RANKS FOR STRONGLY DEPENDENT THEORIES
[COSH:E65]

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Abstract. There is much more known about the family of superstable theories when compared to stable theories. This calls for a search of an analogous “super-dependent” characterization in the context of dependent theories. This problem has been treated in [She6, She8], where the candidates "Strongly dependent", "Strongly dependent 2" and others were considered. These families generated new families when we considering intersections with the stable family. Here, continuing [She6 §2, §5,E,F,G], we deal with several candidates, defined using dividing properties and related ranks of types. Those candidates are subfamilies of “Strongly dependent”, fulfilling some promises from [She6] in particular [She6, 1.4(4)], we try to make this self contained within reason by repeating some things from there. More specifically we fulfill some promises from [She6] to to give more details, in particular: in §4 for [She6, 1.4(4)], in §2 for [She6, 5.47(2)=Ldw5.35(2)] and in §1 for [She6, 5.49(2)]

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1. Strongly dependent theories

Discussion 1.1. The basic property from which this work is derived is strongly dependent\(^1\), it has been studied extensively in [Shelah]. For proofs and more we refer to that article. We quote the necessary minimum in order to build on that.

**Definition 1.** We say that \( \kappa_{\text{ict.1}}(T) := \kappa_{\text{ict}}(T) > \kappa \) if the set

\[
\Gamma_T := \{ \varphi_i(\overline{\alpha}, \overline{\eta}) \mid i < \kappa, j < \omega, \eta \in \kappa \omega \}
\]

is consistent with \( T \), for some sequence of formulas \( \overline{\varphi} = \langle \varphi_i(\overline{\alpha}, \overline{\eta}) : i < \kappa \rangle \). We will say that \( \kappa_{\text{ict}}(T) = \kappa \) iff \( \kappa_{\text{ict}}(T) > \lambda \) holds for all \( \lambda < \kappa \) but \( \kappa_{\text{ict}}(T) > \kappa \) does not.

\( T \) is called strongly dependent \(^1\) if \( \kappa_{\text{ict}}(T) = \aleph_0 \).

Discussion 1.2. The following properties are used in connecting the new properties with the original.

Claim 2. \( T \) is not strongly dependent \(^1\) if there exist sequences \( \overline{\varphi} = \langle \varphi_i(\overline{\alpha}, \overline{\eta}) : i < \kappa \rangle \) and \( \langle \overline{\alpha}_k : i < \kappa, k < \omega \rangle \) such that \( \lg \overline{\alpha}_i = \lg \overline{\alpha}_k \), \( \langle \overline{\alpha}_k : k < \omega \rangle \) an indiscernible sequence over \( \bigcup \{ \overline{\alpha}_k : j \neq i, j < \kappa, k < \omega \} \) for all \( i < \kappa \) it holds that \( \{ \varphi_i(\overline{\alpha}, \overline{\alpha}_i) \wedge \neg \varphi_i(\overline{\alpha}, \overline{\alpha}_i) : i < \kappa \} \) is a type in \( \mathcal{C} \).

**Theorem 3.** For a given (or any) \( \alpha \geq \omega \) the following are equivalent

1. \( T \) is strongly dependent \(^1\)
2. For every \( \overline{\alpha} \subseteq \mathcal{C} \) and indiscernible sequence \( \langle \overline{\alpha}_i : i \in I \rangle \) where \( \lg(\overline{\alpha}_i) = \alpha \) the function \( t \mapsto \text{tp}(\overline{\alpha}_i, \overline{\alpha}) \) divides \( I \) to finitely many convex components.
3. Same as 2 but with \( \lg(\overline{\alpha}) = 1 \)
4. Same as 2 but with \( I = \omega \).

Discussion 1.3. Now we turn to discuss the new properties: strongly dependent \( \ell \) and strongly dependent \( \delta \).

1.1. The dividing properties.

Order-based indiscernible structures, forms and dividing.

**Convention 1.** We fix a set \( A \subseteq \mathcal{P}(\mathcal{M}_{\mu_1, \mu_2}(\mu_3)) \), such that all \( k \in A \) contains at least one \( n \)-ary term, for \( n > 0 \).

**Definition 4.** We call \( k \in A \) a form, and we define

\[
k(I) := \{ \tau(T) : T = \langle t_i : i < \mu \rangle \in \text{incr}(I, \mu), \tau(\mu) \in k, \mu < \mu_3 \}
\]

for a linear order \( I \).
Definition 5. We call $\overline{s}_0, \overline{s}_1$ equivalent in $\mathcal{A}(l)$ iff there exist a term $\overline{r} \subseteq A$ and increasing sequences $\overline{t}_0, \overline{t}_1$ such that $\overline{s}_i = \overline{r}(\overline{t}_i), (i = 0, 1)$.

Let $E$ a convex equivalence relation on $I$ we say that $\overline{s}_0, \overline{s}_1$ are equivalent in $\mathcal{A}(l, E)$ iff $\overline{s}_0, \overline{s}_1$ are equivalent in $\mathcal{A}(l)$ and also $\overline{t}_0, \overline{t}_1$ are equivalent relative to $E$.

Convention 2. We will limit the discussion to the case $A \subseteq \mathcal{P}(M_{\omega}(\omega))$.

Remark 1. Note that a form restricts both the terms which can be used as well as the assignable tuples to those which preserve the same order structure.

Discussion 1.4. We now turn to define the structure classes.

Definition 6. $\mathcal{I}^e$ Denotes the class of linear orders with the dictionary $(I, <)$.

$\mathcal{I}^{or}+(<n)$ Denotes the class of structures $\mathcal{M}(I)$ whose universe is the disjoint union of a linear order $|I|$ with the set of increasing sequences of length $< n$ in $I$, and the dictionary is

$$(I \cup \text{incr}(I, < n), <, S_0 \ldots S_{n-1}, R_0 \ldots R_{n-1})$$

where $<$ is binary, $S_i$ is unary, and $R_i$ binary such that $(I, <)$ is a linear order.

$S_i = \{ \overline{t} \in \text{incr}(I, < n) : \lg(\overline{t}) = i \}$ for all $i < n$, $S_i(\overline{t})$ holds iff $\lg(\overline{t}) = i$. Also $R_i(\overline{t}, \overline{t})$ for all $i < \lg(\overline{t}) (t_i \ell, \overline{t} \in \text{incr}(I, < n))$.

Convention 3. In the above notation, $< n$ can be replaced with $\leq n$ to mean $< n + 1$.

Discussion 1.5. We now turn to define the main properties with which we deal.

Definition 7. We say that the type $p(\overline{r})$ does $\text{ict}^\ell - (\Delta, n)$-divide over $A$ if

For $\ell = 1$: There exist an indiscernible sequence $\langle \overline{r}_t : t \in I \rangle = \overline{r} \in \text{Ind}_\Delta(\mathcal{I}^e, A)$ and $s_0 < t_0 \leq t_1 < \ldots < s_{n-1} \leq s_n$ such that for any $\overline{r}$ which realizes $p$, $\text{tp}_\Delta(\overline{r} \overline{r}_{s_i}, A) \neq \text{tp}_\Delta(\overline{r} \overline{r}_{t_i}, A)$ holds for all $i < n$.

