INNER PRODUCT SPACE WITH NO ORTHO-NORMAL BASIS
WITHOUT CHOICE

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Abstract. We prove in ZF that there is an inner product space, in fact, nicely definable with no orthonormal basis.

§ 1

The theorem below is known in ZFC, but probably not in ZF; really we use the simple black box (see [Sh:309]).

Theorem 1.1. (ZF) There is an inner-product space V over \( \mathbb{R} \) with no orthonormal basis.

Remark 1.2. In fact, nicely definable one, here - Borel.

Proof. Stage A:
Let \( V_1 \) be the Hilbert space over \( \mathbb{R} \) with orthonormal basis \( \{ x_\eta : \eta \in \omega^\omega \} \), so an element \( x \) has a unique representation as \( x = \sum a_{x,\eta} x_\eta \) with \( a_{x,\eta} \in \mathbb{R} \) and norm \( < \infty \) so \( \text{supp}_1(x) := \{ \eta : a_{x,\eta} \neq 0 \} \) is countable and \( \text{supp}_2(x) := \{ \eta : |a_{\eta}| \geq \frac{1}{\eta+1} \} \) finite for every \( k < \omega \) where the norm is \( \Sigma \{ a_{x,\eta}^2 : \eta \in \omega^\omega \} \). The inner product is \( \langle \sum a_{x,\eta} x_\eta, \sum a'_{x,\eta} x_\eta \rangle = \sum a_{\eta} a'_{\eta} : \eta \in \omega^\omega \rangle \in \mathbb{R} \).

For \( \eta \in \omega^\omega \) let \( y_\eta = x_\eta + \sum_{n<\omega} \frac{1}{n} x_\eta[n] \).

Let \( V \) be the subspace of \( V_1 \) generated by \( \{ x_\eta : \eta \in \omega^\omega \} \cup \{ y_\eta : \eta \in \omega^\omega \} \) so as a vector space it is \( \bigoplus_{\eta \in \omega^\omega} \mathbb{R} x_\eta \oplus \bigoplus_{\eta \in \omega^\omega} \mathbb{R} y_\eta \) and it “inherits” the inner product from \( V_1 \).

Toward contradiction assume that \( \{ z_s : s \in S \} \) is an ortho-normal basis of \( V \). So every \( x \in V \) has the unique representation \( \sum b_{x,s} z_s \), where \( b_{x,n} \in \mathbb{R} \) and for \( k \in [1, \omega) \) and \( x \in V \) let \( \text{supp}_2(x) := \{ s \in S : |b_{x,s}| \geq \frac{1}{2^{k+1}} \} \), so finite and \( \text{supp}_3(x) := \{ s \in S : b_{x,s} \neq 0 \} \) so countable.

Stage B:
We choose \( \eta_n \) by induction on \( n \) such that:

\[ \begin{align*}
& (a) \quad \eta_1 \in \omega^\omega \\
& (b) \quad \eta_m = \eta_n | m < n \end{align*} \]

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(c) if \( n = m + 1 \) then \( \eta_n = \eta_m ^{\prec} (i) \) with \( i < \omega \) minimal such that:
if \( \ell \leq m, s \in \text{supp}^2_m (x_{\eta_\ell}) \) and \( \nu \in \text{supp}^1_m (z_s) \subseteq \omega^\omega \) then \( \neg (\eta_m ^{\prec} (i) \subseteq \nu) \).

This is well defined as in clause (c), \( \text{supp}^2_m (x_{\eta_\ell}) \) is a finite subset of \( S \) and for each \( s \in \text{supp}^2_m (x_{\eta_\ell}) \), the set \( \text{supp}^1_m (z_s) \) is a finite subset of \( \omega^\omega \).

Lastly, let

- (a) \( \eta_\omega := \cup \{ \eta_n : n < \omega \} \in \omega^\omega \)
- (b) \( S_1 = \cup \{ \text{supp}_2 (x_\rho) : \rho \prec \eta_\omega \} \)
- (c) \( S_2 = S \setminus S_1 \)
- (d) \( X_\ell \) is the closure inside \( V \) of \( \oplus \{ R z_s : s \in S_\ell \} \) for \( \ell = 1, 2 \)
- (e) \( S_{1,n} := \cup \{ \text{supp}_2^m (x_{\eta_n}) : m, \ell \leq n \} \).

Note

- \( \boxplus_3 V = X_1 \oplus X_2 \), i.e. \( X_1, X_2 \) are orthogonal but \( X_1 + X_2 \) is \( V \)
- \( \boxplus_4 S_1 = \cup \{ \text{supp}_2^m (x_{\eta_n}) : n < \omega, m < n \} = \cup \{ S_{1,n} : n < \omega \} \)
- \( \boxplus_5 \eta_n \in S_1 \) for \( n < \omega \).

Stage C: As \( y_{\eta_n} \in V \) see Stage A and \( \boxplus_2 (a) \) of Stage B, recalling \( \boxplus_3 \)

- \( \odot_1 \) there are \( y^1 \in X_1, y^2 \in X_2 \) such that \( y_{\eta_n} = y^1 + y^2 \).

Also

- \( \odot_2 \{ \rho : \eta_{n+1} \subseteq \rho \in \omega^2 \omega \} \) is disjoint to \( \cup \{ \text{supp}_1^m (z_s) : s \in S_{1,n} \} \) for every \( n < \omega \).

[Why? By the choice of \( \eta_{n+1} \) in \( \boxplus_1 (c) \).]

- \( \odot_3 \eta_n \notin \text{supp}_1^1 (z_s) = \cup \{ \text{supp}_1^m (z_s) : m < \omega \} \) for every \( s \in S_1 \).

[Why? The \( \notin \) by \( \odot_2 \).]

Hence by \( \boxplus_3 \)

- \( \odot_4 \) if \( s \in S_1 \) then \( y_{\eta_s}, z_s \) are orthogonal (in \( V_1 \)).

But

- \( \odot_5 (y_{\eta_n}, x_{\eta_n}) = \frac{1}{2} \).

[Why? By the choice of \( y_{\eta_n} \) is stage N.]

By \( \boxplus_5 + \odot_4 + \odot_5 \) we get contradiction. \( \square_{1,1} \)

References


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