also easy.

\[\square\]

Claim 2.12. 1) There are dense sets \( \mathcal{J}_\alpha \subseteq \mathbb{P} = \mathbb{P}_\bar{\mathfrak{I}}, \alpha < \mu^+ \), such that if \( G \subseteq \mathbb{P} \) is directed and meets every \( \mathcal{J}_\alpha \), then \( G \) is \( \theta^+ \)-directed.

2) If \( \mathfrak{f} \) is \( (\theta, \theta) \)-generic enough and the forcing notion \( \mathbb{R} \) satisfies \((1)_{b_0, \theta} + (1)_{c, \theta} + (2)_{a, 0} \), then in \( V^\mathbb{R} \) there is no \( \theta^+ \)-directed \( G \subseteq \mathbb{P} = \mathbb{P}_\mathfrak{f} \) meeting \( \mathcal{J}_{\bar{\mathfrak{I}}, \delta} \) for every \( \delta < \mu^+ \).

3) Also there is no such \( \mathbb{R} \) satisfying \((2)_{b_D} \) when \( \varepsilon < \mu \) is a limit ordinal of cofinality \( \neq \theta \).

Proof. 1) Let \( \mathcal{J} = \{ \bar{p} : \bar{p} \) is an increasing sequence of conditions in \( \mathbb{P} \) of limit length \( < \theta^+ \} \). Since \( \mu^\omega = \mu \) and \( |\mathbb{P}| = \mu^+, |\mathcal{J}| \leq \mu^\omega \). For each \( \bar{p} = \langle p_i : i < \lambda \rangle \in \mathcal{J} \), let \( \mathcal{J}_\alpha = \{ q \in \mathbb{P} : q \) is either incompatible with \( p_i \) for some \( i < \lambda \) or \( p_i \leq q_i \) for every \( i < \lambda \} \). Since \( \mathbb{P} \) is \( \mu \)-strategically complete (by Claim 1.22(2)), \( \mathcal{J}_\alpha \) is dense. Let \( G \) meet \( \mathcal{J}_\alpha \), for every \( \bar{p} \in \mathcal{J} \). Then \( G \) is \( \theta^+ \)-directed.

2) Towards contradiction, assume \( p_* \Vdash \mathfrak{H} \subseteq \mathbb{P} \) is \( \theta^+ \)-directed meeting \( \mathcal{J}_{\bar{\mathfrak{I}}, \delta} \) for every \( \delta < \mu^+ \). Using \((1)_{b_0, \theta} + (1)_{c, \theta} \), fix a winning strategy \( \text{st} \) for \( \text{COM} \), the completeness player in the game \( \mathcal{O}_{\theta^+}(\mathbb{R}) \). Construct \( \langle (E_\zeta, q_\zeta, \bar{r}_\zeta, h_\zeta, p_\zeta, \alpha_\zeta) : \zeta < \theta^+ \rangle \) such that:

\[
\begin{align*}
(\ast) \quad & (a) \quad q_\zeta = \langle q_{\zeta, \delta} : \delta \in E_\zeta \rangle \text{ and } \bar{r}_\zeta = \langle r_{\zeta, \alpha} : \alpha \in E_\zeta \rangle \\
& (b) \quad p_* \leq q_{\zeta, \delta}, r_{\zeta, \delta} \text{ are from } \mathbb{R} \\
& (c) \quad \langle \langle q_{\zeta, \delta}, r_{\zeta, \delta} : \zeta \leq \zeta \rangle \rangle \text{ is an initial segment of a play of } \mathcal{O}_{\theta^+}(\mathbb{R}) \text{ in which } \text{COM uses } \text{st} \\
& (d) \quad E_\zeta \subseteq \mu^+ \text{ is a club} \\
& (e) \quad h_\zeta \text{ is a regressive function on } S \cap E_\zeta \\
& (f) \quad \text{if } \mathbb{U} \subseteq E_\zeta \cap S, |\mathbb{U}| < \theta \text{ and } h_\zeta \upharpoonright \mathbb{U} \text{ is constant, then } \{r_{\zeta, \delta} : \delta \in \mathbb{U}\} \text{ has a hub in } \mathbb{R} \\
& (g) \quad \bar{p}_C = \langle p_{\zeta, \delta} : \delta \in E_\zeta \rangle \\
& (h) \quad r_{\zeta, \delta} \Vdash \langle p_{\zeta, \delta} \rangle \text{ is above } p_{\zeta, \delta} \text{ for } \zeta < \zeta^+ \\
& (i) \quad \alpha_\zeta = \langle \alpha_{\zeta, \delta} : \delta \in S \cap E_\zeta \rangle \\
& (j) \quad \text{if } \delta \in S \cap E_\zeta, \text{ then } \delta \in v_{p_{\zeta, \delta}} \text{ and } \alpha_{\zeta, \delta} \text{ is the } \zeta \text{-th member of } h_{p_{\zeta, \delta}}(\delta).
\end{align*}
\]