For $\ell = 2$: There exist an indiscernible sequence $\langle \overline{r}_t : t \in I \rangle = \overline{r} \in \text{Ind}_\Delta(\mathcal{I}^e, A)$ and $s_0 < t_0 \leq t_1 < \ldots < s_{n-1} \leq s_n$ such that for any $\overline{r}$ which realizes $p$,

$$\text{tp}_\Delta(\overline{r} \overline{r}_{s_j}, A \cup \{ \overline{r}_{s_j} : j < \ell \}) \neq \text{tp}_\Delta(\overline{r} \overline{r}_{t_j}, A \cup \{ \overline{r}_{s_j} : j < \ell \})$$

holds for all $\ell < n$.

For $\ell = 3$: There exist an indiscernible structure $\langle \overline{r}_t : t \in I \cup \text{incr}(< n, I) \rangle = \overline{r} \in \text{Ind}_\Delta(\mathcal{I}^{or}+(<n), A)$ and $s_0 < t_0 \leq t_1 < \ldots < s_{n-1} < s_n$ such that for any $\overline{r}$ realizing $p$ and $\ell < n$:

$$\text{tp}_\Delta(\overline{r} \overline{r}_{s_j}, A \cup \overline{r}_{s_{j-1} \ldots s_{n-1}}) \neq \text{tp}_\Delta(\overline{r} \overline{r}_{t_j}, A \cup \overline{r}_{s_{j-1} \ldots s_{n-1}})$$

holds.
For \( \ell = A \): For some form \( \lambda \in A \) and indiscernible structure \( \mathfrak{P} = (\mathfrak{P}_t : t \in \lambda(I)) \) over \( A \), \( (\mathfrak{P}_t : t \in \lambda(I), E) \) is not indiscernible over \( A \cup \mathfrak{P} \), for any \( c \) realizing\( p \) and convex equivalence relation \( E \) on \( I \) with \( \leq n \) equivalence classes.

**Observation 1.6.** \( p(\mathfrak{P}) \) does \( \text{ict}^4 - (\Delta, n) \)-divide over \( A \) iff \( p(\mathfrak{P}) \) does \( A - (\Delta, n) \)-divide over \( A \) for \( A = \{ \lambda_\alpha = \{ f_i(0, \ldots, i - 1) : 1 < i < n \} : n < \omega \} \).

**Observation 1.7.** If \( A \subseteq A' \) and \( p(\mathfrak{P}) \) does \( A' - (\Delta, n) \)-divide over \( A \), then \( p(\mathfrak{P}) \) does \( \text{ict}^{A'} - (\Delta, n) \)-divide over \( A \).

**Observation 1.8.** If a type \( p \) does \( \text{ict}^1 - n(*) \)-divide over \( A \) then \( p \) does \( \text{ict}^A - n(*) \)-divide over \( A \).

**Observation 1.9.** If the type \( p \) does \( \text{ict}^\ell - n(*) \)-divide over \( A \) then \( p \) does \( \text{ict}^{\ell+1} - n(*) \)-divide over \( A \) (\( 1 \leq \ell \leq 3 \)).

**Definition 8.** We say that the type \( p(\mathfrak{P}) \) does \( \text{ict}^\ell - (\Delta, n) \)-fork over \( A \) if there exist formulas \( \varphi_i(\mathfrak{P}, \mathfrak{P}_i) \), \( (i < m) \) such that \( p(\mathfrak{P}) \vdash \bigvee_{i < m} \varphi_i(\mathfrak{P}, \mathfrak{P}_i) \) and each \( \varphi_i \) does \( \text{ict}^\ell - (\Delta, n) \)-divide over \( A \).

**Definition 9.** We say that \( \kappa_{\text{ict}, \ell}(T) > \kappa \) if some type \( p \) of \( T \) does \( \text{ict}^\ell - n \)-fork over \( A \), for all \( n < \omega \) and \( A \subseteq \text{Dom}(p) \) of power \( \kappa \).

**Definition 10.** We call \( T \) strongly dependent \( \ell \) (\( A \)) iff \( \kappa_{\text{ict}, \ell}(T) = \aleph_0 \) \( (\kappa_{\text{ict}, A}(T) = \aleph_0) \)

**Observation 1.10.** If \( p(\mathfrak{P}) \) does \( \text{ict}^\ell - (\Delta, n) \)-fork over \( A \) then \( p(\mathfrak{P}) \) does \( \text{ict}^\ell - (\Delta, k) \) fork over \( A \) for all \( k < n \).

**Observation 1.11.** (Finite character) If the type \( p(\mathfrak{P}) \) does \( \text{ict}^\ell - (\Delta, n(*)) \)-divide over \( A \) then \( q \) does \( \text{ict}^\ell - (\Delta, n) \)-divide over \( A \) for some finite \( q \subseteq p \).

**Claim 11.** If \( p(\mathfrak{P}) \) does \( \text{ict}^\ell - (\Delta, n(*)) \)-divide over \( A \), it is possible to find witnesses as follows:

**Case\( \ell = 1 \):** There exist \( \mathfrak{P} = (\mathfrak{P}_n : n < \omega) \in \text{Ind}(\mathfrak{P}^\omega, A) \), \( \mathfrak{P} \) a sequence of length \( n(*) \) from \( \omega \) such that \( s_0 = 0 \), \( 1 \leq s_{n+1} - s_n \leq 2 \) and formulas \( \langle \varphi_i(\mathfrak{P}, \mathfrak{P}_i) : i < i(*) \rangle \in A \) such that

\[
p(\mathfrak{P}) \vdash \bigvee_{i < i(*)} (\varphi_i(\mathfrak{P}_{s_n}, \mathfrak{P}, \mathfrak{P}_i) \land \neg \varphi_i(\mathfrak{P}_{s_{n+1}}, \mathfrak{P}, \mathfrak{P}_i))
\]

for all \( n < n(*) \).
Case $\ell = 2$: There exist $\bar{\pi} = (\bar{\pi}_n : n < \omega) \in \text{Ind}(\bar{\pi}_A, A)$ as in $\ell = 1$ and formulas $(\varphi_i^\prime(\bar{y}_0 \ldots \bar{y}_{n-1}, \bar{x}, \bar{c}) : i < i(\pi), n < n(\pi)) \in \bar{\pi} \in A$ such that
\[
p(\bar{\pi}) \vdash \bigvee_{i < i(\pi)} (\varphi_i^\prime(\bar{\pi}_0 \ldots \bar{\pi}_{n-1}, \bar{\pi}_n, \bar{x}, \bar{c}) \land \neg \varphi_i(\bar{\pi}_0 \ldots \bar{\pi}_{n-1}, \bar{\pi}_n, \bar{x}, \bar{c}))
\]
for all $n < n(\pi)$.

Case $\ell = 3$: There exist $\bar{\pi} = (\bar{\pi}_i : i \in \omega \cup \text{incr}<n(\pi), \omega) \in \text{Ind}(\bar{\pi}_{\text{tor}+\text{or}(<n(\pi)), A})$ as in $\ell = 1$ and formulas $(\varphi_i^\prime(\bar{y}, \bar{x}, \bar{c}) : i < i(\pi), n < n(\pi))$ such that
\[
p(\bar{\pi}) \vdash \bigvee_{i < i(\pi)} (\psi_i^\prime(\bar{\pi}_{(0\ldots n-1)}, \bar{\pi}_n, \bar{x}, \bar{c}) \land \neg \psi_i(\bar{\pi}_{(0\ldots n-1)}, \bar{\pi}_n, \bar{x}, \bar{c}))
\]
for all $n < n(\pi)$.

