ON MODEL COMPLETION OF $T_{\text{aut}}$

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Annotated Content

§0 Introduction

§1

[We characterize stable $T$ for which the model completion of $T_{\text{aut}}$ is stable (i.e., every completion is).]

§2

[We prove that “some completion is stable” is different and characterize it.]

§3

[We prove that if $T$ is stable, $T_{\text{aut}}$ has a model completion, $T_*$ is an unstable complete of $T_{\text{aut}}^\text{mc}$, then $T_*$ satisfies NSOP$_3$. Moreover, simplicity is preserved.]
ON MODEL COMPLETION OF $T_{\text{aut}}$

§0 Introduction

On the subject, history and background see [BlSh 759]. For a complete first order $T$ they dealt with the existence of the model completion $T_{\text{aut}}$ of $T \cup \{\sigma \text{ is an automorphism (for } \tau_T\}$.

We may ask:

0.1 Question: If $T$ is stable and $T_{\text{aut}}$ has model completion $T_{\text{mc}}^{\text{aut}}$, when is (every) completion of $T_{\text{mc}}^{\text{aut}}$ stable?

We answer in 1.6 (observation 1.7 deals with some obvious things).

Section 1 raises some question which we discuss below (assuming $T$ stable, $T_{\text{mc}}^{\text{aut}}$ exists) some of which are answered below.

0.2 Question: 1) Can we in Claim 1.6 below replace “every completion of $T_{\text{mc}}^{\text{aut}}$ is stable” by “some completion of $T_{\text{mc}}^{\text{aut}}$ is stable”?

2) The “unstable” in 1.6 clause (a) can be replaced by “having the independence property”; but can $T_{\text{mc}}^{\text{aut}}$ be completed to a theory with the strict order property? The SOP$_n$’s?

3) What occurs if $T_{\text{mc}}^{\text{aut}}$ does not exist, can we still say something?

4) Point out that (a)($(\equiv (b))$ of 1.6 holds (for some stable $T$ for which $T_{\text{mc}}^{\text{aut}}$ exists) and fails for others.

5) Show for stable $T$ with $T_{\text{mc}}^{\text{aut}}$, that no completion $T_\ast$ of $T_{\text{aut}}$ has the explicit ncp (which means that for some first order $E(\bar{x}, \bar{y}, \bar{z})$, for every $n$ for some $\bar{c} \subseteq \mathcal{C}, E(\bar{x}, \bar{y}, \bar{c})$ is an equivalent relation which has $\geq n$, $< \aleph_0$ equivalence classes); a stronger version is

6) For such $T, T_\ast$ can $T_\ast$ have obstructions (see §4)?

7) What if we use $\sigma_1, \sigma_2$? What about $\sigma_1, \ldots, \sigma_n$? What about pairwise commuting $\sigma_1, \ldots, \sigma_n$? This is like $(T_{\text{aut}})_{\text{aut}}$ for $n = 2$.

8) Is there unstable $T$ such that $T_{\text{aut}}$ has model completion? (A conjecture stating that had been the starting point of Kikyo Shelah [KkSh 748]).

0.3 Discussion: We prove that:

(A) on 0.2(1), for some $T$ (stable with $T_{\text{mc}}^{\text{aut}}$ existing), some completion of $T_{\text{mc}}^{\text{aut}}$ are stable and some are not (still we may wonder on a general characterization, see 2.7 below)

(B) we shall show that for no such $T$ is any completion of $T_{\text{mc}}^{\text{aut}}$ with the strict order property and even have NSOP$_3$, see 3.1

(C) we can look at the class of existentially closed models of $T_{\text{aut}}$ (see [ShUs 789] and references there); the results are similar.

Note
0.4 Observation. [Here?]

(α) for $T = \text{theory of equality, } T_{\text{aut}}$ has a model completion and all completions of $T_{\text{aut}}^{\text{mc}}$ are stable

(β) for $T$ from 2.1, some completions of $T_{\text{aut}}^{\text{mc}}$ are stable and some are not

(γ) for $T = \text{Th}(M \upharpoonright \{E, F_1, F_2, Q\}), M$ from 2.1, we get that all the completions of $T_{\text{aut}}^{\text{mc}}$ are unstable.

I think

0.5 Question: What about getting (in §3) that

(a) $T_{\text{aut}}^{\text{mc}}$ is simple in §3?

(b) even if $T$ is just simple, $T_{\text{aut}}^{\text{mc}} \models \text{NSOP}_3$

(c) non elementary class (true).

See below.
§1 ON THE STABILITY OF MODEL COMPLETION
FOR $T_{\text{aut}} (= T + \sigma$ AN AUTOMORPHISM)

1.1 Hypothesis. 1) $T$ is first order complete and for notational simplicity every formula is equivalent to a relation and $\tau_T$ having only predicates.
2) $\mathcal{C}$ is the monster model of $T$.

1.2 Definition. 1) $T_{\text{aut}}$ is $T \cup \{\sigma$ is an automorphism (for $\tau_T$)\}, so $\sigma$ is a new unary function symbol that is $T_{\text{aut}} = T \cup \{(\forall x_0, \ldots, x_{n-1})[R(x_0, \ldots, x_{n-1}) \equiv R(\sigma(x_0), \ldots, \sigma(x_{n-1}))] : R$ an $n$-place predicate of $\tau_T\}$.
2) $T_{\text{mc}}^\text{aut}$ is the model completion, if it exists.
3) Let $T_*$ denote any completion of $T_{\text{mc}}^\text{aut}$ and $\sigma_*$ or $\sigma^{N+}$ is an automorphism.
4) A completion $T_*$ of $T_{\text{mc}}^\text{aut}$ is cute if it has a model $N^+$ such that for some $M^+ \subseteq N^+$ we have $\sigma^{N+} = \text{id}_{N^+}$.

