A COMMENT ON “$p < t$”

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Abstract. Dealing with the cardinal invariants $p$ and $t$ of the continuum we prove that $m = p = \aleph_2 \Rightarrow t = \aleph_1$. In other words if MA$_{\aleph_1}$ (or a weak version of this) then (of course $\aleph_2 \leq p \leq t$ and) $p = \aleph_2 \Rightarrow p = t$. This is based on giving a consequence.

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§0

The cardinal invariants $p$ measure when a family of infinite subsets of $\omega$, with any finitely many members, has a pseudo intersection (see below). A relative is $t$, which deals with towers, i.e. families well ordered by almost inclusion. They are classical cardinal invariants and closely related. In fact Rothenberg [Ro39], [Ro48] proved (stated in our terminology) that $p = \aleph_2 \Rightarrow p = t$ and ask if $p = t$.

We prove (2.2) that $m = p = \aleph_2 \Rightarrow p = t$; considering that MA$_{\aleph_0}$ is a theorem and $m = \aleph_2 \Rightarrow$ MA$_{\aleph_1}$. The parallelism with Rothenberg is clear. The reader may conclude that probably $m = p \Rightarrow p = t$; not unreasonable but it seemed that (\Sh:769): CON(MA$_{<\lambda} + p = \lambda + t = \lambda^+$). The proof of 2.2 uses a characterization of $p < t$ from §1.
§1 A reduction

1.1 Hypothesis. \( \lambda = p < t \).

Our aim is to prove 1.13, i.e., \( p < t \) iff \((\omega, <^*)\) has a peculiar cut. We give a self-contained proof (except using Bell theorem); this will give the background for a try to prove the consistency of \( \text{CON}(p < t) \) in [Sh:F769]. The results up to 1.8 are well known and essentially covered by [BaJu95, §2.2] in particular \( t \leq \mathcal{M} \) is of Piotrowski and Szymanski.

Note also that Szymanski had proved that \( p \) is regular (see, e.g., Fremlin [Fre], proposition 21K).

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1.2 Convention. \( A, B \) denote members of \([\omega]^{\aleph_0}\).

1.3 Definition. 1) \( \mathcal{B} \) exemplifies \( p \) if:

(a) \( \mathcal{B} \subseteq [\omega]^{\aleph_0} \) has cardinality \( \lambda \)
(b) \( A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B} \)
(c) for no \( A \in [\omega]^{\aleph_0} \) do we have \( B \in \mathcal{B} \Rightarrow A \subseteq^* B \).

1.4 Claim. For any \( \mathcal{B} \) exemplifying \( p \) there are \( \kappa = \text{cf}(\kappa) < \lambda \) and \( \subseteq^* \)-decreasing \( \langle A_i : i < \kappa \rangle \) such that \( \bigwedge_{i < \kappa} A \subseteq^* A_i \Rightarrow \bigvee_{B \in \mathcal{B}} A \cap B \in [\omega]^{<\aleph_0} \), and of course,

\[
\bigwedge_{i < \kappa} \bigwedge_{B \in \mathcal{B}} A_i \cap B \text{ is infinite}.
\]

Proof. Let \( \mathcal{B} = \{ B_i : i < \lambda \} \) and try to choose \( A_i \in [\omega]^{\aleph_0} \) by induction on \( i < \lambda \) such that

(a) \( j < i \Rightarrow A_i \subseteq^* A_j \)
(b) \( B \in \mathcal{B} \Rightarrow |B \cap A_i| = \aleph_0 \)
(c) \( i = j + 1 \Rightarrow A_i \subseteq B_j \).

If we succeed then \( \{ A_i : i < \lambda \} \) has no pseudo-intersection so \( t \leq \lambda \), contradiction. So for some \( i < \lambda \) we cannot choose \( A_i \). Easily \( i \) is a limit ordinal and let \( \kappa = \text{cf}(i) \) so \( \kappa \leq i < \lambda \); choose \( \langle j_\varepsilon : \varepsilon < \kappa \rangle \) which is increasing with limit \( i \). So \( \langle A_{j_\varepsilon} : \varepsilon < \kappa \rangle \) is as required.
1.5 Claim. \( b \geq t > \lambda \).

Proof. Easy.

1.6 Claim. If \( A = \{ A_i : i < \delta \} \) is a sequence of members of \( [\omega]^{\aleph_0} \) (or even \( < t \)) and \( B = \{ B_n : n < \omega \} \) is \( \subseteq^* \)-decreasing and \( i < \delta \) \& \( n < \omega \Rightarrow A_i \cap B_n \subseteq \omega^{\aleph_0} \) and \( i < j < \delta \) then for some \( A \subseteq [\omega]^{\aleph_0} \) we have \( i < \delta \Rightarrow A \subseteq^* A_i \) and \( n < \omega \Rightarrow A \subseteq^* B_n \).