For clauses \((\ast)+(\ast)\), we use condition \((2)_{b_0} \). Since \( \mathfrak{f} \) is \( (\theta, \theta) \)-generic enough, we get \( \langle \delta_i : i < \theta \rangle \) and \( \zeta \) as in Definition 2.9 and let \( \langle \zeta_i : i < \theta \rangle \) be increasing with limit \( \xi \).

By clause \((\text{f})\), for each \( j < \theta \), the set \( \{r_{\zeta_i, \delta_i} : i < j\} \) has a hub \( r_j^\ast \in \mathbb{R} \) - so necessarily \( j_1 < j_2 < \theta \Rightarrow r_{j_1}^\ast \leq r_{j_2}^\ast \). Hence the sequence \( r_j^\ast : j < \theta \) has an upper bound \( r_\ast \) (by \((1)_{b_0} \)). So \( r_\ast \Vdash \langle p_{\zeta_i, \delta_i} : i < j < \theta \rangle \subseteq \mathbb{H} \). As \( r_\ast \Vdash \mathbb{H} \) is \( < \theta^+ \)-directed, we can find some \( p \in \mathbb{P}, r_\ast \geq r_\ast \) such that \( r_\ast \Vdash p \in \mathbb{H} \) is an upper bound for \( \{p_{\zeta_i, \delta_i} : i < j < \theta \} \).
So, on one hand, $g_p(\alpha_\delta, \xi)$ is a subset of $\mu$ of cardinality $\theta$ - by the definition of $\mathbb{P}$. On the other hand, $i < \theta \implies \alpha_{\xi, \delta_i} = \alpha_{\xi, \delta_0}$ and $f_{\delta_i}(\alpha_{\delta_i, \xi}) \in g_p(\alpha_{\delta_i, \xi})$. But by Definition 2.9(B) this is impossible. \hfill \Box
Hypothesis 3.1. 1) \( \mu = \mu^\mu \).
2) \( S \subseteq S_\mu^+ \) stationary.
3) \( \tilde{C} = \langle C_\delta : \delta \in S \rangle, C_\delta \) an unbounded subset of \( \delta \) of order type \( \mu \), listed by
\( \langle \alpha^*_\delta : \zeta < \mu \rangle \) in increasing order.
4) \( \tilde{f} \) as in \( 3.2 \).
5) \( \Theta \subseteq \text{Reg} \cap \mu^+ \), let \( S^\mu_\Theta = \{ \delta < \mu^+ : \text{cf}(\delta) \in \Theta \} \).
6) \( 2 \leq \theta < \mu \) but our main interest is \( \theta = 2 \).

Definition 3.2. \( \tilde{f} \) is a \( (\tilde{C}, \kappa) \)-parameter (or uniformization problem) when \( \tilde{f} = \langle f_\delta : \delta \in S \rangle, f_\delta : C_\delta \to \kappa \).

Definition 3.3. 1) We define \( P_\tilde{f}^1 \) and \( <_{st} \) as in Definition \( 3.2 \), but we change clause
\( (A)(c) \) by:

\( (c)' \) if \( \delta \in v_p \) then
\( (\alpha) \) \( h_p(\delta) \) is a bounded subset of \( C_\delta \) closed only in its supremum,
\( (\beta) \) \( h_p(\delta) \subseteq u_\delta \)
\( (\gamma) \) if \( \beta \in h_p(\delta) \) so \( \delta \in v_p \) then \( \text{cf}(\beta) \in \Theta \Rightarrow f(\beta) \in g(\beta) \) (so really only
\( g_p([u_p \cap S^\mu_\Theta]^{\delta}) \) matters)
\( (\delta) \) if \( \beta \in h_p(\delta) \) and \( \text{cf}(\delta) \notin S^\mu_\Theta \) then \( g_p(\beta) = \emptyset \)
\( (f)' \) for every \( \alpha \) such that \( \text{cf}(\alpha) \in \Theta \) then the set \( w_{p,\alpha} \) has cardinality \( < \theta \)
where \( w_{p,\alpha} = \{ \delta \in v_p : g_p(\delta) \} \).

