CICHÓN’S MAXIMUM

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ABSTRACT. Assuming four strongly compact cardinals, it is consistent that all entries in Cichoń’s diagram (apart from add(\(\mathcal{U}\)) and cof(\(\mathcal{U}\)), whose values are determined by the others) are pairwise different; more specifically that
\[\aleph_1 < \text{add}(\mathcal{U}) < \text{cov}(\mathcal{U}) < b < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) < b < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}) < 2^{\aleph_0}.\]

INTRODUCTION

Independence. How many Lebesgue null sets are required to cover the real line? Obviously countably many are not enough, as the countable union of null sets is null; and obviously continuum many are enough, as \(\bigcup_{r \in \mathbb{R}} \{r\} = \mathbb{R}\).

The answer to our question is a cardinal number called \(\text{cov}(\mathcal{N})\). As we have just seen, 
\[\aleph_0 = |\mathbb{N}| < \text{cov}(\mathcal{N}) \leq |\mathbb{R}| = 2^{\aleph_0}.
\]

In particular, if the Continuum Hypothesis (CH) holds (i.e., if there are no cardinalities strictly between \(|\mathbb{N}|\) and \(|\mathbb{R}|\), or equivalently: if \(\aleph_1 = 2^{\aleph_0}\)), then \(\text{cov}(\mathcal{N}) = 2^{\aleph_0}\); but without CH, the answer could also be some cardinal less than \(2^{\aleph_0}\). According to Cohen’s famous result [Coh63], CH is independent of the usual axiomatization of mathematics, the set theoretic axiom system ZFC. I.e., we can prove that the ZFC axioms neither imply CH nor imply \(\neg\text{CH}\). For this result, Cohen introduced the method of forcing, which has been continuously expanded and refined ever since. Forcing also proves that the value of \(\text{cov}(\mathcal{N})\) is independent. For example, \(\text{cov}(\mathcal{N}) = \aleph_1 < 2^{\aleph_0}\) is consistent, as is \(\aleph_1 < \text{cov}(\mathcal{N}) = 2^{\aleph_0}\).

Cichoń’s diagram. \(\text{cov}(\mathcal{N})\) is a so-called cardinal characteristic of the continuum. Other well-studied characteristics include the following:

- add(\(\mathcal{N}\)) is the smallest number of Lebesgue null sets whose union is not null.
- non(\(\mathcal{N}\)) is the smallest cardinality of a non-null set.
- cof(\(\mathcal{N}\)) is the smallest size of a cofinal family of null sets, i.e., a family that contains for each null set \(\mathcal{N}\) a superset of \(\mathcal{N}\).
- Replacing “null” with “meager”, we can analogously define \(\text{add}(\mathcal{M})\), \(\text{non}(\mathcal{M})\), \(\text{cov}(\mathcal{M})\), and \(\text{cof}(\mathcal{M})\).
- In addition, we define \(b\) as the smallest size of an unbounded family, i.e., a family \(H\) of functions from \(\mathbb{N}\) to \(\mathbb{N}\) such that for every \(f : \mathbb{N} \to \mathbb{N}\) there is some \(h \in H\) which is not almost everywhere bounded by \(f\).

Equivalently, \(b = \text{add}(\mathcal{K}) = \text{non}(\mathcal{K})\), where \(\mathcal{K}\) is the \(\sigma\)-ideal generated by the compact subsets of the irrationals.
- And $b$ is the smallest size of a dominating family, i.e., a family $H$ such that for every $f : \mathbb{N} \to \mathbb{N}$ there is some $h \in H$ such that for every $m \in \mathbb{N}$ $(\exists n \in \mathbb{N}) (\forall m > n) h(m) > f(m)$. Equivalently, $b = \text{cov}(\mathcal{K}) = \text{cof}(\mathcal{K})$.
- For the ideal $\text{ctbl}$ of countable sets, we trivially get $\text{add}(\text{ctbl}) = \text{non}(\text{ctbl}) = \aleph_1$ and $\text{cov}(\text{ctbl}) = \text{cof}(\text{ctbl}) = 2^{\aleph_0}$.

The characteristics we have mentioned so far, and the basic relations between them, can be summarized in Cichoń’s diagram:

\[
\text{cov}(\mathcal{N}) \quad \text{non}(\mathcal{M}) \quad \text{cof}(\mathcal{M}) \quad \text{cof}(\mathcal{N}) \quad 2^{\aleph_0}
\]

An arrow from $\mathfrak{f}$ to $\mathfrak{u}$ indicates that ZFC proves $\mathfrak{f} \leq \mathfrak{u}$. Moreover, $\max(b, \non(M)) = \cof(M)$ and $\min(b, \text{cov}(M)) = \text{add}(M)$. A (by now) classical series of theorems [Bar84, BJS93, CKP85, JS90, Kam89, Mil81, Mil84, RS83, RS85] proves these (in)equalities in ZFC and shows that they are the only ones provable. More precisely, all assignments of the values $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń’s Diagram are consistent with ZFC, provided they do not contradict the above (in)equalities. (A complete proof can be found in [BJ95, ch. 7].)

Note that Cichoń’s diagram shows a fundamental asymmetry between the ideals of Lebesgue null sets and of meager sets (we will mention another one in the context of large cardinals). Any such asymmetry is hidden if we assume CH, as under CH not only all the characteristics are $\aleph_1$, but even the Erdős-Sierpiński Duality Theorem holds [Oxt80, ch. 19]: There is an involution $f : \mathbb{R} \to \mathbb{R}$ (i.e., a bijection such that $f \circ f = \text{Id}$) such that $A \subseteq \mathbb{R}$ is meager iff $f^{-1}A$ is null.

So it is settled which assignments of $\aleph_1$ and $\aleph_2$ to Cichoń’s diagram are consistent. It is more challenging to show that the diagram can contain more than two different cardinal values. For recent progress in this direction see, e.g., [Mej13, GMS16, FGKS17, KTT18].

The result of this paper is in some respect the strongest possible, as we show that consistently all the entries are pairwise different (apart from the two equalities provable in ZFC mentioned above). Of course one can ask more; see the questions in Section 4. In particular, we use large cardinals in the proof.

**Large cardinals.** As mentioned, ZFC is an axiom system for the whole of mathematics. A much “weaker” axiom system (for the natural numbers) is PA (Peano arithmetic).

Gödel’s Incompleteness Theorem shows that a theory such as PA or ZFC can never prove its own consistency. On the other hand, it is trivial to show in ZFC that PA is consistent (as in ZFC we can construct $\mathbb{N}$ and prove that it satisfies PA). We can say that ZFC has a higher consistency strength than PA.

One axiom of ZFC is INF, the statement “there is an infinite cardinal”. If we remove INF from ZFC, we end up with a theory ZFC\(^0\) that can still describe concrete hereditarily finite objects and can be interpreted (admittedly in a not very natural way) as a weak version of

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1There are many other cardinal characteristics, see for example [Bla10], but the ones in Cichoń’s diagram seem to be considered to be the most important ones.
PA which has the same consistency strength as PA.\footnote{More concretely, $ZF_{\text{fin}} = ZFC^0 + \neg \text{INF}$ can be seen to be “equivalent” to PA (i.e., mutually interpretable); this goes back to Ackermann [Ack37], see the survey [KW07].} So we can say that adding an infinite cardinal to $ZFC^0$ increases the consistency strength.

There are notions of cardinals numbers much “stronger” than just “infinite”. Often, such large cardinal assumptions (abbreviated LC in the following) have the following form:

There is a cardinal $\kappa > \aleph_0$ that behaves towards the smaller cardinals in a similar way as $\aleph_0$ behaves to finite numbers.

A forcing proof shows, e.g.,

If $ZFC$ is consistent, then $ZFC + \neg \text{CH}$ is consistent,

and this implication can be proved in a very weak system such as PA. However, we cannot prove (not even in ZFC) for any large cardinal

“if $ZFC$ is consistent, then $ZFC + \text{LC}$ is consistent”;

because in $ZFC + \text{LC}$ we can prove the consistency of $ZFC$. We say: LC has a higher consistency strength than ZFC.

An instance of a large cardinal (in fact a very weak one, a so-called inaccessible cardinal), appears in another striking example of the asymmetry between measure and category: The following statement is equiconsistent with an inaccessible cardinal [Sol70, She84]:

All projective\footnote{This is the smallest family containing the Borel sets and closed under continuous images, complements, and countable unions. In practice, all sets used in mathematics that are defined without using AC are projective. Alternatively we could use the statement: “$ZF$ (without the Axiom of choice) holds and all sets of reals are Lebesgue measurable.”} sets of reals are Lebesgue measurable.

In contrast, according to [She84] no large cardinal assumption is required to show the consistency of

All projective sets of reals have the property of Baire.

So we can assume “for free” that all (reasonable) sets have the Baire property, whereas we have to provide additional consistency strength for Lebesgue measurability.

In the case of our paper, we require (the consistency of) the existence of four compact cardinals to prove our main result. It seems unlikely that any large cardinals are actually required; but a proof without them would probably be considerably more complicated. It is not unheard of that $ZFC$ results first have (simpler) proofs using large cardinal assumptions; an example can be found in [She04].

Annotated Contents. From now on, we assume that the reader is familiar with some basic properties of the characteristics defined above, as well as with the associated forcing notions Cohen, amoeba, random, Hechler and eventually different, all of which can be found, e.g., in [BJ95].

This paper consists of three parts:

In Section 1, we present a finite support ccc iteration $\mathbb{P}^5$ forcing that $\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < b < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = 2^\aleph_0$. This result is not new: Such a forcing was introduced in [GMS16], and we follow this construction quite closely. However, we need GCH in the ground model, whereas [GMS16] requires $2^\lambda \gg \lambda$ for some $\lambda < \kappa$. Also, we describe how the inequalities are “strongly witnessed”, see Definitions 1.8 and 1.15.

In Section 2, we show how to construct (under GCH) for $\kappa$ strongly compact and $\theta > \kappa$ regular a “BUP-embedding” from $\kappa$ to $\theta$, i.e., an elementary embedding $j : V \rightarrow M$ with critical point $\kappa$ and $\text{cf}(j(\kappa)) = |j(\kappa)| = \theta$ such that $M$ is transitive and $<\kappa$-closed and
such that \( j''S \) is cofinal in \( j(S) \) for every \( \leq \kappa \)-directed partial order \( S \). For a ccc forcing \( P \) we investigate \( j(P) \) and show that \( j(P) \) forces the same values to some characteristics in Cichoń’s diagram as \( P \) and different values to others, in a very controlled way; assuming that there were “strong witnesses” for \( P \) forcing the initial values, as described in Section 1.

Section 3 contains the main result of this paper: Assuming four strongly compact cardinals, we let \( k \) be the composition of four such BUP-embeddings, mapping \( \mathbb{P}^5 \) to a ccc forcing \( \mathbb{P}^9 \). We then show that \( \mathbb{P}^9 \) forces

\[
\aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < b < \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) < b < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}) < 2^{\aleph_0},
\]

i.e., we get for increasing cardinals \( \lambda_i \) the constellation of Figure 1.

Boolean ultrapowers as used in this paper were investigated by Mansfield [Man71] and recently applied e.g. by the third author with Malliaris [MS 16] and with Raghavan [RS], where Boolean ultrapowers of forcing notions are used to force specific values to certain cardinal characteristics. Recently the third author developed a method of using Boolean ultrapowers to control characteristics in Cichoń’s diagram. A first (and simpler) application of these methods is given in [KTT18].

We mention some open questions in Section 4.

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ties and typos; and Moti Gitik and Diego Mejía for suggestions to improve the presentation.

1. The initial forcing

1.1. Good iterations and the LCU property. We want to show that some forcing \( \mathbb{P}^5 \) results in \( \mathfrak{r} = \lambda_i \) for certain characteristics \( \mathfrak{r} \). So we have to show two “directions”, \( \mathfrak{r} \leq \lambda_i \) and \( \mathfrak{r} \geq \lambda_i \). For most of the characteristics, one direction will use the fact that \( \mathbb{P}^5 \) is “good”; a notion introduced by Judah and the third author [JS90] and Brendle [Bre91]. We now recall the basic facts of good iterations, and specify the instances of the relations we use.

Assumption 1.1. We will consider binary relations \( R \) on \( X = \omega^\omega \) (or on \( X = 2^\omega \)) that satisfy the following: There are relations \( R^k \) such that \( R = \bigcup_{k<\omega} R^k \), each \( R^k \) is a closed subset (and in fact absolutely defined) of \( X \times X \), and for \( g \in X \) and \( k<\omega \), the set \( \{ f \in X : f R^k g \} \) is nowhere dense (and of course closed). Also, for all \( g \in X \) there is some \( f \in X \) with \( f R g \).

