POWER SET MODULO SMALL, THE SINGULAR OF
UNCOUNTABLE COFINALITY
F1873

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Abstract. Let $\mu$ be singular of uncountable cofinality. If $\mu > 2^{\text{cf}(\mu)}$, we prove
that in $P = ([\mu]^{\mu}, \supseteq)$ as a forcing notion we have a natural complete embedding
of $\text{Levy}(\aleph_0, \mu^+)$ (so $P$ collapses $\mu^+$ to $\aleph_0$) and even $\text{Levy}(\aleph_0, \bigcup_{\text{Ubd}}(\mu))$, well,
when the sup is attained. The “natural” means that the forcing $\langle \{ p \in [\mu]^{\mu} : p \text{ closed} \}, \supseteq \rangle$ is naturally embedded and is equivalent to the Levy algebra. Also
if $P$ fails the $\chi$-c.c. then it collapses $\chi$ to $\aleph_0$ (and the parallel results for the
case $\mu > \aleph_0$ is regular or of countable cofinality).

The 2019 version has more:

(a) instead of just collapsing $\chi$ to $\aleph_0$ we add a generic to $\text{Levy}(\aleph_0, \chi)$
(b) we intend (as promised) to deal also with singular $\mu < 2^{\text{cf}(\mu)}$.

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updated than the one in the mathematical archive.

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§ 0. Introduction

This work tries to confirm the "pcf is effective" thesis.

We may consider the completions of the Boolean Algebras $\mathcal{P}(\mu)/\{u \subseteq \mu : |u| < \mu\} = \mathcal{P}(\mu)/[\mu]^<\mu$. This is equivalent to considering the partial orders $\mathbb{P}_\mu = ([\mu]^\mu, \supseteq)$, viewing them as forcing notions, so actually looking at their completion $\hat{\mathbb{P}}_\mu$, which are complete Boolean Algebras. Recall that forcing notions $\mathbb{P}_1, \mathbb{P}_2$ are equivalent iff their completions are isomorphic Boolean Algebras. The Czech school has investigated them, in particular, (letting $\ell(\mu)$ be 0 if $\text{cf}(\mu) > \aleph_0$ and 1 if $\mu > \text{cf}(\mu) = \aleph_0$, (and $\aleph_0(\mu) = 0$ if $\mu = \aleph_0$) consider the questions:

$(\circ_1)$ is $\hat{\mathbb{P}}_\mu$ isomorphic to the completion of the Levy collapse $\text{Levy}(\aleph_0(\mu), 2^\mu)$?

- (b) which cardinals $\chi$ the forcing notion $\mathbb{P}_\mu$ collapse to $\aleph_0(\mu)$
- (c) is $\mathbb{P}_\mu$ is ($\theta, \chi$)-nowhere dense distributive for $\theta = \aleph_0(\mu)$? this can be phrased as: for some $\mathbb{P}_\mu$-name $\dot{f}$ of a function from $\aleph_0(\mu)$ to $\chi$, for every $p \in \mathbb{P}_\mu$ for some $i < \theta$ the set $\{\alpha < \chi : p \not\Vdash\dot{f}(i) = \alpha\}$ has cardinality $\chi$.

The first, (a) is a full answer, (b) the second seems central for set theories, the last is sufficient if the density is right, to get the first. The case of collapsing seems central (it also implies clause (c) if $\chi$ large enough) so we repeat the summary from Balcar Simon $\text{[BaSi95]}$ of what was known of the collapse of cardinals by $\mathbb{P}_\kappa$, i.e., $(\circ_1)(b)$. Let $\lambda \rightarrow \kappa \mu$ denote the fact that $\lambda$ is collapsed to $\mu$ by $\mathbb{P}_\kappa$.

$\exists_1$ (i) for $\kappa = \aleph_0, 2^{\aleph_0} \rightarrow \kappa \aleph_0$, (but $\mathbb{P}_\kappa$ adds no new sequence of length $< \aleph_0$ so we are done); Balcar Pelant Simon $\text{[BaPS80]}$

(ii) for $\kappa$ uncountable and regular, $\aleph_\kappa \rightarrow \kappa \aleph_0$, (hence $\kappa^+ \rightarrow \kappa \aleph_0$) Balcar Simon $\text{[BaSi88]}$

(iii) for $\kappa$ singular with $\text{cf}(\kappa) = \aleph_0, 2^{\aleph_0} \rightarrow \kappa \aleph_1$, Balcar Simon $\text{[BaSi95]}$

(iv) for $\kappa$ singular with $\text{cf}(\kappa) \neq \aleph_0, \aleph_{\text{cf}(\kappa)} \rightarrow \kappa \aleph_0$, Balcar Simon $\text{[BaSi95]}$.

under additional assumptions for singular cardinals more is known

(v) for $\kappa$ singular with $\text{cf}(\kappa) = \aleph_0$ and $\aleph_0 = 2^\kappa, \aleph_0 \rightarrow \kappa \aleph_1$, Balcar Simon $\text{[BaSi95]}$

(vi) for $\kappa$ singular with $\text{cf}(\kappa) \neq \aleph_0$ and $2^\kappa = \kappa^+, 2^\kappa \rightarrow \kappa \aleph_0$, Balcar Simon $\text{[BaSi95]}$.

Now $\text{[BaSi95]}$ end with the following very reasonable conjecture.

**Conjecture 0.1.** (Balcar and Simon) in ZFC: for a singular cardinal $\kappa$ with countable cofinality, $\kappa^{\aleph_0} \rightarrow \aleph_1$ and for a singular cardinal $\kappa$ with an uncountable cofinality $\kappa^+ \rightarrow \aleph_0$ (here we concentrate on the case $\text{cf}(\mu) > \aleph_0$, see below).

Concerning the other questions they prove

$\exists_2$ (i) Balcar Frank $\text{[BF87]}$:

If $\mu > \text{cf}(\mu) > \aleph_0, 2^{\text{cf}(\mu)} = \text{cf}(\mu)^+$ then $\mathbb{P}_\mu$ is ($\omega; \kappa^+$)-nowhere distributive.
(ii) Balcar Simon [BS97, 5.20, pg.38]:
if \(2^\mu = \mu^+\) and \(2^{cf(\mu)} = cf(\mu)^+\) then \(\mathbb{P}_\mu\) is equivalent to \(\text{Levy}(\mathcal{N}_0, \mu^+)\)
if \(cf(\mu) > \mathcal{N}_0\) and to \(\text{Levy}(\mathcal{N}_1, \mu^+)\) if \(cf(\mu) = \mathcal{N}_0\)

(iii) Balcar Franek [BF87]:
if \(2^\mu = \mu^+, \mu = cf(\mu) > \mathcal{N}_0, J\) a \(\mu\)-complete idea on \(\mu\) and \(J\) nowhere precipitous extending \([\mu]^{<\mu}\) then \(\mathcal{P}(\mu)/J\) is equivalent to \(\text{Levy}(\mathcal{N}_0, \mu^+)\); also the parallel of (ii).

So under G.C.H. the picture was complete; getting clause (ii) of \(\mathcal{E}_2\), and, in ZFC for regular cardinals \(\mu > \mathcal{N}_0\) the picture is reasonable, particularly if we recall that by Baumgartner [Bau76]

\(\mathbb{E}_3\) if \(\kappa = cf(\mu) < \theta = \theta^{<\theta} < \mu < \chi\) and \(V \models \text{G.C.H.}\) for simplicity and \(\mathbb{P}\) is adding \(\chi\)-Cohen subsets to \(\theta\) then
(a) forcing with \(\mathbb{P}\) collapse no cardinal, change no cofinality, adds no new sets of \(< \theta\) ordinals
(b) in \(V^\mathbb{P}\), \([\mu]^\theta, \supseteq\) satisfies the \(\mu^+\)-c.c. where \(\mu_3 = (2^\mu)V\).

Lately Kojman Shelah [KS01] prove the conjecture \(\mathbb{P}_0 \subset \mathbb{P}_1\) for the case when \(\mu > cf(\mu) = \mathcal{N}_0\); moreover

\(\mathbb{E}_4\) (i) if \(\mu > cf(\mu) = \mathcal{N}_0\) then \(\text{Levy}(\mathcal{N}_1, \mu^{\mathcal{N}_0})\) can be completely embeddable into the completion of \(\mathbb{P}_\mu\).

Moreover,

(ii) the embedding is “natural”: \(\text{Levy}(\mathcal{N}_1, \mu^{\mathcal{N}_0})\) is equivalent to \(Q_\mu = \langle \{A \subseteq \mathcal{N} : A \text{ a closed subset of } \mathcal{N} \text{ of cardinality } \mu\}, \supseteq \rangle\).

Here we continue [KS01] in §1, (BS89) in §2 but make it self contained. Naturally we may add to the questions (answered positively for the case \(cf(\kappa) = \mathcal{N}_0\) by [KS01])

\(\otimes_2\) (a) can we strengthen “\(\mathbb{P}_\mu\) collapse \(\chi\) to \(\mathcal{N}_3(\mu)\)” to “\(\text{Levy}(\mathcal{N}_3(\mu), \chi)\) is completely embeddable into \(\mathbb{P}_\mu\) (really \(\mathbb{P}_\mu\))”
(b) can we find natural such embeddings.

We may add that by [BS99] the Baire number of \(\mathcal{V}[\mu]\), the space of all uniform ultrafilters over uncountable \(\mu\) is \(\mathcal{N}_1\), except when \(\mu > cf(\mu) = \mathcal{N}_0\) and in that case it is \(\mathcal{N}_2\) and under some reasonable assumptions. By [KS01] the Baire number of \(\mathcal{V}[\mu]\) is always \(\mathcal{N}_2\) when \(\mu > cf(\mu) = \mathcal{N}_0\).

\(\otimes 0\) § 0(B). The Results.

Here we deal mainly with the case \(\mu > 2^{cf(\mu)}\) (in §1) and prove more on regular \(\mu > \mathcal{N}_0\) (in §2).

Our original aim in this work has been to deal with \(\mu > cf(\mu) > \mathcal{N}_0\), proving the conjecture of Balcar and Simon above (i.e., that \(\mu^+\) is collapsed to \(\mathcal{N}_0\)), first of all when \(2^{cf(\mu)} < \mu\) answering \(\otimes_2(a) + (b)\) using pcf and (replacing \(\mu^+\) by \(pp_{cf(\mu)}(\mu)\)). In fact this seems, at least to me, the best we can reasonably expect. But a posteriori we have more to say.

For \(\mu = \kappa = cf(\mu) > \mathcal{N}_0\), though by the above we know that some cardinal > \(\mu\) is collapsed, (that is \(b_\mu\)) we do not know what occurs up to \(2^\mu\) or when the c.c.
fails. This leads to the following conjecture, (stronger than the Balcar Simon one mentioned above).

Of course, it naturally breaks to cases for according to \( \mu \).

**Conjecture 0.2.** If \( \mu > \aleph_0 \) and \( P_\mu \) does not satisfy the \( \chi \)-c.c., then forcing with \( P_\mu \) collapse \( \chi \) to \( \aleph_{\ell(\mu)} \), see Definition 0.6 below.

Note that

**Observation 0.3.** If conjecture 0.2 holds for \( \mu > \aleph_0 \) then \( P_\mu \) is equivalent to a Levy collapse iff it fails the \( d(P_\mu) \)-c.c. where \( d(P_\mu) \) is the density of \( P_\mu \).

Lastly, we turn to the results; by 1.11, 1.18, 1.19 and 2.7 we have:

**Theorem 0.4.** 1) If \( \mu > \kappa = \text{cf}(\mu) > \aleph_0 \) and \( \mu > 2^\kappa \) then \( Q_\mu \) (a natural complete subforcing of \( P_\mu \), forcing with closed sets) is equivalent to Levy(\( \aleph_0, U_{J^{\text{nc}}}(\mu) \)).

2) Hence if in addition \( U_{J^{\text{nc}}}(\mu) = 2^\mu \), which holds if \( M \) is strong limit then \( P_\mu \) is equivalent to Levy(\( \aleph_0, 2^\mu \)).