Proof. Without loss of generality \( B_{n+1} \subseteq B_n \) and \( \emptyset = \bigcap \{ B_n : n < \omega \} \) (use \( B'_n = \bigcap_{\ell \leq n} B_{\ell}\}). For each \( i < \delta \) let \( f_i \in \omega^\omega \) be defined by \( f_i(n) = \min\{k+1 : k \in B_n \cap A_i \text{ and } k > f(m) \text{ for every } m < n\} \), so by 1.5 there is \( f \in \omega^\omega \) such that \( \bigwedge_{i<\kappa} f_i \leq^* f \) and \( n < f(n) < f(n+1) \) for \( n < \omega \). Let \( B^* = \bigcup\{ (B_{n+1}\cap [0, n)) \cap [0, f(n+1)) : n < \omega \} \) so \( B^* \subseteq [\omega]^{\aleph_0} \) as for \( n \) large enough, \( \min\{ (B_{n+1}\cap [0, n)) \cap [0, f(n+1)) \} \) and \( \exists \in n\{ (B_{n+1}\cap [0, n)) \cap A_0 \neq \emptyset \} \). Clearly \( n < \omega \Rightarrow B^* \setminus [0, f(n)] \subseteq B_n \Rightarrow B^* \subseteq^* B_n \) and also \( i < \kappa \Rightarrow A_i \cap B^* \subseteq^* B_n \) and \( i < \delta \Rightarrow A_i \subseteq^* A \) so apply \( t > \lambda \) to \( \{ A_i \cap B^* : i < \delta \} \) getting \( A^* \), it is as required. □

1.7 Definition. 1) \( S = \{ \bar{n} : \bar{n} \in [\eta_0 : n \in A] \text{ where } A \in [\omega]^{\aleph_0}, \eta_n \in [n, k)^2 \text{ for some } k \in (n, \omega) \} \) and let \( \text{Dom} (\bar{n}) = A \) and let \( \text{set} (\bar{n}) = \cup \{ \text{set} (\eta_n) : n \in \text{ Dom}(\bar{n}) \} \) where \( \text{set}(\eta_n) = \{ \ell : \eta_n(\ell) = 1 \} \).

2) For \( \bar{n}, \bar{\nu} \in S \) let \( \bar{n} \leq^* \bar{\nu} \) mean that for every \( n \) large enough, \( n \in \text{ Dom}(\bar{\nu}) \Rightarrow n \in \text{ Dom}(\bar{n}) \wedge \eta_n \subseteq \nu_n \).

3) For \( \bar{n}, \bar{\nu} \in S \) let \( \bar{n} \leq^{**} \bar{\nu} \) mean that for every \( n \in \text{ Dom}(\bar{\nu}) \) large enough for some \( m \in \text{ Dom}(\bar{n}) \) we have \( \eta_m \subseteq \nu_n \) (as functions).

4) For \( \bar{n} \in S \) let \( C_{\bar{n}} = \{ \nu \in \omega^2 : (\exists \in n)(\eta_n \subseteq \nu) \} \).

1.8 Claim. 1) The union of \( \leq \lambda \) meager subsets of \( \omega^2 \) is meagre.

2) Every \( \leq^* \)-increasing sequence members of \( S \) of length \( \leq \lambda \) (and even \( < t \)) has an \( \leq^* \)-upper bound.

3) \( (S, \leq^*) \) is \( \lambda \)-directed.

4) Similarly for \( \leq^{**} \).

Proof. 1) Can be translated to part (2) as by 1.9 below \( \{ C_{\bar{n}} : \bar{n} \in S \} \) is dense among the co-meagre subsets of \( \omega^2 \).
2) By 1.6. In full, let \( \langle \tilde{\eta}^\alpha : \alpha < \delta \rangle \) be as \( \leq_* \)-increasing sequence and \( \delta < \tau \). Let \( A_\alpha^* := \text{Dom}(\tilde{\eta}^\alpha) \) for \( \alpha < \delta \), so \( A_\alpha^* : \alpha < \delta \) is a \( \subseteq^* \)-decreasing sequence of members of \( [\omega]^\aleph_0 \). As \( \delta < \tau \) there is \( A^* \in [\omega]^\aleph_0 \) such that \( \alpha < \delta \Rightarrow A \subseteq^* A_\alpha \). Now for \( n < \omega \) we define \( B_n = \{ \eta \} : \text{for some } m < k \text{ we have } m \in A^*, n \leq m < k < \omega \) and \( \eta \in [m,k)^2 \), and for \( \alpha < \delta \) we define \( A_\alpha =: \{ \eta \} : \text{for some } n \in \text{Dom}(\tilde{\eta}^\alpha) \) we have \( \eta^\alpha \leq \eta \}. \) Now easily the assumptions of \( ? \) holds (well, replacing \( \omega \) by \( \rightarrow \) scite\{p.4\} undefined \( B_0 \)!!) If \( A \) is as in the conclusion of \( ? \) we let \( A' = \{ n \} : \text{for some } \eta \in A \) we have \( \rightarrow \) scite\{p.4\} undefined \( \eta \in \cup\{ [n,k)^2 : k \in (n, \omega) \} \), it is necessarily infinite and let \( \tilde{\eta}^* = \langle \eta_n : n \in A' \rangle \) where \( \eta_n \) is any member of \( A \cap B_n \setminus B_{n+1} \).

3) Easy, too.

4) Similar to the proof of part (2).

\[ \square_{1.8} \]

1.9 Observation. 1) If \( \tilde{\eta} \leq^* \tilde{\nu} \) and then \( \tilde{\eta} \leq^{**} \tilde{\nu} \) which implies \( C_\rho \subseteq C_\eta \).

2) For every \( \tilde{\eta} \in S \) and meagre \( B \subseteq \omega^2 \) there is \( \tilde{\nu} \in S \) such that \( \tilde{\eta} \leq^* \tilde{\nu} \) and \( C_\rho \cap B = \emptyset \).