2) We define \( \mathcal{A}_{f,\alpha}^1 \subseteq P_\tilde{f}^1 \) as in Definition \( 3.4 \).3

Claim 3.4. \( P_\tilde{f}^1 \) satisfies

\( (a) \) any increasing sequence of length \( \delta < \mu, \text{cf}(\delta) \notin \Theta \) has a lub, i.e. \( (1)_{a,=\theta} \)
for \( \delta \in \Theta \)
\( (b) \) a set of pairwise compatible conditions of cardinality \( < \min(\Theta) \) has a lub -
the union, i.e. \( (1)_{a,=\theta} \) holds.

Proof. Easy. \( \square \)

Claim 3.5. \( P_\tilde{f}^1 \) satisfies:

\( (a) \) we have \( (1)^\delta_{a,=\theta} \), i.e.
\( (\alpha) \) \( \ll_{st} \) is a partial order and \( p_1 \ll p_2 \ll p_3 < p_4 \Rightarrow p_1 <_{st} p_4 \)
\( (\beta) \) any \( <_{st} \)-increasing chains of length \( < \mu \) has an ub
\( (b) \) \( (\alpha) \) we have \( (3)_a \), i.e. if \( p, q \in P_\tilde{f}^1 \) are compatible then they have a lub
\( (\beta) \) \( \{ p_i : i < i(*) \} \) has a lub when \( i(*) < \mu \) and \( \{ p_i : i < i(*) \} \) is a set of
pairwise compatible conditions and for each \( \delta \in S \),
the set \( \{ h_p(\delta) : i < i(*) \) and \( \delta \in v_p \} \) is finite; note that this set
is linearly ordered by being an initial segment.
Claim 3.6.

\( \{ p_i : i < i(\ast) \} \) has a \( \ast \) when \( i(\ast) < \mu \) and \( \{ p_i : i < i(\ast) \} \) is a set of pairwise compatible conditions and if \( \text{cf}(\alpha) \in \Theta \) then

\( |w_{p,\alpha}| < \theta \) where \( w_{p,\alpha} = \{ \delta : \delta \in \bigcup_i v_{p_i} \land \alpha = \sup \{ \text{sup}(g_{p_i}(\delta)) \} + 1 : i < i(\ast) \land \delta \in v_{p_i} \} \)

(c) (\( \alpha \)) (2) \( \ast \) holds

(\( \beta \)) (2) \( \ast \) that \( \ast \) holds if \( \partial < \mu \) is regular and \( \theta \geq 2 \lor \partial \notin \Theta \)

(d) (3) \( \ast \) \( \ast \) holds if \( \kappa = \text{cf}(\varepsilon) \in \mu \setminus \Theta \) is regular.

Proof. Like \( \text{Claim 3.3} \), e.g.

Clause (a): As in \( \text{Claim 3.3} \), (1), (2).

Clause (b): Should be clear.

Clause (c): If \( \theta \geq 2 \) we use (3), i.e. the parallel of \( \text{Claim 3.3} \). If \( \theta = 1 \) and \( \partial \notin \Theta \) use clause (d).

Clause (d): Just recall (\( \ast \)) of Definition \( \text{Claim 3.3} \). \( \square \)

\[ \text{Claim 3.6.} \quad \mathcal{F}_{\bar{f},\alpha} \text{ is a dense open subset of } Q_{\bar{f}} \text{ where} \]

- \( \mathcal{F}_{\bar{f},\alpha} = \{ p \in \mathbb{P}_{\bar{f}} : \alpha \in u_p \land \alpha \in S \Rightarrow \alpha \in v_p \} \).