Case $A$: There exist $A \in A$, $m_* < \omega$, $\bar{\pi} = (\bar{a}_i : i \in A(\omega))$ indiscernible over $A$, sequences $(\bar{s}_0, E, \bar{s}_1, E) \in A(m_*, E) : E \in \text{ConvEquiv}(m_*, n(\pi))$, $\bar{b} \in A$ and formulas $(\psi_{E,i}(\bar{y}, \bar{x}, \bar{c}, \bar{b}) : E \in \text{ConvEquiv}(m_*, n(\pi)), i < i_E)$ such that
\[
p(\bar{\pi}) \vdash \bigvee_{i < i_E} \psi_{E,i}(\bar{y}, \bar{\pi}_{(0\ldots n-1)}, \bar{\pi}_n, \bar{b}) \equiv \neg \psi_{E,i}(\bar{y}, \bar{\pi}_{(0\ldots n-1)}, \bar{\pi}_n, \bar{b})
\]
holds for all $E \in \text{ConvEquiv}(m_*, n(\pi))$.

Proof.

For $\ell = 1, 2, 3$: Easy, so we only give a summary. By 29 it follows that there exists a dense extension $I'$ of $I$ without endpoints such that $(\bar{a}_i : i \in I')$ is an indiscernible structure (for the corresponding $\ell$) over $A$. Let $s_0 < t_0 \leq \ldots \leq s_{n-1} < t_{n-1}$ from $I$ witness the dividing as in the definition. These indices can also be used to show that $I'$ is a witness of dividing. Similarly we can choose an increasing $\langle r_n : n < \omega \rangle$ from $I'$ such that $\{s_i, t_i : i < n - 1 \} < \langle r_n : n < \omega \rangle \subseteq I$, to get a witness based on $\omega$.

For $A$: Assume towards contradiction that the claim does not hold. So we can choose

1. A type $p$ which does $(\Delta, n(\pi))$-fork over $A$
2. A linear order $I$.
3. An indiscernible structure $(\bar{a}_i : i \in A(I))$ over $A$ witnessing $1$.
4. $\bar{\pi}$ realizing $p$

Such that for every finite $S \subseteq I$ there exists a convex equivalence relation $E_S$ on $I$ with $\leq n(\pi)$ equivalence classes such that $tp_{\Delta}(\bar{a}_S, A \cup \bar{\pi}) = tp_{\Delta}(\bar{a}_S, A \cup \bar{\pi})$ holds for any equivalent $\bar{s}_0, \bar{s}_1 \in A(S, E_S)$.
Now let $D$ an ultrafilter on $[I]^{<\omega}$ extending \( \{ G_S : S \in [I]^{<\omega} \} \), where $G_S := \{ T \in [I]^{<\omega} : S \subseteq T \} \in D$. For all $S$ define the 2-sort model (with the sorts $M, I$) $M_S := \langle M, I, E, (f_{\tau,i} : \tau(M) \in \mathbb{A}, i < n_\tau), \bar{\tau} \rangle$ where

1. $M, I$ as defined
2. $E_M^S = E_S$ an equivalence relation.
3. Since $\langle \bar{\tau}_t : t \in \mathbb{A}(I) \rangle$ is indiscernible, for every term $\tau(M)$ $\in \mathbb{A}(\tau)$ we can define $n_\tau < \omega$ such that $n_\tau = \text{lg}(\bar{\tau}_t(M))$ for all $\tau(M) \in [I]^{<\omega}$. We define for each term $\tau(M)$ $\in \mathbb{A}(\tau)$ and $i < n_\tau$:

$$f_{\tau(M)} : \text{lg}(\bar{\tau}_t) [I] \rightarrow M$$

$$\tau(M) \mapsto (a_{\tau(M)})_i$$

Now, consider $N = \left( \prod_{S \in [I]^{<\omega}} M_S \right) / D$. From the properties of ultraproducts it is easy to show that the functions

$$h : M \oplus I \rightarrow N$$

$$a \mapsto \langle a \rangle_{S \in [I]^{<\omega}} / D$$

fulfill

1. $h \upharpoonright (M, \bar{\tau}) : \langle M, \bar{\tau} \rangle \rightarrow N \upharpoonright \mathcal{L}_T \cup \{ \bar{\tau} \}$ is elementary.
2. $h(f_{\tau,i}^M(\bar{\tau})) = f_{\tau,i}^N(h(\bar{\tau})).$
3. $E_N \circ h$ is a convex equivalence relation on $I^N$ with $\leq n(\ast)$ classes.
4. $\text{tp}_\Delta(\bar{\tau}_0, A \cup \bar{\tau}, M) = \text{tp}_\Delta(\bar{\tau}_1, A \cup \bar{\tau}, M)$ holds for every pair of equivalent $\bar{\tau}_0, \bar{\tau}_1 \in \mathbb{A}(I, E, D) \circ h$.

Contradicting that $\langle \bar{\tau}_t : t \in \mathbb{A}(I) \rangle$ witnesses that $p$ does $\Delta, n(\ast)$-divide over $A$.

Now we show that it is possible to choose $I = \omega$. From 29 there exists an extension $J$ of $I$ without endpoints, such that $\langle \bar{\tau}_t : t \in \mathbb{A}(J) \rangle$ is indiscernible, extending $\langle \bar{\tau}_t : t \in \mathbb{A}(I) \rangle$. Let $\langle s_i : i < \omega \rangle$ increasing in $J$ such that $\langle s_0 \ldots s_i \rangle$ enumerates $S$ above. We define $\bar{\tau}_i : \bar{\tau}_i = \bar{\tau}_i(M)$ for all $\bar{\tau}, \tau \in \mathbb{A}$. By the conclusion of the claim it is easy to verify that $\langle \bar{\tau}_i : t \in \mathbb{A}(\omega) \rangle$ is a witness as required.

Now, since for any $\bar{\tau}_0, \bar{\tau}_1 \in \mathbb{A}(S)$ it holds that $\bar{\tau}_0, \bar{\tau}_1$ are equivalent in $\mathbb{A}(I, E)$ if they are equivalent in $\mathbb{A}(S, E \upharpoonright S)$, so for some $m_0, m_1 < \omega$ such that $S \subseteq m_0, m_1$ we can choose two equivalent (in $\mathbb{A}(\omega, E)$) $\bar{\tau}_0, \bar{\tau}_1 \in \mathbb{A}(m_0, m_1)$ with $\bar{\tau}_0, \bar{\tau}_1$ having different types over $A$ based only on $E \upharpoonright m_0$. \( \square \)

We use the following freely
Observation 1.12. If \( p(\bar{x}) \) does \( \text{ict}^\ell - n \) divide over \( A \) then \( p(\bar{x}) \) does \( \text{ict}^\ell - n \)-divide over \( B \) for every \( B \subseteq A \).

1.2. Strongly dependent \( 1 \Rightarrow \text{Strongly dependent} \)

Discussion 1.13. Claim 12 is a connection to \( \text{Sh:863} \).

Claim 12. \( T \) is strongly dependent (Definition 10) \( \Rightarrow T \) is strongly dependent \( \text{1} \) (Definition 1)

Definition 13. For a set of formulas \( \mathbb{Q} \), define the formula

\[
\text{Even} \mathbb{Q} := \bigvee \left\{  \bigwedge_{q \in \mathbb{Q}} q^{i_f(q \in u)} : u \in [\mathbb{Q}]^r, 2| r, r \leq |\mathbb{Q}| \right\}
\]

Remark 2. \( \text{Even} \mathbb{Q} \) is true iff the number of true sentences in \( \mathbb{Q} \) is even.