**Theorem 0.5.** Conjecture 0.2 holds except possibly when \( \aleph_0 < \text{cf}(\mu) < \mu < 2^{\text{cf}(\mu)} \).

We hope in a subsequent paper to prove the Balcar Simon conjecture fully, i.e., in all cases.

**Definition 0.6.** For \( \mu > \aleph_0 \) we define \( \ell(\mu) \in \{0, 1\} \) by:

\[
\ell(\mu) = 0 \text{ if } \text{cf}(\mu) > \aleph_0 \\
\ell(\mu) = 1 \text{ if } \mu > \text{cf}(\mu) = \aleph_0 \\
\text{ and may add} \\
\ell(\mu) = \alpha \text{ when } \mu = \aleph_0, \aleph_\alpha.
\]

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A posteriori ( – 2019) we add the conclusions concerning

**Conjecture 0.7.** If \( P_\mu \) fails the \( \chi \)-c.c., then forcing by \( P_\mu \) adds a generic for Levy(\( \aleph_{\ell(\mu)}, \chi \)).
1.1 Forcing with closed set is equivalent to the Levy algebra

1.1.1 Definition 1.1. 1) For \( f \in {}^\kappa(\text{Ord}\setminus\{0\}) \) and ideal \( I \) on \( \kappa \) let

\[
U_f(I) = \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\sup \text{Rang}(f)]^{\leq \kappa}
\]

such that for every \( g \leq f \) for some \( u \in \mathcal{P} \)
we have \( \{i < \kappa : g(i) \in u\} \in I^+ \} \).

2) Let \( U_f(\lambda) \) means \( U_f(f) \) where \( f \) is the function with domain \( \text{Dom}(I) \) which is
constantly \( \lambda \).

1.1.2 Hypothesis 1.2.
(a) \( \mu \) is a singular cardinal
(b) \( \kappa = \text{cf}(\mu) > \aleph_0 \) (but try to mention when used).

1.1.3 Definition 1.3. 1) \( \mathbb{P}_\mu \) is the following forcing notion

\[
p \in \mathbb{P}_\mu \iff p \in [\mu]^{\mu}
\]

\[
\mathbb{P}_\mu \models p \leq q \iff p \supseteq q.
\]

2) \( \mathbb{P}_\mu' \) is the forcing notion with the same set of elements and with the partial order

\[
\mathbb{P}_\mu' \models p \leq q \iff |p| \setminus q < \mu.
\]

3) \( \mathbb{Q}_\mu = \mathbb{Q}_0^\mu \) is \( \mathbb{P}_\mu \upharpoonright \{p \in \mathbb{P}_\mu : p \) is closed in the order topology of \( \mu \} \).

1.1.4 Choice/Definition 1.4. 1) Let \( \{\lambda_i : i < \kappa\} \) be an increasing sequence of regular cardinals \( \kappa > \mu \) with limit \( \mu \).

2) Let \( \lambda^-_\kappa = \bigcup\{\lambda_i : i < j\} \).

3) For \( p \in \mathbb{P}_\mu \) let \( a(p) = \{i < \kappa : p \cap [\lambda^-_\kappa, \lambda_i) \neq \emptyset\} \).

4) \( \mathbb{Q}_\mu^0 = \{p \in \mathbb{Q}_\mu : \lambda_i < \kappa \Rightarrow |p \cap \lambda_i| < \lambda_i \) and for each \( i \in a(p) \) the set \( p \cap \lambda_i \setminus \lambda^-_\kappa \) has no last element is closed in its supremum and has cardinality \( |p \cap \lambda^-_\kappa| \} \).

5) For \( p \in \mathbb{Q}_\mu^0 \) let \( \text{ch}_p \in \prod_{i \in a(p)} \lambda_i \) be \( \text{ch}_p(i) = \bigcup\{\alpha + 1 : \alpha < p \cap [\lambda^-_\kappa, \lambda_i) \} \) and

\[
\text{cf}_p = \prod_{i \in a(p)} \lambda_i \text{ be } \text{cf}_p(i) = \text{cf}(\text{ch}_p(i)).
\]

6) \( \mathbb{Q}_\mu^1 = \{p \in \mathbb{Q}_\mu^0 : \text{cf}_p(i) > |p \cap \lambda^-_\kappa | \) for \( i \in a(p) \} \).

7) \( p \in \mathbb{P}_\mu \) is \( \lambda_\kappa \) normal when \( \lambda_\kappa = \{\lambda_i : i \in a(p)\} \) and \( \text{otp}(p \cap [\lambda^-_\kappa, \lambda_i)) = \lambda_i \) for \( i \in a(p) \).

1.1.5 Claim 1.5. 1) \( \mathbb{Q}_\mu^0, \mathbb{Q}_\mu^1, \mathbb{Q}_\mu^2 \) are complete sub-forcing of \( \mathbb{P}_\mu \).

2) For \( \ell = 0, 1, 2 \) and \( p, q \in \mathbb{Q}_\mu^\ell \) we have \( p \vdash_{\mathbb{Q}_\mu^\ell} " q \in \mathbb{G} " \) iff \( |q| \setminus p < \mu \) and similarly for \( \mathbb{P}_\mu \).

3) \( \mathbb{Q}_\mu = \mathbb{Q}_0^\mu, \mathbb{Q}_1^\mu, \mathbb{Q}_\mu^2 \) are equivalent, in fact \( \mathbb{Q}_\mu^2 \) a dense subset of \( \mathbb{Q}_1^\mu \) which is dense in \( \mathbb{Q}_0^\mu \).

Proof. Easy.

Recall
Claim 1.6. 1) $\mathbb{P}_\kappa$ can be completely embedded into $\mathbb{P}_\mu$ (naturally).
2) $Q_\mu$ can be completely embedded into $\mathbb{P}_\mu$ (naturally).
3) $\mathbb{P}_\kappa$ is completely embedded into $Q_\mu$ (naturally).

Proof. 1) Known: just $a \in [\kappa]^\kappa$ can be mapped to $\bigcup(\lambda_i^-, \lambda_i) : i \in a \rangle$.
2) By [KS94, Ch.III, 2.2].
3) Should be clear (again map $A \in [\kappa]^\kappa$ to $\bigcup(\lambda_i^-, \lambda_i) : i \in A \rangle$).

1.4.2
Claim/Definition 1.7. 1) $\lambda_\ast = U_{J_{\mu, \ast}}(\mu)$.
2) $\chi$ is, e.g., $(\_\star(\ast_\mu))^+, \gamma^+$ a well ordering of $\mathcal{H}(\chi)$.
3) $\mathcal{B}$ is an elementary submodel of $(\mathcal{H}(\chi), \in, <_\chi)$ of cardinality $\lambda_\ast$ such that $\lambda_\ast + 1 \subseteq \mathcal{B}$.

Recall
Claim 1.8. Assume $\mu > 2^\kappa$.
1) $\lambda_\ast = \sup\{ \text{pp}_{J_{\mu, i}}(\mu') : \kappa < \mu' \leq \mu, \text{cf}(\mu') = \kappa \} = \sup\{ \text{tcf}(\prod_{i<\kappa} \chi_i^{J_{\mu, i}^{\text{bd}}} : \lambda_i \in \text{Reg} \cap (\kappa, \mu) \} $ and $\prod_{i<\kappa} \chi_i^{J_{\mu, i}^{\text{bd}}} \text{ has true cofinality}.\}
2) For every regular cardinal $\theta \in [\mu, \lambda_\ast]$, for some increasing sequence $\langle \lambda_i^* : i < \kappa \rangle$ of regulars $\in (\kappa, \mu)$ we have $\theta = \text{tcf}(\prod_{i<\kappa} \lambda_i^*, <_{J_{\mu, i}^{\text{bd}}})$.

Proof. 1) Note that $J_{\mu, i}^{\text{bd}}| A \approx J_{\mu}^{\text{bd}}$ if $A \in (J_{\mu, i}^{\text{bd}})^+$, we use this freely. By their definition the second and third terms are equal. Also by the definition the second is smaller or equal to the first.

By [Sh:589, 1.1], the first, $\lambda_\ast = U_{J_{\mu, i}^{\text{bd}}}(\mu)$ is greater than the second number (well it speaks on $T_{J_{\mu, i}^{\text{bd}}}(\mu)$, instead $U_{J_{\mu, i}^{\text{bd}}}(\mu)$ but as $2^\kappa < \mu$ they are the same).
2) As $\lambda_\ast$ is regular by [Sh:589, 1.1] we actually get the stronger conclusion.

1.4.3
Convention 1.9. We fix $\mathcal{C}^\ast$ as in $1.4.4$.

Claim/Definition 1.10. 1) There is $\mathcal{C}^\ast = \{ C_\alpha^\ast : \alpha < \mu \} \in \mathcal{B}$ such that:

(a) $C_\alpha^\ast$ is a subset of $\langle \lambda^-_i, \lambda_i \rangle$ closed in its supremum when $\alpha \in \langle \lambda^-_i, \lambda_i \rangle$
(b) if $i < \kappa$, $\gamma < \lambda_i$, $C$ is a closed subset of $\langle \lambda^-_i, \lambda_i \rangle$ of order type $\langle \gamma \rangle$ then for some $\alpha \in \langle \lambda^-_i, \lambda_i \rangle$, $C_\alpha^\ast \subseteq C$ and $\text{otp}(C_\alpha^\ast) = \gamma$.

2) $Q^2_{\mu, C} = \{ p \in Q^2_\mu : i \in a(p) \text{ then } p \cap [\lambda^-_i, \lambda_i] \in \{ C^\ast_\alpha : \alpha \in \langle \lambda^-_i, \lambda_i \rangle \} \}$ is a dense subset of $Q^1_{\mu, i} \cap Q^2$ hence of $Q_\mu$ as we are fixing $\mathcal{C}^\ast$, we may write $Q^2_{\mu, C}$.
3) For $p \in Q^2_{\mu, C}$ let $\text{cd}_p = \prod_{i<\kappa} \lambda_i$ be such that $\text{cd}_p(i) \in \langle \lambda^-_i, \lambda_i \rangle$ is the minimal $\alpha \in \langle \lambda^-_i, \lambda_i \rangle$ such that $p \cap [\lambda^-_i, \lambda_i] = C^\ast_\alpha$.

Proof. 1) It is enough, for any limit $\delta \in (\lambda^-_i, \lambda_i]$ and regular $\theta, \theta^+ < \text{cf}(\delta)$, to find a family $P_{\delta, \theta}$ of closed subsets of $\lambda^-_i, \delta$ of order type $\theta$ such that any club of $\delta$ contains (at least) one of them. This holds by guessing clubs, see [Sh:599, Ch.III, §2]; in fact also singular $\theta$ is O.K.
2) By the definitions.

Claim 1.11. 1) If $\mu > 2^\kappa$ (or just $\lambda_\ast \geq 2^\kappa$) then $Q^2_{\mu, i}$ (hence $Q^1_{\mu, i}$) has a dense subset of cardinality $\lambda_\ast$.
2) If $\mu > 2^\kappa$ then $Q^1_{\mu, i} \cap \mathcal{B}$ is a dense subset of $Q^1_{\mu, i}$ and has cardinality $\lambda_\ast$. 

\(*\)
Hypothesis 1.12. $2^\kappa < \mu$. (in addition to Lemma 1.12.) Recall (Claim 1.15(1) is Balcar Simon [BS89, 1.15] and 1.14(2) is a variant).

1.6 Definition 1.13. A forcing notion $\mathbb{P}$ is $(\theta, \lambda)$-nowhere distributive when there are maximal antichains $\mathbb{P} = \{p_\alpha : \alpha < \alpha_\varepsilon\}$ of $\mathbb{P}$ for $\varepsilon < \theta$ such that for every $p \in \mathbb{P}$ for some $\varepsilon < \theta, \lambda \leq \{|\alpha < \alpha_\varepsilon : p, p_\alpha \text{ are compatible}|\}$.

Recall that

Lemma 1.14. 1) If

(a) $\mathbb{P}$ is a forcing notion, $(\theta, \lambda)$-nowhere distributive
(b) $\mathbb{P}$ has density $\lambda$
(c) $\theta > \aleph_0 \Rightarrow \mathbb{P}$ has a $\theta$-complete dense subset

then $\mathbb{P}$ is equivalent to $\text{Levy}(\theta, \lambda)$.