Proof. Should be clear. \( \square \)

\[ \text{Definition 3.7.} \quad \text{For } (\mu, \theta, \partial, D, \bar{f}) \text{ as in clause (A) below we define a game } \mathcal{G}_{\bar{g}_n}(\bar{f}, \theta, \partial, D) \text{ in clause (B) below where:} \]

(A) (a) \( \mu = \mu^+ > \partial = \text{cf}(\partial) \geq \aleph_0 \) and

(b) \( S \subseteq S^0_\mu = \mathcal{C} = (C_\delta : \delta \in S) \) a club sytem

(c) \( D \) is a normal filter on \( \mu^+ \) to which \( S \) belongs

(d) \( \bar{f} = (f_\delta : \delta \in S) \), \( f_\delta \) a function from \( C_\delta \) to \( \partial \)

(B) (a) a play last \( \partial \) moves

(b) in the \( \zeta \)-th move, \( S^0_\zeta \subseteq D \) such that \( S^0_\zeta \subseteq S^1_\zeta \subseteq \land (\forall \zeta < \zeta) (S^1_\zeta \subseteq S^2_\zeta) \)

and \( \alpha^\zeta = (\alpha^\zeta, \delta : \delta \in S^0_\zeta), \alpha^\zeta, \delta \subseteq C_\delta, \alpha^\zeta > \alpha^\zeta, \delta > \sup \{ \alpha^\zeta, \delta : \zeta < \delta \} \) and \( h^\zeta \) pressing down functions on \( S^\zeta \)

(c) in the \( \zeta \)-th anti-generic player chooses \( S^1_\zeta, \alpha^1_\zeta, h^1_\zeta \) and then the generic play chooses \( S^2_\zeta, \alpha^2_\zeta, h^2_\zeta \)

(d) in the end of the play the generic player wins when for some \( \delta_1 < \delta_2 \) from \( \cap \{ S^\zeta : \zeta < \partial \} \) we have \( \sup \{ \alpha^\zeta, \delta_1 : \zeta < \partial, \ell = 1, 2 \} = \sup \{ \alpha^\zeta, \delta_2 : \zeta < \partial, \ell = 1, 2 \} \), call it \( \alpha \) and \( f_{\delta_1}(\alpha) \neq f_{\delta_2}(\alpha) \), \( \bigwedge_{k<\partial} h_k(\delta_1) = h_k(\delta_2) \).

\[ \text{Theorem 3.8.} \quad \text{If } \sigma \in \Theta, \theta = 2 \text{ and } \bar{f} \text{ is such that in the game } \mathcal{G}_{\bar{g}_n}(\bar{f}, \theta, \partial, D) \text{ from Definition } \text{Claim 3.7} \text{ the generic player wins or just does not lose, so } D \text{ a normal filter} \]

on \( \mu^+ \), \( S^\mu_\mu \in D \text{ then:} \]

(a) \( P^n_{\bar{f}} \) fails \( \ast_\mu \).

(b) no forcing satisfying \( \ast_\mu \) add a generic to \( P^n_{\bar{f}} \), moreover
(c) no forcing satisfying $*_{\mu,D}$ add a $(\leq \mu)$-directed or just $(\sigma^+)$-directed $G \subseteq \mathbb{P}_i^\mu$ meeting $\mathcal{J}_{f,\alpha}$ for every $\alpha < \mu^+$ (defined in (3.8)).

Proof. As in the proof of (3.26) (1), e.g.

Clause (c):

In the proof of (3.26), (2), we replace $\mathbf{st}$ by a winning strategy of the completeness player an the game for $(2)_i^\mu$ and toward contradiction assume $f$ is an $(S,C,\theta)$-parameter, $P_* \in \mathbb{P}_i^\mu$ and $P_* \Vdash \langle H \subseteq \mathbb{P}_i^\mu \rangle$ is a $(\leq \sigma^+)$-directed and meet every $\mathcal{J}_{f,\alpha,\sigma} < \mu^+$.