1.3 Definition. For $T$ as in 0.2 let:
1) $K_{\text{aut}}(T) =$ the class of models of $T_{\text{aut}}$.
2) $K_{\text{ec}}^\text{aut}(T) =$ the class of e.c. models of $T_{\text{aut}}$.
3) $K_*(T)$ is a subclass of $K_{\text{ec}}^\text{aut}(T)$ such that $M \equiv N \in K_* \Rightarrow M \in K_*$ and if $M \subseteq N$ are from $K_{\text{ec}}^\text{aut}$ then $M \in K_* \Leftrightarrow N \in K_*$; there are $\leq 2^{|T|}$ such classes.
4) $K_*$ is cute, etc.
5) $\mathcal{C}_{\text{aut}}$ is a monster model for $K_{\text{ec}}^\text{aut}$, i.e., a member of $K_{\text{ec}}^\text{aut}$ which is $\bar{\kappa}$-saturated of cardinality $\bar{\kappa}$; it is unique if $K_{\text{aut}}(T)$ has the JEP.
6) A class $K_*$ is stable\,\footnote{this is for classes as above, for general non first order classes this does not fit} if for some $\lambda < \bar{\kappa}$ there is no model $M \in K_*, m < \omega, \bar{a}_i \in mM, i < \lambda$ and q.f. formula $\varphi(\bar{x}, \bar{y})$ which order $\{\bar{a}_i : i < \lambda\}$.
7) $K_*$ is simple if there is a q.f. formula $\varphi(\bar{x}, \bar{y})$ and $m$ such that for every $\lambda, \kappa$ we can find $M \in K_*, \bar{a}_\eta \in f^(\bar{y})M$ for $\eta \in ^\kappa \lambda$, $\bar{b}_\nu \in f^g(\bar{x})M$ for $\nu \in ^\kappa \lambda$ such that:
   \begin{enumerate}
   \item[(i)] $M \models \varphi(\bar{b}_\eta, \bar{a}_\eta | ^\alpha) \text{ for } \alpha < \kappa, \eta \in ^\kappa \lambda$
   \item[(ii)] no sequence in $m$ realizes $\geq m$ of the formulas $\langle \varphi(\bar{x}, \bar{a})_{\eta \cdot <1} : i < \lambda\rangle$.
   \end{enumerate}

On such models see [Sh 54], [xx], [xx].

1.4 Fact: If $T_{\text{mc}}^\text{aut}$ exists then $K_{\text{aut}}^\text{ec}(T)$ is the class of its models.

1.5 Claim. In the claims below we can replace ‘$T$ has model completion’ by dealing with the class $K_{\text{aut}}^\text{ec}(T)$, and replace $T^*$ is a model completion by dealing with $K_*$. 

On such models see [Sh 54], [xx], [xx].
1.6 Claim. Let $T$ be stable, $T_{\text{aut}}^{\text{mc}}$ exists. The $(a) \iff (b)$ where

(a) $T_{\text{aut}}^{\text{mc}}$ is stable (i.e., every completion is stable)

(b) if $M_0 \prec M_\ell \prec \mathcal{C}$ for $\ell = 1, 2$ and $M_1 \bigcup M_2$ then in $\mathcal{C}^{\text{eq}}$, $\text{acl}_{\mathcal{C}^{\text{eq}}}(M_1 \cup M_2) = \text{dcl}_{\mathcal{C}^{\text{eq}}}(M_1 \cup M_2)$

(c) $T_{\text{aut}}^{\text{mc}}$ is dependent (i.e., every completion does not have the independence property).

Proof. $(b) \Rightarrow (a)$

We work in $\mathcal{C}^{\text{eq}}$ and use observation 1.7 below. Suppose $\mathcal{C}_* = (\mathcal{C}, \sigma_*)$ is an expansion of $\mathcal{C}^{\text{eq}}$ to a model of $T_{\text{aut}}^{\text{mc}}$ and let $\sigma_*^{\text{eq}}$ be the unique extension of $\sigma_*$ to an automorphism of $\mathcal{C}^{\text{eq}}$. Let $\lambda = |T|$, $M^+ \prec (\mathcal{C}^{\text{eq}}, \sigma_*^{\text{eq}}), |M^+| = \lambda$ (note $|T| \geq \aleph_0$ here (by 1.1(1))).

Now for every $p \in \mathcal{S}(M^+, \mathcal{C}_*)$ let $a_p \in \mathcal{C}$ realize $p$ in $(\mathcal{C}, \sigma_*)$ and let $M^+_p, N^+_p$ be such that

$$M^+_p \prec M^+, |M^+_p| = |T| + \aleph_0$$

$$M^+_p \prec N^+_p \prec \mathcal{C}_*, |N^+_p| = |T|$$

$$a_p \in N^+_p$$

$$N^+_p \upharpoonright \tau_T \bigcup M^+_p \upharpoonright \tau_T.$$

Let $A_p = \text{acl}_{\mathcal{C}^{\text{eq}}}(|N^+_p| \cup |M^+_p|)$. We define a two-place relation $E$ on $\mathcal{S}(M^+, \mathcal{C}_*)$ as follows:

$\circ$ $pEq$ iff $M^+_p = M^+_q$ and there is an isomorphism $f$ from $N^+_p$ onto $N^+_q$ which is the identity on $M^+_p$ and satisfying $f_p(a_p) = a_q$.