We will actually use another space as well, the space \( C \) of strictly positive rational sequences \( (q_n)_{n<\omega} \) such that \( \sum_{n<\omega} q_n \leq 1 \). It is easy to see that \( C \) is homeomorphic to \( \omega^\omega \), when we equip the rationals with the discrete topology and use the product topology. Let us fix one such (absolutely defined) homeomorphism.

We use the following instances of relations \( R \) on \( X \); it is easy to see that they all satisfy the assumption (for \( X_4 = C \) we use the homeomorphism mentioned above):
Definition 1.2. 1. \( X_1 = C : f \in R_1 \) if \( (\forall n \in \omega) f(n) \leq g(n) \).

(We use \( \forall^*n \) as abbreviation for \((\exists n_0) (\forall n > n_0)\).)

2. Fix a partition \((I_n)_{n \in \omega}\) of \( \omega \) with \( |I_n| = 2^{n+1} \).

3. \( X_2 = 2^\omega : f \in R_2 \) if \( (\forall n \in \omega) f \upharpoonright I_n \neq g \upharpoonright I_n \).

4. \( X_3 = \omega^\omega : f \in R_3 \) if \( (\forall n \in \omega) f(n) \leq g(n) \).

5. \( X_4 = \omega^\omega : f \in R_4 \) if \( (\forall n \in \omega) f(n) \neq g(n) \).

Note that Assumption 1.1 is satisfied, witnessed by the relations \( R_k^\ell \) defined by replacing \((\forall n \in \omega)\) with \((\forall n \geq k)\).

We say “\( f \) is bounded by \( g^*\) if \( f R g \); and, for \( \mathcal{Y} \subseteq \omega^\omega\), “\( f \) is bounded by \( \mathcal{Y}\)” if \((\exists y \in \mathcal{Y}) f R y \). We say “unbounded” for “not bounded”. (I.e., \( f \) is unbounded by \( \mathcal{Y}\) if \((\forall y \in \mathcal{Y}) \neg f R y \).) We call \( \mathcal{X} \) an \( R\)\(\omega\)-unbounded family if \( (\neg (\exists g) (\forall x \in \mathcal{X}) x R g) \) and an \( R\)-dominating family if \( (\forall f) (\exists x \in \mathcal{X}) f R x \).

- Let \( b_1 \) be the minimal size of an \( R_1\)-unbounded family.
- and let \( b_2 \) be the minimal size of an \( R_1\)-dominating family.

We only need the following connections between \( R \) and the cardinal characteristics:

Lemma 1.3. 1. \( \text{add}(\mathcal{N}) = b_1 \) and \( \text{cof}(\mathcal{N}) = b_1 \).

2. \( \text{cov}(\mathcal{N}) \leq b_2 \) and \( \text{non}(\mathcal{N}) \geq b_2 \).

3. \( b = b_2 \) and \( \bar{b} = b_3 \).

4. \( \text{non}(\mathcal{M}) = b_4 \) and \( \text{cov}(\mathcal{M}) = b_4 \).

Proof. (3) holds by definition. (1) can be found in [BJ95, 6.5.B]. (4) is a result of [Mil82, Bar87], cf. [BJ95, 2.4.1 and 2.4.7].

To prove (2), note that for fixed \( f \in 2^\omega \) the set \( \{ g \in 2^\omega : \neg f R_2 g \} \) is a null set, call it \( N_f \). Let \( F \) be an \( R_2\)-unbounded family. Then \( \{ N_f : f \in F \} \) covers \( 2^\omega \). Fix \( g \in 2^\omega \). As \( g \) does not bound \( F \), there is some \( f \in F \) unbounded by \( g \), i.e., \( g \in N_f \). Let \( X \) be a non-null set. Then \( X \) is \( R_1\)-dominating: For any \( f \in 2^\omega \) there is some \( x \in X \setminus N_f \), i.e., \( f \upharpoonright R_2 x \).

We will also use:

Lemma 1.4. [BJ95] 
Amoeba forcing \( \mathcal{A} \) adds a dominating element \( \bar{b} \) of \( C \), i.e., \( \mathcal{A} \models \bar{q} R_1 \bar{b} \) for all \( \bar{q} \in \mathcal{C} \cap V \).

Proof. Let us define a slalom \( S \) to be a function \( S : \omega \to [\omega]^{<\omega} \) such that \( |S(n)| > 0 \) and \( \sum_{n=1}^{\infty} \frac{|S(n)|}{n^2} < \infty \).

Amoeba forcing will add a null set covering all old null sets, and therefore (according to [BJ95, 2.3.3]) a slalom \( S \) covering all old slaloms. Set \( a_n := \frac{|S(n)|}{n^2} \), \( M := \sum_{n=1}^{\infty} a_n \), set \( M' \) the smallest natural number \geq M, and set \( b_n := \frac{a_{n+1}}{M'} \). Then it is easy to see that \( (b_n)_{n \in \omega} \in C \) dominates every old sequence \( (q_n)_{n \in \omega} \) in \( C \).

Definition 1.5. [JS90] Let \( P \) be a ccc forcing, \( \lambda \) an uncountable regular cardinal, and \( R \) as above. \( P \) is \( (R, \lambda)\)-good, if for each \( P\)-name \( r \in \omega^\omega \) there is \( (\text{in } V) \) a nonempty set \( \mathcal{Y} \subseteq \omega^\omega \) of size \( < \lambda \) such that every \( f \) (in \( V \)) that is \( R\)-unbounded by \( \mathcal{Y} \) is forced to be \( R\)-unbounded by \( r \) as well.

Note that \( \lambda\)-good trivially implies \( \mu\)-good if \( \mu \geq \lambda \) are regular.

How do we get good forcings? Let us just note the following results:

Lemma 1.6. A finite support (henceforth abbreviated FS) iteration of Cohen forcing is good for any \((R, \lambda)\), and the composition of two \((R, \lambda)\)-good forcings is \((R, \lambda)\)-good.
Assume that \((P_\alpha, Q_\alpha)_{\alpha<\delta}\) is a FS ccc iteration. Then \(P_\alpha\) is \((R, \lambda)\)-good, if each \(Q_\alpha\) is forced to satisfy the following:

1. For \(R = R_1\): \(|Q_\alpha| < \lambda\), or \(Q_\alpha\) is \(\sigma\)-centered, or \(Q_\alpha\) is a sub-Boolean-algebra of the random algebra.
2. For \(R = R_2\): \(|Q_\alpha| < \mu\), or \(Q_\alpha\) is \(\sigma\)-centered.
3. For \(R = R_3\): \(|Q_\alpha| < \lambda\).
4. For \(R = R_4\): \(|Q_\alpha| < \mu\).

(Remark: For \(R_3\) the same holds as for \(R_4\), which however is of no use for our construction.)

Proof. \((R, \lambda)\)-goodness is preserved by FS ccc iterations (in particular compositions), as proved in [JS90], cf. [BJ95, 6.4.11–12]. Also, ccc forcings of size \(<\lambda\) are \((R, \lambda)\)-good [BJ95, 6.4.7]; which takes care of the case \(|Q_\alpha| < \lambda\) (and in particular of Cohen forcing). So it remains to show that (for \(i = 1, 2\)) the “large” iterands in the list are \((R, \lambda)\)-good.

For \(R_1\) this follows from [JS90] and [Kam89], cf. [BJ95, 6.5.17–18]. For \(R_2\), this is proven in [Bre91], and as the proof is very short, we give it here: Write \(Q_\alpha\) as union \(\bigcup_{k \in \omega} Q^k\) of centered sets. Given the \(Q_\alpha\)-name \(p\), pick a countable elementary submodel \(N\) containing \(r\) and \(Q_\alpha\), and let \(\mathcal{V} = N \cap 2^\omega\). Assume towards a contradiction that \(f\) is unbounded by \(\mathcal{V}\), but is forced by \(p_0\) to be bounded by \(r\), i.e., \(p_0\) forces \((\forall n > n_0) f \upharpoonright I_n \neq r \upharpoonright I_n\). Now \(p_0\) may not be in \(N\), but there is some \(k_0 \in \omega\) such that \(p_0 \in Q^{k_0}\). In \(N\), we can pick for each \(n \in \omega\) some \(s_n \in 2^\omega\) such that no \(q \in Q^{k_0}\) forces \(r \upharpoonright I_n \neq s_n\).

(There are only finitely many \(s \in 2^\omega\): if each \(s\) is forbidden by some \(q\), then the common stronger element would prevent all possibilities for \(r \upharpoonright I_n\).) So in \(N\), we get some \(g \in 2^\omega\) such that \(g \upharpoonright I_n = s_n\). As \(f\) is unbounded by \(\mathcal{V}\) (or equivalently: by \(N\)), there is some \(n > n_0\) such that \(f \upharpoonright I_n = g \upharpoonright I_n = s_n\), which implies that \(p_0\) (as an element of \(Q^{k_0}\)) does not force \(r \upharpoonright I_n \neq f \upharpoonright I_n\), a contradiction.

\(\square\)

Lemma 1.7. Let \(\lambda \leq \kappa \leq \mu\) be uncountable regular cardinals. After forcing with \(\mu\) many Cohen reals \((c_\alpha)_{\alpha \in \omega'}\), followed by an \((R, \lambda)\)-good forcing, we get: For every real \(r\) in the final extension, the set \(\{a \in \kappa : c_\alpha\text{ is unbounded by }r\}\) is cobounded in \(\kappa\). I.e., \((\exists \alpha \in \kappa)(\forall \beta \in \kappa \setminus \alpha) c_\beta \notin Rr\).

(The Cohen real \(c_\beta\) can be interpreted both as Cohen generic element of \(2^\omega\) and as Cohen generic element of \(\omega^\omega\); we use the interpretation suitable for the relation \(R\).)

Proof. Work in the intermediate extension after \(\kappa\) many Cohen reals; let us call it \(V_\kappa\). The remaining forcing (i.e., \(\mu \setminus \kappa\) many Cohens composed with the good forcing) is good; so applying the definition we get in \((V_\kappa)\) a set \(\mathcal{V}\) of size \(<\lambda\).

As the initial Cohen extension is ccc, and \(\kappa \geq \lambda\) is regular, we get some \(a \in \kappa\) such that each element \(y\) of \(\mathcal{V}\) already exists in the extension by the first \(\alpha\) many Cohens, call it \(V_\alpha\). The set of reals \(M_y\) bounded by \(y\) is meager (and absolute). Any \(c_\beta\) for \(\beta \in \kappa \setminus \alpha\) is Cohen over \(V_\alpha\), and therefore not in \(M_y\), i.e., not bounded by \(y\), i.e., not by \(\mathcal{V}\). So according to the definition of good, each such \(c_\beta\) is unbounded by \(r\) as well, for the given \(r\).

\(\square\)

In the light of this result, let us revisit Lemma 1.3 with some new notation, the “linearly coinfinitely unbounded” property LCU:

Definition 1.8. For \(i = 1, 2, 3, 4, \gamma\) a limit ordinal, and \(P\) a ccc forcing notion, let \(\mathrm{LCU}(P, \gamma)\) stand for:

There is a sequence \((x_\alpha)_{\alpha \in \gamma}\) of \(P\)-names of elements of \(X_1\) (the domain of the relation \(R_j\)) such that for every such \(P\)-name \(y\)

\[\exists \alpha \in \gamma \forall \beta \in \gamma \setminus \alpha P \vdash \neg x_\beta R_j y.\]
Lemma 1.9.  \(\bullet\) \(\mathrm{LCU}_1(P, \delta)\) is equivalent to \(\mathrm{LCU}_1(P, \operatorname{cf}(\delta))\).

\(\bullet\) If \(\lambda\) is regular, then \(\mathrm{LCU}_1(P, \lambda)\) implies \(b_1 \leq \lambda\) and \(b_1 \geq \lambda\).

In particular:

1. \(\mathrm{LCU}_1(P, \lambda)\) implies \(P \Vdash (\operatorname{add}(\mathcal{N}) \leq \lambda \& \operatorname{cof}(\mathcal{N}) \geq \lambda)\).
2. \(\mathrm{LCU}_2(P, \lambda)\) implies \(P \Vdash (\operatorname{cov}(\mathcal{N}) \leq \lambda \& \operatorname{non}(\mathcal{N}) \geq \lambda)\).
3. \(\mathrm{LCU}_3(P, \lambda)\) implies \(P \Vdash (b \leq \lambda \& b \geq \lambda)\).
4. \(\mathrm{LCU}_4(P, \lambda)\) implies \(P \Vdash (\operatorname{non}(\mathcal{M}) \leq \lambda \& \operatorname{cov}(\mathcal{M}) \geq \lambda)\).