1.10 Definition. For \( \mathcal{A} \subseteq [\omega]^\aleph_0 \) let \( S_\mathcal{A} = \{ \tilde{\eta} \in S : (\forall B \in \mathcal{A})[\text{set}(\tilde{\eta}) \subseteq^* B] \) and for every \( n, \text{set}(\eta_n) \neq \emptyset \}. \) If we write \( A' = \langle A'_i : i < \alpha \rangle \) instead \( \mathcal{A} \) we mean \( \{ A'_i : i < \alpha \} \).

1.11 Claim. If \( \mathcal{A} \subseteq [\omega]^\aleph_0 \) with f.i.p. and \( |\mathcal{A}| < \lambda \) then \( S_\mathcal{A} \neq \emptyset \).

Proof. Let \( A \in [\omega]^\aleph_0 \) be such that \( B \in \mathcal{A} \Rightarrow A \subseteq^* B \), exists as \( |\mathcal{A}| < \lambda \). Let \( k_n = \text{Min}\{ k : k > n \text{ and } k \in A \} \), and let \( \eta_n \in [n,k_n+1)^2 \) be defined by \( \eta_n(\ell) = 0 \) if \( \ell \in [n,k_n) \) and is 1 if \( \ell = k_n \). Now \( \langle \eta_n : n < \omega \rangle \) is as required. \[ \square_{1.11} \]

1.12 Claim. Let \( \bar{A} = \langle A_i : i < \kappa \rangle \) be \( \subseteq^* \)-decreasing, \( \kappa < \lambda \) and \( \mathcal{B} = \{ B_\alpha : \alpha < \lambda \} \) exemplifying \( p \) as in 1.4 and let \( \text{pr}: \lambda \times \lambda \to \lambda \) be one to one onto satisfying \( \text{pr}(\alpha_1, \alpha_2) \geq \alpha_1, \alpha_2 \). Then we can find \( \langle \tilde{\eta}^\alpha : \alpha \leq \lambda \rangle \) such that

(a) \( \tilde{\eta}^\alpha \in S_{\bar{A}} \) for \( \alpha < \lambda \) and \( \tilde{\eta}^\lambda \in S \) (sic!)

(b) \( \langle \tilde{\eta}^\alpha : \alpha \leq \lambda \rangle \) is \( \leq^* \)-increasing

(c) if \( \alpha < \lambda \) and \( n \in \text{Dom}(\tilde{\eta}^{\alpha+1}) \) large enough then \( \text{set}(\eta_n^{\alpha+1}) \cap B_\alpha \neq \emptyset \) (hence \( (\forall^* n \in \text{Dom}(\tilde{\eta}^\beta))(\text{set}(\eta_n^\beta) \cap B_\alpha \neq \emptyset) \) holds for every \( \beta \in [\alpha + 1, \lambda] \))

(d) if \( \alpha = \text{pr}(\beta, \gamma) \) then the truth value of \( \text{Min}(\text{set}(\eta_n^{\alpha+1}) \cap B_\beta) \) and of set \( (\eta_n^{\alpha+1}) \cap B_\beta \neq \emptyset \), and of set \( (\eta_n^{\alpha+1}) \cap B_j \neq \emptyset \) are the same for all \( n \in \text{Dom}(\tilde{\eta}^{\alpha+1}) \)

(e) in (d), if \( \beta < \kappa \) we can replace \( B_\beta \) by \( A_\beta \); similarly with \( \gamma \) and \( \beta, \gamma < \kappa \) we can replace both.
Proof. We choose $\bar{\eta}^\alpha$ by induction on $\alpha$. For $\alpha = 0$ trivial. For $\alpha$ limit $< \lambda$ use 1.7(2) theorem (and $|\alpha| < \lambda$). For $\alpha$ successor first choose $B'_\alpha \in [\omega]^\lambda_0$ such that $B'_\alpha \subseteq B_\alpha$ and $i < \kappa \Rightarrow B'_\alpha \subseteq^* A_i$. Second, for $n \in \text{Dom}(\bar{\eta}^{\alpha-1})$ choose $\eta'_n$ such that $\eta^{\alpha-1}_n < \eta'_n$ and $\{ \ell : \eta'_n(\ell) = 1 \text{ and } \ell g(\eta^{\alpha-1}_n) \leq \ell < \ell g(\eta'_n) \}$ is not empty and is included in $B'_\alpha$. Thirdly, let $\bar{\eta}^\alpha = \langle \eta'_n : n \in \text{Dom}(\bar{\eta}^{\alpha-1}) \rangle$. By shrinking the domain of $\bar{\eta}^\alpha$ there is no problem to take care of clauses (d), (e).