Now for $\zeta < \sigma$ let $Y_\zeta$ be the set of $(\tilde{q}_\zeta, \tilde{r}_\zeta, h_\zeta, E_\delta, \tilde{p}_\zeta, \tilde{h}_\zeta)$ such that:

$\forall (a) \langle \tilde{q}_\zeta, \tilde{r}_\zeta, h_\zeta : \zeta \leq \zeta \rangle$ is an initial segment of a play of the game from Definition 2.13 in which the player COM uses the strategy $\mathbf{st}$

(b) so $\tilde{q}_\zeta = (q_\zeta, S_\zeta : \zeta \in S_\zeta), \tilde{r}_\zeta = (r_\zeta, S_\zeta : \zeta \in S_\zeta), S_\zeta \subseteq \{ S_\zeta : \zeta < \zeta \}

(c) \tilde{h}_\zeta = (h_\zeta, \delta : \zeta \in S_\zeta)$ and $\tilde{p}_\zeta = \delta \subseteq \mathbb{P}_i^1$

(d) $r_\zeta \in \mathbb{P}_i^\mu$ and $r_\zeta \in \mathbb{P}_i$

(e) $\delta \in \mathbb{P}_i$

(f) $\sup(\operatorname{dom}(h_\delta) : \delta \leq \zeta)$ is strictly increasing.

Now we use the definition of the game $\Omega_{\mathbb{G}_\theta}(f, \theta, \sigma, D)$ to finish as in (3.17). $\square_{3.19}$

The above theorem helps for further problem as

Claim 3.9. 1) If a forcing notion $\mathbb{P}$ satisfies $(1)_b + (2)_a$ and $\sigma \in \operatorname{Reg} \cap \mu$ then $\mathbb{P}$ satisfies $*_{\mu}^\sigma$, i.e. $(2)_a^\sigma$.

2) If $\mathbb{Q}$ is adding $\mu^+$, $\mu$-Cohen $(\eta_\alpha : \alpha < \mu^+), \eta_\alpha \in \mu^\theta$ and $\theta \leq \mu, \aleph_1 \leq \sigma = \operatorname{cf}(\sigma) < \mu, D$ a normal filter on $\mu$ such that $S_\alpha \subseteq D$ then $\mathbb{Q} \Vdash \langle \eta_\alpha : \alpha < \mu^+ \rangle$ is a $(\mathbb{C}, \mu)$-parameter and is $(\theta, \sigma)$-generic enough and also the generic player wins in the game $\Omega_{\mathbb{G}_\theta}(\eta_\sigma, 2, \sigma, D)$, pedantically replaces $D$ by the normal filter it generates.

Explain 3.9(2).

Conclusion 3.10. Assume $\aleph_0 \leq \sigma = \operatorname{cf}(\sigma) < \mu = \mu^\sigma$ and $\mathbb{Q}$ is the forcing notion of adding $\mu^+$, $\mu$-Cohen.

1) In $\mathbb{V}_\mathbb{G}^\mathbb{Q}$, there is a forcing notion $\mathbb{P}$ satisfying $(1)_a + (2)_b$ for $\theta \in \operatorname{Reg} \cap \mu \setminus \{ \sigma \}$ but not $(2)_a^\sigma$.

2) Moreover in $\mathbb{V}_\mathbb{Q}$, if $\mathbb{P}$ is a forcing notion satisfying $(1)_a, (2)_b^\sigma$ then it adds no generic to $\mathbb{P}$, in fact $|\mathbb{P}| = \mu^+$ and we should demand $G \subseteq \mathbb{P}$ is $\sigma^+$-directed, $G \cap \mathcal{J}_\alpha \neq \emptyset$ for $\alpha < \mu^+$ for some dense $\mathcal{J}_\alpha \subseteq \mathbb{P}$ for $\alpha < \mu^+$.

3) So for some $(\leq \mu)$-complete $\mu^+$-c.c. forcing notion (satisfying $(1)_a + (2)_b^\sigma$), in $(\mathbb{V}_\mathbb{Q})^\mathbb{P}$ we have $\mathbb{N}_\mu^\sigma$ but no $G \subseteq \mathbb{P}$ as above.

Proof. In $\mathbb{V}_\mathbb{G}$ let $\mathbb{P}$ be from (3.9), $G$ be from Definition 3.5.

Now (1) follows from (2). For (2) use 3.8 and (3.4, 3.5, 3.6). For part (3) use the forcing from 3.6. $\square_{3.10}$
FORCING AXIOMS FOR $\lambda$-COMPLETE $\mu^+$.C.C.

References


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