\textbf{Proof.} Assume that \((\alpha_\beta)_{\beta \in \delta}^{\mathcal{G}}\) is increasing continuous and cofinal in \(\delta\). If \((x_\alpha)_{\alpha \in \delta}^{\mathcal{G}}\) witnesses \(\mathrm{LCU}_1(P, \delta)\), then \((x_\alpha)_{\alpha \in \delta}^{\mathcal{G}}\) witnesses \(\mathrm{LCU}_1(P, \operatorname{cf}(\delta))\). And if \((x_\alpha)_{\alpha \in \delta}^{\mathcal{G}}\) witnesses \(\mathrm{LCU}_1(P, \operatorname{cf}(\delta))\), then \((y_\beta)_{\beta \in \delta}^{\mathcal{G}}\) witnesses \(\mathrm{LCU}_1(P, \delta)\), where \(y_\beta = x_\delta\) for \(\alpha \in [\alpha_\beta, \alpha_{\beta+1})\).

The set \(\{x_\alpha : \alpha \in \lambda\}\) is certainly forced to be \(\mathcal{R}\)-unbounded; and given a set \(Y = \{y_j : j < \theta\}\) of \(\theta < \lambda\) many \(P\)-names, each has a bound \(a_j \in \lambda\) so that \((\forall \beta \in \lambda \setminus a_j) P \Vdash \neg x_\beta \mathcal{R}_j y_j\), so for any \(\beta \in \lambda\) above all \(a_j\) we get \(P \Vdash \neg x_\beta \mathcal{R}_j y_j\) for all \(j\); i.e., \(Y\) cannot be dominating.

\textbf{Remark 1.10.} \(b_1 \leq \lambda\) is equivalent to the existence of a sequence \((x_\alpha : \alpha \in \lambda)\) with the property \((\forall y) (\exists \alpha)(x_\alpha \mathcal{R}_y)\); such a sequence might be called a “\(\lambda\)-witness” for \(b_1 \leq \lambda\). In \(\mathrm{LCU}\) we demand a stronger property; a sequence \((x_\alpha : \alpha < \lambda)\) with this stronger property could informally be called a “\(\lambda\)-witness” for \(b_1 \leq \lambda\). Similarly, the next subsection introduces a different notion, \(\mathrm{COB}\), corresponding to “\(\lambda\)-witnesses” for \(b_1 \leq \mu\).

1.2. \textbf{The initial forcing} \(\mathbb{P}^5\): \textbf{Partial forcings and the \(\mathrm{COB}\) property.} Assume we have a forcing iteration \((P_\beta, Q_\beta)_{\beta \in \omega}\) with limit \(P_\omega\), where each \(Q_\beta\) is forced by \(P_\beta\) to be a set of reals such that the generic filter of \(Q_\beta\) is determined (in a Borel way)\(^4\) from some generic real \(\eta_\beta\). Fix some \(w \subseteq \alpha\). We define the \(P_\alpha\)-name \(Q_\alpha\) to consist of all random forcing conditions that can be Borel-calculated from generics at \(w\) alone.

More explicitly:

\textbf{Definition 1.11.} \(1\) \(q\) is in \(Q_\alpha\) if there are in the ground model \(V\) a countable subset \(u \subseteq w\) and a Borel function \(B : \mathbb{R}^u \to \mathbb{R}\) such that \(q = B(\eta_\beta)_{\beta \in u}\) is a random condition.

Being a random condition is a Borel property (if we fix some suitable representation of random forcing). Accordingly, we can restrict ourselves to the case that \(B\) is a Borel function whose image consists of random conditions only.

\(2\) We call a pair \((B, u)\) as above “\(w\)-groundmodel-code” or just “code”. Note that this code is a ground model object. \(Q_\alpha\) consists exactly of the evaluations of such codes.

\(3\) We call a condition \((p, q) \in P_\alpha \ast Q_\alpha\) “determined at position \(a^*\)”, if there is a code \((B, u)\) such that \(p\) forces that \((B, u)\) is a code for \(q\). (Note that generally we only have a \(P_\alpha\)-name for a code.) Given some \((p, q)\), we can obviously find \(p' \leq p\) such that \((p', q)\) is determined at \(a\).

\(4\) We will later also consider so-called “groundmodel-code-sequences” for elements of \(Q_\alpha\), that is (in \(V\)) a sequence \((B_n, u_n)_{n \in \omega}\) of codes, where \(u_n\) in \(w_\alpha\). Of course not every \(w\)-sequence of \(Q_\alpha\)-conditions in the \(P_\omega\)-extension is described

\(^4\)More specifically, we require that the Borel function for \(Q_\beta\) is already fixed in the ground model. For example, assume \(Q_\beta\) is random forcing, defined as the set of all positive pruned trees \(T\), i.e., \(T \subseteq 2^{<\omega}\) without leaves such that \([T]\) has positive measure. Then the generic filter \(G\) for this forcing is determined by the generic real \(\eta\) (the random real), and \(G\) consists of those trees \(T\) such that \(\eta \in [T]\), which is a Borel relation. See [KTT18, Sec. 1.2] for a formal definition and more details.
by a ground model sequence. (In particular, there will only be few ground model sequences, but many new ω-sequences in the extension.)

Clearly, in the $P_\alpha$ extension, $Q_\alpha$ is a subforcing (not necessarily a complete one) of the full random forcing, and if $p, q$ in $Q_\alpha$ are incompatible in $Q_\alpha$ then they are incompatible in random forcing. (Two compatible conditions $p, q$ have a canonical conjunction $p \land q$ (the intersection), and if $p$ and $q$ are both Borel-calculated from $w$, then so is the intersection.) In particular $Q_\alpha$ is ccc.

We call this forcing “partial random forcing defined from $w$”. Analogously, we define the “partial Hechler”, “partial eventually different” and “partial amoeba” forcings (and the same argument shows that these forcings are also ccc).

Assume that $\lambda$ is regular uncountable and that $\mu < \lambda$ implies $2^{<\mu} < \lambda$. Then $|\mu| < \lambda$ implies that the sizes of the partial forcings defined by $w$ are $<\lambda$.

We will assume the following throughout the paper:

**Assumption 1.12.** $\aleph_1 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$ are regular cardinals such that $\mu < \lambda_i$ implies $2^{<\mu} < \lambda_i$. Furthermore, $\lambda_5$ is the successor of a regular cardinal $\chi$ with $2^{<\chi} = \chi$, and $\lambda_5^+ = \lambda_5$.

We set $\delta_5 = \lambda_5 + \lambda_5$, and partition $\delta_5 \setminus \lambda_5$ into unbounded sets $S^1, S^2, S^3$ and $S^4$. Fix for each $\alpha \in \delta_5 \setminus \lambda_5$ some $w_\alpha \subseteq \alpha$ such that each $\{w_\alpha : \alpha \in S^i\}$ is cofinal in $[\delta_5]^{<\lambda_5}$.

The reader can assume that $(\lambda_i)_{i=1,\ldots,5}$, $(S^i)_{i=1,\ldots,4}$ as well as $(w_\alpha)_{\alpha \in S^i}$ for $i = 1, 2, 3$ have been fixed once and for all (let us call them “fixed parameters”), whereas we will investigate various possibilities for $\bar{w} = (w_\alpha)_{\alpha \in S^i}$ in the following Subsections 1.3 and 1.4. (We will call such a $\bar{w}$ that satisfies the assumption a “cofinal parameter”).

**Definition 1.13.** Let $P^5 = (P_\alpha, Q_\alpha)_{\alpha \in \delta_5}$ be the FS iteration where $Q_\alpha$ is Cohen forcing for $\alpha \in \lambda_5$, and

$Q_\alpha$ is the partial

\[
\begin{cases}
\text{amoeba} & \text{random} \\
\text{Hechler} & \text{eventually different}
\end{cases}
\]

forcing defined from $w_\alpha$ if $\alpha$ is in

\[
\begin{cases}
S^1 \\
S^2 \\
S^3 \\
S^4
\end{cases}
\]

According to Lemma 1.6 $P^5$ is $(\lambda_i, R_i)$-good for $i = 1, 2, 4$, so Lemmas 1.7 and 1.9 give us:

**Lemma 1.14.** $LCU(P^5, \kappa)$ holds for $i = 1, 2, 4$ and each regular cardinal $\kappa$ in $[\lambda_i, \lambda_5]$.

So in particular, $P^5$ forces $\text{add}(\mathcal{N}) \leq \lambda_1$, $\text{cof}(\mathcal{N}) \leq \lambda_2$, $\text{non}(\mathcal{M}) \leq \lambda_4$ and $\text{cov}(\mathcal{M}) = \text{cov}(\mathcal{N}) = \text{aleph}(\mathcal{N}) = \lambda_5 = 2^{<\lambda_5}$; i.e., the respective characteristics in the left half of Cichoń’s diagram are small enough. It is easy to see that they are also large enough:

For example, the partial amoebas and the fact that $(w_\alpha)_{\alpha \in S^1}$ is cofinal ensure that $P^5$ forces $\text{add}(\mathcal{N}) \geq \lambda_1$: Let $(N_k)_{k \in \mu}$, $\mathcal{N}_1 \leq \mu < \lambda_1$ be a family of $P^5$-names of null sets. Each $N_k$ is a Borel-code, i.e., a real, i.e., a sequence of natural numbers, each of which is decided by a maximal antichain (labeled with natural numbers). Each condition in such an antichain has finite support, hence only uses finitely many coordinates in $\delta_5$. So all in all we get a set $w^*$ of size $\leq \mu$ that already decides all $N_k$. (I.e., for each $k \in \mu$ there is a Borel function

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$^5$See 1.22 for the definition.

$^6$I.e., if $\alpha \in S^i$ then $|w_\alpha| < \lambda_i$, and for all $\alpha \subseteq \delta_5$, $|\alpha| < \lambda_i$ there is some $\alpha \subseteq S^i$ with $w_\alpha \supseteq u$. 

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There is some $\beta \in S^1$ such that $w_\beta \supseteq w^*$, and the partial amoeba forcing at $\beta$ sees all the null sets $N_k$ and therefore covers their union.

We will reformulate this in a slightly cumbersome manner that can be conveniently used later on, using the “cone of bounds” property COB:

**Definition 1.15.** For a ccc forcing notion $P$, regular uncountable cardinals $\lambda, \mu$ and $i = 1,3,4$, let $\text{COB}(P, \lambda, \mu)$ stand for:

There are a $<\lambda$-directed partial order $(S, \prec)$ of size $\mu$ and a sequence $(g_s)_{s \in S}$ of $P$-names for reals such that for each $P$-name $f$ of a real $(\exists s \in S)(\forall t > s) P \Vdash f R_s g_s$.

So $s$ is the tip of a cone that consists of elements bounding $f$.

**Lemma 1.16.** For $i = 1,3,4$, $\text{COB}(P, \lambda, \mu)$ implies $P \Vdash (b_i \geq \lambda$ & $b_i \leq \mu$).

*Proof.* $b_i \leq \mu$, as the set $(g_s)_{s \in S}$ is a dominating family of size $\mu$. To show $b_i \geq \lambda$, assume $(f_a)_{a \in \theta}$ is a sequence of $P$-names of length $\theta < \lambda$. For each $f_a$ there is a cone of upper bounds with tip $s_a \in S$, i.e., $(\forall t > s_a) P \Vdash f_a R_t g_t$. As $S$ is $<\lambda$-directed, there is some $t$ above all tips $s_a$. Accordingly, $P \Vdash f_a R_t g_t$ for all $a$, i.e., $\{f_a : a \in \theta\}$ is not unbounded.

So for example, $\text{COB}_2(P, \lambda, \mu)$ implies $\lambda_1 \leq b_1 = \text{add}(\mathcal{N})$, etc. The definition and lemma would work for $i = 2$ as well, but would not be useful\(^7\) as we do not have $b_2 \leq \text{cov}(\mathcal{N})$. So instead, we define $\text{COB}_2$ separately:

**Definition 1.17.** For $P$, $\lambda$ and $\mu$ as above, let $\text{COB}_2(P, \lambda, \mu)$ stand for:

There are a $<\lambda$-directed partial order $(S, \prec)$ of size $\mu$ and a sequence $(g_s)_{s \in S}$ of $P$-names for reals such that for each $P$-name $f$ of a null set $(\exists s \in S)(\forall t > s) P \Vdash g_s \notin f$.

**Lemma 1.18.**

1. $\text{COB}_1(P, \lambda, \mu)$ implies $P \Vdash (\text{add}(\mathcal{N}) \geq \lambda \& \text{cov}(\mathcal{N}) \leq \mu$).
2. $\text{COB}_2(P, \lambda, \mu)$ implies $P \Vdash (\text{cov}(\mathcal{N}) \geq \lambda \& \text{non}(\mathcal{N}) \leq \mu$).
3. $\text{COB}_3(P, \lambda, \mu)$ implies $P \Vdash (b \geq \lambda \& b \leq \mu$).
4. $\text{COB}_4(P, \lambda, \mu)$ implies $P \Vdash (\text{non}(\mathcal{M}) \geq \lambda \& \text{cov}(\mathcal{M}) \leq \mu$).