For $\alpha = \lambda$, use 1.8. \hfill \Box_{1.12}

1.13 Theorem. In 1.12, we can find $\kappa = \text{cf}(\kappa) < \lambda$ and $f_i, f^\alpha \in A^\omega$ for $i < \kappa, \alpha < \lambda$ satisfies

\begin{enumerate}
    \item[(a)] $i < j < \kappa \Rightarrow f_j \leq^* f_i$ (and without loss of generality $<^*$)
    \item[(b)] $\alpha < \beta < \lambda \Rightarrow f^\alpha \leq^* f^\beta$ (and without loss of generality $<^*$)
    \item[(c)] $i < \kappa \wedge \alpha < \lambda \Rightarrow f^\alpha \leq^* f_i$ (clearly $<^*$ holds)
    \item[(d)] if $f : A^* \rightarrow \omega$ and $\bigwedge_{i < \kappa} f^i \leq^* f_i$ then $\bigvee_{\alpha < \lambda} f \leq^* f^\alpha$
    \item[(e)] if $f : A^* \rightarrow \omega$ and $\bigwedge_{\alpha < \lambda} f^\alpha \leq^* f$ then $\bigvee_{i < \kappa} f_i \leq^* f$.
\end{enumerate}

Proof of 1.13. It is enough to find such $f_i(i < \kappa), f^\alpha(\alpha < \lambda)$ from $A^\omega$ for some infinite $A^* \subseteq \omega$ (so by renaming, it is $\omega$) Let $\langle A_i : i < \kappa \rangle$, $\langle B_\alpha : \alpha < \lambda \rangle$, $\langle \bar{\eta}^\alpha : \alpha \leq \lambda \rangle$ be as in 1.12. Let

1. $\bar{\eta}^\alpha = \text{Dom}(\eta^\lambda)$
2. for $i < \kappa$ let $f_i : A^* \rightarrow \omega$ be $f_i(n) = \min\{ \ell : \eta^\lambda(n + \ell) = 1, n + \ell \notin A_i \}$ or $\text{Dom}(\eta^\lambda) = [n, n+\ell)$
3. for $\alpha < \lambda$ let $f^\alpha : A^* \rightarrow \omega$ be $f^\alpha(n) = \min\{ \ell + 1 : \eta^\lambda(n + \ell) = 1, n + \ell \in B_\alpha \}$ or $\text{Dom}(\eta^\lambda) = [n, n+\ell)$.

Now $\kappa = \text{cf}(\kappa)$ by 1.4 and $f_i, f^\alpha \in A^\omega$ by their definitions (remembering that $\eta^\lambda_\alpha \in \bigcup\{[n,k) : k \in (n,\omega)\}$ by the definition of $S$).

Note that (by the choice of $f_i$, i.e., clause (b)):

\begin{enumerate}
    \item[(*)] $\cup\{[n, n + f_i(n)) \cap \text{set}(\eta^\lambda_i) : n \in A^*\} \subseteq^* A_i$ for every $i < \kappa$.
\end{enumerate}

\hfill \Box_{1.12} \text{ Clause $(a)$ holds.}

Why? Let $i < j < \kappa$ so (by 1.4) $A_j \subseteq^* A_i$ hence for some $n^*, A_j \setminus n^* \subseteq A_i$, hence for every $n \in A^* \setminus n^*$ in the definition of $f_i, f_j$ in clause (b), if $\ell$ can serve as a candidate for $f_i(n)$ then it can serve for $f_j(n)$ so (as we use the minimum there) $f_j(n) \leq f_i(n)$. Hence $f_j \leq^* f_i$. 


To have “without loss of generality $f_j <^* f_i$”, it is enough to show that for every $i < \kappa$ for some $j \in (i, \kappa)$ we have $f_j <^* f_i$, so assume toward contradiction that for some $i(*) < \kappa$ we have $(\forall j)(i(*) < j < \kappa \rightarrow \neg(f_j <^* f_{i(*)}))$ hence for $j < \kappa$ let $B_j^* := \{ n \in A^* : f_j(n) \geq f_{i(*)}(n) \}$ so $B_j^* \in [A^*]^{\aleph_0}$ is $\subseteq$-decreasing, so there is a pseudo-intersection $B^*$ of $\langle B_j^* : j < \kappa \rangle$; i.e., $B^* \in [\omega]^{\aleph_0}$ and $j < \kappa \Rightarrow B^* \subseteq^* B_j^*$. Now letting $A' = \cup \{ \text{set}(\eta_n^\alpha) \cap \{ n, n + f_{i(*)}(n) : n \in B^* \} : j < \kappa \}$ it satisfies

(i) it is an infinite subset of $\omega$

[Why? By recalling that by clause (a) of 1.12 we have $\bar{\eta}^0 \in S_A$ hence (see Definition 1.10), we have $n \in \text{Dom}(\bar{\eta}^0) \Rightarrow \text{set}(\eta_n) \neq \emptyset$ and set($\bar{\eta}^0$) $\subseteq^* A_{i(*)}$. By clause (b) of @0 for every $n$ large enough, $n \in \text{Dom}(\bar{\eta}^\lambda) \Rightarrow n \in \text{Dom}(\bar{\eta}^0)$ & $\eta_n^\lambda \leq \eta_n^\alpha$. Since for every $n$ large enough set($\bar{\eta}^0$)$\setminus \{ 0, \ldots, n - 1 \} \subseteq A_{i(*)}$ we know that for $n \in \text{Dom}(\bar{\eta}^\lambda)$ large enough $\eta_n^\lambda \leq \eta_n^\alpha \land \emptyset \neq \text{set}(\eta_n^\alpha) \subseteq \text{set}(\eta_n^\lambda)$ so $[n, f_{i(*)}(n)] \cap \text{set}(\eta_n^\alpha) \neq \emptyset$ so we are done.]