Clearly

$\circ_0$ $E$ is an equivalence relation on $\mathcal{S}(M^+, \mathcal{C}_*)$

$\circ_1$ $|\mathcal{S}(M^+, \mathcal{C}_*)/E| \leq |T|$.
Hence it is enough to prove that
\[ \otimes_2 pE_q \Rightarrow p = q. \]

**Proof of \( \otimes_2 \).** Let \( f \) witness \( pE_q \).

Let \( f^+ : A_p = \text{dcl}_{C^{eq}}(\{ |N_p^+| \cup |M^+| \}) \rightarrow A_q \) extends \( f \cup \text{id}_M \) and be an elementary mapping (in \( C^{eq} \)); by non forking calculus it exists and is unique. Obviously it commutes with \( \sigma_* \). Also \( A_p \) (and \( A_q \)) are algebraically closed sets in \( C^{eq} \) by our hypothesis (that is, clause (b)) applied to \( |M_p^+|, |N_p^+|, |M^+| \) hence by 1.7(4), 1.8(4) below, \( f^+ \) can be extended to an automorphism of \( C^{eq} \). So by properties of model completion (and the obvious 1.8(1) below) we are done.

\( \neg(b) \Rightarrow \neg(a) \):

Let \( M_0, M_1, M_2 \) form a counterexample to \( (b) \). So let \( e \in \text{acl}_{C^{eq}}(M_1 \cup M_2) \) hence we can find \( a \in \omega > (M_1), b \in \omega > (M_2) \) and \( n < \omega, \varphi(x, b, a) \) such that

\( \otimes(i) \ C^{eq} \models \varphi[e, b, a] \)

\( (ii) \ \models (\exists \bar{n} x) \varphi(x, b, a) \)

\( (iii) \ n \ \text{minimal under } (i) + (ii) \).

We know \( \varphi(x, b, a) \vdash \text{tp}(e, M_1 \cup M_2) \) and let \( \{ e_0, \ldots, e_{n-1} \} \) list \( \varphi(C^{eq}, b, a) \).

Let \( \bar{e} = \langle e_0, \ldots, e_{n-1} \rangle \). Possibly increasing \( a, b \) for some formula \( \psi = \psi(\bar{x}, b, a) \) with \( \bar{x} = \langle x_\ell : \ell < n \rangle \) we have \( \models \psi(\bar{e}, \bar{b}, \bar{a}) \) and \( \psi(\bar{x}, b, a) \vdash \text{tp}(\bar{e}, M_1 \cup M_2) \).

So we can find \( f \) such that

\( \otimes f \) is an elementary mapping in \( C \)

\[ \text{Dom}(f) = M_1 \cup M_2 \cup \bar{e} \]

\[ f \upharpoonright (M_1 \cup M_2) \text{ is the identity} \]

\[ f(e_0) \neq e_0 \] (but of course \( f \) permutes \( \{ e_\ell : \ell < n - 1 \} \)).

Let \( f(\bar{e}) = \bar{e}'. \) Let \( \bar{e}_0 = \bar{e}, \bar{e}_1 = f(\bar{e}) \).

We can find a sequence of \( C^{eq} \)-elementary mapping \( \langle g_i : i < |T|^+ \rangle \) such that

\[ \text{Dom}(g_i) = \text{acl}_{C^{eq}}(M_1 \cup M_2) \]

\[ g_i \upharpoonright M_2^{eq} = \text{id} \]

\[ \bigcup \{ \text{Rang}(g_i) : i < |T|^+ \}. \]

Now
\[ \text{if } k < \omega, i_0 < \ldots < i_{k-1} < \omega \text{ and } \eta \in {}^2 \] 

\[ p_\eta = \text{tp}(g_{i_0}(\bar{e}_{\eta(0)})^{g_{i_1}(\bar{e}_{\eta(1)})} \ldots g_{i_{k-1}}(\bar{e}_{\eta(k)}), \bigcup_{i < \lvert T \rvert} \text{Rang}(g_i)) \text{ does not depend on } \eta. \]

[Why? By induction on \( k \), hence by transitivity of equality it is enough to prove \( p_\eta = p_\nu \) when \( 1 = \lvert \{ \xi : \eta(\xi) \neq \nu(\xi) \} \rvert \).

By an indiscernible sequence = indiscernible set (= symmetry of nonforking, etc.) without loss of generality \( \eta(0) \neq \nu(0) \). As \( \text{Rang}(\bar{e}_0) = \text{Rang}(\bar{e}) \), without loss of generality \( \bigwedge_{\ell < k-1} \eta(1 + \ell) = 0 = \nu(1 + \ell) \). Lastly, \( \text{tp}(\bigcup_{i > 0} \text{Rang}(g_i), \text{Rang}(g_0)) \) is finitely satisfiable in \( M_2 \) so by the choice of \( \psi \) we are done.]