Proof. Assume that \( T \) is not strongly dependent \( \text{1} \): by \( \alpha(2)^\omega \) of theorem 3 there exist an indiscernible sequence \( \langle \pi_n : n < \omega \rangle \) (\( \lg \pi_n = \omega \)) and an element \( c \) such that \( \text{tp}(\pi_n, c) \neq \text{tp}(\pi_{n+1}, c) \) for all \( n < \omega \). Consider \( p(x) := \text{tp}(c, \cup \{ \pi_n : n < \omega \}) \). Fix a finite \( A \subseteq \text{Dom}(p) \). We need to show that \( p \) does \( \text{ict}^\ell - n(*) \)-fork over \( A \) for some \( n(*) \), however we can prove this for any \( 1 < n(*) < \omega \). Fix \( n(*) \) and let \( \mathbf{m} \subseteq I \) increasing and finite such that \( A \subseteq \cup \{ \pi_i : i < \lg \mathbf{m} \} \). Let \( m = \max \mathbf{m} + 1 \). So \( \langle \pi_n : m \leq n < \omega \rangle \) is indiscernible over \( A \). Since for all \( n \geq m \) there exists \( \varphi_n(\mathbf{m}, y) \) such that \( \models \varphi_n(\pi_n, c) \land \lnot \varphi_n(\pi_{n+1}, c) \), we get that \( \varphi_n(\pi_n, x) \land \lnot \varphi_n(\pi_{n+1}, x) \in p(x) \).

Define a map \( f : [\omega]^2 \rightarrow \{ t, f \}^4 \) as follows \( f(\{i, j\}) = (s_0, s_1, s_2, s_3) \) where w.l.o.g \( i < j \) and \( s_k(k < 4) \) are truth values such that

\[
\models \varphi_{m+2i}(\pi_{m+2j})^{s_0} \land \varphi_{m+2i}(\pi_{m+2j+1})^{s_1} \land \varphi_{m+2j}(\pi_{m+2i})^{s_2} \land \varphi_{m+2j}(\pi_{m+2i+1})^{s_3}
\]

By Ramsey’s theorem, there exists an infinite \( S \subseteq \omega \) such that \( f \upharpoonright [S]^2 \) is constant with value \( (s_0, s_1, s_2, s_3) \). Let \( \langle i_n : n < n(*) \rangle \) enumerate \( S \) in increasing order.

Define \( \psi(x, \mathbf{y}) \) as follows:

if \( s_0 = s_1 \land s_2 = s_3 \) let \( \psi(x, \mathbf{y}) := \text{Even} \{ \varphi_{m+2i_n}(\mathbf{y}, x) : n < n(*) \} \).
if \( s_0 \neq s_1 \) let \( \psi(x, \mathbf{y}) := \varphi_m(\mathbf{y}, x) \).
if \( s_0 = s_1 \land s_2 \neq s_3 \) let \( \psi(x, \mathbf{y}) := \varphi_{m+2i_{n-1}}(\mathbf{y}, x) \).

Now let \( \vartheta(x) := \bigwedge_{n < n(*)} \psi(x, \pi_{m+2i_n}) \Delta \psi(x, \pi_{m+2i_{n+1}}) \). It is easy to verify that \( \models \psi(c, \pi_{m+2i_n}) \equiv \lnot \psi(c, \pi_{m+2i_{n+1}}) \) holds for any \( n < n(*) \), so

\( p \vdash \vartheta \). Now \( \vartheta \) does \( \text{ict}^\ell - (\psi, n(*)-\text{divide over } A \).
Choose a finite $u \subseteq \lg \pi$ and let $\psi'(x, \overline{y} \mid u) = \psi(x, \overline{y})$. So $\psi(x) \vdash \psi'(x, \pi_{m+2i_n} \mid u) \equiv \neg \psi'(x, \pi_{m+2i_n+1} \mid u)$ holds for the indiscernible sequence $\langle \pi_n \mid u : m \leq n < \omega \rangle$ and elements $s_n = m + 2i_n, t_n = m + 2i_n + 1$. 

2. Ranks

**Definition 14.** We define the ranks $\text{ict} - r_{\text{rk}}^m \ (P \in \{\text{fork, div}\})$ on the class of $m$-types of $T$ ( $m < \omega$ ) as follows:

- $\text{ict} - r_{\text{rk}}^m(p(\overline{z})) \geq 0$ for all $m$-types.
- For a given ordinal $\alpha$, $\text{ict} - r_{\text{rk}}^m(p(\overline{z})) \geq \alpha$ if for all $q \subseteq p$, $A \subseteq \text{Dom}(p)$ and $n < \omega$ ( $q, A$ finite) and $\beta < \alpha$, for some extension $q' \supseteq q$ it holds that $\text{ict} - r_{\text{rk}}^m(q') \geq \beta$ and also:
  - For $P = \text{fork}$: $q'$ does $\text{ict} - (\mathcal{L}, n)\text{-fork over } A$.
  - For $P = \text{div}$: $q'$ does $\text{ict} - (\mathcal{L}, n)\text{-divide over } A$.
- If $P = \text{fork}$ we omit $P$.

**Observation 2.1.** $\text{ict} - r_{\text{rk}}^m(p) \geq \text{ict} - r_{\text{div}}^m(p)$ for any $m$-type $p$.

**Observation 2.2.** For an $m$-type $p$ over $B$ such that $\text{ict} - r_{\text{rk}}^m(p) = \alpha$ there exists an extension $p \subseteq q \in S^m(B)$, a complete type of the same rank.

**Proof.** Identical to [Sh:90, Theorem II.1.6, p.24].

**Convention 4.** We denote for the rest of this section

$$
\lambda_\ell = |T| \\
\lambda_A = |T| + \sum_{A \in \mathcal{A}} \lambda_0^{|A|}
$$

**Lemma 15.** If $\text{ict} - r_{\text{rk}}^m(\overline{x} = \overline{y}) \geq \lambda_\ell^+$ then there exists $p \in S^m(A)$ which does $\text{ict} - n(*)\text{-divide over } B$ for all $n(*) < \omega$, $B \in [A]^{<\omega}$.