*Proof.* The cases $i \neq 2$ are direct consequences of Lemmas 1.3 and 1.16. The proof for $i = 2$ is analogous to the proof of Lemma 1.16.

**Lemma 1.19.** $\text{COB}_1(P^5, \lambda, \lambda_3)$ holds (for $i = 1, 2, 3, 4$).

*Proof.* Set $S = S^1$ and $s < t$ if $w_s \nsubseteq w_t$. As $\lambda_i$ is regular, $(S, \prec)$ is $<\lambda_i$-directed. Let $g_s$ be the generic added at $s$ (e.g., the partial random real in case of $i = 2$, etc). A $P^5$-name $f$ depends (in a Borel way) on the subsequence of generics indexed by a countable set $w^* \subseteq \delta$. Fix some $s \in S^1$ such that $w_s \supseteq w^*$. Pick any $t > s$. Then $w_t \supseteq w_s$, so $w_t$ contains all information to calculate $f$, so we can show that $P \Vdash f R_t g_t$. Let us list the possible cases: $i = 2$: A partial random real $g_t$ will avoid the null set $f$. $i = 3$: A partial Hechler real $g_t$ will dominate $f$. $i = 4$: A partial eventually different real $g_t$ will be eventually different from $f$. As for $i = 1$, we use\(^8\) Lemma 1.4.

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\(^7\)More specifically: this definition would give us the property $g_t \notin f$ only for the null sets of the specific form $f = \{ h : \neg \text{R}_r h \} = N_r$ for some $r \in 2^\omega$, whereas we will define $\text{COB}_2$ to deal with all names $f$ of null sets.

\(^8\)Alternatively, we could use, instead of amoeba, some other Suslin ccc forcing that more directly adds an $R_1$-dominating element of $\mathcal{C}$. 

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\[ \]
So to summarize what we know so far about $\mathbb{P}^5$:
- COB holds for $i = 1, 2, 3, 4$. So the left hand characteristics are large.
- LCU holds for $i = 1, 2, 4$. So the left hand characteristics other than $b$ are small.

However, LCU$_3$ (corresponding to “$b$ small”) is missing; and we cannot get it by a simple “preservation of $(R_3, \lambda_3)$-goodness” argument. Instead, we will argue in the following two
sections that it is possible to choose the parameter $(w_a)_{a \in S^4}$ in such a way that LCU$_3$ holds as well.

1.3. **Dealing with $b$ without GCH.** In this section, we follow (and slightly modify) the main construction of [GMS16].

In this section (and this section only) we will assume the following (in addition to Assumption 1.12, i.e., in particular to the assumption $\lambda_3 = \chi^+$):

**Assumption 1.20.** (This section only.) $2^\chi = |\delta_3| = \lambda_3$.

Set $S^0 = \lambda_3 \cup S^1 \cup S^2 \cup S^3$. So $\delta_3 = S^0 \cup S^4$, and $\mathbb{P}^5$ is a FS ccc iteration along $\delta_3$ such that $a \in S^0$ implies $|Q_a| < \lambda_3$, i.e., $|Q_a| \leq \chi$. Let us fix $P_a$-names

\[(1.21) \quad i_a : Q_a \rightarrow \chi \text{ injective}
\]

(for $a \in S^0$). Note that we can strengthen each $p \in \mathbb{P}^5$ to some $q$ such that $a \in \text{supp}(q) \cap S^0$ implies $q \upharpoonright a \Vdash i_a(q(a)) = j$ for some $j \in \chi$.

For $a \in S^4$, $Q_a$ is a partial eventually different forcing. At this point, we should specify which variant of this forcing we actually use.\(^9\)

**Definition 1.22.**
- Eventually different forcing $E$ consists of all tuples $(s, k, \varphi)$, where $s \in \omega^{< \omega}$, $k \in \omega$, and $\varphi : \omega \rightarrow [\omega]^{< \omega}$ satisfies $s(i) \notin \varphi(i)$ for all $i \in \text{dom}(s)$.
- We define $(s', k', \varphi') \preceq (s, k, \varphi)$ if $s' \subseteq s'$, $k \leq k'$, and $\varphi(i) \subseteq \varphi'(i)$ for all $i$.
- The generic object $g^* = \bigcup_{(s, k, \varphi) \in G^*} s$ is a function such that each condition $(s, k, \varphi)$ forces that $s$ is an initial segment of $g^*$, and $g^*(i) \subseteq \varphi'(i)$ for all $i$.
- We call $s \in \omega^{< \omega}$ the “stem” of $(s, k, \varphi)$ and $k \in \omega$ the “width”.

A density argument shows that $g^*$ will be eventually different from all functions $f : \omega \rightarrow \omega$ from $V$.

The following is easy to see:
- If $p, q \in E$ are compatible, then they have a greatest lower bound.
- Any finite set of conditions with the same stem has a lower bound (again with the same stem). So $E$ is $\sigma$-centered.
- If $q = (s', k', \varphi')$ and $p = (s, k, \varphi)$ and $s'$ extends $s$, then $p$ and $q$ are compatible iff $s'(i) \not\in \varphi(i)$ for all $i \in \text{dom}(s')$.
- If a condition $q^* = (s^*, k^*, \varphi^*)$ is compatible with each condition in a finite set $B \subseteq E$, and $s^*$ extends $s$ for each $(s, k, \varphi) \in B$, then the set $B \cup \{q^*\}$ has a lower bound. (Use $s^*$ as stem, and take the pointwise union of all $\varphi$ that occur in $B \cup \{q^*\}$.)

We will not force with $E$, but with a partial version of $E$. In the $P_a$-extension (for $a \in S^4$), this partial forcing $Q_a = E'$ is a (generally not complete) sub-forcing of $E$ which is easily seen to be closed under conjunctions (i.e., under the partial operation “greatest lower bound” of finite sets of conditions). Note that this implies that compatibility is absolute between $E$ and $E'$, and that the previous items also hold for $E'$. For later reference, let us explicitly state the last item:\(^9\)In the previous section it did not matter which variant we use.
Fact 1.23. Assume $E' \subseteq E$ is closed under conjunctions. If a condition $q^* = (s^*, k^*, \varphi^*)$ in $E'$ is compatible with each condition in a finite set $B \subseteq E'$, and $s^*$ extends $s$ for each $(s, k, \varphi) \in B$, then the set $B \cup \{q^*\}$ has a lower bound in $E'$.

Definition 1.24. Let $D$ be a non-principal ultrafilter on $\omega$, and let $\bar{p} = (p_n)_{n\in\omega} = (s, k, \varphi_n)_{n\in\omega}$ be a sequence of conditions in $E$ with the same stem and the same width. We define $\lim_D \bar{p}$ to be $(s, k, \varphi_\infty)$, where for all $i$ and all $j$ we have $j \in \varphi_\infty(i) \iff \{n : j \in \varphi_n(i)\} \in D$.

The following is easy to see: $\lim_D \bar{p} \in E$ and if $q \leq \lim_D \bar{p}$, then the set $B := \{n \in \omega : p_n \text{ compatible with } q\}$ is in $D$.  
(Proof: $q = (s', k', \varphi') \leq \lim_D \bar{p} = (s, k, \varphi_\infty)$. So for each $i \in \text{dom}(s')$, $s'_i \notin \varphi_\infty(i)$, and by the definition of the limit, $A'_i := \{n : s'_i \notin \varphi_n(i)\} \in D$. If $n \in \bigcap_{i\in\text{dom}(s')} A'_i$, then $p_n$ is compatible with $q$.)

As $B$ is defined using only compatibility, the statement still holds for compatibility preserving subforcings. We state it for later reference in the following form:

Fact 1.25. Assume that $E'$ is a subforcing of $E$ closed under conjunctions, let $\bar{p}$ be a sequence of $E'$ conditions with the same stem and width, and assume that $\lim_D \bar{p} \in E'$ and that $q \leq_{E'} \lim_D \bar{p}$. Then $B := \{n \in \omega : p_n \text{ compatible with } q\}$ is in $D$.

Definition 1.26.  
- A "partial guardrail" is a function $h$ defined on a subset of $\delta_5$ such that $h(a) \in \chi$ for $a \in S^0 \cap \text{dom}(h)$, and $h(a) \in \omega^{\omega \times \omega}$ for $a \in S^4 \cap \text{dom}(h)$.
- A "countable guardrail" is a partial guardrail with countable domain. A "full guardrail" is a partial guardrail with domain $\delta_5$.

We will use the following lemma, which is a consequence of the Engelking-Karłowicz theorem [EK65] on the density of box products (cf. [GMS16, 5.1]):

Lemma 1.27. (As $|\delta_5| \leq 2^{\omega}$ and $\chi^{\aleph_0} = \chi$.) There is a family $H^*$ of full guardrails with $|H^*| = \chi$, such that each countable guardrail is extended by some $h \in H^*$. We will fix such an $H^*$ and enumerate it as $(h^*_a)_{a \in \chi}$.

Note that the notion of guardrail (and the density property required in Lemma 1.27) only depends on $\chi$, $\delta_5$, $S^0$ and $S^4$, i.e., on fixed parameters; so we can fix an $H^*$ that will work for all cofinal parameters $\vec{\omega} = (\omega_a)_{a \in S^4}$.

Once we have decided on $\vec{\omega}$, and thus have defined $\mathbb{P}_5$, we can define the following:

Definition 1.28. A condition $p \in \mathbb{P}_5$ follows the full guardrail $h$, if
- for all $\alpha \in S^0 \cap \text{dom}(p)$, the empty condition of $P_\alpha$ forces that $p(\alpha) \in Q_\alpha$ and $i_a(p(\alpha)) = h(\alpha)$ (where $i_a$ is defined in (1.21)), and
- for all $\alpha \in S^4 \cap \text{dom}(p)$:
  - $p|\alpha$ forces that the pair of stem and width of $p(\alpha)$ is equal to $h(\alpha)$, and moreover
  - $p$ is determined at $\alpha$. (This was defined in 1.11(3): We already know in $V$ a code $(B, u)$ that evaluates to $p(\alpha)$.)

As we are dealing with a FS iteration, the set of conditions $p$ determined at each position $\alpha \in \text{dom}(p)$ is easily seen to be dense (by induction). So note that
- the set of conditions $p$ such that there is some guardrail $h$ such that $p$ follows $h$, is dense; while
- for each fixed guardrail $h$, the set of all conditions $p$ following $h$, is centered (i.e., each finitely many such $p$ are compatible).
Definition 1.29.  
- A “Δ-system with root V following the full groundrail h” is a family $\bar{\rho} = \{p_i\}_{i \in I}$ of conditions all following $h$, where $(\text{dom}(p_i) : i \in I)$ is a Δ-system with root $V$ in the usual sense (so $V \subseteq \delta_5$ is finite).  
- We will be particularly interested in countable Δ-systems. Let $(p_n : n \in \omega)$ be such a Δ-system with root $V$ following $h$, and assume that $D = (D_\alpha : \alpha \in \omega)$ is a sequence such that $\alpha \geq \nu \cap S^4$ and each $D_\alpha$ is a $P_\alpha$-name of an ultrafilter on $\omega$. Then we define the $\lim_{\bar{\rho}} \bar{\rho}$ to be the following function with domain $V$:  
  - If $\beta \in \nu \cap S^0$, then $\lim_{\bar{\rho}} \bar{\rho}(\beta)$ is the common value of all $p_\alpha(\beta)$. (Recall that this value is already determined by the groundrail $h$.)  
  - If $\alpha \in \nu \cap S^4$, then $\lim_{\bar{\rho}} \bar{\rho}(\alpha)$ is (forced by $P_\alpha^5$ to be) $\lim_{D_\alpha}(p_\alpha(\alpha))_{\alpha \in \omega}$.  

Note that in general $\lim_{\bar{\rho}} \bar{\rho}$ will not be a condition in $P_\alpha^5$: For $\alpha \in S^4 \cap \nu$, the object $\lim_{\bar{\rho}} \bar{\rho}(\alpha)$ will be forced to be in the eventually different forcing $E$, but not necessarily in the partial eventually different forcing $Q_\alpha \subseteq E$.  

Also note the following: If $\bar{\rho}$ is a countable Δ-system, and $\alpha \in \nu \cap S^4$, then $(p_\alpha(\alpha))_{\alpha \in \omega}$ is a ground-model-code-sequence (see Definition 1.11(4)). This follows trivially from the definition of “$\lim_{\bar{\rho}} \bar{\rho}$” and the fact that $\bar{\rho}$ is in $V$.  

Recall that we assume all of the parameters defining $P_\alpha^5 = (P_\alpha, Q_\alpha)_{\alpha \in \delta_5}$ to be fixed, apart from $(u_\alpha)_{\alpha \in S^4}$. Once we fix $u_\alpha$ for $\alpha \in S^4 \cap \nu$, we know $P_\alpha$.

