CONSISTENCY OF “THE IDEAL OF NULL RESTRICTED TO SOME $A$ IS $\kappa$–COMPLETE NOT $\kappa^+$–COMPLETE, $\kappa$ WEAKLY INACCESSIBLE AND $\text{cov}(\text{meager}) = \aleph_1$”

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In this note we give an answer to the following question of Grinblat (Moti Gitik asked about it in the Oberwolfach meeting:

Grinblat’s Question 1. Is it consistent that

(**) for some set $X$, $\text{cov}(\text{Null } \uparrow X) = \lambda$ is a weakly inaccessible cardinal (so $X$ not null of course) while $\text{cov}(\text{Meager})$ is small, say it is $\aleph_1$.

A. THE FORCING:

Starting with a universe $\mathbf{V}$ and a cardinal $\lambda$ of cofinality $> \aleph_0$, regular for simplicity (otherwise the only difference is that $J$ consists of “bounded subsets”), in fact weakly inaccessible for Grinblat’s question.

Let $\mathbb{P} = \mathbb{P}_\lambda$ be the result of FS iteration $\langle \mathbb{P}_i, \mathbb{Q}_i : i < \lambda \rangle$ with $\mathbb{Q}_i$ being the random real forcing, and $\mathbb{Q}_{2i+1}$ being the Cohen forcing notion. Let $\mathbb{R}$ be a $\mathbb{P}$–name for the forcing notion adding $\aleph_1$ random reals (i.e., forcing with the measure algebra of Borel subsets of $^{\omega_1}/2$ of positive Lebesgue measure).

We claim that $\mathbf{V}_2 = \mathbf{V}^{\mathbb{P} \times \mathbb{R}}$ is as required.

Let $\mathbf{V}_1 = \mathbf{V}^\mathbb{P}$.

As the whole forcing satisfies the ccc, no cardinal is collapsed etc

B. WHY $\text{cov}(\text{Meager}) = \aleph_1$ ?

As forcing by $\mathbb{R}$ does it (well known).

C.

Let $\eta_i$ be the $\mathbb{Q}_i$–generic real for $i < \lambda$. Clearly they are pairwise distinct. Let

$X \overset{\text{def}}{=} \{ \eta_i : i < \lambda \}$.

This is a set of cardinality $\lambda$. Let $J$ be the ideal of subsets of $X$ of cardinality $< \lambda$ (it is a $\lambda$–complete ideal on $X$).

It is enough to prove

(*) $J$ is equal to the ideal of null subsets of $X$.

C1.

Now, for every $\alpha < \lambda$ the set $\{ \eta_i : i < \alpha \}$ is null in $\mathbf{V}_2$. Why? Because $\eta_{2^\alpha + 1}$ is Cohen over $\mathbf{V}_2^{2^{\omega_1}}$ the universe to which the above set belongs and is an inner model of $\mathbf{V}_2$.

This is enough to show that every member of $J$ is null.

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C2.

For the other direction, let $Y$ be a $\mathbb{P} \ast \mathbb{R}$ name of an unbounded subset of $\lambda$. We shall prove that

$$\{\eta_{2i} : i \in Y\}$$

is forced to be non-null (this clearly suffices).

Let $p$ be a condition in $\mathbb{P} \ast \mathbb{R}$ forcing the inverse, so for some $\mathbb{P} \ast \mathbb{R}$-name $Z$ of a null Borel subset of $\mathbb{R}$, we have

$$p \downarrow \{\eta_{2i} : i \in Y\} \subseteq Z$$

We can find $\alpha < \lambda$ such that, in $V^{P_{\alpha}}$, $Z$ becomes an $\mathbb{R}^{V_{\alpha}}$-name and $p$ is a member of $\mathbb{R}^{V_{\alpha}}$.

Now for every $i$, if $\alpha < 2i < \lambda$ then $\eta_{2i}$ is random over $V^{P_{\alpha}}$. Hence, by the Fubini theorem (i.e., random reals commute), it is also random over $(V^{P_{\alpha}})^{\mathbb{R}^{V_{\alpha}}}$. Consequently it does not belongs to $Z$, so we are done.

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