Now for any \( \eta \in (\lvert T \rvert^+)^2 \) we define the function \( h_\eta \):

\[ \text{Dom}(h_\eta) = M_2^{eq} \cup \bigcup \{ g''_i(M_1^{eq}) : i < \lvert T \rvert^+ \} \cup \{ g_i(\bar{e}) : i < \lvert T \rvert^+ \} \]

\[ h_\eta \upharpoonright M_2^{eq} = \text{identity} \]

\[ h_\eta \upharpoonright g''_i(M_1^{eq}) = \text{identity} \]

\[ h_\eta(g_i(\bar{e})) = \begin{cases} 
  g_i(\bar{e}) = g_i(\bar{e}_0) & \text{if } \eta(i) = 0 \\
  g_i(\bar{e}_1) & \text{if } \eta(i) = 1
\end{cases} \]

We can find \( M_3, M_4, \sigma \) such that

\[ \bigcup \{ g_i(M_1) : i < \lvert T \rvert^+ \} \subseteq M_3 < M_4 < C \]

\[ M_2 \sqcup M_4 \]

\[ M_4 \text{ is saturated of cardinality } > \| M_3 \| \]

\[ \sigma \in \text{Aut}(M_4), \sigma \upharpoonright M_3 = \text{identity} \]

\( (M_4, \sigma) \) is a model of \( T_{\text{aut}}^{\text{inc}}. \)

Now for every \( \eta \in (\lvert T \rvert^+)^2 \) we can find \( (M_5^{\eta}, \sigma) \models T_{\text{aut}} \) such that \( (M_4, \sigma) \subseteq (M_5, \sigma) \) and \( \bar{b}_\eta \) realizing \( \text{tp}_{\text{eq}}(\bar{b}, M_0, C) \) such that
ON MODEL COMPLETION OF $T_{\text{aut}}$ E34

\[ \eta(i) = 0 \iff (\exists \bar{x})(\psi(\bar{x}, \bar{b}, g_i(\bar{a})) \land \sigma(\bar{x}) = x). \]

So \( \{(\exists \bar{x})(\psi(\bar{x}, \bar{y}, g_i(\bar{a})) : i < |T|^+) \) is an independent set of formulas in \((M_4, \sigma)\) hence \(T_{\text{mc}}^{\text{aut}}\) is unstable.

\[(a) \Rightarrow (d): \]

Trivial.

\[\neg(b) \Rightarrow \neg(c): \]

Included in the proof of \(\neg(b) \Rightarrow \neg(a)\). \(\square_{1.6}\)

1.7 Observation. Assume \(T_{\text{mc}}^{\text{aut}}\) exists, \(T_\ast\) any completion of it.
1) If \(\mathfrak{C}\) is a saturated model of \(T\) of cardinality \(\kappa = \kappa^{<\kappa}\), can be expanded to a model \(\mathfrak{C}_\ast\) of \(T_\ast\).
2) If \(M \models T, \sigma \in \text{Aut}(M)\), let \(\sigma^\text{eq}\) be the natural extension of \(\sigma\) to an automorphism of \(M^\text{eq}\), then (it exists and is unique) \((M^\text{eq}, \sigma^\text{eq}) \models (T^\text{eq})_{\text{aut}}\).
3) \((T^\text{eq})_{\text{aut}}\) has a model completion \(T\) and there is a natural one to one correspondence between the completions of the model completions of \((T^\text{eq})_{\text{aut}}\) and \(\{T_{\ast*} : T_{\ast}\) a model completion of \(T_{\text{mc}}^{\text{aut}}\}\) any one of the former is essentially bi-interpretable with the corresponding one of the latter (but we have the elements not in any \(P_{E(\bar{x}, \bar{y})}\)).
4) Let \(\mathfrak{C}_\ast = (\mathfrak{C}, \sigma_\ast)\) be a \(\kappa\)-saturated model of \(T_\ast\) expanding \(\mathfrak{C}\). If \(A_\ell \subseteq \mathfrak{C}^\text{eq}, A_\ell = \text{acl}_{\mathfrak{C}^\text{eq}}(A_\ell), A_\ell\) closed under \(\sigma_\ast, f\) is an \(\mathfrak{C}^\text{eq}\)-elementary mapping from \(A_1\) onto \(A_2\) commuting with \(\sigma\) then \(f\) can be extended to an automorphism of \((\mathfrak{C}^\text{eq})_{\text{aut}}\) (it is \(\mathfrak{C}^\text{eq}\) expanded by \(\sigma\) naturally extended to \(\sigma^+\)).

1.8 Observation. 1) \(M\) is a model of \(T, \sigma_\ast \in \text{Aut}(M) \iff (M, \sigma_\ast)\) is a model of \(T_{\text{aut}}\).
2) If \(M \prec \mathfrak{C}\) and \((M, \sigma_\ast)\) as a model of \(T_{\text{aut}}\) then for one and only one \(\sigma^\text{eq}_\ast \in \text{Aut}(M^\text{eq})\) extend \(\sigma_\ast\).
3) If \(M \prec \mathfrak{C}, \sigma^\text{eq}_\ast \in \text{Aut}(M^\text{eq})\) then \(\sigma^\text{eq}_\ast \upharpoonright M \in \text{Aut}(M)\).
4) If \(A_\ell \subseteq \mathfrak{C}^\text{eq}\) and \(A_0 = \text{acl}_{\mathfrak{C}^\text{eq}}(A_0)\) and \(f_\ell\) is an \(\mathfrak{C}^\text{eq}\)-elementary mapping from \(A_\ell\) onto \(A_\ell\) for \(\ell = 0, 1, 2\) and \(f_0 \subseteq f_1, f_0 \subseteq f_2\) then for some automorphism \(F\) of \(\mathfrak{C}^\text{eq}\) \(F \upharpoonright A_0 = \text{id}_{A_0}\) and \(f_2 \cup Ff_1F^{-1}\) is an elementary mapping in \(\mathfrak{C}^\text{eq}\) (hence can be extended to an automorphism of \(\mathfrak{C}^\text{eq}\); if \(A_1 \bigcup\bigcup A_2\) then without loss of generality \(F \upharpoonright (A_1 \cup A_2) = \text{id}_{A_1 \cup A_2}\)).
2.1 Example: There is $T$ such that:

(a) $T$ is as in 1.1, stable $T_{\text{mc aut}}$ exists. Moreover $T$ is superstable, countable $I(8_\alpha, T) \leq 2^{\lfloor \alpha \rfloor}$ for $\alpha \geq 2^{\aleph_0}$ (hence NDOP, NOTOP, shallow with small depths, with $\leq 2^{\aleph_0}$ dimensions)

(b) $T_{\text{mc aut}}$ exist

(c) some completions of $T_{\text{mc aut}}$ are stable and some are not.