Lemma/Construction 1.30. We can construct by induction on $\alpha \in \delta_5$ the sequences $(D_\alpha^5)_{\alpha \in S^4}$ and, if $\alpha \in S^3$, also $u_\alpha$, such that:  
(a) Each $D_\alpha^5$ is a $P_\alpha$-name of a nonprincipal ultrafilter extending $\bigcup_{\beta < \alpha} D_\beta^5$.
(b) For each countable Δ-system $\bar{\rho}$ in $P_\alpha$ which follows the groundrail $h_\alpha^5 \in H^*$: 
$$\lim_{D_\alpha^5} \bar{\rho} \in P_\alpha \cdots$$  
(c) ... and forces that $A_\alpha := \{n \in \omega : p_n \in G_\alpha\}$ is in $D_\alpha^5$.
(d) (If $\alpha \in S^3$) $u_\alpha \subseteq \alpha$, $|u_\alpha| < \lambda_\delta$, and for all ground-model-code-sequences $\alpha$ for elements of $Q_\alpha$, the $D_\alpha^5$-limit is forced to be in $Q_\alpha$ as well (for all $\epsilon \in \chi$).

Actually, the set of $u_\alpha$ satisfying this is an $\omega_1$-club set in $[\alpha]^{\lambda_\delta \cdot 11}$.

Proof. (b) for a limit: The root of a Δ-system is finite and therefore below some $\beta < \alpha$, so the limit exists (by induction) already in $P_\beta$.

(a+c) for a limit: It is enough to show, for each $\epsilon \in \chi$, that $P_\alpha$ forces that the following generates a proper filter (i.e., any finite intersection of elements of this set is nonempty): 
$$\bigcup_{\beta < \alpha} D_\beta^5 \cup \{A_\alpha : \bar{\rho} \in \text{a countable Δ-system following } h_\alpha^5 \text{ and } \lim_{D_\alpha^5} \bar{\rho} \in G_\alpha\}.$$  

(Then we let $D_\alpha^5$ be any ultrafilter extending this set.)

So assume towards a contradiction that $q \in P_\alpha$ forces that $A \cap A_\beta \cap \cdots \cap A_{\beta-1} = \emptyset$, where $A \subseteq D_\beta^5$ for some $\beta_0 < \alpha$ (we can assume $\beta_0$ is already decided in $V$) and $\bar{\rho}$ as above with $\alpha \leq \lim_{D_\alpha^5} \bar{\rho}$ for $i < n$. Let $\beta_1 < \alpha$ be the maximum of the union of the roots of the $\bar{\rho}^i$, and set $\beta_2 := \max(\text{supp}(q))$ and $\gamma := \max(\beta_0, \beta_1, \beta_2) + 1$. By the induction hypothesis, $q$ forces $A' := A \cap \bigcap_{\gamma \in \alpha} A_\gamma \cap A_\gamma^5 \in D_\gamma^5$ (as $\lim_{D_\gamma^5} \bar{\rho}^\gamma$). Since the root lies below $\gamma$). As $A'$ is a $P_\gamma$-name, we can find $q' \leq q$ in $P_\gamma$ and $\epsilon' \in \omega$ such that $q' \Vdash \epsilon' \in A'$. We now find $q'' \leq q'$ in $P_\gamma$ by defining $q''(\bar{\rho})$ for each element $\bar{\rho}$ of the finite

---

10See Definition 1.11(4).
11I.e., for each $u^* \in [\alpha]^{\lambda_\delta}$ there is a $u^* \supseteq u^*$ satisfying (d), and if $(u^*)_{\epsilon \in \omega}$ is an increasing sequence of sets satisfying (d), then the limit $u^* := \bigcup_{\epsilon \in \omega} u^*$ satisfies (d) as well.
set $\bigcup_{i<n} \text{supp} (p^*_i) \setminus \gamma$: For such $\beta$ in $S^0$, the guardrail gives a specific value $h^*_\beta \in Q_\beta$, which we use for $q''(\beta)$ as well. For $\beta \in S^4$, all conditions $p^*_i(\beta)$ (where defined) have the same stem and width $h^*_\beta(\beta)$; hence there is a common extension $q''(\beta)$.

Clearly $q''$ forces that $\ell'$ is in the allegedly empty set, the desired contradiction.

(b) for $\alpha = \gamma + 1$ successor: Assume the nontrivial case, $\gamma \in S^4$: Write the $\Delta$-system as $(p_i, q_i)_{i<n}$ with $(p_i, q_i) \in P_\gamma \ast Q_\gamma$. As noted above, $(q_n)_{n \in \omega}$ is a ground-model-code-sequence, and by induction (d) holds for $w_i$. So it is forced that the $D^*_\beta$-limit of the $q_n$ is in $Q_\gamma$. Again by induction, the limit $p^*$ of the $p_n$ exists as well; and $(p^*, q^*)$ is the required limit.

(a+c) for $\alpha = \gamma + 1$ successor: We again have to show that $P_\alpha$ forces that the following is a filter base, for each $\epsilon \in \gamma$:

$$D^*_\epsilon \cup \{ A_\beta : \beta \in \text{a countable } \Delta \text{-system following } h^*_\epsilon \text{ and } \lim_{i<n} p^*_i(\beta) \in G_\alpha \}.$$  

As above, assume that $q$ forces $A \cap A^*_\beta \cap \cdots \cap A^*_{\beta^i} = \emptyset$.

We can assume that $q \Vdash \gamma$ forces that $q(\gamma)$ is stronger than the limit of all $p(\gamma)$ (for $i < n$).

Thus, by Fact 1.25, each $B_i := \{ \ell' \in \omega : q(\gamma) \text{ compatible with } p_i(\gamma) \}$ is forced to be in $D^*_\epsilon$.

By induction, $q \Vdash \gamma$ forces that $A' := A \cap \bigcap_{i<n} A^*_{\beta^i} \in D^*_\epsilon$, and therefore also forces that $B' := A' \cap \bigcap_{i<n} B_i$ is in the ultrafilter and in particular nonempty. Work in the $P_\gamma$-extension by some generic filter containing $q \Vdash \gamma$. Fix some $\ell' \in B'$. By the definition of $B_i$, $q(\gamma)$ is compatible with $p_i(\gamma)$ for $i < n$. According to Fact 1.23 there is a common lower bound $q''$.

$$q \Vdash \gamma \upharpoonright P \ast q'' \Vdash Q, \ell' \in A^*_\beta. \text{ I.e., } q \Vdash \gamma \ast q'' \leq q \text{ forces that } \ell' \text{ is an element of the allegedly empty set.}$$

(d) For any $w \subseteq \alpha$, let $Q^{w'}$ be the ($P_\alpha$-name for) the partial eventually different forcing defined using $w$. Start with some $u^0 \subseteq \alpha$ of size $<\lambda_4$. There are $|u^0|^{|\omega|}$ many ground-model sequences in $Q^{u^0}$. For any $\epsilon$ and any such sequence, the $D^*_\epsilon$-limit is a real; so we can extend $u^0$ by a countable set to some $u'$ such that $Q^{u'}$ contains the limit. We can do that for all $\epsilon \in \gamma$ and all sequences, resulting in some $u^1 \supseteq u^0$ still of size $<\lambda_4$. We iterate this construction and get $u'$ for $i \leq \omega_1$, taking the unions at limits. Then $w^*_u := u^{|\omega|}_u$ is as required, as $Q_u := Q^{u^*_u} = \bigcup_{i<n} Q^{u^i}$.

So this proof actually shows that the set of $w_u$ with the desired property is an $\omega_1$-club.

We now carry out the construction of this lemma, getting a forcing notion $\mathbb{P}^5$ satisfying the following:

**Lemma 1.31.** $\text{LCU}_\gamma (\mathbb{P}^5, \kappa)$ for $\kappa \in [\lambda_3, \lambda_5]$, witnessed by the sequence $(c_\alpha)_{\alpha \in \kappa}$ of the first $\kappa$ many Cohen reals.

**Proof.** We want to show that for every $\mathbb{P}^5$-name $y$ there are coboundedly many $\alpha \in \kappa$ such that $\mathbb{P}^5 \Vdash \neg \varphi_\alpha \leq y$.

Assume that $p^*$ forces that there are unboundedly many $\alpha \in \kappa$ with $c_\alpha \leq y$, and enumerate them as $(\alpha_i)_{i \in \omega}$ in increasing order (so in particular $\alpha_i \geq i$). Pick $p_i \leq p^*$ deciding $\alpha_i$ to be some $\beta_i$, and also deciding $n_i$ such that $(\forall m \geq n_i) e_\alpha(m) \leq \gamma(m)$. We can assume that $\beta_i \in \text{dom}(p_i)$. Note that $\beta_i$ is a Cohen position (as $\beta_i < \kappa \leq \lambda_5$), and we can assume that $p_i(\beta_i)$ is a Cohen condition in $V$ (and not just a $P_{\beta_i}$-name for such a condition). By thinning out, we may assume:

- All $n_i$ are equal to some $n^*$. 

\begin{enumerate}
\item \((p_i)_{i \in \kappa}\) forms a \(\Delta\)-system with root \(U\).
\item \(\beta_i \not\in \mathbb{V}\), hence all \(\beta_i\) are distinct.
\quad (For any \(\beta \in \kappa\), at most \(|\beta|\) many \(p_i\) can force \(\alpha_i = \beta\), as \(p_1\) forces that \(\alpha_i \geq i\) for all \(i\).)
\item \(p_i(\beta_i)\) is always the same Cohen condition \(s\), without loss of generality of length \(n^{**} \geq n^*\).
\end{enumerate}

(Otherwise extend \(s\).)

Pick the first \(\omega\) many elements \((p_i)_{i < \omega}\) of this \(\Delta\)-system. Now extend each \(p_i\) to \(p'_{i}\) by extending the Cohen condition \(p_i(\beta_i) = s\) to \(s^{-i}\) (i.e., forcing \(c_{\beta_i}(n^{**}) = i\)). Note that \((p'_{i})_{i < \omega}\) is still a countable \(\Delta\)-system, following some new countable guardrail and some full guardrail \(h_{s}^* \in H^{*}\).

Accordingly, the limit \(\lim_{\delta_{1} \rightarrow \varnothing} \beta^*\) forces that infinitely many of the \(p'_{i}\) are in the generic filter. But each such \(p'_{i}\) forces that \(c_{\beta_i}(n^{**}) = i \leq \gamma(n^{**})\), a contradiction. \(\square\)

1.4. Recovering GCH. For the rest of the paper we will assume the following for the ground model \(V\) (in addition to Assumption 1.12):

**Assumption 1.32.** GCH holds.

(Note that this is incompatible with Assumption 1.20.)

Recall that all parameters used to define \(\mathbb{P}^2\) are fixed, apart from \(\bar{w} = (w_{a})_{a \in S^1}\).

**Lemma 1.33.** We can choose \(\bar{w}\) such that \(\text{LCU}_3(\mathbb{P}^2, \kappa)\) holds for all regular \(\kappa \in [\lambda_3, \lambda_3^*]\).

For the proof, we will use the following easy observation:

**Lemma 1.34.** Assume \(\chi\) is a cardinal and \(B\) a set and \(X^0 \in [B]^\chi\), \(\mathbb{R}\) is a \(\chi^+\)-cc forcing notion, and \(C\) is an \(\mathbb{R}\)-name such that the empty condition forces that \(C\) is an \(\omega_1\)-club subset of \([B]^\chi\). Then there is a set \(X \supseteq X^0\) (in the ground model) such that the empty condition forces \(X \in C\).