(ii) $A' \subseteq^* A_j$ for $j \in (i(*)$, $\kappa)$ (hence for $j < \kappa$)

[as $f_j \upharpoonright B^* =^* f_{i(*)} \upharpoonright B^*$ for $j \in (i(*)$, $\kappa)$]

(iii) $A' \cap B_\alpha$ is infinite for $\alpha < \lambda$

[Why? By clauses (c) + (a) of 1.12, we have: for every $n$ large enough $n \in \text{Dom}(\bar{\eta}^{\alpha+1})$ we have set($\eta_n^{\alpha+1}$)$\cap B_\alpha \neq \emptyset$ and set($\eta_n^{\alpha+1}$) $\subseteq A_{i(*)}$.]

Together $A'$ contradicts 1.4, hence $(\forall i < \kappa)(\exists j < \kappa)(f_j <^* f_i)$, so we are done proving $\odot_1$

$\odot_2$

(i) the set (of function) $\{ f_i : i < \kappa \} \cup \{ f^\alpha : \alpha < \lambda \}$ is linearly ordered by $\leq^*$; moreover

(ii) in fact if $f'$, $f''$ are in the family then $f' = f'' \text{ mod } J_\omega^{bd}$ or $f' < f'' \text{ mod } J_\omega^{bd}$ or $f'' < f' \text{ mod } J_\omega^{bd}$.

[Why? By clause (d) + (e) of 1.12.]

So we can choose $\langle \alpha(\varepsilon) : \varepsilon < \varepsilon^* \rangle$ such that:

$\odot_3$

(i) $\alpha(\varepsilon)$ is the minimal $\alpha \in \lambda \setminus \{ \alpha(\zeta) : \zeta < \varepsilon \}$ satisfying $i < \kappa \Rightarrow f^\alpha <^* f_i$ and $\zeta < \varepsilon \Rightarrow f^\alpha(\zeta) <^* f^\alpha$

(ii) we cannot choose $\alpha(\varepsilon^*)$.

We ignore (till $\odot_7$) the question of the value of $\varepsilon^*$. Now

$\odot_4$ $\langle f_i : i < \kappa \rangle$, $\langle f^\alpha(\varepsilon) : \varepsilon < \varepsilon(*) \rangle$ satisfies clauses (\beta), (\gamma).

[Why? By clause (i) of $\odot_3$.]

$\odot_5$ $\langle f_i : i < \kappa \rangle$, $\langle f^\alpha(\varepsilon) : \varepsilon < \varepsilon(*) \rangle$ satisfies clause (\delta).
[Why? Assume toward contradiction that \( f : A^* \to \omega \) and \( i < \kappa \Rightarrow f \leq^* f_i \) but \( \varepsilon < \varepsilon^* \Rightarrow \neg(f \leq^* f^{\alpha(\varepsilon)}) \). Clearly without loss of generality \( n \in A^* \Rightarrow [n, n + f(n)) \subseteq \text{Dom}(\eta^\lambda_n) \). Let \( A' = \bigcup\{[n, n + f(n)) \cap \text{set}(\eta^\lambda_n) : n \in A^*\} \). Now for every \( i < \kappa, A' \subseteq^* A_i \) because \( f \leq^* f_i \) and the definition of \( f_i \).

Also, for every \( \alpha < \lambda \), the set \( A' \cap B_\alpha \) is infinite. Why? Because for some \( \varepsilon < \varepsilon^* \), \( f^{\alpha}_{\leq^*} f^{\alpha(\varepsilon)} \) (otherwise by \( \oplus_2 \) we have \( \varepsilon < \varepsilon^* \Rightarrow f^{\alpha(\varepsilon)} <^* f^{\alpha} \), so \( \alpha \) is a candidate to being \( \alpha(\varepsilon^*) \) contradiction to clause (ii) of \( \oplus_3 \)). Also we have \( \neg(f \leq^* f^{\alpha(\varepsilon)}) \) (by the assumptions toward contradiction on \( f \)). Hence if \( n \in A^* \) is large enough then \( f^{\alpha}(n) \leq f^{\alpha(\varepsilon)}(n) \) and for infinitely many \( n \in A^* \) we have \( f^{\alpha}(n) \leq f^{\alpha(\varepsilon)}(n) < f(n) \leq f_0(n) \leq |\text{dom}(\eta^\lambda_n)| \). For any such \( n \) let \( \ell^*_n = \min\{\ell : \eta^\lambda_n(n + \ell) = 1, n + \ell \in B_\alpha \) or \( \text{Dom}(\eta^\lambda_n) = [n, n + \ell)\} \), then \( \ell^*_n + 1 \leq f^{\alpha}(n) \leq f^{\alpha(\varepsilon)}(n) \leq f(n) \leq f_0(n) \leq |\text{Dom}(\eta^\lambda_n)| \) hence \( n + \ell^*_n \in A' \cap B_\alpha \).