**Proof.** We prove for $\ell = 1$ and $\ell = A$ (the cases $\ell = 2, 3$ are analogous to $\ell = 1$).

We choose, for each $\eta \in d\lambda(\lambda_\ell^+)$, by induction on $\lg(\eta)$ the following objects:

**Case $\ell = 1$:**

$p_\eta, k_\eta, \overline{t}_\eta, \overline{v}_\eta$

$$
\langle \varphi_{\eta, k}(\overline{x}, \overline{y}_\eta), \overline{t}_0 \rangle = \langle \overline{u}_{\eta, k, l} : l \in \omega \rangle, \overline{v}_{\eta, k} : k < k_\eta \rangle \\
\langle \overline{w}_{\eta, k, l}(\overline{t}_{\eta, k, i}, \overline{y}_\eta, \overline{t}) : k < k_\eta, i < \lg(\overline{t}_{\eta, k}) \rangle
$$
Case $\ell = A$: 

$$
\langle \varphi_{n,k}(x, y), \overline{\pi}_{n,k} = \langle \pi_{n,k,t} : t \in \mathbb{A}_{n,k}(\omega) \rangle, \ m_{n,k} : k < k_n \rangle
$$

such that

- $p(\gamma) = 0, \overline{b}(\gamma) = (\gamma), k(\gamma) = 0$.
- $\overline{\pi}_{\eta}$ realizes $p_{\eta}$.
- $p_{\eta}$ is a finite type, $\text{ict}^\ell - \text{rk}^m(p_{\eta}) \geq \min \left(\text{Rang}(\eta) \cup \{\lambda^{+}_\eta\}\right)$ for all $\eta \in \text{ds}(\lambda^{+}_\eta)$.
- $p_{\eta} \vdash \bigvee_{k < k_n} \varphi_{n,k}(x, \overline{b}_{\eta})$.
- For $\eta = \nu^\gamma(\alpha)$:
  - $p_{\eta^\gamma(\alpha)} \supseteq p_{\eta}$
  - $\overline{b}_\nu \prec \overline{b}_{\eta}$.
  - $p_{\eta}$ does $\text{ict}^\ell - \text{lg}(\eta)$-fork over $\overline{b}_\nu$. In particular $\varphi_{n,k}(x, \overline{b}_{\eta})$ does $\text{ict}^\ell - \text{lg}(\eta)$-divide over $\overline{b}_\nu$ for $k < k_{\eta}$. Moreover,
  - Case $\ell = 1$: $\overline{\pi}_{n,k,1}$ is a finite sequence of formulas, and
    $$
    \varphi_{n,k}(x, \overline{b}_{\eta}) \vdash \bigvee_{\psi \in \varphi_{n,k,i}} \left[ \psi(\pi_{n,k,s_i, i}, \overline{b}_{\eta}, x) \equiv -\psi(\pi_{n,k,s_i+1, i}, \overline{b}_{\eta}, x) \right]
    $$
    holds for $i < \text{lg}(\eta) = \text{lg}(\varphi_{n,k})$.
  - Case $\ell = A$: $\overline{\pi}_{n,k,E}$ is a finite sequence of formulas, and
    $$
    \varphi_{n,k}(x, \overline{b}_{\eta}) \vdash \bigvee_{\psi \in \varphi_{n,k,E}} \left[ \psi(\pi_{n,k,s_i, i}, \overline{b}_{\eta}, x) \equiv -\psi(\pi_{n,k,s_i+1, i}, \overline{b}_{\eta}, x) \right]
    $$
    holds for every $E \in \text{ConvEquiv}(m_{n,k}, \text{lg}(\eta))$ for some equivalent sequences $\overline{\pi}_{n,k,E}^0, \overline{\pi}_{n,k,E}^1 \in \mathbb{A}_{n,k}(m_{n,k}, E)$.

Choice of a tree of types with descending ranks. For $\eta = (\gamma)$ - clear. Now let $\eta \in \text{ds}(\lambda^{+}_\ell), \alpha < \min \left(\text{Rang}(\eta) \cup \{\lambda^{+}_\ell\}\right)$, and $p_{\eta}$ a finite rank such that $\text{ict}^\ell - \text{rk}^m(p_{\eta}) \geq \min \left(\text{Rang}(\eta) \cup \{\lambda^{+}_\ell\}\right)$. By the definition of rank and since $p_{\eta}$, $\text{Dom}(p_{\eta})$ are finite, there exists $q \supseteq p_{\eta}$ which does $\text{ict}^\ell - (\text{lg}(\eta) + 1)$-fork over $\text{Dom}(p_{\eta})$ with rank $\geq \alpha$. By the finite character of forking, there exists a finite $p_{\eta^\gamma(\alpha)} \subseteq q$ which does $\text{ict}^\ell - \text{lg}(\eta)$-fork over $\overline{b}_\nu$ extending $p_{\eta}$. On the other hand,

$$
\text{ict}^\ell - \text{rk}^m(p_{\eta^\gamma(\alpha)}) \geq \text{ict}^\ell - \text{rk}^m(\eta) \geq \alpha
$$
holds, since $q \supseteq p_{\eta^\omega}(\alpha)$. By the definition of forking and 11 we get $\langle \varphi_{\eta^\omega}(\alpha), k(\tau, \eta^\omega(\alpha)) : k < k_{\eta^\omega(\alpha)} \rangle$. (We choose w.l.o.g $\overline{b}_{\eta^\omega(\alpha)} > \overline{b}_\eta$) and the witnesses for $\text{ict}' - \lg(\eta)$-dividing of each formula. This completes the iterated choice.

Choosing an infinite sequence. We define for every $\eta \neq ( )$:

Case $\ell = 1$:

\[ \rho_\eta := (k_\eta, \langle \varphi_{\eta,k}(\tau, \eta_\eta), l_{\eta,k}, \overline{\eta}_{\eta,k}, \overline{\varphi}_{\eta,k}(\overline{\eta}_{\eta,k}, \eta_\eta, \tau) : k < k_\eta \rangle) \]

where $l_{\eta,k} = \lg (\overline{\eta}_{\eta,k})$ for all $n \in \omega$.

Case $\ell = A$:

\[ \rho_\eta := (k_\eta, \langle \varphi_{\eta,k}(\tau, \eta_\eta), l_{\eta,k} : \eta_{\eta,k} \rightarrow \omega, m_{\eta,k} : k < k_\eta \rangle) \]

\[ \langle \overline{\eta}_{0,E}, \overline{\eta}_{1,E}, \overline{\varphi}_{\eta,k,E}(\overline{\eta}_{\eta,k,E}, \eta_\eta, \tau) : E \in \text{ConvEquiv}(m_{\eta,k}, \lg(\eta)) \rangle \]

where $l_{\eta,k}$ is a function, mapping to each term $\tau(\eta) \in \eta_{\eta,k}$ the length of $\overline{\eta}_{\eta,k,k,\tau(\eta)}$.

Now, there are at most $\lambda_\ell$ possibilities for the choice of $\rho_\eta$ since:

Case $\ell = 1$: $k_\eta$, $l_{\eta,k}$, $\overline{\eta}_{\eta,k}$, $\lg (\eta_\eta)$, $\lg (\overline{\eta}_{\eta,k,i})$, $\lg (\overline{\varphi}_{\eta,k,i}) < \omega$ and so $\rho_\eta$ has at most $|T|$ possibilities.

Case $\ell = A$: $k_\eta$, $m_{\eta,k} < \omega$. $l_{\eta,k}$ has at most $\sum_{k \in A} \aleph_0^{\aleph_0}$ possibilities and $\overline{\eta}^{0,E}$, $\overline{\eta}^{1,E}$ have at most $\sum_{k \in A} \aleph_0$ possibilities. The formulas contain a finite number of variables, so there are at most $|T|$ possibilities.