2) If $\mathbb{P}$ is a forcing notion of density $\lambda$ collapsing $\lambda$ to $\aleph_0$, then $\mathbb{P}$ is equivalent to $\text{Levy}(\aleph_0, \lambda)$.

3) If $\mathbb{P}$ is a forcing notion of density $\lambda$ and is nowhere $(\theta, \lambda)$-distributive, then $\mathbb{P}$ collapses $\lambda$ to $\theta$ (and may or may not collapse $\theta$).

1.7.3 Claim 1.15. Assume $(b_\varepsilon : \varepsilon < \kappa)$ is a sequence of pairwise disjoint members of $[\kappa]^{< \kappa}$ with union $b$. Then we can find anti-chain $\mathcal{I}$ of $Q_\mu^3$ such that:

$*$ if $q \in Q_\mu^3$ and $(\forall \varepsilon < \kappa)(a(q) \cap b_\varepsilon \in [\kappa]^{< \kappa})$, then $q$ is compatible with $\lambda_* =$: $\bigcup_{p_{\alpha}(\mu)}$ of the members of $\mathcal{I}$.

Proof. Let

$\mathcal{I}^* = \{p \in Q_\mu^3 : p \in \mathfrak{B} \text{ and we can find an increasing sequence} \ <i_\varepsilon : \varepsilon < \kappa> \text{ such that} i_\varepsilon \in b_\varepsilon \setminus (\varepsilon + 1) \}
 p(a(p)) \subseteq \{i_\varepsilon : \varepsilon < \kappa\}$ and $i_\varepsilon \in a(p) \Rightarrow p \cap [\lambda^r_\varepsilon, \lambda^r_\varepsilon) \text{ has order type } \lambda_\varepsilon$.\]

Let $\mathcal{I}^* = \{p \in Q_\mu^3 : \text{for every } \varepsilon < \kappa \text{ we have } a(p) \cap b_\varepsilon \in [\kappa]^{< \kappa}\}.$

Clearly

(a) $|\mathcal{I}^*| \leq \lambda_* = \bigcup_{p_{\alpha}(\mu)}$.

[Why? As $\mathcal{I}^* \subseteq \mathfrak{B}.$]
Claim 1.16. Let \( \theta = |\mathcal{F}| + \mu \) be such that \( \theta \in \theta^+ \) and \( \mu \in \theta \). If \( q \in \mathcal{F}^* \) and \( q \) is compatible with every \( p \in \mathcal{F} \), then there is \( r \) such that \( q \leq r \in \mathcal{F}^* \) and \( r \) is incompatible with every \( p \in \mathcal{F} \).

[Why? Let \( \theta = \{\mathcal{F}\} + \mu \) be such that \( \theta \in \theta^+ \) and \( \mu \in \theta \). If \( q \) is compatible with every \( p \in \mathcal{F} \), then there is \( r \) such that \( q \leq r \in \mathcal{F}^* \) and \( r \) is incompatible with every \( p \in \mathcal{F} \).]

Proof. Let \( \mathcal{F} = \{\mathcal{F}\} + \mu \in \theta \). If \( q \) is compatible with every \( p \in \mathcal{F} \), then there is \( r \) such that \( q \leq r \in \mathcal{F}^* \) and \( r \) is incompatible with every \( p \in \mathcal{F} \).

1.7.6 Conclusion 1.17. 1) If \( 2^\kappa < \mu \) (and \( \kappa = \text{cf}(\mu) < \mu \), of course) then \( \mathbb{Q}_\mu \) is equivalent to \( \text{Levy}(\kappa, \lambda_\mu) \), i.e., they have isomorphic completions (recalling \( \mathbb{Q}_\mu \) is naturally completely embeddable into the completion of \( \mathbb{P}_\mu = (\mu^\kappa, \sup) \)).

2) If \( (\forall \alpha < \mu)(|\alpha|^\kappa < \mu) \) then \( \mathbb{Q}_\mu \) is equivalent to \( \text{Levy}(\kappa, \mu^\kappa) \).

3) If \( \mu \) is strong limit (singular of uncountable cofinality \( \kappa \)), then \( \mathbb{P}_\mu \) is equivalent to \( \text{Levy}(\kappa, \mu^\kappa) = \text{Levy}(\kappa, 2^\mu) \).
Proof. 1) By \(1.11\) \(Q^3_\mu\) has density (and even cardinality) \(\lambda_s\) and by \(1.7.6\) it is \((b_\kappa, \lambda_s)\)-no-where distributive hence by \(1.14(3)\), we know that \(Q^3_\mu\) collapse \(\lambda_s\) to \(b_\kappa\). But \(P_\kappa\) is completely embeddable into \(Q^3_\mu\) (see \(1.16(3)\)) and \(P_\kappa\) collapse \(b_\kappa\) to \(\kappa_0\) (see \(1.2\)) and \(Q^3_\mu\) is dense in \(Q^2_\mu\). Together forcing with \(Q^3_\mu\) collapse \(\lambda_s\) to \(\kappa_0\). As \(Q^3_\mu\) has density \(\lambda_s\), by \(1.14(2)\) we get that \(Q^3_\mu\) is equivalent to \(\text{Levy}(\kappa_0, \lambda_s)\).

Lastly \((Q^3_\mu, b_\kappa)\) are equivalent by \(1.9.3\) \(1.10(3)\) \(1.10(2)\) so we are done.

2) Recalling \(\kappa_0\) by \([\text{She94}\] we have \(\lambda_s = \mu^\kappa\) (alternatively directly as in \([\text{She97}, \S 3]\)). Now apply part (1).

3) By easy cardinal arithmetic \(\mu^\kappa = 2^\kappa\). Enough to check the demands in \(1.14(2)\).

Now as \(Q_\mu\) collapse \(\lambda_s\) to \(\kappa_0\) by part (1) and \(Q_\mu\) can be completely embeddable into \(P_\mu\) (see \(1.6(2)\)) clearly \(P_\mu\) collapse \(\lambda_s\) to \(\kappa_0\). But \(|P_\mu| \leq |\mu|^\kappa = 2^\kappa\), so \(P_\mu\) has density \(\leq 2^\kappa\).

Lastly \(\lambda_s = 2^\kappa\) by \([\text{She94}, \text{Ch.VIII}]\). So we are done.

\(1.11\) Claim 1.18. Assume that \(P_\mu\) does not satisfy the \(\chi\)-c.c. Then forcing with \(P_\mu\) collapse \(\chi\) to \(\kappa_0\).

Proof. By the nature of the conclusion without loss of generality \(\chi\) is regular. Now we can find \(X\) such that:

\[\begin{align*}
\{\text{(*)}_1\} & \quad (a) \quad \bar{X} = \langle X_\xi : \xi < \chi \rangle \\
\text{or} & \\
(b) & \quad X_\xi \in P_\mu \\
(c) & \quad X_\xi \cap X_\zeta \in [\mu]^\zeta \text{ for } \zeta \neq \xi < \chi.
\end{align*}\]

As \(Q_\mu < P_\mu\), by the earlier proof (e.g., \(1.9.1\)) it suffices to prove that \(P_\mu\) collapses \(\chi\) to \(\lambda_s\). Let \(P = \langle \hat{A} : \hat{A} = \langle A_\alpha : \alpha < \mu \rangle, \text{ the } A_\alpha \text{'s are pairwise disjoint and each } A_\alpha \text{ belongs to } [\mu]^\alpha \rangle\), such that \(|P| = \lambda_s\) and \[(*)_2\) for every \(p \in P_\mu\) there is a \(\hat{A} \in P \cap \mathfrak{B}\) such that \((\forall \alpha < \mu)[|A_\alpha \cap p| = \mu].\]

[Why? By induction on \(\varepsilon < \kappa\) we can find \(\delta \mu < \mu\) of cofinality \(\lambda_s^{++}\) such that \(p \cap \delta \alpha \delta\) is unbounded in \(\delta \alpha \delta\) and \(\delta_\alpha \mu < \alpha \). There is a club \(C_\mu \in \mathfrak{B}\) of \(\delta_\alpha \mu\) of order type \(\lambda_s^{++}\) with \(\min(C_\mu) = \cup(\delta_\alpha \mu : \alpha < \varepsilon)\). Let \(C_\alpha = \{\delta \in C_\mu : \text{otp}(\delta \cap C_\mu) = \text{otp}(\alpha \cap C_\mu) = (\alpha, \text{Min}(C_\mu \setminus (\alpha + 1)) \setminus \nu \neq \emptyset)\}\). It is a club of \(\delta_\alpha \mu\) but in general not from \(\mathfrak{B}\).

But by the club guessing see \(1.10\) there is \(C_\alpha \subseteq \mathfrak{B}\) such that \(C_\alpha \subseteq C_\mu \subseteq \mathfrak{B}\) and \(\text{otp}(C_\mu) = \lambda_s\). We can find in \(\mathfrak{B}\) also \(\langle (W_{\varepsilon, \alpha} : \alpha < \lambda_s) : \varepsilon < \kappa \rangle\) a sequence such that \(\langle W_{\varepsilon, \alpha} : \alpha < \lambda_s \rangle\) is a partition of \(\lambda_s\) into \(\lambda_s\) (pairwise disjoint) sets each of cardinality \(\lambda_s\).

As \(\lambda_s = \cup_{\mu < \kappa}(\mu)\) and \(2^\kappa < \mu\), there is \(a \in [\kappa]^\kappa \subseteq \mathfrak{B}\) such that \(C_\alpha \subseteq [\kappa]^\kappa \subseteq \mathfrak{B}\) and \(\text{otp}(C_\alpha) = \lambda_s\). Lastly, let us define \(A = \langle A_\alpha : \alpha < \mu \rangle\) by

\[A_\alpha = \cup\{[\beta, \text{min}(C_\alpha \setminus (\beta + 1)) : \varepsilon \in a \text{ satisfies } \alpha < \lambda_s \text{ and } \beta \in C_\alpha \text{ and } \text{otp}(C_\alpha \setminus (\beta + 1) \in W_{\varepsilon, \alpha})\}.
\]

Easily \(\langle A_\alpha : \alpha < \mu \rangle \subseteq \mathfrak{B}\) is as required in \((*)_2\).

Now for \(A \in P \cap \mathfrak{B}\) we define a \(P_\mu\)-name \(\tau_A\) as follows: for \(G \in P_\mu\) generic over \(V\),

\[\begin{align*}
\langle \tau_A || G \rangle & = \xi \iff \xi \text{ is minimal such that } \cup\{A_\alpha : \alpha < \chi \} \in G.
\end{align*}\]

Clearly
\( * \) for every \( p \in \mathbb{P}_\mu \) for some \( \bar{A} \in \mathbb{P} \cap \mathcal{B} \) for every \( \xi < \chi \) we have \( p \upharpoonright \bar{A} \neq \xi \).

[Why? Let \( \bar{A} \in \mathbb{P} \cap \mathcal{B} \) be such that \((\forall \alpha < \mu) (\mu = |p \cap A_\alpha|\), it exists by \((*)_2\). Now we can find \( q \) satisfying \( p \leq q \in \mathbb{P}_\mu \) such that \((\forall \alpha < \mu) (q \cap A_\alpha \text{ is a singleton})\) and for each \( \xi < \chi \) let \( q_\xi = \cup \{A_\alpha \cap q : \alpha \in X_\xi\} \). Clearly \( \zeta < \xi \Rightarrow |X_\zeta \cap X_\xi| < \mu \Rightarrow \cup \{A_\alpha : \alpha \in X_\zeta\} \cap q_\xi \subseteq \cup \{A_\alpha \cap q_\xi : \alpha \in X_\zeta\} = \cup \{A_\alpha \cap q : \alpha \in X_\zeta \cap X_\xi\} \in [\mu]^{< \mu}, \) hence \( q_\xi \models \text{“} \xi = \bar{A}[G] \text{”} \).