Together \( A' \) contradict the choice of \( \langle A_i : i < \kappa \rangle, \langle B_\alpha : \alpha < \lambda \rangle \) from 1.4.]

\[ \oplus_6 \langle f_i : i < \kappa \rangle, \langle f^{\alpha(\varepsilon)} : \varepsilon < \varepsilon^*(\ast) \rangle \text{ satisfies clause } (\varepsilon). \]

[Why? Assume toward contradiction that \( f : A^* \to \omega \), and \( \varepsilon < \varepsilon^* \Rightarrow f^{\alpha(\varepsilon)} \leq^* f \) but \( i < \kappa \Rightarrow \neg(f_i \leq^* f) \). As \( i < j < \kappa \Rightarrow f_j <^* f_i \) and we are assuming \( i < \kappa \Rightarrow \neg(f_i \leq^* f) \) there is an infinite \( A^{**} \subseteq A^* \) such that \( i < \kappa \Rightarrow (f \upharpoonright A^{**}) <^* (f_i \upharpoonright A^{**}) \). Let\(^1\) \( A' = \bigcup\{[n, n + f(n)) \cap \text{set}(\eta^\lambda_n) : n \in A^{**}\} \), so \( A' \subseteq \omega \) and if \( \alpha < \lambda \) then by \( \oplus_2 \) for some \( \varepsilon, f^{\alpha} \leq^* f^{\alpha(\varepsilon)} \) and by our assumption toward contradiction, \( f^{\alpha(\varepsilon)} \leq^* f \), so by the definition of \( f^{\alpha} \), \( f^{\alpha(\varepsilon)} \) we get \( A' \cap B_\alpha \) is infinite. As \( i < \kappa \Rightarrow (f \upharpoonright A^{**}) <^* (f_i \upharpoonright A^{**}) \) we get \( i < \kappa = A' \subseteq A_i \), so we have gotten a contradiction to the choice of \( \langle A_i : i < \kappa \rangle, \langle B_\alpha : \alpha < \lambda \rangle \).]

\[ \oplus_7 \varepsilon^* \leq \lambda, \]

[Why? As we have chosen \( \alpha(\varepsilon) \) “the minimal \( \alpha < \lambda \) such that ...”, the sequence \( \langle \alpha(\varepsilon) : \varepsilon < \lambda \rangle \) is an increasing sequence of ordinals \( < \lambda \), hence \( \varepsilon^*(\ast) \leq \lambda \).]

\[ \oplus_8 \varepsilon^* = \lambda. \]

[Why? If \( \varepsilon^* < \lambda \) by Bell theorem we get contradiction to \( \oplus_4 \) above.]

So \( \langle f_i : i < \kappa \rangle, \langle f^{\alpha} : \alpha < \lambda \rangle \) are as required (in detail, clause (\( \alpha \)) by \( \oplus_1 \), clause (\( \beta \)) by \( \oplus_4 \), clause (\( \gamma \)) also by \( \oplus_4 \), clause (\( \delta \)) by \( \oplus_5 \), clause (\( \varepsilon \)) by \( \oplus_6 \). \( \Box_{1.13} \)

An obvious compliment to 1.13 is

**1.14 Claim.** Assume \( \kappa \leq \lambda \) are regular and \( A^* \subseteq \omega \) is finite and \( \langle f_i : i < \kappa \rangle, \langle f^{\alpha} < \lambda \rangle \) satisfies clauses (\( \alpha \)) – (\( \delta \)) from Theorem 1.13.

**Then**

\(^1\)alternatively we could have replaced \( \bar{\eta}^\lambda \) by \( \bar{\eta}^\lambda \upharpoonright A^{**} \) and use \( \oplus_5 \)
(a) MA_{\kappa+\lambda}(\sigma\text{-centered}) fail

(b) for some \sigma\text{-centered forcing notion }Q\text{ of cardinality }\kappa\text{ and sequence }\langle\mathcal{I}_\alpha : \alpha < \lambda\rangle\text{ of dense subsets of }Q,\text{ there is no directed }G \subseteq Q\text{ such that }\alpha < \lambda \Rightarrow G \cap \mathcal{I}_\alpha \neq \emptyset.

Remark. In 1.13 by restricting the \(f_i, f^\alpha\)’s to \(A^*\) and renaming without loss of generality \(A^* = \omega\).

Proof. We define the set of member of \(Q\) as the set of pairs \(p = (\rho, u)\) where \(\rho \in \omega > \omega\) and \(u \subseteq \kappa\) is finite.

The order is:

\[(\rho_1, u) \leq_Q (\rho_2, u) \text{ iff (both are in } Q \text{ and)(a) } \rho_1 < \rho_2 \]
\[(b) \ u_1 \subseteq u_2 \]
\[(c) \text{ if } n \in [\ell g(\rho_1), \ell g(\rho_2)) \text{ and } i \in u_1 \text{ then } f_i(n) \leq \rho_2(n).\]

For \(\alpha = 2j < \kappa\) let \(\mathcal{I}_\alpha = \{(\rho, u) \in Q : j \in u\}\).

For \(\alpha = 2j \in [\kappa, \lambda)\) let \(\mathcal{I}_\alpha = \mathcal{I}_0\).

For \(\alpha = \omega\beta + 2n + 1 < \lambda\) let

\[\mathcal{I}_\alpha = \{(\rho, u) : \text{ for some } m < \ell g(\rho) \text{ we have } m \geq n, m \in A^* \text{ and } \rho(m) < f^\beta(m)\} \]

Clearly each \(\mathcal{I}_\alpha\) is a dense open subset of \(Q\). Suppose toward contradiction there \(G \subseteq Q\) is directed non-disjoint to \(\mathcal{I}_\alpha\) for every \(\alpha < \lambda\). So \(g = \cup\{\rho : (\rho, u) \in G\}\) is a function, its domain is \(\omega\) (as \(G \cap \mathcal{I}_{2n+1} \neq \emptyset\) for \(n < \omega\) and \(f_i \leq^* g\) (as \(G \cap \mathcal{I}_{2i} \neq \emptyset\)) and \(\{n \in A^* : g(n) < f^\alpha(n)\}\) is infinite as \(G \cap \mathcal{I}_{\omega \alpha + 2n + 1} \neq \emptyset\) for every \(n\).