So by claim 28 it follows that we can find a sequence $(\rho_j : j < \omega)$ such that for any $j_* < \omega$ there exists $\eta_{j_*} \in \text{ds}(\lambda_{j_*}^\omega)$ and $\rho_{\eta_{j_*}, 1, j} = \rho_j$ holds for all $j \leq j_*$. We denote the chosen objects as follows:

Case $\ell = 1$:

\[ \rho_j := (k_j, \langle \varphi_{j,k}(\tau, \eta_j), l_{j,k}, \overline{\eta}_{j,k}, \overline{\varphi}_{j,k}(\overline{\eta}_{j,k}, \eta_j, \tau) : k < k_j \rangle) \]

Case $\ell = A$:

\[ \rho_j := (k_j, \langle \varphi_{j,k}(\tau, \eta_j), l_{j,k} : \eta_{j,k} \rightarrow \omega, m_{j,k} : k < k_j \rangle) \]

\[ \langle \overline{\eta}_{0,E}^{j,k}, \overline{\eta}_{1,E}^{j,k}, \overline{\varphi}_{j,k,E}(\overline{\eta}_{j,k,E}, \eta_j, \tau) : E \in \text{ConvEquiv}(m_{j,k,E}) \rangle \]
Using compactness to choose a new object. We define a new dictionary $\tau_*$ by adding the constant symbols to $\tau_M$: $\text{lg}\overline{b}_j = \text{lg}\overline{b}_j$, $\text{lg}\overline{r}$ = $\text{lg}\overline{r}$ and also

Case $\ell = 1$: $\text{lg}(\overline{\pi}_{j,k,t}) = l_{j,k}$

$$\tau_* = \tau_M \cup \{\overline{\pi}_{j,k,t} : t \in \omega, k < j, \lambda < \omega\} \cup \{\overline{b}_j : j < \omega\} \cup \overline{r}$$

Case $\ell = A$: $\text{lg}(\overline{\pi}_{j,k,\tau(\overline{r})}) = l_{j,k}(\tau(\overline{r}))$

$$\tau_* = \tau_M \cup \{\overline{\pi}_{j,k,t} : t \in \lambda_{j,k}(\omega), k < j, \lambda < \omega\} \cup \{\overline{b}_j : j < \omega\} \cup \overline{r}$$

We now define families of formulas in $L(\tau_*)$, for every $1 \leq j < \omega$:

$$\Delta^\text{type}_j = \left\{ \bigvee_{k < j} \varphi_{j,k}(\overline{r}, \overline{b}_j) \right\}$$

Case $\ell = 1$:

$$\Delta^\text{div}_j := \cup \left\{ \text{Ind}(\overline{\pi}_{j,k}, \overline{b}_{j-1}) : k < j \right\} \cup \left\{ (\forall \overline{r}) \varphi_{j,k}(\overline{r}, \overline{b}_j) \rightarrow \bigvee_{i < \text{lg}(\overline{b}_{j,k,i})} \left( \psi_{j,k,i}(\overline{\pi}_{j,k,s_{j,k,i}}, \overline{v}_{j-1}, \overline{r}) \equiv \neg \psi_{j,k,i}(\overline{\pi}_{j,k,s_{j,k,i}+1}, \overline{v}_{j-1}, \overline{r}) : E \in \text{ConvEquiv}(m_{j,k,j}) \right) \right\}$$

Case $\ell = A$:

$$\Delta^\text{div}_j := \cup \left\{ \text{Ind}(\overline{\pi}_{j,k}, \overline{b}_{j-1}) : k < j \right\} \cup \left\{ (\forall \overline{r}) \varphi_{j,k}(\overline{r}, \overline{b}_j) \rightarrow \bigvee_{i < \text{lg}(\overline{b}_{j,k,i})} \left( \psi_{j,k,E,i}(\overline{\pi}_{j,k,s_{j,k,i}}, \overline{v}_{j-1}, \overline{r}) \equiv \neg \psi_{j,k,E,i}(\overline{\pi}_{j,k,s_{j,k,i}+1}, \overline{v}_{j-1}, \overline{r}) : E \in \text{ConvEquiv}(m_{j,k,j}) \right) \right\}$$

And define $\Delta_j = \Delta^\text{type}_j \cup \Delta^\text{div}_j$. The collection $\Delta := \bigcup_{j < \omega} \Delta_j$ is consistent with $T$, since for all $j_*$, the assignment

$$\overline{\pi}_{j_*}, \overline{b}_{j_*}, \overline{r}_{j_*} : j_* \mapsto \overline{\pi}_{j,k}, \overline{b}_j, \overline{r} \quad (j \leq j_*)$$

realizes $\bigcup_{j < j_*} \Delta_j$.

Proving the chosen object is a counterexample. finishing the proof. Now, let $\overline{\pi}_{j^*, k^*}, \overline{b}_{j^*} \subseteq \mathcal{C}_T$ realizing $\Delta$ (recall that $\mathcal{C}$ is sufficiently saturated) and work again in $\tau_T$. To complete the proof we note the following:
• $p_0(\overline{x}) = \left\{ k \leq k_j \mid k < k_j \right\}$ is a type in $T$.

• The formula $\varphi_j,k(\overline{x},\overline{y}_j)$ does $\text{ict}^\ell - (\Delta, j)$-divide over $\overline{y}_{j-1}$ for all $k < k_j$, $0 < j < \omega$.

• For $S^m(\bigcup_{j<\omega} \overline{y}_j) \ni p \geq p_0$, $n < \omega$ and finite $A \subseteq \text{Dom}(p)$, there exists $n \leq j < \omega$ such that $A \subseteq \overline{y}_{j-1}$. Since $p$ is complete, $p \vdash \bigvee_{k < k_j} \varphi_j,k(\overline{x},\overline{y}_j)$ and $\text{Dom}(p)$ contains the constants on the right hand, there exists $k < k_j$ such that $p \vdash \varphi_j,k$. Since $\Delta_j^{\text{div}}$ is realized, we get that $\varphi_j,k(\overline{x},\overline{y}_j)$ does $\text{ict}^\ell - j$-divide over $\overline{y}_{j-1}$, and by monotonicity of dividing we get that $\varphi_j,k$ does $\text{ict}^\ell - n$-divide over $A$. Therefore $p$ does also $\text{ict}^\ell - n$ divide over $A$.

\begin{corollary}
\end{corollary}

\begin{corollary}
\end{corollary}

\begin{proof}
\end{proof}

\begin{theorem}
\end{theorem}

\begin{proof}
\end{proof}

3. Equivalent definitions of “strongly dependent $\ell(\mathcal{A})$” using automorphisms

\begin{discussion}
\end{discussion}

\begin{discussion}
\end{discussion}
**Definition 18.** The model $M$ is strongly $\kappa$-saturated if $\text{tp}(\overline{a}, M) = \text{tp}(\overline{b}, M)$, implies that $f(\overline{a}) = \overline{b}$ for some $f \in \text{Aut}(M)$, for all $\overline{a}, \overline{b} \in M$, $\gamma < \kappa$.

**Claim 19.** Let $M$ be strongly $(\kappa + |L_M|)^+$-saturated. Then $\text{Th}(M)$ is strongly independent $^1$ iff for some finite sequence $\overline{a}$ and $\langle \overline{a} \rangle$ is indiscernible over $\{ \overline{a} : i < \omega, \alpha \neq \alpha(*) \}$ but $\pi(\overline{a}, \alpha) \neq \overline{a}$, for all $\pi \in \text{Aut}(M/\overline{a})$, $\alpha < \kappa$.