So

\( \mathbb{P}_\mu \models \text{“} \chi = \bigcup \{\bar{A}[G] : \bar{A} \in \mathbb{P} \cap \mathcal{B}\} \text{”}. \)

Together clearly \( \mathbb{P}_\mu \) collapses \( \chi \) to \( \lambda_* + |\mathbb{P} \cap \mathcal{B}| \) which is \( \leq \|\mathcal{B}\| = \lambda_* \), so as said above we are done.

Lastly, concerning the singular \( \mu_* \) of cofinality \( \aleph_0 \) so we forget the hypothesis \( 1.1.2 \).

Claim 1.19. If \( \mu_* > \text{cf}(\mu_*) = \aleph_0 \) and \( \mathbb{P}_\mu \) fails the \( \chi \)-c.c., then \( \mathbb{P}_\mu \) collapse \( \chi \) to \( \aleph_1 \); note that in this case \( Q_{\mu_*} \) is equivalent to \( \text{Levy}(\aleph_1, \mu_{\aleph_0}^*) \) by [KS01].

Proof. Let \( \lambda_* = \mu_{\aleph_0}^* \).

By Kojman Shelah [KS01], \( \mathbb{P}_\mu \) collapse \( \lambda_* \) to \( \aleph_1 \) hence it suffices to prove that \( \mathbb{P}_\mu \) collapse \( \chi \) to \( \lambda_* \) assuming \( \chi > \lambda_* \) (otherwise the conclusion is known). Let \( \langle \lambda_n : n < \omega \rangle \) be a sequence of regular uncountable cardinals with limit \( \mu_* \). Now repeat the proof of \( 1.1.8 \).
§ 2. The regular uncountable case

We prove that (for $\kappa$ regular uncountable), $\mathbb{P}_\kappa$ collapse $\lambda$ to $\aleph_0$ iff $\mathbb{P}_\kappa$ fail the $\lambda$-c.c. This continues Balcar Simon [BS88, 2.8] so we first re-represent what they do; the proof of [BS88, 2.8] is made to help later. In the present notation they let $\lambda = b_\kappa$ (rather than $\lambda \in b^\text{spc}_\kappa$ as below a minor point); let $(f_\alpha : \alpha < b_\kappa)$ be a sequence exemplifying it; let $C_\alpha = \{\delta < \kappa : (\forall \beta < \delta)(f_\alpha(\beta) < \delta), \delta$ a limit ordinal $\}$ and let $B_\alpha = \kappa \setminus C_\alpha$, so $(B_\alpha : \alpha < \lambda)$ is a $(\kappa, \lambda)$-sequence (see Definition 2.3(1)), derive a good $(\kappa, \omega^\omega \lambda)$-sequence from it (see [BS88, 2.8]), define $\alpha_n(A), \beta_n(A)$ and used the $A$'s to define the $\mathbb{P}_\kappa$-names $\beta_\alpha$ and prove $\mathbb{P}_\kappa$, “$\{g^n(\beta_\alpha) : n < \omega\} = b_\kappa$” (see 2.6). Then comes the major point we prove the new result: if $\mathbb{P}_\kappa$ fail the $\chi$-c.c. then it collapses $\chi$ to $\aleph_0$.

A major point is that the proof splits to two cases: when $\lambda > \kappa^+$ and when $\lambda = \kappa^+$. In the 2019 revision we strengthen the result to “adding a generic to $\text{Levy}(\aleph_0, \chi)$”.

4.0. Context 2.1. $\kappa$ is a fixed regular uncountable cardinal.

4.1. Definition 2.2. 1) Let $b^\text{spc}_\kappa$ be the set of regular $\lambda > \kappa$ such that there is a $<\rho^{\text{acc}}$-increasing sequence $(f_\alpha : \alpha < \lambda)$ of members of $\kappa^\kappa$ with no $<\rho^{\text{acc}}$-upper bound in $\kappa^\kappa$.

2) Let $b_\kappa = \text{Min}(b^\text{spc}_\kappa)$.

3) Note that in Definition 2.3(1) we do not require that the $B_\alpha$'s define a MAD family.

4a.1. Definition 2.3. 1) We say $\bar{B}$ is a $(\kappa, \lambda)$-sequence when:

(a) $\bar{B} = \langle B_\alpha : \alpha < \lambda \rangle$
(b) $B_\alpha \in [\kappa]^\kappa$ and $\kappa \setminus B_\alpha \in [\kappa]^\kappa$ and $B_{\alpha + 1} \setminus B_\alpha \in [\kappa]^\kappa$
(c) for every $B \subseteq [\kappa]^\kappa$ for some $\alpha, B \cap B_\alpha \subseteq [\kappa]^\kappa$
(d) $B_\alpha \subseteq^* B_\beta$ when $\alpha < \beta < \lambda$, i.e., $B_\alpha \setminus B_\beta \subseteq [\kappa]^\kappa$.

2) We say that $\bar{B}$ is a $(\kappa, \omega^\omega \lambda)$-sequence when:

(a) $\bar{B} = \langle B_\eta : \eta \in \omega^\omega \lambda \rangle$
(b) $B_\eta \in [\kappa]^\kappa$
(c) if $\eta_1 < \eta_2 \in \omega^\omega \lambda$ then $B_{\eta_2} \subseteq^* B_{\eta_1}$ which means $B_{\eta_2} \setminus B_{\eta_1} \subseteq [\kappa]^\kappa$
(d) $B_\kappa = \kappa$
(e) if $\eta \in \omega^\omega \lambda$ and $A \in [B_\eta]^\kappa$ then for some $\alpha < \kappa$ we have $A \cap B_{\eta^\alpha} \subseteq [\kappa]^\kappa$
(f) if $\eta \in \omega^\omega \lambda$ and $\alpha < \beta < \lambda$ then $B_{\eta^\alpha} \subseteq^* B_{\eta^\beta}$ and $B_\alpha \setminus B_{\eta^\alpha} \subseteq [\kappa]^\kappa$.

3) For a $(\kappa, \omega^\omega \lambda)$-sequence $\bar{B}$ and $A \subseteq [\kappa]^\kappa$ we try to define an ordinal $\alpha_k(A, \bar{B})$ by induction on $k < \omega$. If $\eta = (\alpha_k(A, \bar{B}) : \ell < k)$ is well defined (which trivially holds for $k = 0$) and there is an $\alpha < \lambda$ such that $\alpha \subseteq^* B_{\eta^\alpha} \setminus B_{\eta^\beta} \subseteq [\kappa]^\kappa$ then we let $\alpha_k(A, \bar{B}) = \alpha$; note that $\alpha$, if exists, is unique.

3A) Let $n(A, \bar{B})$ be the $n \leq \omega$ such that $\alpha_k(A, \bar{B})$ is well defined iff $\ell < n$.

4) We say that $(\bar{B}, \nu)$ is a $(\kappa, \omega^\omega \lambda)$-parameter when:
(a) \( \bar{B} = \{ B_\eta : \eta \in \omega^>\lambda \} \) is a \((\kappa, \omega^>\lambda)\)-sequence

(b) \( \bar{\nu} \) is an \( S^\lambda_\alpha \)-ladder which means that \( \bar{\nu} = \{ \nu_\delta : \delta \in S^\lambda_\alpha \} \), \( \nu_\delta \) is an increasing sequence of ordinals of length \( \kappa \) with limit \( \delta \), where \( S^\lambda_\alpha = \{ \delta < \lambda : cf(\delta) = \kappa \} \).

5) We say \((\bar{B}, \bar{\nu})\) is a good \((\kappa, \omega^>\lambda)\)-parameter when (a)+(b) of part (4) holds and

(c) if \( A \in [\kappa]^\kappa \) then for some \( n < \omega, \eta \in \eta^\lambda, \delta \in S^\lambda_\alpha \) and \( A' \in [A]^\kappa \) we have

\( \alpha_\ell(A', \bar{B}) = \eta(\ell) \) for \( \ell < n \)

\( \beta \) for \( \kappa \) many ordinals \( \zeta < \kappa \) we have \( (\forall \zeta < \zeta)(A' \cap B_{\eta, \zeta} \cap B_{\eta, \zeta}) \) belongs to \( [\kappa]^{\zeta} \).

6) \( \bar{B} \) is a good \((\kappa, \omega^>\lambda)\)-sequence if clause (a) of part (4) and clause (c) of part (5) hold for some \( S^\lambda_\alpha \)-ladder (see above). We say \((\bar{B}, \bar{\nu})\) is a weakly good \((\kappa, \omega^>\lambda)\)-parameter if clause (a) of part (4) and clause (c)'- of part (5) which means that we ignore subclause (a) there. Similarly \( \bar{B} \) is a weakly good sequence.

4a.1d

Observation 2.4. 1) In \( \frac{4a.1b}{4a.1b}(\nu)(\beta) \), the “for \( \kappa \) many ordinals \( \zeta < \kappa \)” implies “for club many ordinals \( \zeta < \kappa \)”.

2) In \( \frac{4a.1b}{4a.1b}(6) \) it does not matter which \( S^\lambda_\alpha \)-ladder you choose.

Proof. Notice, if \( \nu_1, \nu_2 \in \eta^\delta \) are increasing and \( sup(\nu_1) = sup(\nu_2) = \delta \), then \( \{ i < \kappa : \bigcup_{j < i} \nu_1(j) = \bigcup_{j < i} \nu_2(j) \} \) is a club of \( \kappa \), so it doesn’t matter which \( S^\lambda_\alpha \)-ladder you choose.

\( \square \) 4a.1d

Note that for \( \S \) (except \( \frac{11}{11} \), \( \frac{12}{12} \)) we need no more than Claim 4a.2 (actually the weakly good version is enough for \( \S \) except presenting the proof that \( \bar{B}_\kappa \) is collapsed).

4a.2

Claim 2.5. 1) Assume \( \lambda = \bar{b}_\kappa \) or just \( \lambda \in \bar{b}_\kappa^{\omega^\kappa} \). Then \( \lambda \) is regular > \( \kappa \) and there is a \( \subseteq^\omega \)-decreasing sequence \( \langle C_\alpha : \alpha < \lambda \rangle \) of clubs of \( \kappa \) such that for no \( \alpha < \lambda \) ⇒ no \( A \in [\kappa]^\kappa \) do we have \( \alpha < \lambda \Rightarrow A \subseteq^\omega C_\alpha \) and for \( |C_\alpha \setminus C_{\alpha+1}| = \kappa \). Hence \( \langle \kappa \setminus C_\alpha : \alpha < \lambda \rangle \) is a \((\kappa, \lambda)\)-sequence.

2) Assume \( \bar{C} = \langle C_\alpha : \alpha < \lambda \rangle \) is as above and \( \bar{\nu} = \{ \nu_\delta : \delta \in S^\lambda_\alpha \} \) is an \( S^\lambda_\alpha \)-ladder, see Definition 4a.4, clause (b) (such \( \bar{\nu} \) always exists). Then \( \bar{B} = \bar{B}_\bar{C} : f = f_\bar{C} \) are well defined and \( (\bar{B}, \bar{\nu}) \) is a good \((\kappa, \omega^>\lambda)\)-parameter when we define \( \bar{B} \) and \( f \) as follows:

\( \odot \)

(a) \( \bar{B} = \{ B_\eta : \eta \in \omega^>\lambda \} \)

(b) \( \bar{f} = \{ f_\eta : \eta \in \omega^>\lambda \} \)

(c) \( B_{\nu^\omega} = \kappa, f_{\nu^\omega} = id_\kappa \)

(d) \( B_\eta \in [\kappa]^\kappa, f_\eta \) is a function from \( B_\eta \) onto \( \kappa \), non-decreasing, and not eventually constant

(e) if the pair \( (B_\eta, f_\eta) \) is defined and \( \alpha < \lambda \) we let

\( B_\eta^{-\omega^\omega} = \{ \gamma \in B_\eta : f_\eta(\gamma) \in \kappa \setminus C_\alpha \} \)

(f) if \( \eta = \rho^{-}(\alpha) \) and \( B_\rho, f_\rho \) are defined and \( B_\eta \) is defined as in clause (e), then we let \( f_\eta : B_\eta \to \kappa \) be defined by \( f_\eta(i) = otp\{ j : \text{for some } i_1 < i_2 < i \text{ we have } j = f_\rho(i_1) < f_\rho(i_2) \text{ and } f_\rho(i_1) \in C_\alpha \text{ and } f_\rho(i_2) \in C_\alpha \} \) for each \( i < \kappa \).
We have defined by $f_\eta(i) > 0 \Rightarrow f_\eta(i) > f_\eta(<\eta>(i))$.