Proof. Let us define $M, I$

$|M|$ is $\{(\eta, k, n, \ell) : k, n < \omega, \ell < 2$ and $\eta \in \omega^2\}$ and $k = n \Rightarrow \ell = 0$

$E_n^M$, a two-place relation is $\{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2) \in |M| \times |M| : \eta_1 \upharpoonright n = \eta_2 \upharpoonright n\}$

$E^M$, a two-place relation is $\{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2) \in |M| \times |M| : \eta_1 = \eta_2\}$

$Q^M$, a one-place relation is $\{(\eta, k, n, \ell) \in |M| : k = n\}$

$F^M_1$, a one-place relation is: $F^M_1(\eta, k, n, \ell) = (\eta, k, k, 0)$

$F^M_2$, a one-place relation is: $F^M_2(\eta, k, n, \ell) = (\eta, n, n, 0)$

Let $T = \text{Th}(M)$. Clearly it satisfies (a):

\[ \text{\dag} \]

$T_{\text{mc aut}}$ exists.

[Why? Check that there are no obstructions.]

\[ \text{\dag} \]

$T_{\text{mc aut}}$ has an unstable completion.

[Why? By 1.6, or more specifically, see below.]

We shall now prove

\[ \text{\dag} \]

for $T_*$ a completion of $T_{\text{mc aut}}, T_*$ is unstable if:

for some $M^+ \models T_*$, for some $a \in M^+$ we have $\bigwedge_n aE_n(\sigma^{M^+}(a))$ or just

$(\exists m) \bigwedge_{n<\omega} aE_n((\sigma^{M^+})^m(a))$, i.e., for some $m^* \in [1, \omega)$ we have $\bigwedge_n aE_na_{M^+}$

where $a_0 = a, a_{\ell+1} = \sigma^{M^+}(a_{\ell})$ for $\ell < \omega$.

Let $m^*, a, (a_\ell : \ell < \omega)$ be as above. We define $N$ a model of $T_*$: let $|N|$, the universe of $N$ be

$|M^+| \cup \{(m, k, n, \ell) : m < m^*, k, n < \omega, \ell < 2, k = n \Rightarrow \ell = 0\}$

we assume no incidental identification.
ON MODEL COMPLETION OF $T_{\text{aut}}$ E34

$E_n^N : \begin{cases} 
E_n^N \text{ is an equivalence relation} \\
E_n^N \upharpoonright |M^+| = E_n^M \\
every (m,k,n,\ell) \in |N|\downarrow |M^+| \text{ is } E_n \text{-equivalent to } a_m 
\end{cases}$

$\begin{cases} 
E^N \text{ is an equivalence relation} \\
E^N \upharpoonright |M^+| = E^N \\
\{(m,k,n,\ell) \in |N|\downarrow |M^+| : k, n < \omega, \ell < 2, k = n \Rightarrow \ell = 0\} \\
is an E^N \text{-equivalence class (for each } m < m^*) \\
Q^N = Q^N \cup \{(m,k,k,0) : k < \omega\} \\
\end{cases}$

$F_1^N \text{ extends } F_1^{M^+}, F_1^N((m,k,n,\ell)) = (m,k,k)$

$F_2^N \text{ extends } F^{M^+}, F_2^N((m,k,n,\ell)) = (m,n,n)$.

Easily

$\Box_1 M^+ \upharpoonright \tau_T \prec N$.

Now we define an automorphism $\sigma^+$ of $N$:

$\Box_2 \sigma^+ \upharpoonright |M^+| = \sigma^{M^+}$

$\Box_3$ if $m_1, m_2 < m^*, m_2 = m_1 + 1 \mod m^*$ then

$\sigma(m_1, n, k, \ell)$ is:

$(m_2, n, k, 1 - \ell)$ if $m_1 = m^* - 1 \& n < k$

$(m_2, n, k, \ell)$ otherwise.

Easy to check that $\sigma^+ \in \text{Aut}(N)$, so $(N, \sigma) \supseteq M^+$ is a model of $T_{\text{aut}}$. As $T_{\text{aut}}^{\text{mc}}$ exists and $M^+ \models T_{\text{aut}}^{\text{mc}}$ there is a model $N^+ \models T_{\text{aut}}^{\text{mc}}$ such that $M^+ \prec M^+, (N, \sigma) \subseteq N^+$.

Let

$\varphi(x, y) = Q(x) \& Q(y) \& xEy \& (\exists z)(F_1(z) \& F_2(z) = y \& (\sigma^{m^+}(z) \neq z))$

This is a first order formula in $L(\tau_{\text{Th}(M^+)}) = L(\tau_{T_{\text{aut}}})$ and $N^+ \models \varphi[b_n, b_k]$ iff $n < \omega$ where $b_n = (0, n, n, 0) \in N \subseteq N^+$, so this formula has the order property in $\text{Th}(N^+) = \text{Th}(M^+)$. So $\text{Th}(M^+)$ is unstable as required in $\Box_2^+$.
\( \iff \) if \( T \) is a completion of \( T_{\text{mc}}^{\text{aut}} \) not satisfying the demand in \( \iff^+ \) then \( T \) is stable.

[Why? As any model \( M^+ \) of \( T, \sigma^M \) acts as a permutation of \( |M^+|/E^{M^+} \) which has no fix point and even no finite cycle. Now reflect.]