**Proof.** We use claim 2. Indeed, assume that $\text{Th}(M)$ is not strongly dependent $^1$. Therefore we can find $\overline{a} := \langle \varphi_i(\overline{a}, \overline{b}) : i < \kappa \rangle$ such that the union of the set of formulas in the variables $\langle \overline{a} \rangle$ is indiscernible sequence over $\{ \overline{a} : i < \omega, \alpha \neq \alpha(*) \}$ and $\{ \varphi_\alpha(\overline{a}, \overline{a}, 0) \land \varphi_\alpha(\overline{a}, \overline{a}, 1) : \alpha < \kappa \}$ is consistent, this is a family of formulas in $\kappa$ which is realized in $M$, by saturation. Clearly no elementary map over $\overline{a}$ maps $\overline{a}, 0$ to $\overline{a}, 1$, for any $\alpha < \kappa$. Conversely, if we can find $\langle \overline{a} \rangle$ as above, it clearly follows by the strong saturation that $\text{tp}(\overline{a}, \overline{a}, 0, M) \neq \text{tp}(\overline{a}, \overline{a}, 1, M)$ for all $\alpha < \kappa$.

**Discussion 3.2.** We now turn to strongly dependent $^1, (\overline{a})$. By Theorem 17, being strongly independent $(\overline{a})$ is equivalent to existence of $A, \overline{a}$ such that $\text{tp}(\overline{a}, B, C)$ does $\text{ict}^\ell - n$-divide over $B$ for any finite $B \subseteq A$ , $n < \omega$. From this it follows that finding a characterization by automorphisms for dividing is sufficient.

**Claim 20.** Let $M$ be a strongly $\kappa$-saturated model. For some $\overline{a}, A \in M, |\overline{a}| + |A| < \kappa$ it holds that $\text{tp}(\overline{a}, A, M)$ does $\text{ict}^\ell - n$-divide (ict$^4 - n$-divide) strongly over $B$ if and only if:

- **Case $\ell = 1$:** There exists an indiscernible sequence $\langle \overline{a}_t : t \in \omega \rangle$ over $B$ and a sequence $\overline{a}$ of length $n$ such that $1 \leq s_{i+1} - s_i < 2$ and for all $f \in \text{Aut}(M/A)$, $g \in \text{Aut}(M/B \cup f(\overline{a}))$ and $i < n$, it holds that $g(\overline{a}_i) \neq \overline{a}_{i+1}$.
- **Case $\ell = 2$:** There exists an indiscernible sequence $\langle \overline{a}_t : t \in \omega \rangle$ over $B$ and a sequence $\overline{a}$ of length $n$ such that $1 \leq s_{i+1} - s_i < 2$ and for all $f \in \text{Aut}(M/A)$, $i < n$ and $g \in \text{Aut}(M/B \cup f(\overline{a})) \land g(\overline{a}_i) \neq \overline{a}_{i+1}$.
- **Case $\ell = 3$:** There exists an indiscernible structure $\langle \overline{a}_t : t \in \omega \rangle$ over $B$ and a sequence $\overline{a}$ of length $n$ such that $1 \leq s_{i+1} - s_i < 2$ and for all $f \in \text{Aut}(M/A)$, $i < n - 1$ and $g \in \text{Aut}(M/B \cup f(\overline{a})) \land g(\overline{a}_i) \neq \overline{a}_{i+1}$.
- **Case $\ell = 4$:** There exists an indiscernible structure $\langle \overline{a}_t : t \in \omega \rangle$ over $B$, $m < \omega$ and equivalent sequences $\overline{a}_{E, 0}, \overline{a}_{E, 1} \in \omega$ for all $E \in \text{ConvEquiv}(m, n)$ such that for all $f \in \text{Aut}(M/A)$ and $g \in \text{Aut}(M/B \cup f(\overline{a}))$ it holds that $g(\overline{a}_{E, 0}) \neq \overline{a}_{E, 1}$.
4. Preservation of strongly dependent under sums

**Fact 21.** For a cardinal \( \kappa \), there exist a cardinal \( \mu \) and a ultrafilter \( \mathcal{D} \) on \( \mu \) such that for any model \( M \), the ultrapower \( M^\mu / \mathcal{D} \) is strongly \( \kappa^+ \)-saturated.

**Definition 22.** Let \( M, N \) be models in the same relational dictionary (i.e. no functions or constants) \( \tau \). We define new models \( M \oplus N \) and \( M + N \) as follows

- The universe of \( M \oplus N \) is \( |M| \cup |N| \) (w.l.o.g \(|M \cap N| = 0\)). the dictionary \( \tau \cup \{ L, R \} \) where \( L, R \) are unary relation, interpreting \( S^{M \oplus N} = S^M \cup S^N \) for every relation \( S \in \tau \), and \( L^{M \oplus N} = |M|, R^{M \oplus N} = |N| \).
- \( M + N = M \oplus N \restriction \tau \)

**Claim 23.** For \( \mathcal{D} \) an ultrafilter on \( I \) it holds that \( (M \oplus N)^I / \mathcal{D} \simeq M^I / \mathcal{D} \oplus N^I / \mathcal{D} \)

**Theorem 24.** Let \( M_1, M_2 \) models in a relational dictionary \( \tau \). If \( \text{Th}(M_1), \text{Th}(M_2) \) are strongly dependent, then \( \text{Th}(M_1 \oplus M_2) \) is also strongly dependent.

**Proof.** By Claim 23 and 21 it follows that w.l.o.g \( M_1, M_2, M_1 \oplus M_2 \) are strongly \( \kappa^+ \)-saturated. By Claim 19 there exist \( (\overline{\tau}_{\alpha,i} : \alpha < \kappa, j < \omega) \) witnessing \( \kappa^{\text{ict}}(\text{Th}(M_1 \oplus M_2)) > \kappa \). W.l.o.g \( \overline{\tau} = \overline{\tau}^1 \setminus \overline{\tau}^2 \), \( \overline{\tau}_{\alpha,j} = \overline{\tau}^1_{\alpha,j} \setminus \overline{\tau}^2_{\alpha,j} \) such that \( \overline{\tau}^1_{\alpha,j} \in M_i \). Recall that \( \langle \overline{\tau}^1_{\alpha(j)}, j < \omega \rangle \) is an indiscernible sequence over \( \langle \overline{\tau}^1_{\alpha(j)} : j < \omega, \alpha \neq \alpha(\ast) \rangle \) for \( \alpha(\ast) < \kappa \), therefore \( \langle \overline{\tau}^1_{\alpha(j)}, j < \omega \rangle \) is indiscernible over \( \overline{\tau} \). Also, \( f \in \text{Aut}(M_1 \oplus M_2) \iff \) there exist \( f_i \in \text{Aut}(M_i) \) such that \( f = f_1 \cup f_2 \) (as functions). Therefore, for some \( i = 1, 2 \) and unbounded \( S \subseteq \kappa \) it holds for all \( \alpha \in S \) and for all \( f_i \in \text{Aut}(M_i/\overline{\tau}) \) that \( f_i(\overline{\tau}_{\alpha,0}) \neq \overline{\tau}_{\alpha,1} \). By Claim 19 it follows that the sequences \( \langle \overline{\tau}^1_{\alpha(j)} : j < \omega, \alpha \in S \rangle \) are witnesses for \( \kappa^{\text{ict}}(M_i) > \text{otp}(S) = \kappa \).

**Theorem 25.** (Case \( \ell = 1, 2, 3 \)) \( \text{Th}(M^1 \oplus M^2) \) is strongly dependent if \( \text{Th}(M^1), \text{Th}(M^2) \) are strongly dependent.