Proof. 1) Recall $S^\lambda_\delta := \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$.

By the definition of $b^\kappa$, there is an increasing sequence $(f^*_\alpha : \alpha < \lambda)$ of members of $^*\kappa$ with no $<_{j^\kappa}$-upper bound from $^*\kappa$. Let $C_\alpha := \{\delta < \kappa : \delta \text{ is a limit ordinal such that } (\forall \gamma < \delta)(f^*_\alpha(\gamma) < \delta)\}.$

Clearly $(*)_1$ $C_\alpha$ is a club of $\kappa$.

[Why? As $\kappa$ is regular uncountable]

$(*)_2$ if $\alpha < \beta < \lambda$ then $C_\beta \subseteq ^* C_\alpha$; i.e., $C_\beta\backslash C_\alpha \in [\kappa]^\kappa$.

[Why? As if $\alpha < \beta$ then $f^*_\alpha <_{j^\kappa} f^*_\beta$, i.e., for some $\varepsilon < \kappa, (\forall \zeta)(\varepsilon \leq \zeta < \kappa \Rightarrow f^*_\alpha(\zeta) < f^*_\beta(\zeta < \kappa))$; without loss of generality $\varepsilon \in C_\alpha \cap C_\beta$ hence $C_\beta\backslash (\varepsilon + 1) \subseteq C_\alpha$ as required.]

$(*)_3$ for every club $C$ of $\kappa$ for some $\zeta < \lambda$ we have $C\backslash C_\zeta \in [\kappa]^\kappa$.

[Why? Toward contradiction assume $C$ is a counterexample. Let $f : \kappa \rightarrow \kappa$ be defined by $f(i) = (i + 1)$-th member of $C$, clearly for every $\alpha < \lambda$ for some $j < \kappa$, we have $C\backslash j \subseteq C_\alpha$ so possibly increasing $j$, without loss of generality $\text{otp}(C \cap j) = \text{otp}(C_\alpha \cap j)$. Hence easily $i < (j, \kappa) \Rightarrow f_\alpha(i) < \min(C_\alpha \setminus i) \leq \min(C \setminus i) \leq f(i)$, hence $f_\alpha < f \ mod j^\kappa$. As this holds for every $\alpha < \theta|f$ contradicts the choice of $f$, i.e. $f$ has no $<_{j^\kappa}$-bound in $^*\kappa$].

Hence $(*)_4$ for every unbounded subset $A$ of $\kappa$ for some $\zeta < \lambda$ we have $A\backslash C_\zeta \in [\kappa]^\kappa$.


Clearly without loss of generality $\alpha < \lambda \Rightarrow (C_\alpha \setminus C_{\alpha + 1}) = \kappa$ hence $(C_\alpha : \alpha < \lambda)$ is as required.

Lastly, let $B_\kappa = \kappa\backslash C_\alpha$, it is easy to check that $(B_\kappa : \alpha < \lambda)$ is a $(\kappa, \lambda)$-sequence. 2) Clearly $B_\kappa, f_\kappa$ are well defined and $(B, f)$ is a $(\kappa, ^*\lambda)$-parameter and clauses (a)-(l) of $\odot$ holds. Why is it good? Toward contradiction assume that it is not, so choose $A \in [\kappa]^\kappa$ which exemplify the failure of clause (c) of Definition 2.3(5) and so define

$\mathcal{T}_0 = \mathcal{P}_A^0 = \{\eta \in ^*\lambda : \text{ there is } A' \in [A]^\kappa \text{ such that } (\alpha_\ell(A', B) : \ell < \ell(g(\eta))) \text{ is well defined and equal to } \eta\}$.

and define

$\mathcal{T}_1 = \mathcal{P}_A^1 := \{\eta \in \mathcal{P}_A^0 : \text{ for every } k < \ell(g(\eta)) \text{ there are } < \kappa \text{ ordinals } \alpha < \eta(k) \text{ such that } \eta^\sim(\alpha) \in \mathcal{T}_0\}.$

Clearly $(*)_1 \mathcal{T}_0 \supseteq \mathcal{T}_1$ are non-empty subsets of $^*\lambda$ (in fact $<> \in \mathcal{T}_1 \subseteq \mathcal{T}_0$)

$(*)_2 \mathcal{T}_0, \mathcal{T}_1$ are closed under initial segments.
For $\eta \in \mathcal{R}$ let $\text{Suc}_{\mathcal{R}}(\eta) = \{ \rho \in \mathcal{R} : \ell g(\rho) = \ell g(\eta) + 1 \text{ and } \eta \triangleleft \rho \}$. We try to choose $A_\eta \in [B_\eta]^\kappa$ for $\eta \in \mathcal{R}_1$ by induction on $\ell g(\eta)$:

\[(*)_3 \quad (a) \quad A_{<\gamma} = A_{<\gamma}
\]

(b) if $A_\nu$ is defined and $\nu \sim \langle \alpha \rangle \in \mathcal{R}_1$ then we let $A_{\nu \sim \langle \alpha \rangle} = A_\nu \cap B_{\nu \sim \langle \alpha \rangle} \cup \{ B_{\nu \sim \langle \beta \rangle} : \beta < \alpha \text{ and } \nu \sim \langle \beta \rangle \in \mathcal{R}_1 \}.$

Now

\[(*)_4 \quad \text{if } \nu \in \mathcal{R}_1 \text{ then }
\]

(a) if $B \in [A]^{\kappa}$ and $\langle \alpha_\gamma(B, B) : \ell < \ell g(\nu) \rangle$ is well defined and equal to $\nu$ then $A_\nu$ is well defined and $B \subseteq A_\nu.$

(b) if $j \in \{ 0, 1 \}$ and $\text{Suc}_j(\nu)$ has cardinality $< \kappa$ then $A_{\nu} \cup \{ A_\rho : \rho \in \text{Suc}_{\mathcal{R}_1}(\nu) \}$ has cardinality $< \kappa.$

[Why? First we can prove clause (a) by induction on $\ell g(\nu)$ using the definition of $\mathcal{R}_1$ and clause (c) of $\mathcal{R}_1^{\ell g(2)}.$ Second, we can prove clause (b) from it.]

\[(*)_5 \quad |\mathcal{R}_1| \geq \kappa.
\]

[Why? Otherwise by $(*)_4(b)$ the set $A' := \bigcup\{ A_\nu \cup \{ A_\rho : \rho \in \text{Suc}_0(\nu) \} : \nu \in \mathcal{R}_1 \}$ is a subset of $\kappa$ of cardinality $< \kappa$ and by clause (d) of $\mathcal{R}_1$ of the present claim also $A'' = \bigcup\{ f_\nu^{-1}\{ 0 \} : \nu \in \mathcal{R}_1 \}$ is a subset of $\kappa$ of cardinality $< \kappa.$ So we can choose $j \in A \setminus (A' \cup A'').$ Now we try to choose $\nu_n \in \mathcal{R}_1$ by induction on $n$ such that $\ell g(\nu_n) = n, \nu_{n+1} \in \text{Suc}_{\mathcal{R}_1}(\nu_n),$ and $j \in A_{\nu_n}.$

First, $\nu_0 = <\gamma$ belongs to $\mathcal{R}_1$ by clause (a). Second, assume $\nu_n$ is well defined, then $\text{Suc}_{\mathcal{R}_1}(\nu_n) = \text{Suc}_{\mathcal{R}_1}(\nu_n).$

[Why? Otherwise, as $\mathcal{R}_1 \subseteq \mathcal{R}_0$ there is an $\alpha$ with $\nu_n \sim \langle \alpha \rangle \in \text{Suc}_{\mathcal{R}_1}(\nu_n) \setminus \text{Suc}_{\mathcal{R}_1}(\nu_n),$ hence by the definition of $\mathcal{R}_1$ the set $u := \{ \beta < \alpha : \nu_n \sim \langle \beta \rangle \in \mathcal{R}_1 \}$ has cardinality $\geq \kappa$ but then $\beta \in u \cap \beta^n \kappa \subseteq \kappa \Rightarrow \nu_n \sim \langle \beta \rangle \in \mathcal{R}_1$ which implies that $|\text{Suc}_{\mathcal{R}_1}(\nu_n)| \geq \kappa,$ contradiction to the “otherwise”].

Now $j \notin A'$ and $A' \supseteq A_{\nu_n} \cup \{ A_\rho : \rho \in \text{Suc}_{\mathcal{R}_1}(\nu_n) \}$ but $j \in A_{\nu_n}$ hence clearly $j \in \bigcup \{ A_\rho : \rho \in \text{Suc}_{\mathcal{R}_1}(\nu_n) \}$ so we can choose $\nu_{n+1}$ as required. As $j \in A_{\nu_n} \subseteq B_{\nu_n}$ by $(*)_3(b)$ above clearly $f_{\nu_n}(j)$ is well defined (for each $n < \omega$). For each $n,$ as $j \notin A''$ and $f_\nu^{-1}\{ 0 \} \subseteq A''$ so $j \notin f_\nu^{-1}\{ 0 \},$ necessarily $f_{\nu_n}(j) \neq 0$ and so $f_{\nu_n}(j) > f_{\nu_{n+1}}(j)$ by the choice of $f_{\nu_{n+1}}$ in clauses (d) of $\mathcal{R}_1.$ Hence $(f_{\nu_n}(j) : n < \omega)$ is decreasing, contradiction. So $(*)_5$ holds.]

Let $n < \omega$ be maximal such that $|\mathcal{R}_1 \cap n \geq \lambda| < \kappa,$ it exists as $|\mathcal{R}_1| \geq \kappa = \text{cf}(\kappa) > \aleph_0$ and $n = 0 \Rightarrow |\mathcal{R}_1 \cap n \geq \lambda| = 1 < \kappa,$ and let $\eta \in \mathcal{R}_1 \cap n \lambda$ be such that $\text{Suc}_{\mathcal{R}_1}(\eta) \geq \kappa$ members; it exist as $\kappa$ is regular. We can choose an increasing sequence $\langle \alpha_i : i < \kappa \rangle$ of ordinals such that $\alpha_i$ is the $i$-th member of the set $\{ \alpha < \lambda : \eta \prec \langle \alpha \rangle \in \mathcal{R}_1 \}$ and let $A_i \in [A]^{\kappa}$ be such that $\langle \alpha_{<\lambda}(A_i, B) : \ell \leq n \rangle = \eta \prec \langle \alpha_i \rangle$ and let $\delta = \bigcup \{ \alpha_i : i < \kappa \}$ so $\delta \in S_\kappa^\lambda.$

Let

$$A_\ast = \bigcup \{ A_i : i < \kappa \} \cap B_\eta \setminus A_\ast$$

where

$$A_\ast = \bigcup \{ A_{\eta \langle \ell \rangle} \sim < \gamma_\ast : \ell < \ell g(\eta) \text{ and } \gamma \prec \eta \langle \ell \rangle \text{ and } (\eta \mid \ell) \prec \langle \gamma \rangle \in \mathcal{R}_1 \}$$

(note that the number of pairs $(\ell, \gamma)$ as mentioned above is $< \kappa$).
Claim 2.6. for some γ

1) Let (h, δ, i < κ, ε < κ) and A ∩ h = A for i < κ, ℓ < n so clause (α) of (c) of Definition 2.3 holds (recalling 2.5(3)), as well as clause (β) because αn(A ∩ h, B, δ) = α1 for i < κ.

So (B, ν) is a good (κ, ℰ(λ))-parameter indeed, hence we are done proving 2.6(2).

For later use note that we have:

(*) for every A ∈ [κ]κ there are n, γ, δ, i = Aδ,i : i < κ, A, as above.

Claim 2.6. 1) If there is a good (κ, ℰ(λ))-parameter and λ1 ∈ b_{κ,λ}, then the forcing notion Pκ collapses λ1 to κ0.