Lastly, trivially \(Q\) is \(\sigma\text{-centered}\) as for each \(\rho \in \omega \omega\), the subset \(\{(\eta, u) \in Q : \eta = \rho\}\) is directed. \(\square\)

\(1.14\) scite\{y.17\} ambiguous
2.1 Claim. In Theorem 1.13:

(a) \(\aleph_1 \leq \kappa = \text{cf}(\kappa) < \lambda = p = \text{cf}(\lambda)\)
(b) if \(\text{MA}_{\aleph_1}\) then \(\kappa = \aleph_1\) is impossible.

2.2 Conclusion. If \(\text{MA}_{\aleph_1}\) then \(p = \aleph_2\) \(\iff\) \(t = \aleph_0\). In other words \(m = p = \aleph_1 \Rightarrow p = \aleph_2\).

Remark. The proof of (b) actually uses Hausdorff gaps on which much is known, see (xxxx).

Proof. Clause (a):

By 1.6 clearly \(\kappa \neq \aleph_0\) and by its choice \(\kappa\) is regular \(< \lambda = p = \text{cf}(\lambda)\).

Clause (b):

Assume \(\kappa = \aleph_1\). We define a forcing notion \(Q\) as follows:

\((*)_1\) \(p \in Q\) iff

(a) \(p = (u, \bar{\rho}) = (u_p, \bar{\rho}_p)\)
(b) \(u \subseteq \omega_1\) is finite
(c) \(\bar{\rho}_0 = \langle \rho_\alpha : \alpha \in u \rangle\) and let \(\rho_\alpha = \rho_\alpha^p\)
(d) for some \(n = n_p\) we have \(\alpha \in u \Rightarrow \rho_\alpha \in n_\omega\)
(e) \(f_\alpha \upharpoonright n_p \leq \rho_\alpha\) for \(\alpha \in u_p\), i.e. \(n < n_p \Rightarrow f_\alpha(n) \leq \rho_\alpha(n)\)
(f) \(\langle f_\alpha \upharpoonright [n_p, \omega) : \alpha \in u \rangle\) is increasing

\((*)_2\) \(p \leq_Q q\) iff

(a) \(u_p \subseteq u_q\)
(b) \(\rho_\alpha^p \preceq \rho_\alpha^q\) for every \(\alpha \in u_p\)
(c) if \(\alpha < \beta\) are from \(u_p\) then \(\rho_\alpha^p \upharpoonright [n_p, n_q) < \rho_\beta^q \upharpoonright [n_p, n_q)\)
(d) if \(\alpha < \beta, \alpha \in u_q \backslash u_p\) and \(\beta \in u_p\) then for some \(n \in [n_p, n_q)\) we have \(\rho_\beta^q(n) < \rho_\alpha^q(n)\).

\((*)_3\) \(Q\) is a quasi-order of cardinality \(\aleph_1\).
A COMMENT ON "\( \mathfrak{B} \prec \mathfrak{T} \)"

[Why? Obvious.]

\((*)_4\) \( \mathbb{Q} \) satisfies the c.c.c.

[Why? Let \( p_\varepsilon \in \mathbb{Q} \) for \( \varepsilon < \omega_1 \). Without loss of generality \( \langle p_\varepsilon : \varepsilon < \omega_1 \rangle \) is without repetition. We can find an unbounded \( \mathcal{U} \subseteq \omega_1 \) and \( n(*) < \omega \)

\((a)\) if \( \varepsilon \in \mathcal{U} \) then \( |u_\varepsilon| = n(*) = \bar{\rho}_* \) so let \( u_\varepsilon = \{ \alpha_{\varepsilon,\ell} : \ell < n(*) \} \) and \( \alpha_{\varepsilon,\ell} \)

increases with \( \ell \).

By pigeon-hole pre? for some \( m(*) \leq n(*) \)

\((b)\) \( \alpha_{\varepsilon,\ell} = \alpha_{\ell,\varepsilon}, \rho_{\varepsilon,\ell} = \rho_{\ell}^* \) for \( \ell < m(*) \).

If \( m(*) = n(*) \) then \( p_\varepsilon = p_\zeta \) for \( \varepsilon, \zeta \in \mathcal{U} \), contradiction any assumption, so \( m(*) < n(*) \) so by the \( \Delta \)-system lemma

\((c)\) if \( \varepsilon < \zeta \) are from \( \mathcal{U}, \rho \in [m(*)], n(*) \), \( k \in [m(*)], n(*) \) then \( \alpha_{\varepsilon,\ell} < \alpha_{\zeta,k} \).

We can find \( \delta(*) < \omega_1 \) such that

\((d)\) for every \( \zeta \in \mathcal{U} \) and \( k < \omega \) there is \( \varepsilon \in \delta(*) \cap \mathcal{U} \) such that \( \ell < n(*) \Rightarrow \)

\( f_{\alpha_{\varepsilon,\ell}} \upharpoonright k = f_{\alpha_{\ell,\varepsilon}} \upharpoonright k \).