**Proof.** By induction, choose (in the ground model) sequences \(X^a, \bar{X}^a\) for \(a < \omega_1\) such that \(X^a = [B]^\chi\), the sequence of the \(X^a\) is increasing with \(\alpha, \bar{X}^a\) is an \(\mathbb{R}\)-name, and the empty condition forces: “\(\bar{X}^a\) is in \(C\) and is a \(\omega_1\)-cc subset of \(X^a\); and the sequence of the \(\bar{X}^a\) is increasing (not necessarily continuous).” Moreover, the empty condition forces \(\bar{X}^a \subseteq X^{a+1}\). (In a limit step \(\gamma\), we set \(X^\gamma = \bigcup_{a < \gamma} X^a\), and in a successor step \(a+1\) we use \(\chi^+\)-cc to cover the name \(\bar{X}^a\).) Then \(X = \bigcup_{a \in \omega_1} X^a\) is as required. \(\square\)

**Proof of Lemma 1.33.** Let \(\mathbb{R}\) be a \(<\chi\)-closed \(\chi^+\)-cc p.o. that forces \(2^\chi = \lambda_3\).

In the \(\mathbb{R}\)-extension \(V^*\), Assumption 1.20 holds; and Assumption 1.12 still holds for the fixed parameters.\(^{12}\)

So in \(V^*\), we can perform the inductive Construction 1.30, where now “ground model” refers to \(V^*\), not \(V\) (e.g., when we talk about determined positions, or ground-model-code-sequences, etc.). Actually, we can construct in \(V\) the following, by induction on \(a \in \delta,\) and starting with some cofinal \(\bar{\omega}_{\text{initial}} = (\bar{\omega}_{a}^\text{initial})_{a \in S^1}\) in \(V\):

\begin{itemize}
\item An \(\mathbb{R}\)-name \((D^a_{a})_{a \in S^1}\) (forced to be constructed) according to 1.30(a,b,c).
\item If \(a \in S^1\), some \(w_{a} \supseteq \bar{\omega}_{a}^\text{initial}\) in \(V\) such that \(R\) forces \(\bar{\omega}_{a}\) satisfies 1.30(d).
\end{itemize}

(We can do this by Lemma 1.34, as the set of potential \(w_{a}\)’s is an \(\omega_1\)-club subset of \([\delta]\)^{<\lambda_3^+}\).

\(^{12}\)In particular, \((w_{a})_{a \in S^1}\) is still cofinal in \([\delta_1]^{<\chi}\): For \(i = 1, 2\), the forcing \(\mathbb{R}\) doesn’t add any new elements of \([\delta_i]^{<\chi}\) as \(R\) is \(\lambda_i\)-closed; for \(i = 3\) any new subset of \(\delta_3\) of size \(\theta < \lambda_3\) is contained in a ground model set of size at most \(\theta \times \chi < \lambda_3\), as \(R\) is \(\chi^+\)-cc.
Assuming GCH and given Theorem 1.35. 

\[ \mathcal{P}^5 \] is basically the same as \( \mathbb{P}^5 \). More formally:

In the \( \mathbb{R} \)-extension \( V^* \), \( \mathbb{P}^5 = (P_\alpha, Q_\alpha)_{\alpha < \delta_5} \) (the iteration constructed in \( V \)) is canonically densely embedded into \( \mathbb{P}^{s, 5} = (P_\alpha, Q_\alpha)_{\alpha < \delta_5} \) (the iteration constructed in \( V^* \) using the same parameters).

Proof: By induction, we show (in the \( \mathbb{R} \)-extension) that \( P^\alpha \) forces that \( Q^\alpha \) (evaluated by the \( P^\alpha \)-generic) is equal to \( Q_\alpha \) (evaluated by the induced \( P_\alpha \)-generic, as per induction hypothesis): Every element of \( Q^\alpha \) is a Borel function (which already exists in \( V \)) applied to the generics at a countable sequence of indices in \( w_\alpha \) (which also already exists in \( V \)).

This implies:

In \( V \), \( \text{LCU}_1(\mathbb{P}^5, \kappa) \) holds for all \( \kappa \in [\lambda_3, \lambda_5] \), witnessed by the first \( \kappa \) many Cohen reals.

Proof: Let \( y \) be a \( \mathbb{P}^5 \)-name of a real. In \( V^* \), we can interpret \( y \) as \( \mathbb{P}^{s, 5} \)-name, and as \( \text{LCU}_1(\mathbb{P}^{s, 5}, \kappa) \) holds, we get (\( \exists \alpha \in \kappa \))(\( \forall \beta \in \kappa \)) \( \mathbb{P}^{s, 5} \models c_\beta \not\in y \), where \( c_\beta \) is the Cohen added at \( \beta \). As \( \beta < \kappa \), there is in \( V \) an upper bound \( \alpha^* < \kappa \) for the possible values of \( \alpha \). For any \( \beta \in \kappa \setminus \alpha^* \), we have (in \( V \)) \( \mathbb{P}^5 \models c_\beta \not\in y \) (by absoluteness).

To summarize:

**Theorem 1.35.** Assuming GCH and given \( \lambda_i \) as in Assumption 1.12, we can find parameters\(^{13}\) such that the FS ccc iteration \( \mathbb{P}^5 \) as defined in 1.13 satisfies, for \( i = 1, 2, 3, 4 \):

- \( \text{LCU}_1(\mathbb{P}^5, \kappa) \) holds for any regular cardinal \( \kappa \) in \([\lambda_i, \lambda_5]\).
- \( \text{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5) \) holds.

So in particular \( \mathbb{P}^5 \) forces \( \text{add}(\mathcal{N}) = \lambda_1, \text{cov}(\mathcal{N}) = \lambda_2, b = \lambda_3, \text{non}(\mathcal{M}) = \lambda_4 \) and \( \text{cov}(\mathcal{M}) = d = \text{non}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \lambda_5 = 2^{\aleph_0} \).

For the rest of the paper we fix these parameters and thus the forcing \( \mathbb{P}^5 \).

2. **BOOLEAN ULTRAPOWERS**

In Subsections 2.1 and 2.2 we describe how to get an elementary embedding (which we call a BUP-embedding) \( j : V \rightarrow M \) with \( \text{cr}(j) = \kappa \) and \( \text{cf}(j(\kappa)) = |j(\kappa)| = \theta \), assuming \( \kappa \) is strongly compact and \( \theta > \kappa \) is a regular cardinal with \( \theta^\kappa = \theta \).

In Subsections 2.3 and 2.4 we show how to use such embeddings to transform a ccc forcing \( P \) to \( j(P) \) while preserving some of the values forced to the entries of Cichoń’s diagram (and changing others).

2.1. **Boolean ultrapowers.** Boolean ultrapowers generalize ordinary ultrapowers by using arbitrary Boolean algebras instead of the power set algebra.

We assume that \( \kappa \) is strongly compact and that \( B \) is a \( \kappa \)-distributive, \( \kappa^+ \)-cc, atomless complete Boolean algebra. Then every \( \kappa \)-complete filter in \( B \) can be extended to a \( \kappa \)-complete ultrafilter \( U \).\(^{14}\) Also, there is a maximal antichain \( A_0 \) in \( B \) of size \( \kappa \) such that \( A_0 \cap U = \emptyset \) (i.e., \( U \) is not \( \kappa \)-complete).\(^{15}\)

For now, fix some \( \kappa \)-complete ultrafilter \( U \).

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\(^{13}\) i.e., we set \( \delta_1 = \lambda_3 + \lambda_5 \), and find \( (S^\gamma)_{\gamma \in \alpha} \) and \( w = (w_\gamma)_{\gamma \in \delta_5} \).

\(^{14}\) For this, neither \( \kappa^+ \)-cc nor atomless is required, and \( \kappa \)-complete is sufficient. The proof is straightforward; the first proof that we are aware of has been published in [KT64].

\(^{15}\) Proof: Let \( A \) be a maximal antichain in the open dense set \( B \setminus U \); by \( \kappa^+ \)-cc \( |A| \leq \kappa \). And \( A \) cannot have size \( \kappa \), as otherwise it would meet the \( \kappa \)-complete \( U \).
The Boolean algebra $B$ can be used as forcing notion. As usual, $V$ (or: the ground model) denotes the universe we “start with”. In the following, we will not actually force with $B$ (and in this subsection and the following subsection, we will not force with anything, we always remain in $V$); but we still use forcing notation. In particular, we call the usual $B$-names “forcing names”.

A BUP-name (or: labeled antichain) $x$ is a function $A \to V$ whose domain is a maximal antichain of $B$. We may write $A(x)$ to denote $A$.

Each BUP-name corresponds to a forcing name\(^\text{16}\) for an element of $V$. We will identify the BUP-name and the corresponding forcing name. In turn, every forcing name $\tau$ for an element of $V$ has a forcing-equivalent BUP-name. In particular there is a standard BUP-name $\delta$ for each $\tau \in V$.

We can calculate, for two BUP-names $x$ and $y$, the Boolean value $[x = y]$. We call $x$ and $y$ equivalent, if $[x = y] \in U$ (the $\kappa$-complete ultrafilter fixed above).

For example, any two standard BUP-names for the same $\tau \in V$ trivially are equivalent (as $1_B \in U$). So we can speak (modulo equivalence) of the standard BUP-name for $\tau$.

The Boolean ultrapower $M^-$ consists of the equivalence classes $[x]$ of BUP-names $x$; and we define $[x] \in^\equiv [y]$ by $[x \in y] \in U$. We are interested in the $\in$-structure $(M^-, \in^-)$. We let $j^- : V \to M^-$ map $\tau$ to $[\delta]$.

Given BUP-names $x_1, \ldots, x_n$ and an $\in$-formula $\varphi$, the truth value $[\varphi^V(x_1, \ldots, x_n)]$ is well defined (it is the weakest element of $B$ forcing that in the ground model $\varphi(x_1, \ldots, x_n)$ holds, which makes sense as $x_1, \ldots, x_n$ are guaranteed to be in the ground model).

A straightforward induction (which can be found in [KTT18, Sec. 2]) shows:

- Lö"{s}’s theorem: $(M^-, \in^-) \vDash \varphi([x_1], \ldots, [x_n])$ iff $[\varphi^V(x_1, \ldots, x_n)] \in U$.
- $j^- : (V, \in) \to (M^-, \in^-)$ is an elementary embedding.
- In particular, $(M^-, \in^-)$ is a ZFC model.

As $U$ is $\sigma$-complete, $(M^-, \in^-)$ is wellfounded. So we let $M$ be the transitive collapse of $(M^-, \in^-)$, and let $j : V \to M$ be the composition of $j^-$ with the collapse. We denote the collapse of $[x]$ by $x^U$. So in particular $\delta^U = j(\delta)$.

Facts 2.1.

- $M \vDash \varphi([x_1^U], \ldots, [x_n^U])$ iff $[\varphi^V(x_1, \ldots, x_n)] \in U$. In particular, $j : V \to M$ is an elementary embedding.
- If $|Y| < \kappa$, then $j(Y) = j^Y$. In particular, $j$ restricted to $\kappa$ is the identity. $M$ is closed under $<\kappa$-sequences.
- $j(\kappa) \neq \kappa$, i.e., $\kappa = \text{cf}(j)$.

As we have already mentioned, an arbitrary forcing name for an element of $V$ has a forcing-equivalent BUP-name, i.e., a maximal antichain labeled with elements of $V$. If $\tau$ is a forcing name for an element of $Y$ ($Y \in V$), then without loss of generality $\tau$ corresponds to a maximal antichain labeled with elements of $Y$. We call such an object $y$ a “BUP-name for an element of $j(Y)$” (and not “for an element of $Y$”, for the obvious reason: unlike in the case of a forcing extension, $y^U$ is generally not in $Y$, but, by definition of $\in^-$, it is in $j(Y)$).

Lemma 2.2. If the partial order $(S, \leq)$ is $\leq \kappa$-directed, then $j^\kappa S$ is cofinal in $j(S)$.

Proof. Let $x^U$ be some element of $j(S)$; without loss of generality we can assume that $x$ is a labeled antichain which only uses elements of $S$ as labels. The size of the antichain is at most $\kappa$, so all labels have some common upper bound $s_0$. Then $[x \leq s_0] \in 1_B$, and thus in $U$; so $(M^-, \in^-) \vDash [x] \leq s_0$, i.e., $j(s_0) \geq x^U$ as required.

\(^{16}\)more specifically, to the forcing name $\{\langle x(a), a \rangle : a \in A(x)\}$.
For later reference, let us summarize what we know about $j$ in the form of a definition:

**Definition 2.3.** A BUP-embedding is an elementary embedding $j : V \rightarrow M$ ($M$ transitive) with critical point $\kappa$, such that $M$ is $\prec\kappa$-closed and such that $j''(S)$ is cofinal in $j(S)$ for every $\leq\kappa$-directed partial order $S$.

So the embedding $j$ defined as above for a $\kappa$-distributive, $\kappa^+$-cc atomless complete Boolean algebra and a $\kappa$-complete ultrafilter $U$ is a BUP-embedding.

**Lemma 2.4.** Let $j$ be a BUP-embedding with $\text{cr}(j) = \kappa$.

- If $|A| < \kappa$, then $j''(A) = j(A)$.
- If $S$ is a $\prec\lambda$-directed partial order for some regular $\lambda < \kappa$, then $j(S)$ is $\prec\lambda$-directed.
- If $\text{cf}(\alpha) \neq \kappa$, then $j''(\alpha)$ is cofinal in $j(\alpha)$, so in particular $j''(\alpha) = \text{cf}(\alpha)$.