\( \iff \) there is a completion \( T \) of \( T_{\text{mc}}^{\text{aut}} \) which is stable.

Why? Let \( f \) be a permutation of \( \omega^2 \) such that

\begin{enumerate}
  \item \( \eta, \nu \in \omega^2 \land \eta \upharpoonright n = \nu \upharpoonright n \Rightarrow f(\eta) \upharpoonright n = f(\nu) \upharpoonright n \)
  \item for every \( m < \omega \geq 2 \) for some \( n < \omega \) we have if \( \eta \in \omega^2 \) then \( \eta, f^m(\eta) \) are not \( E_n \)-equivalent.
\end{enumerate}

Easy to construct (or use \( \prod_{n<\omega} (n+1) \) instead \( \omega^2 \) and define \( M^+ \), a \( \tau_{T_{\text{aut}}} \)-expansion of \( M \) by defining

\[ \sigma^{M^+}(\langle \eta, k, n, \ell \rangle) = (f(\eta), k, n, \ell). \]

So if \( M^+ \models N^+ \models T_{\text{mc}}^{\text{aut}} \) then \( T \) is \( \text{Th}(N^+) \) fail the demand in \( \iff^+ \) hence by \( \iff \) it is stable as required (and it is uniquely determined by \( M^+ \), really just the action on \( \text{acl}_{C^\text{eq}}(\emptyset) \), suffice. So \( \iff \) holds. \( \square_{2.1} \)

### 2.2 Discussion
It seems reasonable that we can characterize when this occurs thus answering fully 0.1; see below.

A closely related example is

#### 2.3 Claim. There is \( T \) such that:

\begin{enumerate}
  \item \( T \) is stable (complete countable first order theory) and has elimination of quantifiers for simplicity
  \item \( T \) is superstable and small, i.e., with countable \( D(T) \)
  \item \( T_{\text{aut}} \) has no model completion
  \item some \( T_{\text{aut}}(M^+) \) has a model completion where
\end{enumerate}

#### 2.4 Definition. 1) For a model \( M^+ = (M, \sigma^M) \) of \( T_{\text{Aut}} \) let \( T_{\text{aut}}(M^+) = T_{\text{aut}} \cup \text{Th}(M, c)_{c \in M} \cup \{ \sigma(c_1) = c_2 : \sigma^M(c_1) = c_1 \} \).

2.5 Remark. Actually we can use any completion of \( T_{\text{aut}} \cup \text{(the action of \( \sigma \) on \( \text{acl}_{C^\text{eq}}(\emptyset, C_T) \) (i.e., on the \( E \)-equivalence classes for each \( n \))} \).
Proof. Define $M$

(a) $\tau_M = \{E_n, P_n : n < \omega\} \cup \{E, E_4\}$

(b) $|M| = \{(\eta, k, n, \ell) : \eta \in \omega^2, k < \omega, n < \omega, \ell < 2\}$

(c) $E_n^M = \{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2) \in |M| \times |M| : \eta_1 \equiv n = n_2 \equiv n\}$

(d) $E^M = \{(\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2) \in |M| \times |M| : \eta_1 = \eta_2 \text{ and } k_1 = k_2\}$

(e) $E^*_M = \{((\eta_1, k_1, n_1, \ell_1), (\eta_2, k_2, n_2, \ell_2)) \in |M| \times |M| : \eta_1 = \eta_2, k_1 = k_2, n_1 = n_2\}$

(f) $P_n^M = \{(\eta, k, n, \ell) \in |M| : n = m\}$.

We choose $\sigma^M$ such that $\sigma(\eta, k, n, \ell) = (\eta', k, n, \ell)$ and $(\eta, \eta')$ are as in the proof of 2.1.

Remark. If we let $(d)'$ be as in 2.8 below we add $\sigma =$ the identity then $(a) + (c) + (d)'$ is impossibly by [BlSh 759].

Actually the case $\sigma$ is the identity on some $M$ is the real one because

2.6 Claim. For any first order complete $T_1$ (with $\tau_{T_1}$, a set of predicates for simplicity) there is $T$ such that:

(a) $T$ is first order complete

(b) if $a \in M, M \models T$ then we can interpret $T_1$ in $(M, a)$

(c) $\tau_T \setminus \tau_{T_1}$, countable

(d) some $T_{\text{aut}}(M^+)$ has a model completion.

Proof. As in 2.3 without $E_4, P_n(n < \omega)$ in any $E^M$-equivalence class we “plant” a model of $T_1$.

2.7 Claim. Let $T_*$ be a completion of $T_{\text{aut}}^\text{mc}$.

The following are equivalent:

Condition (a): $T_*$ is stable.

Condition (b): If $T$ is stable and $(\alpha) + (\beta) + (\gamma)$ below holds, then $(\ast)$ below holds where

\[
\begin{align*}
(\alpha) \quad M_0^+ < M_1^+ < M_3^+ \text{ for } \ell = 1, 2, M_0 \models T_*, M_\ell \models T_{\text{aut}} \text{ for } \ell = 1, 2, 3 \text{ and } M_3 \\
(\beta) \quad M_\ell = M_\ell \upharpoonright \tau_T \text{ and } M_1 \bigcup_{M_0} M_2 \text{ without loss of generality } M_3 < \mathcal{C} = \mathcal{C}_T
\end{align*}
\]
(γ) if $f$ is an elementary mapping from $\acl_{\mathcal{E}_{\mathcal{E}}}(M_1 \cup M_2)$ onto itself extending $\sigma^{M_1} \cup \sigma^{M_2}$

(∗) there is an elementary mapping $h$ from $\acl_{\mathcal{E}_{\mathcal{E}}}(M_1 \cup M_2)$ onto itself such that $h \upharpoonright (M_1 \cup M_2) = \text{identity}_{M_1 \cup M_2}$ and $hf h^{-1} = \sigma^{M_3} \upharpoonright \acl_{\mathcal{E}_{\mathcal{E}}}(M_1 \cup M_2)$.