2) Moreover, forcing with Pκ adds a generic to ℧(κ, λ1).

Proof. 1) Let (B, ν) be a good (κ, ℰ(λ))-parameter.

Note

 oval when A1 ⊆ A2 are from [κ]κ and αg(A2, B) is well defined then αg(A1, B) is well defined and equal to αg(A2, B), recalling Definition 2.3.

Let h = |h| ∈ ℵ(λ, γ < λ1) exemplify λ1 ∈ b_{κ,λ}, i.e., is as in Definition 2.1 and (as in the proof of 2.3) without loss of generality [i < j < κ ⇒ i < h(i) < h(j)]. For each δ ∈ Sκ, i < κ and η < ℰ(λ) let Aη,δ,i = Bη,δ,i \ {Bη,δ,i+j : j < i} for i < κ so {Aη,δ,i : i < κ} are pairwise disjoint subsets of κ (each of cardinality κ).

For A ∈ [κ]κ and n < ω we try to define an ordinal βn(A, B, ν, h) as follows:

 βn(A, B, ν, h) = γ = n for some n < ω, η ∈ ℰ(λ) and δ ∈ Sκ we have αg(A, B) : ℓ ≤ n = η(δ) so in particular is well defined and A ⊆ αg ∪ {Aη,δ,i \ h(i) : i < κ} but for every β < γ we have A ∩ {Aη,δ,i \ h(i) : i < κ} ∈ [κ]κ.\n
Next we define a Pκ-name βn = βn(A, B, ν, h) by:

 βn(A, B, ν, h) = γ = n for some n < ω.

Why is this really a well defined name? Because

• if A1, A2 ∈ [κ]κ and βn(A1, B, ν, h) = γ > 0 and A2 ⊆ A1 then βn(A2, B, ν, h) = γ.

Now

 βn(A, B, ν, h) = γ.

Why? We know that w = {i < κ : A ∩ Aη,δ,i ∈ [κ]κ} has cardinality κ.

First, why is u “unbounded”? For any γ1 < λ1, we define a function h ∈ [κ]κ as follows, h(i) is the minimal i1 < κ such that for some i0, i = i0 < i1 the set A ∩ Aη,δ,i \ h(i0) is not empty, clearly h is well defined because |w| = κ. So for some γ2 ∈ (γ1, λ1) the set u := {i < κ : h(i) < hγ2(i)} has cardinality κ. Let C be the club {δ < κ : δ is a limit ordinal and i < δ ⇒ h(i) < δ ∩ hγ2(i) < δ} and let (αε : ε < κ) list C \ {0} increasing order κ and let A′ = {A ∩ Aη,δ,i \ αε+1 : i < κ, αε < κ and αε ≤ i < αε+1}, now A′ ∈ [κ]κ. So Pκ | A ≤ A′ and A′ | | βn(A, B, ν, h) ∈ (γ1, γ2], recalling that the hγ2’s are increasing.

4b.3
Second, why “the set $u$ is $\kappa$-closed” (that is the limit of any increasing sequence of length $\kappa$ of members belong to it)? Easy, too.

Let $\langle S_\varepsilon : \varepsilon < \lambda_1 \rangle$ be pairwise disjoint stationary subsets of $\lambda_1$ included in $S^\kappa_\kappa$ and define $g^\ast : \lambda_1 \to \lambda_1$ by $g^\ast(\gamma) = \varepsilon$ if $\gamma \in S_\varepsilon \wedge (\gamma \in \lambda_1 \setminus \bigcup_{\zeta < \lambda_1} S_\zeta \wedge \varepsilon = 0)$.

So

$\oplus_5$ for every $p \in \mathbb{P}_n$ for some $n$, for every $\varepsilon < \lambda_1, p \not\vDash \text{"}g^\ast(\beta_n) \neq \varepsilon\text{"}$

so we are done.

2) Let $\langle \rho_\varepsilon : \varepsilon < \lambda_1 \rangle$ list $\omega^\omega(\lambda_1)$ and we define the $\mathbb{P}_\kappa$-name $\rho$ by:

$\oplus \rho$ is the concatenation of $\rho^g(\beta_0) \land \rho^g(\beta_1) \land \ldots$

Clearly $\Vdash_{\mathbb{P}_\kappa} \text{"}\rho \in \langle (\lambda_1) \rangle\text{"}$ and by $\oplus_4$ above.

$\oplus_5$ if $A \in [\kappa]^\kappa$ then for some $n, \eta, \delta$ we have:

(a) $A$ forces a value to $\beta_0, \ldots, \beta_{n-1}$ hence to $g^\ast(\beta_0) \ldots g^\ast(\beta_{n-1})$ call them $\gamma_0, \ldots, \gamma_{n-1}$

(b) for some club $C$ of $\lambda_1$, for every $\delta \in C \cap S^\lambda_\kappa$, $A \not\vDash \text{"}\beta_n \neq \delta\text{"}$

(c) for every $\varepsilon < \lambda_1, A \not\vDash p_{\kappa, n} \text{"}g^\ast(\beta_n) \neq \varepsilon\text{"}$

(d) for every $\nu \in \omega^\omega(\lambda_1)$ there is a $A'$ such that $\mathbb{P}_\kappa \vDash \text{"}A \leq A'\text{"}$ and $A' \Vdash p_{\kappa, n} \text{"}\nu \vDash \rho\text{"}$.

This clearly suffices.

Now we arrive to the main point.

4b.4

**Main Claim 2.7.** 1) If $\mathbb{P}_\kappa$ does not satisfy the $\chi$-c.c., then forcing with $\mathbb{P}_\kappa$ collapses $\chi$ to $\theta_0$. Morever, if $b^\omega_\lambda \neq \{\kappa\}$ then $\Vdash_{\mathbb{P}_\kappa} \text{"}there is } \rho \in \langle \chi \rangle \text{ generic for } \text{Levy}(\mathbb{N}_\kappa, \chi) \text{ over } V\text{"}.$

2) There is $\langle \check{A}_\alpha : \alpha < b_\kappa \rangle$ such that $\check{A}_\alpha = \langle A_{\alpha,i} : i < \kappa \rangle$ is a sequence of pairwise disjoint subsets of $\kappa$ (without loss of generality each is a partition of $\kappa$ to sets of cardinality $\kappa$) such that for every $B \in [\kappa]^\kappa$ for some $\alpha < b_\kappa$ we have $i < \kappa \Rightarrow \kappa = |A_{\alpha,i} \cap B|$. i.e., for every $i < \kappa$ not just of $\kappa$ many $i < \kappa$.

3) In part (2) we can replace $b_\kappa$ by $\lambda \in b^\omega_\kappa$ (so $\lambda = \kappa^+ \Rightarrow b_\kappa = \kappa^+$).

**Proof.**

The proof is divided to two cases:

**Case 1:** $\lambda \in b^\omega_\kappa, \lambda > \kappa^+$.

So $\lambda$ is regular $> \kappa^+$ and a good $(\kappa, \omega^\omega)\lambda$ sequence $B$ exists (by $\mathfrak{L}_2$).

Let $\nu = \langle \nu_\delta : \delta \in S_\lambda^\kappa \rangle$ be such that $\nu_\delta \in \kappa^\delta$ is increasing continuous with limit $\delta$ and $\nu$ guess clubs (i.e. for every club $C$ of $\lambda$, for stationarily many $\delta \in S_\lambda^\kappa$ we have $\text{Rang}(\nu_\delta) \subseteq C$); exists by [Sh:94], Ch.III,[2] because $\lambda = \text{cf}(\lambda) > \kappa^+$. As $B$ be a good $(\kappa, \omega^\omega)\lambda$-sequence, $(\check{B}, \check{\nu})$ is a good $(\kappa, \omega^\omega)\lambda$-parameter.

Let $\langle h_\alpha : \alpha < \lambda \rangle$ exemplify $\lambda \in b^\omega_\kappa$; without loss of generality $i < j < \kappa \Rightarrow i < h(i) < h(j).

For $\eta \in \omega^\omega \lambda, \delta \in S_\lambda^\kappa$ and $i < \kappa$, recall that $A_{\eta,\delta,i} = B_{\eta \cap <\nu_\delta(i+1)} \setminus \bigcup \{ B_{\eta \cap <\nu_\delta(i+1)} : j < i \}$ and let $\beta_{\eta,\delta,i} = \beta_{\eta,\delta,i}(B, \nu, h)$ be defined as in the proof of $\mathfrak{L}_6$. For $\eta \in \omega^\omega \lambda, \delta \in S_\lambda^\kappa, \delta^* \in S_\lambda^\kappa$ and $i < \kappa$ and $\gamma < \lambda$ let $\beta_{\eta,\delta,i}^\gamma : = \bigcup \{ A_{\eta,\delta,i} \cap h_\gamma(i) : i < \kappa \}$. So clearly (for each $\eta \in \omega^\omega \lambda, \delta \in S_\lambda^\kappa$ the sequence $\langle \beta_{\eta,\delta,i}^\gamma : \gamma < \lambda \rangle$ is $\kappa^+\text{}-increasing. Let } A_{\eta,\delta,i}^\gamma := \bigcup \{ B_{\eta,\delta,i}^\gamma : j < i \}$. So $\langle A_{\eta,\delta,i}^\gamma : i < \kappa \rangle$ are
pairwise disjoint subsets of $\kappa$. Note that (by the proof of 2.6 but not used) for each pair $(\eta, \delta)$ as above for some club $E_{\eta, \delta}$ of $\lambda$, for every $\delta^* \in S^\lambda_{\eta, \delta}$ satisfying $\text{Rang}(\nu_{\delta^*}) \subseteq E_{\eta, \delta}$ we have $i < \kappa \Rightarrow A^*_{\eta, \delta, \delta^*, i}$ has$^1$ of cardinality $\kappa$. We shall show during the proof of (1) that $\{\langle A^*_{\eta, \delta, \delta^*, i} : i < \kappa \rangle : \eta < \omega, \delta \in S^\lambda_{\eta, \delta}, \delta^* \in S^\lambda_{\eta, \delta}\}$ is as required in part (2), so this will prove part (2) when $b_\kappa > \kappa^+$ and also part (3) when $\lambda > \kappa^+$.

Let $\langle X^*_\xi : \xi < \chi \rangle$ be an antichain of $\mathbb{P}_\kappa$, exist by the assumption of $\mathbb{P}_\kappa$. We now, for $\eta, \delta, \delta^*$ as above define $\mathbb{P}$-names $\gamma_{\eta, \delta, \delta^*}$ if an ordinal $< \chi$: for $G \subseteq \mathbb{P}_\kappa$ generic over $\mathbb{V}$ we let:

\[ \oplus_0 \text{ for } \xi \in (0, \chi), n < \omega \text{ and } \eta \in \omega \lambda \text{ and } \delta, \delta^* \in S^\lambda_{\eta, \delta} \text{ we have:} \]

\[ \bullet \quad \gamma_{\eta, \delta, \delta^*}[G] = \xi \text{ iff for some } A \in G \]

\[ (a) \quad \langle \alpha_\ell(A, B) : \ell < n \rangle = \eta \text{ so in particular is well defined} \]

\[ (b) \quad \alpha_\ell(A, B) = \delta \in S^\lambda_{\eta, \delta} \]

\[ (c) \quad \beta_\ell(A, B, \nu, \bar{h}) = \delta^* \in S^\lambda_{\eta, \delta} \]

\[ (d) \quad A \cap A^*_{\eta, \delta, \delta^*, i} \text{ has at most one member for each } i < \kappa \]

\[ (e) \quad A \subseteq \bigcup \{A^*_{\eta, \delta, \delta^*, i} : i \in X^*_\xi\}. \]

Note that demands (a),(b),(c) are natural but actually not being used for proving just the first half of $\mathbb{P}_\kappa(1)$; with them we can define the $\mathbb{P}_\kappa$-names $\gamma$ which is $\gamma_{\eta, \delta, \delta^*}$ when defined, see below.