**Proof.** For the second item, use that $M$ believes that $j(S)$ is $\prec\lambda$-directed and that $M$ is $\prec\kappa$-closed. For the last item, assume $\text{cf}(\alpha) = \lambda \neq \kappa$, witnessed by some strictly increasing cofinal function $f : \lambda \rightarrow \alpha$. If $\lambda < \kappa$, then $M$ thinks that $j(f)$ is strictly increasing cofinal from $j(\lambda) = \lambda$ to $j(\alpha)$, which is absolute. If $\lambda > \kappa$, then $\alpha$ is a $\leq \kappa$-directed (linear) order, so $j''(\alpha)$ is cofinal in $j(\alpha)$. So $j''(f)$, i.e. $(j(f(\zeta)))_{\zeta \in \lambda}$, witnesses that $\text{cf}(\lambda) = \text{cf}(\alpha) = \text{cf}(j(\alpha))$, and $\text{cf}(\alpha) = \text{cf}(\lambda) = \lambda$ (as these orders are isomorphic). $\square$

**2.2. The algebra and the filter.** For a strongly compact cardinal we can get large $\text{cf}(j(\kappa))$:

**Lemma 2.5.** Let $\kappa$ be strongly compact, $\theta > \kappa$ and $\text{cf}(\theta) > \kappa$. Then there is a BUP-embedding $j$ with $\text{cr}(j) = \kappa$ such that

1. $\text{cf}(j(\kappa)) = \text{cf}(\theta)$ and $j(\kappa) \geq \theta$.
2. $|j(\mu)| \leq \max(\mu, \theta)^{\kappa}$ for any $\mu$.
3. In particular, if $\theta^\kappa = \theta$ and $\kappa \leq \mu \leq \theta$ then $|j(\mu)| = \theta$.

We will use this in the following form:

**Definition 2.6.** A “BUP-embedding from $\kappa$ to $\theta^\kappa$” is a BUP-embedding $j$ with critical point $\kappa$ such that $\text{cf}(j(\kappa)) = |j(\kappa)| = \theta$ (in particular $\kappa$ and $\theta$ are regular).

The lemma immediately implies:

**Corollary 2.7.** Assume $\kappa$ is strongly compact and $\theta > \kappa$ is a regular cardinal such that $\theta^\kappa = \theta$. Then there is a BUP-embedding $j$ from $\kappa$ to $\theta$. (And $|j(\mu)| = \theta$ whenever $\kappa \leq \mu \leq \theta$.)

**Proof of Lemma 2.5.** Let $B$ be the complete Boolean algebra generated by the forcing notion $P_{\kappa, \theta}$ consisting of partial functions from $\theta$ to $\kappa$ with domain of size $\prec\kappa$, ordered by extension. Clearly $B$ is $\prec\kappa$-distributive (as $P_{\kappa, \theta}$ is even $\prec\kappa$-closed) and $\kappa^+$-cc.

The forcing adds a canonical generic function $f^* : \theta \rightarrow \kappa$. So for each $\delta \in \theta$, $f^*(\delta)$ is a forcing name for an element of $\kappa$, and thus a BUP-name for an element of $j(\kappa)$.

Let $x$ be some other BUP-name for an element of $j(\kappa)$, i.e., an antichain $A$ of size $\kappa$ labeled with elements of $\kappa$. As $P_{\kappa, \theta}$ is dense in $B \setminus \{0_B\}$, we can assume that $A \subseteq P_{\kappa, \theta}$. Let $\delta \in \theta$ be bigger than the supremum of the domain of $a$ for each $a \in A$. We call such a pair $(x, \delta)$ “suitable”, and set $b_{x, \delta} := \{f^*(\delta) > x\}$. We claim that these elements generate a $\kappa$-complete filter. To see this, fix suitable pairs $(x_i, \delta_i)$ for $i < \mu < \kappa$; we have to show that $\bigwedge_{i < \mu} b_{x_i, \delta_i} \neq \emptyset$. Enumerate $\{\delta_i : i \in \mu\}$ increasing (and without repetitions) as $\delta^\ell$ for $\ell < \mu$. Set $A_\ell = \{i : \delta_i = \delta^\ell\}$. Given $q_\ell$, define $q_{\ell+1} \in P_{\kappa, \theta}$ as follows: $q_{\ell+1} \leq q_\ell$;

$\delta^\ell \in \text{supp}(q_{\ell+1}) \subseteq \delta^\ell \cup \{\delta_i \}$; and $q_{\ell+1} \upharpoonright \delta^\ell$ decides all $i \in A_\ell$ the values of $x_i$ to be...
some \(a_i\); and \(q_{\ell+1}(\delta') = \sup_{i \in A_\ell}(a_i) + 1\). This ensures that \(q_{\ell+1}\) is stronger than \(b_{\kappa, \delta}\) for \(i \in A_\ell\). For \(\ell \leq \gamma\) limit, let \(q_\ell\) be the union of \(\{q_k : k < \ell\}\). Then \(q_\ell\) is stronger than each \(b_{\kappa, \delta}\).

As \(\kappa\) is strongly compact, we can extend the \(\kappa\)-complete filter generated by all \(b_{\kappa, \delta}\) to a \(\kappa\)-complete ultrafilter \(U\). Then the sequence \(f^*(\delta)_{\delta \in \theta}\) is strictly increasing (as \((f^*(\delta), \delta')\) is suitable for all \(\delta < \delta'\) and cofinal in \(j(\kappa)\) (as we have just seen); so \(\text{cf}(j(\kappa)) = \text{cf}(\theta)\) and \(j(\kappa) \geq \theta\).

To get an upper bound for \(j(\mu)\) for any cardinal \(\mu\), we count all possible BUP-names for elements of \(j(\mu)\). As we can assume that the antichains are subsets of \(P_{\kappa, \theta}\), which has size \(\theta^{<\kappa}\), we get the upper bound \(|j(\mu)| \leq [\theta^{<\kappa}]^\ast \times \mu^\kappa = \max(\theta, \mu)^\kappa\). \(\square\)

2.3. The ultrapower of a forcing notion. We now investigate the relation of a forcing notion \(P \in V\) and its image \(j(P) \in M\), which we use as forcing notion over \(V\). (Think of \(P\) as being one of the forcings of Section 1; it has no relation with the Boolean algebra \(B\) used to construct \(j\).)

Note that as \(j(P) \in M\) and \(M\) is transitive, every \(j(P)\)-generic filter \(G\) over \(V\) is trivially generic over \(M\) as well, and we will use absoluteness between \(M[G]\) and \(V[G]\) to prove various properties of \(j(P)\).

Lemma 2.8. Let \(j : V \rightarrow M\) be elementary, \(M\) transitive and \(<\kappa\)-closed with \(\text{cr}(j) = \kappa\).

Assume that \(P\) is \(v\)-cc for some \(v < \kappa\).

(1) \(j(P)\) is \(v\)-cc.
(2) If \(\tau\) is (in \(V\)) a \(j(P)\)-name for an element of \(M[G]\), then there is a \(j(P)\)-name \(\sigma\) in \(M\) such that the empty condition forces \(\tau = \sigma\).
(3) In particular, every \(j(P)\)-name for a real, a Borel-code, a countable sequence of reals, etc., is in \(M\) (more formally: has an equivalent name in \(M\)).
(4) \(M[G]\) is \(<\kappa\)-closed in \(V[G]\).
(5) If \(\exists \lambda < \kappa\) and \(P\) forces \(2^\lambda = \lambda\), then \(j(P)\) forces \(2^\lambda = |j(\lambda)|\).
(6) \(j''P\), which is isomorphic to \(P\) via \(j\), is a complete subforcing of \(j(P)\).

Proof. (1): If \(A \subseteq j(P)\) has size \(v\), then \(A \in M\), and by elementarity \(M\) thinks that \(A\) is not an antichain, which is absolute.

(2): \(\tau\) corresponds to \((A, f)\) where \(A \subseteq j(P)\) is a maximal antichain and \(f : A \rightarrow M\) maps \(a\) to a \(j(P)\)-name in \(M\). As \(j(P)\) is \(v\)-cc and \(M\) \(<\kappa\)-closed, \((A, f)\) is in \(M\) and we can interpret in \(M\) \((A, f)\) as a \(j(P)\)-name \(\sigma\).

This immediately implies (3) and (4): Given a \(j(P)\)-name \(\tau\) for a \(\zeta\)-sequence of elements of \(M[G]\), \(\zeta < \kappa\), we can interpret \(\tau\) as a \(\zeta\)-sequence of names \((\tau_\zeta)_{\zeta < \zeta}\), and find for each \(\tau_\zeta\) an equivalent \(j(P)\)-name \(\sigma_\zeta\) in \(M\). As \(M\) is \(<\kappa\)-closed, the sequence \((\sigma_\zeta)_{\zeta < \zeta}\) is in \(M\) and defines a \(j(P)\)-name in \(M\) equivalent to \(\tau\).

(And if \(\tau\) is a \(j(P)\)-name for a \(<\kappa\)-sequence in \(M[G]\), we can use the fact that \(\kappa\) is regular and that \(j(\kappa)\) is \(\kappa\)-cc to get a bound \(\zeta < \kappa\) for the length of \(\tau\).)

(5) \(M[G]\) thinks that \(|2^\lambda| = |j(\lambda)|\), and \(2^\lambda \cap M[G] = 2^\lambda \cap M[G]\).
(6): It is clear that \(j''P\) is a compatibility-preserving subforcing of \(j(P)\): \(j(p) \leq j(q)\) in \(j''P\) iff \(p \leq q\) in \(P\) (by definition) iff \(M\) thinks that \(j(p) \leq j(q)\) in \(j(P)\) (by elementarity) iff this holds in \(V\) (by absoluteness); and the same argument works for compatibility instead of \(\leq\). Similarly, assume \(A \subseteq j''P\) is a maximal antichain. By definition, \(B := j^{-1}(A) \subseteq P\) is one as well, and in particular of size \(<v\). Therefore \(j(B) = B\), and by elementarity \(M\) thinks that \(B \subseteq j(P)\) is maximal, which holds in \(V\) by absoluteness. \(\square\)
To round off the picture, let us mention the following fact (which is however, not required for the rest of the paper):

**Lemma 2.9.** If \( P = (P^a, Q^a)_{a < \delta} \) is a finite support (FS) ccc iteration of length \( \delta \), then \( j(P) \) is a FS ccc iteration of length \( j(\delta) \) (more formally: it is canonically equivalent to one).

**Proof.** \( M \) certainly thinks that \( j(P) = (P^a, Q^a)_{a < j(\delta)} \) is a FS iteration of length \( j(\delta) \).

By induction on \( a \) we define the FS ccc iteration \( (P^a, Q^a)_{a < j(\delta)} \) and show that \( P^a \) is a dense subforcing of \( \hat{P}^a \): Assume this is already the case for \( P^a \). \( M \) thinks that \( Q^a \) is a \( P^a \)-name, so we can interpret it as \( \hat{Q}^a \)-name and use it as \( \hat{Q}^a \). Assume that \( (p, q) \) is an element (in \( V \)) of \( \hat{P}^a * \hat{Q}^a \). So \( p \) forces that \( q \) is a name in \( M \); we can strengthen \( p \) to some \( p' \) that decides \( q \) to be the name \( q' \in M \). By induction we can further strengthen \( p' \) to \( p'' \) \( P^{a+1} \), then \( (p'', q') \in P^{a+1} \) is stronger than \( (p, q) \). (At limits there is nothing to do, as we use FS iterations.)

\( j(P) \) is ccc according to Lemma 2.8(1). \( \square \)

### 2.4. Preservation of values of characteristics.

Recall Definition 1.8 of \( \text{LCU} \); and Definitions 1.15 and 1.17 of \( \text{COB} \).

**Lemma 2.10.** Assume\(^\text{17}\) that \( P \) is ccc and that \( j \) is a \( \text{BUP} \)-embedding with critical point \( \kappa \).

1. \( \text{LCU}_j(P, \delta) \) implies \( \text{LCU}_j(j(P), j(\delta)) \).
2. If \( \lambda \not= \kappa \) regular, then \( \text{LCU}_j(P, \lambda) \) implies \( \text{LCU}_j(j(P), \lambda) \).
3. If \( \kappa > \lambda \), then \( \text{COB}_j(j(P), \lambda, j(\kappa)) \); if \( \kappa < \lambda \), then \( \text{COB}_j(j(P), \lambda, j(\kappa)) \).

**Proof.** (1) Let \( \bar{x} = (x_a)_{a < \delta} \) be the sequence of \( P \)-names witnessing \( \text{LCU}_j(P, \delta) \). So \( M \) thinks: For every \( j(P) \)-name \( y \) of a real \( (\exists a \in j(\delta)) (\forall b \in j(\delta) \setminus a) \neg ((j(\bar{x}))_b = R_y) \). This is absolute, so \( j(\bar{x}) \) witnesses \( \text{LCU}_j(j(P), j(\delta)) \).

The second claim follows from the fact that \( \text{LCU}_j(j(P), j(\delta)) \) is equivalent to \( \text{LCU}_j(j(P), \text{cf}(j(\delta))) \) and that \( \text{cf}(j(\lambda)) = \lambda \) for regular \( \lambda \not= \kappa \).