Proof. $(b) \Rightarrow (a)$:
As in the proof of 1.6.

$\neg (b) \Rightarrow \neg (a)$:
We can use compactness to replace $\neg (b)$ by a finite failure, and continue as in the proof of 1.6.

2.8 Remark. We can make $\neg (b)$ more explicit as in the proof of 2.7.
As by [KkSh 748], if $T_{\text{mc}}^\text{aut}$ exists, then $T$ fails the strict order property. It seems reasonable to ask if any $T_{\text{mc}}^\text{aut}$, which exists, can have the strict order property. As we understand the stable case, it seems reasonable to deal with it. In fact, more turn out to hold.

3.1 Claim. [T as in 1.1.] If $T$ is stable, any completion $T_*$ of $T_{\text{mc}}^\text{aut}$ satisfies $\text{NSOP}_3$ (see [Sh 500, §2] and [ShUs 789]).

Proof. 1) Clause (a):

Let $T_*$ be completion of $T_{\text{mc}}^\text{aut}$ and $\varphi(x, y)(\ell g(x) = \ell g(y) = n^* < \omega)$ a first order formula in $L(\tau_T)$ and for some $M \models T_*$ we have $M \models \varphi(a_n, a_m)_n^{m(n<m)}$. Hence we can find an E.M.-template $\Phi$ such that $\tau_\Phi \supseteq \tau_T = \tau_T \cup \{\sigma\}$ and for linear orders $I \subseteq J$, $\text{EM}(I, \Phi) < \text{EM}(J, \Phi) \neq T_*$, with skeleton $\langle a_t : t \in J \rangle$ such that $\text{EM}(J, \Phi) \models \varphi[a_s, a_t]^{[s<t]}$ for $s, t \in J$ (so $a_t \in \text{EM}(\{t\}, \Phi)$ (see, e.g., [Sh:c, VII] or [Sh:e, III]). Now (recalling that $\text{EM}_T(I, \Phi) = \text{EM}(I, \Phi) \upharpoonright \tau_j$) without loss of generality

- if $I_1, I_2 \subseteq J, I_0 = I_1 \cap I_2$ and if $t \in I_1 \setminus I_0$ then there is $s \in I_0$ such that $s < t$ & $[s, t] \cap I_0 \subseteq I_0$ or $t < s$ & $[t, s] \cap I_2 \subseteq I_0$ then $\text{tp}_{\tau_T}(\text{EM}_{\tau_T}(I_1, \Phi), \text{EM}_{\tau_T}(I_2, \Phi))$ is f.s. (finitely satisfiable) in $\text{EM}_{\tau_T}(I_0, \Phi)$

[Why? Let $I \times \mathbb{Z}$ be ordered lexicographically, choose $\Phi'$ such that $\text{EM}(I, \Phi') = \text{EM}(I \times \mathbb{Z}, \Phi)$, with skeleton $\bar{a}_t = \bar{a}_{(t,0)}$; can look at [Sh 394].]

For $u \subseteq \{0, 1, 2\}$ let $M_u^2 = \text{EM}(u, \Phi)$ and if $|u| = |v|$ both subsets of $\{0, 1, 2\}$ let $f_{v, u}$ be the canonical isomorphism from $M_u$ onto $M_v$. Let $M_u^1 = M_u^2 \upharpoonright \tau_T$, $M_u^0 = M_u^2 \upharpoonot \tau_T$. Let $N$ be such that $M_{\{0,1,2\}}^0 < N$, $N$ is $\|M_{\{0,1,2\}}^0\|^{+}$-saturated

- in $N$, $\bigcup_{M_{\{0\}}^0, M_{\{1\}}^0, M_{\{2\}}^0} M_{\emptyset}^0$

[Why? By @1 and nonforking calculus.]

Let $g_0 =: f_{\{0\}, \{2\}} \cup f_{\{2\}, \{0\}}$

- $g_0$ is an elementary mapping (inside $N$)

[Why? Nonforking calculus.]

Let $g_1$ be an elementary mapping inside $N$ extending $g_0$ with domain $M_{\{0,2\}}^0$.

Let $M_{\{0,2\}}^{0,*} = g(M_{\{0,1\}}^0)$. 

\[\text{ON MODEL COMPLETION OF } T_{\text{AUT}} \text{ E34} 15\]
Let $M^{1,*}_{\{0,2\}}$ be an expansion of $M^{0,*}_{\{0,2\}}$ by an automorphism $\sigma^{M^{1,*}_{\{0,2\}}}$ such that $g_1$ is an isomorphism from $M^{1}_{\{0,2\}}$ onto $M^{1,*}_{\{0,2\}}$, clearly exists.

As $N$ is a model of the stable theory $T$ without loss of generality $tp_{L^\ast(\tau)}([M^{1,*}_{\{0,2\}}], |M^0_{\{0,1,2\}}|)$ does not fork over $|M^0_{\{0\}}| \cup |M^0_{\{2\}}|$. Now the point is that

\[
\exists h = \sigma^{M^{1}_{\{0,1\}} \cup \sigma^{M^{1,*}_{\{0,2\}}} \cup \sigma^{M_{\{1,2\}}}} is a permutation of $|M^{1,*}_{\{0,1\}}| \cup |M^1_{\{0,1\}}| \cup |M^1_{\{1,2\}}|$

and is an elementary mapping.