Now clearly

\[ \oplus_1 \gamma_{\eta, \delta, \delta^*} \text{ is a } \mathbb{P}_\kappa\text{-name of an ordinal } < \chi \text{ (may have no value)} \]

\[ \oplus_2 \text{ for every } p \in \mathbb{P}_\kappa \text{ for some } n, \eta \in \omega \lambda \subseteq \omega \lambda \text{ and } \delta, \delta^* \in S^\lambda_{\eta, \delta}, \text{ for every } \varepsilon < \chi \text{ there is } q \text{ such that } p \leq q \in \mathbb{P}_\kappa \text{ and } q \Vdash_{\mathbb{P}_\kappa} \langle \gamma_{\eta, \delta, \delta^*} = \varepsilon \rangle \text{ and } \eta = \langle \alpha_\ell(p, B) : \ell < n \rangle. \]

[Why? We start as in the proof of 2.6. First, possibly increasing $p$, there are $n < \omega, \eta \in \omega \lambda$ and $\delta \in S^\lambda_{\eta, \delta}$ such that $p \cap A^*_{\eta, \delta, \delta^*, i} \in [\kappa]$ for $\kappa$ many ordinals $i < \kappa$ and $\eta = \langle \alpha_\ell(p, B) : \ell < n \rangle$.]

Second, the set $W_p \subseteq \lambda$ is unbounded in $\lambda$ (by the proof of 2.6) where

\[ W_p = \{\beta < \lambda : \text{ for some } \gamma \in (\beta, \lambda) \text{ we have } p \cap B^*_{\eta, \delta, \gamma} \setminus B^*_{\eta, \delta, \beta} \text{ is from } [\kappa]. \}

Third, the club $C_p$ of $\lambda$ defined by $C_p = \{\delta < \lambda : \delta \text{ is a limit ordinal suchthat } \delta = \sup(W_p \cap \delta)\}$ satisfies:

\[ \oplus_{2.1} \text{ if } \beta < \gamma < \lambda \text{ are from } C_p \text{ then } p \cap B^*_{\eta, \delta, \gamma} \setminus B^*_{\eta, \delta, \beta} \in [\kappa]. \]

Now by the choice of $\bar{h}$, i.e., its being club guessing, there is $\delta^* \in \text{acc}(C_p) \cap S^\lambda_{\eta, \delta}$ such that $(\forall i < \kappa)(\nu_{\delta^*}(i) \in C_p)$. So (note that we have used $\nu_{\delta^*}(i + 1), \nu_{\delta^*}(j + 1)$ in the definition of $A^*_{\eta, \delta, \delta^*, i}$)

\[ \oplus_{2.2} \text{ if } \nu_{\delta^*}(i) \in C_p \text{ then } p \cap A^*_{\eta, \delta, \delta^*, i} \in [\kappa]. \]

---

$^1$and also $i < \kappa \Rightarrow A^*_{\eta, \delta, \delta^*, i} \in [\kappa]$
This fulfills the promise needed for proving part (2) in the present case. Choose $\zeta \in p \cap A^*_\eta,\delta,\gamma,i$ for $i < \kappa$. Now for every $\xi < \chi$ let $q_\xi = \langle \zeta_i : i \in X_\xi \rangle$. Recall that $\langle X_\xi : \zeta < \chi \rangle$ is an antichain in $P_\kappa$. Clearly for $\xi < \chi$ we have $P_\kappa \models \langle p \leq q_\xi \rangle$ and $q_\xi \models \langle \gamma_\eta,\delta,\gamma,i = \xi \rangle$; so we have finished proving $\odot_2$.]

This is enough for proving $\odot_3$ forcing with $P_\kappa$ collapse $\chi$ to $\aleph_0$.

[Why? By $\odot_1 + \odot_2$ we know that $P_\kappa \models \langle \gamma_\eta,\delta,\gamma,i : i < \kappa \rangle$ for each $\xi < \chi$; so it is forced that $|\chi| \leq |\lambda|$. As we already have by $\odot_3$, we are done.]

For proving the “moreover” in $\odot_4$

For $n < \omega$ we define the $P_\kappa$-name $\gamma_n$ of an ordinal $< \chi$ by: for $G \subseteq P_\kappa$ generic over $V$ and $\xi \in G(0, \chi)$ we have $\gamma_n[G] = \xi$ if $\{\xi\} = \{\gamma_\eta,\delta,\gamma,i : i < \kappa, \gamma < \alpha \} \subseteq \omega>\lambda, \delta \in S^\lambda_k$ and $\eta < \kappa$. Now for every $\xi < \chi$ for some $A' \in [\kappa]^\omega$ we have:

1. $A' \models \langle \beta_\ell = \beta_\ell' \rangle$ for $\ell < n$
2. $A' \models \langle \gamma_\ell = \xi_\ell \rangle$ for $\ell < n$
3. $A' \models \langle \gamma_\ell = \delta, \gamma_\ell = \xi \rangle$

This is proved as in the proof of $\odot_3$.

Now as there:

$\odot_6$ in $V^{P_\kappa}$ there is $\eta < \chi$ which is a generic for $\text{Levy} (\aleph_0, \chi)$ over $V$.

So we are done with proving $\odot_4$ in Case 1, i.e. when “there is $\lambda \in b^\omega_{\kappa^+}, \lambda > \kappa^+$”.

Case 2: $b^\kappa = \kappa^+$.

Let $\lambda = \kappa^+$ and $B$ be a good $(\kappa, \omega>\lambda)$-sequence. Let $\langle S_\xi : \xi < \kappa \rangle$ be a partition of $S^\lambda_k$ to (pairwise disjoint) stationary sets. For $\alpha < \kappa^+$ let $\langle u^\alpha_i : i < \kappa \rangle$ be an increasing continuous sequence of subsets of $\alpha$ each of cardinality $< \kappa$ with union $\alpha$ and without loss of generality $\alpha < \beta \Rightarrow (\forall i)(i < \kappa)(u^\alpha_i = u^\beta_i \cap \alpha)$. Let $h = \langle h_\beta : \beta < \kappa^+ \rangle$ exemplifying $\kappa^+ \in b^\omega_{\text{spc}}$ be such that each $h_\beta$ is strictly increasing, $(\forall i)(i < \delta \in \xi < \delta$ is a limit ordinal and for every $i < \delta$ we have $h_\beta(i) < \delta$ and $B(i, \hat{v})$ be a good $(\kappa, \omega>\lambda)$-parameter; exists by $\odot_3$. Now for $\eta < \kappa^+ \lambda$ and $\delta \in S^\lambda_k$ and define $A_{\eta,\delta,\gamma}(i < \kappa), B^*_\eta,\delta,\gamma, \langle \gamma < \alpha \rangle$ as in Case 1. Now for $\eta < \omega>\lambda, \delta \in S^\lambda_k, \alpha < \kappa^+$ and $\beta < \kappa^+$ we define the sequence $\langle Y_{\eta,\delta,\alpha,\beta,\gamma} : \gamma < \alpha \rangle$ by

\[ Y_{\eta,\delta,\alpha,\beta,\gamma} = \bigcup \{B^*_\eta,\delta,\gamma_i \cap [i, \text{Min}(C_\beta \setminus (i + 1))] \cup \{B^*_\eta,\delta,\gamma_1 : \gamma \in \gamma \cap u^\alpha_i \} : i < C_\beta \text{ satisfy } \gamma \in u^\alpha_i \}. \]

Actually, we can make this case cover Case 1 too: for $\delta \in S^\lambda_k$ choose $C_\delta$, a club of $\delta$, of order type $\kappa^+$. Now for each $a$ we can repeat the construction of names from the proof of Case 2, for each $p \in P_\kappa$ for some $\delta$, we succeed to show $\odot_6$.
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So \( \langle Y_{\eta, \delta, \alpha, \beta, \gamma} : \gamma < \alpha \rangle \) is a sequence of pairwise disjoint subsets of \( \kappa \) and for \( \varepsilon < \kappa \) let

\[
Z_{\eta, \delta, \alpha, \beta, \varepsilon} = \bigcup \{ Y_{\eta, \delta, \alpha, \beta, \gamma} : \gamma \in S_{\varepsilon} \cap \alpha \}.
\]

Clearly

- \( \Box_1 \) (a) for \( (\eta, \delta, \alpha, \beta) \) as above, \( (Z_{\eta, \delta, \alpha, \beta, \varepsilon} : \varepsilon < \kappa) \) is a sequence of pairwise disjoint subsets of \( \kappa \)
- (b) for \( (\eta, \delta, \alpha) \) as above such that \( (\forall \varepsilon < \chi)(S_{\varepsilon} \cap \alpha \neq \emptyset) \), for every large enough \( \beta < \kappa^+ \) we have \( \varepsilon < \kappa \Rightarrow Z_{\eta, \delta, \alpha, \beta, \varepsilon} \in [\kappa]^\kappa \). is a sequence of pairwise disjoint subsets of \( \kappa \)

We shall show during the proof of (1) that

\[
\langle \eta, \delta, \alpha, \beta, \varepsilon \rangle < (Z_{\eta, \delta, \alpha, \beta, \varepsilon} : \varepsilon < \kappa) : \eta < \omega^> \lambda, \delta \in S^\lambda_{\kappa}, \alpha < \lambda, \beta < \lambda >
\]

exemplify part (2) and in the present case, part (3) is equivalent to part (2).

Let \( \langle X^*_\xi : \xi < \chi \rangle \) be a family of sets from \([\kappa]^\kappa\) such that the intersection of any two have cardinility \( < \kappa \), it exists as \( P_\kappa \) fail the \( \chi \)-c.c.. For each \( \eta < \omega^> \lambda, \delta \in S^\lambda_{\kappa}, \alpha < \kappa^+ \) and \( \beta < \kappa^+ \) we define a \( \mathbb{P}_\kappa \)-name \( \tau_{\eta, \delta, \alpha, \beta} \) as follows:

- \( \Box_2 \) for \( G \subseteq \mathbb{P}_\kappa \) generic over \( V, \tau_{\eta, \delta, \alpha, \beta}[G] = \xi \) iff:
  - (a) for some \( A \in G \) we have
    - (a) \( \varepsilon < \kappa \Rightarrow A \cap Z_{\eta, \delta, \alpha, \beta, \varepsilon} \) has at most one member
    - (b) \( A \subseteq \bigcup \{ Z_{\eta, \delta, \alpha, \beta, \varepsilon} \ : \varepsilon \in X^*_\xi \} \)
  - (\beta) if no \( A \in G \) does (a)+(b) hold then \( \xi = 0 \).

Clearly

- \( \Box_3 \) \( \tau_{\eta, \gamma, \alpha, \beta} \) is a well defined \( (\mathbb{P}_\kappa \)-name \) (by \( \Box_2 \)).

Now

- \( \Box_4 \) for every \( p \in \mathbb{P}_\kappa \), for some \( \eta < \omega^> \lambda, \delta \in S^\lambda_{\kappa}, \alpha < \kappa^+, \beta < \kappa^+ \) we have: for every \( \xi < \chi \) for some \( q \in \mathbb{P}_\kappa \) above \( p \) we have \( q \Vdash "\tau_{\eta, \delta, \alpha, \beta} = \xi" \).

As in Case 1, this is enough for proving that \( P_\kappa \) collapse \( \chi \) to \( \lambda = \kappa^+ \). But by \( \Box_{\eta, \gamma, \beta} \) we already know that forcing with \( P_\kappa \) collapses \( \kappa^+ \) to \( \aleph_0 \) and so we are done except the “Moreover” in part (1).

Note: we can eliminate \( \eta \) from the \( \tau_{\eta, \delta, \alpha, \beta} \), but not worth it. So we are left with proving \( \Box_4 \).

Why does \( \Box_4 \) hold? First, as in the earlier cases, find \( \eta < \omega^> \lambda \) and \( \delta \in S^\lambda_{\kappa} \) such that \( p \cap A_{\eta, \delta, i} \in [\kappa]^\kappa \) for \( \kappa \) ordinals \( i < \kappa \). Second, as in the previous proof \( W_\rho \subseteq \lambda = \text{sup}(W_\rho) \) where \( W_\rho := \{ \beta < \lambda : \text{for some } \gamma \in (\beta, \lambda) \text{ we have } p \cap B^*_\eta, \delta, \gamma \setminus B^*_\eta, \delta, \beta \in [\kappa]^\kappa \} \). Third, for some club \( C_\rho \) of \( \lambda \) we have \( \beta < \gamma \wedge \beta, \gamma \in C_\rho \Rightarrow p \cap B^*_\eta, \delta, \gamma \setminus B^*_\eta, \delta, \beta = B^*_\eta, \delta, \beta \in [\kappa]^\kappa \).