**Proof.** Only if\' direction - assume w.l.o.g that \( \text{Th}(M^1) \) is not strongly dependent. By Lemma 15 there exist \( \overline{\tau} \in M^1 \) and a set \( A \subseteq M^1 \) such that \( \text{tp}(\overline{\tau}, A, M^1) \) does not divide over \( B \) for any finite \( B \subseteq A \) and \( n < \omega \). This easily implies that \( \text{tp}(\overline{\tau}, A, M^1 \oplus M^2) \) does not divide over \( B \) for any finite \( B \subseteq A \) and \( n < \omega \), and so \( \text{Th}(M^1 \oplus M^2) \) is not strongly dependent.

If\' direction - By 15, there exist \( \overline{\tau} \in M^1 \) and sets \( A^i \subseteq M^i \) such that \( \text{tp}(\overline{\tau}^i, A^1 \cup A^2, M^1 \oplus M^2) \) does not divide over \( B^1 \) for any finite \( B^1 \subseteq A^1 \) and \( n < \omega \). If \( \text{tp}(\overline{\tau}, A^1, M^1) \) does not divide over \( B^1 \) for all \( n < \omega \) and finite \( B^1 \subseteq A^1 \) this concludes the proof. Otherwise, there exist \( n_0 < \omega \) and finite \( B^1 \subseteq A^1 \) such that \( \text{tp}(\overline{\tau}, A^1, M^1) \) does not divide over \( B^1 \). Since for all finite \( B^2 \subseteq A^2, n > n_0 \) it holds that \( \text{tp}(\overline{\tau}^2, A^1 \cup A^2, M^1 \oplus M^2) \) does not divide over \( B^1 \cup B^2 \), we
get by claim 27 that \( \text{tp}(\bar{\pi}, A^2, M^2) \) does necessarily \(\text{ict}^\ell - n \) divide over \( B^2 \). Thus, once again by 15, \( \text{Th}(M^2) \) is not strongly dependent \( \ell \).

\[ \square \]

**Fact 26.** \( M \oplus N \equiv M' \oplus N' \) for models \( M \equiv M', N \equiv N' \).

**Claim 27.** (Cases \( \ell = 1, 2, 3 \)) Let \( \bar{\pi}^i, A^i, B^i \subseteq |M^i|, \ (i \in \{1, 2\}) \), then \( \text{tp}(\bar{\pi}^1 - \bar{\pi}^2, A^1 \cup A^2, M^1 \oplus M^2) \) does \(\text{ict}^\ell - 2n \)-divide over \( B^1 \cup B^2 \) if \( \text{tp}(\bar{\pi}^i, A^i, M^i) \) does \(\text{ict}^\ell - n \)-divide over \( B^i \), for some \( i \in \{1, 2\} \).

**Proof.** The proof for all the cases is analogous and the “if” direction is easy so we only give here the “only if” case of case \( \ell = 1 \): w.l.o.g \( M^1, M^2, M^1 \oplus M^2 \) are strongly \( \kappa^+ \)-saturated and \( |A^1 \cup A^2| \leq \kappa \). By 20 we can find \( \langle \bar{\pi}^i_\ell : t \in \omega \rangle \), an indiscernible sequence over \( B^1 \cup B^2 \) and a sequence \( \bar{\pi} \) of length \( 2n \) such that \( 1 \leq s_{j+1} - s_j \leq 2 \) for all \( j < 2n \) and that \( g(\bar{\pi}^i_{s_j} - \bar{\pi}^i_{s_j+1}) \neq \bar{\pi}^i_{s_{j+1}} - \bar{\pi}^i_{s_{j+1}+1} \) holds for all \( f \in \text{Aut}(M^1 \oplus M^2/A^1 \cup A^2) \), \( g \in \text{Aut}(M^1/B^1 \cup B^2 \cup f(\bar{\pi}^1 \cup \bar{\pi}^2)) \) and \( j < 2n \).

Now, assume towards contradiction that \( f_1 \in \text{Aut}(M^1/A^1) \ (i = 1, 2) \) and that \( g_i \in \text{Aut}(M^i/B^i \cup f(\bar{\pi}^i)) \) are such that \( g_i(\bar{\pi}^i_{s_j}) = \bar{\pi}^i_{s_{j+1}} \) holds for some \( j < 2n \). By the bijection \( \Phi : \text{Aut}(M^1) \times \text{Aut}(M^2) \to \text{Aut}(M^1 \oplus M^2) \), we get that \( f = f_1 \cup f_2 \in \text{Aut}(M^1 \oplus M^2/A^1 \cup A^2) \) and that \( g = g_1 \cup g_2 \in \text{Aut}(M^1/B^1 \cup B^2 \cup f(\bar{\pi}^1 \cup \bar{\pi}^2)) \) - a contradiction. Thus, for all \( j < 2n \) there exists \( i \in \{1, 2\} \) such that \( g(\bar{\pi}^i_{s_j}) \neq \bar{\pi}^i_{s_{j+1}} \) holds for all \( f \in \text{Aut}(M^i/A^i), \ g \in \text{Aut}(M^i/B^i \cup f(\bar{\pi}^i)) \). Denote by \( i(j) \), the appropriate \( i \) for every \( j < 2n \). Let \( i_0 \in \{1, 2\} \) be such that \( S_{i_0} = \{ i(j) = i_0 : j < 2n \} \) has at least \( n \) elements. It now follows easily from 20 that \( \langle \bar{\pi}^n_\ell : t \in \omega \rangle \) are witnessing that \( \text{tp}(\bar{\pi}^n_\ell, A^n, M^n) \) does \(\text{ict}^\ell - n \)-divide over \( B^n \). \( \square \)

### 5. Appendix - Various Claims.

**Claim 28.** Let \( \kappa \) be a cardinal, \( f : \text{ds}(\kappa^+) \to \kappa \). We can find a sequence \( \langle \alpha_k : k < \omega \rangle \subseteq \kappa \) such that for every \( k_0 < \omega \) there exists \( \eta \in \text{ds}(\kappa^+) \) of length \( k_0 \) such that \( f(\eta \upharpoonright k) = \alpha_k \) holds for all \( k < k_0 \).

**Corollary 29.** If \( M \) is \( \kappa \)-homogeneous and \( \kappa \)-saturated, and \( I' \supseteq I \) are linear orders such that \( |I'| < \kappa, A \subseteq M, |A| < \kappa \) then:

1. Every \( \langle \bar{\pi}_t : t \in I \rangle \in \text{Ind}(\text{for}, A, M) \) can be extended to \( \langle \bar{\pi}_t : t \in I' \rangle \in \text{Ind}(\text{for}, A, M) \).
2. Every \( \langle \bar{\pi}_t : t \in I \cup \eta I \rangle \in \text{Ind}(\text{for} + \text{ort}(\lessgtr n), A, M) \) can be extended to \( \langle \bar{\pi}_t : t \in I' \cup \eta I' \rangle \in \text{Ind}(\text{for} + \text{ort}(\lessgtr n), A, M) \).
3. Every \( \langle \bar{\pi}_t : t \in \lessm I \rangle \in \text{Ind}(\text{for}(\lessm n), A, M) \) can be extended to \( \langle \bar{\pi}_t : t \in \lessm I' \rangle \in \text{Ind}(\text{for}(\lessm n), A, M) \).
4. Every structure \( \langle \bar{\pi}_t : t \in \text{A}(I) \rangle \) indiscernible over \( A \) can be extended to \( \langle \bar{\pi}_t : t \in \text{A}(I') \rangle \) also indiscernible over \( A \).
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