[Why? Let $B_0 = |M^0_{\{0\}}| \cup |M^0_{\{2\}}|$, $B_1 = |M^0_{\{0,1\}}| \cup |M^0_{\{2,2\}}|$. By [Sh:c, XII], the pair $(B_0, B_1)$ satisfies the T.V. condition inside $N$ (i.e., if $\varphi(x, y) \in L(\tau_T)$, $N \models \varphi[\bar{a}, \bar{b}]$, $\bar{a} \subseteq B_1, \bar{b} \subseteq B_0$ then for some $\bar{a}' \subseteq B_0, N \models \varphi[\bar{a}', \bar{b}]$. Moreover, we can allow $\bar{b} \subseteq |M^{0,*}_{\{0,2\}}|$ then this follows.]

So for some $N', N < N' \models T$ and there is an automorphism $h'$ of $N'$ extending $h$ and we can extend $(N', h')$ to a model $(N'', h'')$ of $T_*$. By this model clearly

\[
(N'', h'') \models \varphi[\bar{a}_0, a_1] \text{ using } M^1_{\{0,1\}}
\]

\[
(N'', h'') \models \varphi[\bar{a}_1, \bar{a}_2] \text{ using } M^1_{\{1,2\}}
\]

\[
(N'', h'') \models \varphi[\bar{a}_2, \bar{a}_0] \text{ using } M^{1,*}_{\{0,2\}} \text{ and}
\]

$g_1$ being an isomorphism from $M^{1}_{\{0,2\}}$ onto $M^{1,*}_{\{0,2\}}$.

This is enough to show $T_* \models \text{NDOP}_3$.

3.2 Claim. $T$ is stable or just simple then any $T_*$ (assuming it exists, $K_*$ in general) is simple.

Proof. We write it for $K_*$. Choose $\kappa = \text{cf}(\kappa) > |T|$ and $\mu$ a strong limit singular cardinal of cofinality $\kappa$. Let $\langle \lambda_i : i < \kappa \rangle$ be increasing with limit $\mu$, $\lambda_0 > \kappa, \lambda_\kappa = \mu$. $\langle s M^+_i : i < \kappa \rangle$ is an increasing sequence of elementary submodels of $\mathcal{E}_{K_*}$ (check notation), $\|s M^+_i\| = 2^{\lambda_i}, s M^+_i$ is $\lambda^+_i$-homo universal (in $K_{\text{aut}}^\text{ec}(T)$), $M^+ = \cup \{s M^+_i : i < \kappa\}$. Let $\langle p^+_i : i < \mu^+ \rangle$ be a sequence of existential types in $L(\tau \cup \{\sigma\})$ each of cardinality $\leq \kappa$ with domain $\subseteq M$, and we shall prove that for some $\alpha < \beta < \mu^+, p^+_\alpha \cup p^+_\beta$ is realized in $\mathcal{E}_{K_*}$, this suffices.
For each $\alpha < \mu^+$, we can find $a_\alpha \in \mathfrak{C}_{K^*}$ realizing $p_i$ and $N_{3,\alpha}^+ < \mathfrak{C}_{K^*}$ of cardinality $\kappa$ to which $a_i$ belongs and $N_{2,\alpha}^+ = N_{3,\alpha}^+ \cap M^+ < M^+$ and $\text{tp}_{\mathfrak{C}}(\langle |N_{3,\alpha}^+|, |M^+| \rangle)$ does not fork over $|N_{2,\alpha}^+|$. Let $N_{1,\alpha}^+ < N_{3,\alpha}^+$ be of cardinality $|T|$ such that $a_i \in N_{1,\alpha}^+$, $\text{tp}_{\mathfrak{C}}(\langle |N_{1,\alpha}^+|, |M^+| \rangle)$ does not fork over $|N_{0,i}^+|$ where $N_{0,i}^+ = N_{1,i}^+ \cap M^+ < M^+$. Without loss of generality $\alpha < \mu^+ \Rightarrow N_{0,\alpha}^+ = N_0^+$ and for every $\alpha, \beta < \mu^+$ there is an isomorphism $h_{\beta,\alpha}$ from $N_{3,\alpha}^+$ onto $N_{3,\beta}^+$ mapping $a_\alpha, N_{1,\alpha}^+, N_{2,\alpha}^+$ to $a_\beta, N_{1,\beta}^+, N_{2,\beta}^+$ respectively and $h_{\beta,\alpha} \mid N_0^+ = \text{id}_{N_0^+}$. Moreover, without loss of generality for some well ordering $<^*$ all $h_{\beta,\alpha}$ are order preserving.

Let $\kappa > \bar{\kappa}$, $\mathfrak{B}$ be an elementary submodel of $(\mathcal{H}(\chi), \in)$ of cardinality $2^\kappa$ such that $T, \kappa, \mu, \mathfrak{C}, \mathfrak{C}_{K^*}, M^+, \langle N_i^+ : i < \mu^+ \rangle$ belongs and such that $[\mathfrak{B}] \leq \kappa \subseteq \mathfrak{B}$. Now choose $\alpha(2) \in \mu^+ \setminus \mathfrak{B}$, and let $M_0^+ = N_{1,\alpha}^+ \upharpoonright \mathfrak{B}$. Clearly $M_0^+ < M^+$ and there is $\alpha(1) \in \mu^+ \cap \mathfrak{B}$ such that $h_{\alpha(1),\alpha(2)}$ is the identity on $M_0^+$. [FILL?]
REFERENCES.