As \( S_\varepsilon \) (for \( \varepsilon < \kappa \)) is a stationary subset of \( \lambda \) and \( C_\rho \) a club of \( \lambda \) for each \( \varepsilon < \kappa \) we can choose \( \gamma^*_\varepsilon \in S_\varepsilon \cap C_\rho \). Hence there is \( \alpha^* < \kappa^+ \) large enough such that \( \varepsilon < \kappa \Rightarrow \gamma^*_\varepsilon < \alpha^* \in C_\rho \). Now define a function \( h : \kappa \to \kappa \) by induction on \( i \), as follows:
\[ h(i) = \text{Min}\{ j : j \in (i, \kappa) \text{ and } i_1 < i \Rightarrow h(i_1) < j \}\]

and if
\[ \gamma \in u_i^\alpha \cap S \text{ then } \]
\[ p \cap (i, j) \cap B_{\bar{n}, \delta, \gamma}^* \setminus \{ B_{\bar{n}, \delta, \beta}^* : \beta \in \gamma \cap u_i^\alpha \} \text{ is not empty}. \]

It is well defined as for a given \( i < \kappa \) the number of pairs \((\gamma, \varepsilon)\) such that \( \gamma \in u_i^\alpha \cap S \) is \( < \kappa \) and is increasing; next we define

\[ C = \{ j < \kappa : j \text{ is a limit ordinal such that } i < j \Rightarrow h(i) < j \}. \]

Clearly \( C \) is a club of \( \kappa \) and let \( h' \in \kappa^+ \) be defined by \( h'(i) = h(\text{Min}(C \setminus (i + 1)) \). By the choice of \( \bar{h}_\beta : \beta \in \lambda \) there is \( \beta < \lambda \) such that for \( \kappa \) many ordinals \( i < \kappa \) we have \( h'(i) < h_\beta(i) \). Recall that \( C_\beta = \{ \delta < \kappa : \delta \text{ is a limit ordinal and for every } i < \delta \text{ we have } h_\beta(i) < \delta \}. \)

So \( W_1 = \{ i < \kappa : h'(i) < h_\beta(i) \} \in [\kappa]^\kappa \). Let \( \langle i_0^j : j < \kappa \rangle \) be an enumeration of \( C \cap C_\beta \) in increasing order, so clearly \( \mathcal{U} = \{ j < \kappa : W_1 \cap [i_0^j, i_{j+1}^0) \neq \emptyset \} \) is unbounded in \( \kappa \). For each \( j \in \mathcal{U} \) let \( i_j^j \in W_1 \cap [i_0^j, i_{j+1}^0) \), and

\[ (*) \quad i_0^j \leq i_j^j \leq h(i_j^j) < h'(i_j^j) < h_\beta(i_j^j) < i_{j+1}^0. \]

Now for each \( \varepsilon < \kappa \) we know that \( \gamma_\varepsilon^\kappa \in \alpha^\varepsilon \cap S \cap C_p \subseteq \alpha^\kappa = \cup \{ u_i^\alpha : i < \kappa \} \) and \( \langle u_i^\alpha : i < \kappa \rangle \) is \( \subseteq \)-increasing hence for some \( j(\varepsilon) < \kappa \) if \( j \in \mathcal{U}(\varepsilon) \) then \( \gamma_\varepsilon^\kappa \in u_{i_j^j}^\varepsilon \) hence by the choice of \( h(i_j^j) \) and \( (*) \) we have \( p \cap (i_0^j, i_{j+1}^0) \cap B_{\bar{n}, \delta, \gamma_\varepsilon^\kappa}^* \setminus \{ B_{\bar{n}, \delta, \beta}^* : \beta \in \gamma_\varepsilon^\kappa \cap u_{i_j^j}^\alpha \} \) is not empty which by the definition of \( Y_{\eta, \delta, \alpha^\varepsilon, \beta, \gamma_\varepsilon^\kappa} \) implies that \( p \cap Y_{\eta, \delta, \alpha^\varepsilon, \beta, \gamma_\varepsilon^\kappa} \cap [i_0^j, i_{j+1}^0) \neq \emptyset. \)

As this holds for every large enough \( j \in \mathcal{U} \setminus j(\varepsilon) \) and \( \mathcal{U} \in [\kappa]^\kappa \) it follows that \( p \cap Y_{\eta, \delta, \alpha^\varepsilon, \beta, \gamma_\varepsilon^\kappa} \in [\kappa]^\kappa \). By the definition of \( Z_{\eta, \delta, \alpha^\varepsilon, \beta, \varepsilon} \) it follows that \( p \cap Z_{\eta, \delta, \alpha^\varepsilon, \beta, \varepsilon} \in [\kappa]^\kappa \).

Choose \( \zeta_\varepsilon \in p \cap Z_{\eta, \delta, \alpha^\varepsilon, \beta, \varepsilon} \). Now for each \( \xi < \chi \) let
\[ q_\xi = \{ \zeta_\varepsilon : \varepsilon \in X^*_\xi \}. \]

So clearly:
\[ \xi < \chi \Rightarrow \mathbb{P}_\kappa \models "p \leq q_\xi" \quad \text{and} \quad q_\xi \models "\tau_{\eta, \delta, \alpha^\varepsilon, \beta} = \xi". \]

What about the “Moreover” (in part (1)’)?

It is not clear that for every \( n \) the sequence \( \langle \tau_{\eta, \delta, \alpha, \beta} : \eta \in \alpha, \delta \in S^\kappa_\alpha, \alpha < \lambda, \beta < \lambda \rangle \) has at most one non-zero entry. Clearly the following suffixes (we could have used it in both cases).

\( \mathbb{E}_0 \) If (A) then (B) where:

(A) (a) \( B = \langle B_\alpha : \alpha < \lambda \rangle \)
(b) \( B_\alpha \in [\kappa]^{\alpha^\varepsilon} \) is \( \subseteq \)-increasing
(c) \( (\forall \alpha < \lambda)(\exists \beta)(\alpha < \beta < \lambda) \)

(B) there is a sequence \( \langle A_\alpha : \alpha < \kappa' \rangle \) such that:

(a) \( A_\alpha = \langle A_{\alpha_i} : i < \lambda \rangle \) is a partition of \( \chi \)
(b) if $A \in [\kappa]^\kappa$ and $(\forall \beta < \lambda)[A \setminus (\forall \beta < \lambda)(\exists \gamma) \beta < \gamma < \lambda \land A \cap (B_\beta \setminus B_\gamma) \in [\kappa]^\kappa]$ then for stationarily many $\delta \in S^\kappa_\kappa$ we have $(\forall \gamma < \kappa)(A_\delta \cap A_{\alpha \cdot i} \in [\kappa]^\kappa)$

(c) if $\delta$ is as in xyz, then the assumption of clause B means $\delta < \lambda \land A \setminus B_\beta \in [\kappa]^\kappa$

(d) if $\lambda = \kappa^+$, then in clause (b) the conclusion holds for every $\delta < \kappa^+$ large enough.

[Why? The main case is $\lambda = \kappa^+$.

$\oplus_1$ For $\delta \in S^\kappa_\kappa$ and $\gamma < \lambda$ we define $A^\delta_{\lambda, \gamma} = \cup \{B_{\nu 2(j)} \cap B_{\delta} \cup \{B_{\nu 2(j)} : j < i\} \cap h_\gamma(i) : i < \kappa\}$

$\oplus_2$ (a) if $\delta_1 < \delta_2$ are from $S^\kappa_\kappa$ and $\gamma < \lambda$ then $A^\delta_{\lambda, \gamma} \cap A^{\delta_2}_{\lambda, \gamma} \in [\kappa]^\kappa$

(b) if $\delta \in S^\kappa_\kappa$ and $\gamma_1 < \gamma_2 < \lambda$ then $A^\delta_{\lambda, \gamma_1} \subseteq A^\delta_{\lambda, \gamma_2}$ mod $[\kappa]^\kappa$.

[Why? For clause (a), because $A^\delta_{\lambda, \gamma} \subseteq B_\delta$ whereas $A^{\delta_2}_{\lambda, \gamma} \cap B_{\delta_1} \cup \{B_{\nu 2(j)} : j < i\} \cap h_\gamma(i) = (i < \kappa\}$ (for $\delta_i < \delta_2$ from $S^\kappa_\kappa$).

Now the last set is the union of $\cup \{i(i, \delta_2)\} < \kappa$ sets, the i-th set of cardinality $\leq |h(i)| < \kappa$, so $A^{\delta_1}_{\lambda, \gamma} \cap A^{\delta_2}_{\lambda, \gamma} \in [\kappa]^\kappa$. Clause (b) is easy, too.]

$\oplus_3$ for $\gamma < \lambda$ there is a function $g_\gamma : S^\kappa_\kappa \cap \gamma \rightarrow \kappa$ such that $\langle A^\delta_{\lambda, \gamma} \setminus g_\gamma(\delta) : \delta \in S^\kappa_\kappa \cap \gamma\rangle$ is a sequence of pairwise disjoint sets.

Lastly,

$\oplus_4$ for $\gamma \in S^\kappa_\kappa$ we let $\bar{A}_\gamma = \langle A_{\gamma, \varepsilon} : \varepsilon < \kappa\rangle$ be defined by $A_{\gamma, \varepsilon} = \cup \{A^\delta_{\lambda, \gamma} \setminus g_\gamma(\delta) : \delta \in S^\kappa_\kappa \cap \gamma\}$. Note that by $\oplus_2 + \oplus_3$

$\oplus_5$ $\langle A_{\gamma, \varepsilon} : \varepsilon < \kappa\rangle$ is a sequence of pairwise disjoint subsets of $\kappa$.

Now the main point is:

$\oplus_6$ if $A \in [\kappa]^\kappa$ is as in clause (b) of $\oplus$ then for a club of $\gamma \in S^\kappa_\kappa$, $(\forall \varepsilon < \kappa)(A \cap A_{\gamma, \varepsilon} \in [\kappa]^\kappa)$.

Why? First as earlier, there is a club $E_1$ of $\lambda$ such that: $\beta < \gamma \in E \Rightarrow A \cap B_{\gamma} \setminus B_\beta \in [\kappa]^\kappa$ and let $E_2 = \{\delta \in E_1 : \delta = \sup(\delta \cap E_1)\}$.

Second, for each $\varepsilon < \kappa$ choose $\delta(\varepsilon) \in E_2 \cap S^\kappa$ and let $\gamma(\varepsilon) < \lambda$ be such that $\gamma(\varepsilon) \leq \gamma < \lambda \Rightarrow A_{\delta, \gamma} \cap A \in [\kappa]^\kappa$.

Third, let $\gamma_\ast = \cup \{\gamma_\varepsilon + 1 : \varepsilon < \kappa\} < \lambda$; and we shall show:

- if $\delta \in S^\kappa_\kappa \setminus \gamma_\ast$ then $(\forall \varepsilon < \kappa)(A_{\delta, \gamma} \cap A \in [\kappa]^\kappa)$.

Clearly this suffices.

Why $\bullet$ holds? For $\varepsilon < \kappa$, $A^{\delta_{\varepsilon, \gamma}} \subseteq A_{\delta, \varepsilon}$ mod $[\kappa]^\kappa$ and $A^{\delta_{\varepsilon, \gamma}} \cap A \in [\kappa]^\kappa$ hence $A_{\delta, \varepsilon} \cap A \in [\kappa]^\kappa$, as promised.

$\Box$

4b.5 Conclusion 2.8. If $\kappa$ is regular uncountable and $\mathbb{P}_\kappa$ fail the $2^\kappa$-c.c. then $\text{comp}(\mathbb{P}_\kappa)$ is isomorphic to the completion of Levy$(\mathbb{N}_0, 2^\kappa)$. 
REFERENCES