Now choose \( \zeta_1 < \zeta_2 \) from \( \mathcal{U} \setminus \delta(*) \), hence we can find \( k < \omega \) such that \( \langle f_{\alpha}(k) : \alpha \in \{ \alpha_{\varepsilon,\ell} : \ell < n(*) \} \cup \{ \alpha_{\varepsilon,\ell} : \ell < n(*) \} \rangle \) is a strictly increasing sequence of natural numbers (in fact every \( k \) large enough is O.K. as \( \langle f_{\alpha} : \alpha < \omega_1 \rangle \) is \( <^* \)-increasing).

Apply clause \((d)\) to \((\zeta_2, k + 1)\) and get \( \varepsilon \in \delta(*) \cap \mathcal{U} \). Now define \( q = (u_1, \rho_q) \) as follows:

\[ u_q = u_{p_\varepsilon} \cup u_{p_{\zeta_1}} \]

\[ \rho_q^* \text{ is :} \rho_{p_\varepsilon}^* \cup (f_{\alpha} \upharpoonright [n(*)], k + 1) \text{ if } \alpha \in u_{p_\varepsilon} \]

\[ \rho_{p_{\zeta_1}}^* \cup (f_{\alpha} \upharpoonright [n(*)], k + 1) \text{ if } \alpha \in u_{p_{\zeta_1}}. \]

It is well defined (as \( \rho_{p_{\alpha_{\varepsilon,\ell}}}^* \subseteq \rho_{p_{\alpha_{\ell,\varepsilon}}}^* \) for \( \ell < m(*) \)).

Also \( q \in \mathbb{Q} \).

Lastly, \( p_\varepsilon \leq \mathbb{Q} q, p_{\zeta_2} \leq q \), easily checked.

\((*)_5\) for each \( \alpha < \omega_1 \) and \( n < \omega \) the set \( \mathcal{I}_{\alpha,n} \) is a dense open subset of \( \mathbb{Q} \) where \( \mathcal{I}_{\alpha,n} = \{ p : u_p \not\subseteq \alpha \text{ or for no } q, p \leq \mathbb{Q} q \land u_q \not\subseteq \alpha \} \)

\((*)_6\) for each \( \alpha \) there is \( p_\alpha^* \in \mathbb{Q} \) such that \( u_{p_\alpha} = \{ \alpha \} \)

\((*)_7\) for some \( \alpha(*) \), \( p_{\alpha(*)}^* \models \mathbb{Q} \langle \beta : p_\beta \in G \rangle \) is unbounded in \( m \).
[Why? By (\(\ast\))_4.]

\((\ast)_8\) there is a directed \(G \subseteq \mathbb{Q}\) such that \(p_{\alpha(\ast)}^* \in G\) and \(n < \omega \land \alpha < \omega_1 \Rightarrow I_{\alpha,n} \cap G \neq 0\).

[Why? As MA\(\aleph_1\) holds +\((\ast)_4\).]

\((\ast)_9\) \(U := \{u_p : p \in G\}\) is unbounded in \(\omega_1\).

[Why? As \(p_{\alpha(\ast)}^* \in G\) and \(G \cap I_\alpha \neq \emptyset\) for \(\alpha < \omega_1\).]

For \(\alpha \in \mathbb{W}\) let \(g_\alpha = \cup \{p_\alpha^p : p \in G\}\), clearly \(g_\alpha \in \omega_\omega\) (as \(G\) is directed, \(I_{\alpha,n} \cap G \neq \emptyset\) for \(\alpha < \omega_1, n < \omega\)).

Also \(f_\alpha \leq g_\alpha\) by clause (x) of the definition of \(\mathbb{Q}\). Also \((g_\alpha : \alpha \in \mathbb{W})\) is \(<_{j_{\beta\alpha}}\)-decreasing by clause (y) of the definition of \(\leq_{\mathbb{Q}}\). Hence for each \(\alpha < \omega_1\) we have \(\beta < \omega_1 \Rightarrow f_\beta <^* g_\alpha\) hence by clause (?) of Theorem 1.13 there is \(\gamma(\alpha) < \lambda\) such that \(f^{\gamma(\alpha)} <^* g_\alpha\). Let \(\gamma(*) = \sup \{\gamma(\alpha) : \alpha < \omega_1\}\), so \(\gamma(*) < \lambda\) (as \(\lambda = \text{cf}(\lambda) > \kappa = \aleph_1\)) hence \(\beta < \omega_1 \land \alpha < \omega_1 \Rightarrow f_\beta <^* f^{\gamma(*)} <^* g_\alpha\). Let \(n_\alpha\) be minimal such that \(n \in [n_\alpha, \omega) \Rightarrow f_\alpha(n) < f^{\gamma(*)} < g_\alpha(n)\). So for some \(n^*\) the set \(\mathbb{W}_* = \{\alpha \in \mathbb{W} : n_\alpha = n_*\}\) is unbounded in \(\omega_1\).

Let \(\alpha(*) \in \mathbb{W}_*\) be such that \(\mathbb{W}_* \cap \alpha(*)\) is infinite and let \(\langle q_n : n < \omega \rangle\) be an \(\leq_{\mathbb{Q}}\)-increasing sequence of members of \(G\) such that \(\mathbb{W}_* \cap (\alpha(*) + 1) \subseteq \cup \{u_{q_n} : n < \omega\}\). By clause (z) of the definition of the order \(\leq_{\mathbb{Q}}\) we get a contradiction. \(\Box\)

\(2.1\)
REFERENCES.


[Sh:F769] Saharon Shelah. Large continuum, oracles, and a try at $\mathfrak{p} < \mathfrak{t}$.