(2) Let \( (S, <) \) and \( \bar{g} \) witness \( \text{COB}_j(P, \lambda, \mu) \). \( M \) thinks that

\[ (*) \quad \text{for each } j(P) \text{-name } f : (\exists s \in j(S))(\forall t \in j(S))(t > s \Rightarrow j(P) \Vdash R_j(j(\bar{g}), j(s))) \]

(\( j(P) \) is a \( P \)-name for a real, then we can assume \( f \in M \), and so we can find \( s \in j(S) \) such that for all \( t > s \), \( M[G] \Vdash f \in R_j(j(\bar{g})) \), which holds in \( V[G] \) as well, as \( R_j \) is absolute.

If \( \lambda < \kappa \), then \( j(\lambda) = \lambda \), and \( j(S) \) is \( \lambda \)-directed in \( M \) and therefore in \( V \) as well, so we get \( \text{COB}_j(P, \lambda, j(\mu)) \).

So assume \( \lambda > \kappa \). We claim that \( j''(S) \) and \( j''(\bar{g}) \) witness \( \text{COB}_j(P, \lambda, \mu) \). \( j''(S) \) is isomorphic to \( S \), so directedness is trivial. Given a \( j(P) \)-name \( f \) of a real, without loss of generality in \( M \), there is in \( M \) a cone with tip \( s \in j(S) \) as in (\( * \)). As \( j''(S) \) is cofinal in \( j(S) \) there is some \( s' \in S \) such that \( j(s') > s \). Then for all \( t > s' \), i.e., \( j(t) > j(s') \), we get \( j(P) \Vdash f \in R_j(j(g_s)) \). (Or, in case \( i = 2 \), \( j(P) \Vdash f \notin R_j(j(g_s)) \).

We list the specific cases that we will use:

**Corollary 2.11.** Let \( j \) be a \( \text{BUP} \) embedding from \( \kappa \) to \( \theta \).

\(^{17}\)For most of the Lemma, the requirements of Lemma 2.8 are sufficient: We use \( \text{ccc} \) only to simplify notation as we do not have to indicate where we calculate cofinalities (in \( V \) or the \( j(P) \) extensions \( V[G] \)); and we need \( \text{BUP} \)-embedding for the last part of (2) only.
Assume GCH and that Theorem 3.1.

3. A Fine Iteration of BUP Embeddings

We now have everything required for the main result:

**Theorem 3.1.** Assume GCH and that $\mathfrak{K}_1 < \kappa_0 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_9$ are regular, $\lambda_5$ a successor of a regular cardinal, $\lambda_i$ not successor of a cardinal with countable cofinality for $i = 1, 2, 4, 5$, and $\kappa_i$, strongly compact for $i = 6, 7, 8, 9$. Then there is a ccc forcing notion $\dot{P}^9$ resulting in:

\[
\text{add}(\mathcal{N}) = \lambda_1 < \text{cov}(\mathcal{N}) = \lambda_2 < b = \lambda_3 < \text{non}(M) = \lambda_4 < \text{add}(M) = \lambda_5 < b = \lambda_6 < \text{non}(\mathcal{M}) = \lambda_7 < \text{cof}(\mathcal{N}) = \lambda_8 < 2^{\aleph_0} = \lambda_9.
\]

**Proof.** For $i = 6, \ldots, 9$, let $j_i$ be a BUP-embedding from $\kappa_i$ to $\lambda_i$, i.e., $\text{cf}(j_i(\kappa_i)) = |j_i(\kappa_i)| = \lambda_i$. (Such an embedding exists according to Corollary 2.7.)

We use $P^5$ of Theorem 1.35, and set $P^{i+1} := j_{i+1}(P^i)$ for $i = 5, 6, 7, 8$. In particular, $P^9 = j_0(j_8(j_7(j_6(P^5))))$.

We enumerate the relevant characteristics of Cichoń’s diagram as $\mathfrak{f}_1, \ldots, \mathfrak{f}_8$ in the desired increasing order as displayed in Figure 1. For $i = 1, \ldots, 4$ (i.e., $\mathfrak{f}_i$ in the left half) we set $i^* = 9 - i$ (so $\mathfrak{f}_{i^*}$ is the dual of $\mathfrak{f}_i$ in the right half).

Recall that according to Lemmas 1.9 and 1.18, $\text{LCU}_i(\lambda_i)$ implies $\mathfrak{f}_i \leq \lambda$ and $\mathfrak{f}_{i^*} \geq \lambda$; and $\text{COB}_i(\lambda_i, \mu)$ implies $\mathfrak{f}_i \geq \lambda$ and $\mathfrak{f}_{i^*} \leq \mu$.

**Claim:** $P^9$ forces $\aleph_0 = \lambda_9$.

**Proof:** By induction on $i = 5, \ldots, 8$ each $P^{i+1}$ forces $2^{\aleph_0} = j_{i+1}(\lambda_i) = \lambda_{i+1}$ (according to Lem. 2.8(5) and Cor. 2.7).

**Claim:** $\text{LCU}_i(P^9, \lambda_i)$ holds for $i = 1, 2, 4$; as well as $\text{LCU}_i(P^9, \lambda_5)$.

**Proof:** The statements hold for $P^5$ by Thm. 1.35 and are preserved by Cor. 2.11(a).

This implies $\mathfrak{g}_i \leq \lambda_i$ for $i = 1, \ldots, 4$; as well as $\mathfrak{g}_5 = \text{cov}(M) \geq \lambda_5$.

**Claim:** $\text{LCU}_i(P^9, \lambda_{i^*})$ holds for $i = 1, 2, 3$.

**Proof:** Note that $\kappa_i^+ < \lambda_i < \kappa_8^+ < \lambda_3$. So $\text{LCU}_i(P^5, \kappa_i)$ holds (Thm. 1.35). This implies $\text{LCU}_i(P^5, \kappa_8)$ for $\ell = 5, \ldots, i^* - 1$ (Cor. 2.11(a)), then $\text{LCU}_i(P^9, \lambda_{i^*})$ for $\ell = i^*$ (Cor. 2.11(b)), and then again $\text{LCU}_i(P^9, \lambda_{i^*})$ for $\ell = i^* + 1, \ldots, 9$ (again Cor. 2.11(a)).

This implies $\mathfrak{g}_{i^*} \geq \lambda_{i^*}$ for $\ell = 6, 7, 8$.

**Claim:** $\text{COB}_i(P^9, \lambda_i, \lambda_{i^*})$ holds for $i = 1, 2, 3, 4$.

**Proof:** $\text{COB}_i(P^5, \lambda_i, \lambda_{i^*})$ holds by Theorem 1.35 and implies $\text{COB}_i(P^9, \lambda_i, \lambda_{i^*})$ for $\ell = 5, \ldots, i^*$ (while $\kappa_8^+ > \lambda_5$) (Cor. 2.11(c)), then $\text{COB}_i(P^9, \lambda_i, \lambda_{i^*})$ for $\ell = i^* + 1, \ldots, 9$ (Cor. 2.11(d)).

This implies $\mathfrak{g}_i \geq \lambda_i$ for $i = 1, \ldots, 4$ as well as $\mathfrak{g}_{i^*} \leq \lambda_{i^*}$ for $\ell = 5, \ldots, 8$.

4. Questions

The result poses some obvious questions. (Since the initial submission of the paper, some of the questions found partial answers which we mention in the following.)

(a) Can we prove the result without using large cardinals?

It would be quite surprising if compact cardinals are needed, but a proof without them will probably be a lot more complicated.

**Partial answers:**
Gitik [Git19] points out that certain extender embeddings are BUP-embeddings, and that a variation of superstrongs is sufficient to construct the BUP-embeddings required in our construction.

As mentioned, we think that the result does not require any large cardinals. The proof in this paper obviously does: Gitik (ibid.) notes that a measurable $\kappa$ with Mitchell order $\geq \kappa^{++}$ is required to get a BUP-embedding from $\kappa$ to some regular $\lambda > \kappa$. More generally, an easy argument given in [GKMS19, Sec. 3.1] (following a deeper observation pointed out by Mildenberger [Mil98]), shows that at least $0^\#$ is required to get a constellation of models of the type used in our proof.\footnote{More specifically, for $j_6 : V \rightarrow M$ and $G$, $j_6(\mathcal{G})$-generic over $V$, we know that a $M[G]$ is a $\kappa$-closed transitive subclass of $V[G]$, cf. Lemma 2.8(4). And we have $M[G] \vDash \text{non}(\mathcal{M}) = j_6(\lambda_4)$ and $V[G] \vDash \text{non}(\mathcal{M}) = \lambda_4 < j_6(\lambda_4) = \lambda_6$, which implies at least $0^\#$ (and probably a measurable).}

In [BCM18] (building on [Mej19a]), a construction for the left half of Cichoń’s diagram is introduced that additionally forces $\text{non}(\mathcal{M}) < 2^{\aleph_0}$. Accordingly, three strongly compact cardinals (or: subcompacts) are sufficient to get the ten different values.

(b) Does the result still hold for other specific values of $\lambda_i$, such as $\lambda_i = \aleph_{i+1}$?

In our construction, the regular cardinals $\lambda_i$ for $i = 4, \ldots, 9$ can be chosen quite arbitrarily (above the compact $\kappa_0$, that is). However, $\aleph_1, \lambda_1, \lambda_2$ and $\lambda_3$ each have to be separated by a compact cardinal (and furthermore $\lambda_3$ has to be a successor of a regular cardinal).

Partial answer: In [GKMS19], it is shown that we can choose the values quite freely. E.g., $\lambda_1 = \aleph_{i+1}$ is possible; as is basically “any choice” of successor cardinals. We also show that we can replace any number of instances of $<$ by $=$.\footnote{In fact, we counted 57 in addition to the 4 that are compatible with FS ccc.}

(c) Are other linear orders between the characteristics of Cichoń’s diagram consistent?

Note that in this paper, we use a FS ccc iteration of length $\delta$ with uncountable cofinality, cf. 2.9, which always results in $\text{non}(\mathcal{M}) \leq \text{cof}(\delta) \leq \text{cov}(\mathcal{M})$. Under these restrictions, there are only four possible assignments. Of course there are a lot more\footnote{\label{footnote1}In fact, we counted 57 in addition to the 4 that are compatible with FS ccc.} possibilities to assign $\lambda_1, \ldots, \lambda_9$ to Cichoń’s diagram in a way that satisfies the known ZFC-provable (in)equalities. Figure 2.8 is an example. Such orders require entirely different methods. (Even to get just the five different values $\aleph_1 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ in this figure turned out to be rather involved [FGKS17, Sec. 11].)

Partial answer: Another of the orders compatible with FS ccc iterations, the one of Figure 2A, is consistent [KST19]. See also [Mej19b]. (A different initial forcing gives

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (N1) {$\aleph_1$};
\node (N2) [below of=N1] {$\lambda_1$};
\node (N3) [below of=N2] {$\lambda_5$};
\node (N4) [below of=N3] {$\lambda_6$};
\node (N5) [right of=N4] {$\lambda_8$};
\node (N6) [above of=N5] {$\lambda_9$};
\node (N7) [below of=N5] {$\lambda_2$};
\node (N8) [below of=N7] {$\lambda_7$};
\node (N9) [below of=N8] {$\lambda_3$};
\node (N10) [below of=N9] {$\lambda_4$};
\node (N11) [right of=N10] {$\lambda_5$};
\node (N12) [above of=N11] {$\lambda_6$};
\node (N13) [below of=N11] {$\lambda_8$};
\node (N14) [below of=N13] {$\lambda_9$};

\draw [->] (N1) -- (N2);
\draw [->] (N2) -- (N3);
\draw [->] (N3) -- (N4);
\draw [->] (N4) -- (N5);
\draw [->] (N5) -- (N6);
\draw [->] (N6) -- (N7);
\draw [->] (N7) -- (N8);
\draw [->] (N8) -- (N9);
\draw [->] (N9) -- (N10);
\draw [->] (N10) -- (N11);
\draw [->] (N11) -- (N12);
\draw [->] (N12) -- (N13);
\draw [->] (N13) -- (N14);
\end{tikzpicture}
\caption{Alternative orderings of the cardinal characteristics.}
\end{figure}
the modified ordering of the left hand side; then the same construction and proof as in this paper gives us the whole diagram.)

(d) Is it consistent that other cardinal characteristics that have been studied,\(^{20}\) in addition to the ones in Cichoń’s diagram, have pairwise different values as well?

Partial answer: In [GKMS19], it is forced that additionally \(\mathfrak{m}_1 < \mathfrak{m} < \mathfrak{p} < \mathfrak{h} < \text{add}(\mathcal{N})\) holds.

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\(^{20}\)The most important ones are described in [Bla10].


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